Introduction to p-adic q-difference equations of rank 1

Pierre Houedry

Abstract

We present the article *q-difference equations and p-adic local monodromy* written by Lucia Di Vizio and Yves André. We only expose the first part of the article which aims to introduce a p-adic theory of q-difference equations of rank 1. This report was achieved under the supervision of Bernard Le Stum whom I thank for his kindness and his advice.

Contents

2010 Mathematics Subject Classification 39A13,12H25

1. Introduction

The study of q-differences and q-analogues started with the work of Euler, in the 18th century, on combinatorial problems in number theory. The q-analogues then appear in the work of Heine, Jacobi, Poincaré, Picard, Ramanujan and Birkhoff to name only famous mathematicians that have been interested in this concept. However, this topic has been forgotten until 30 years ago. For various reasons, q-analogues now appear in various emerging research topics such as arithmetic geometry (in the work of Scholze for example), quantum groups, combinatorics and even physics. Let us try in this introduction to give a naive idea of how q-analogues arise in mathematics. Let $n \in \mathbb{N}$ we define

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + ... + q^{n-1}.$$

Thinking of q as variable it is easily seen that $[n]_q \to n$ when taking the limit as $q \to 1$. We can also define the q-factorial

$$[n]_q^! = \prod_{i=1}^{n} [i]_q.$$

and the q-binomial coefficient

$$\binom{n}{k}_q = \frac{[n]_q^!}{[k]_q^! [n-k]_q^!}.$$

When q is a power of a prime we have that $\binom{n}{k}_q$ is the number of k-dimensional subvectorspaces of $\mathbb{F}_q^n$. Thus it seems that q-analogues have good arithmetic properties but they can also be useful for analytical problems. Let f be a "good" function such that we can define

$$d_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

We have $d_q \to \frac{d}{dx}$ as $q \to 1$. Moreover, it satisfies a twisted Leibniz formula

$$d_q(fg)(x) = f(qx) d_q(g)(x) + d_q(f)(x) g(x).$$

Many mathematical objects have a q-analogue such that we recover the usual object by taking the limit as $q \to 1$. On the other side, we are interested in p-adic differential equations. The study of p-adic differential equations was motivated by the study of de Rham cohomology over certain varieties and the study of zeta functions of varieties over finite fields. This research topic started with mathematicians such as Tate, Dwork and Robba. This is still an active field of research nowadays. In this report, we study an article that brings together the q-analogues and the p-adic differential equations. We give below a description of the set-up in which we work.

1.1. Let K be a field of characteristic zero complete with respect to a non archimedean absolute value $| \ |$, with residue field k of characteristic $p > 0$. Note that it implies that $| \ |_{\mathbb{Q}}$ is the p-adic absolute value by the Ostrowski theorem ([Rob00, Section 1.2.1]). We assume that the absolute value is normalized by $|p| = p^{-1}$. For any interval $I \subset \mathbb{R}_{\geqslant 0}$ we consider the K-algebra $\mathcal{A}_K(I)$ of analytic function in K, on the annulus $\mathcal{C}_K(I) = \{x \in K \ : \ |x| \in I\}$,

$$\mathcal{A}_K(I) = \Big\{ \sum_{n \in \mathbb{Z}} a_n x^n \ : \ a_n \in K, \ \lim_{n \to \pm\infty} |a_n| \rho^n = 0 \ \forall \rho \in I \Big\}.$$

We denote by $\mathcal{M}_K(I)$ it's field of fractions and by $\mathcal{B}_K(I)$ the subring of bounded elements of $\mathcal{A}_K(I)$. From now on, we denote by q an element of K such that $|q - 1| < 1$ and that q is not a root of unity. We choose such a q because we don't want to have $[n]_q = 0$ for a given integer n. It also implies that $|q| = 1$ and that q is "small". We have a K-algebra isomorphism

$$\begin{aligned} \sigma_q \ : \ \mathcal{A}(I) &\to \mathcal{A}(I) \\ f(x) &\mapsto f(qx) \end{aligned}$$

We will mainly be interested by *q-difference equation of rank 1* with coefficient in $\mathcal{M}(I)$:

$$y(qx) = a(x)y(x), \ a(x) \in \mathcal{M}(I).$$

We shall write the above expression as

$$d_q(y)(x) = g(x)y(x), \text{ with } g(x) = \frac{a(x)-1}{(q-1)x}.$$

From this form, it arise a sequence of equations

$$d_q^n(y)(x) = g_n(x)y(x),$$

if $g(x)$ is analytic at 0, then one can check that $\sum_{n\geqslant 0} \frac{g_n(0)}{[n]_q^!} x^n$ is a formal solution of $y(ax) = a(x)y(x)$. It automatically brings the problem of convergence of a formal solution. For any $u \in \mathcal{M}(I)^*$, we say that

$$y(qx) = u(qx)^{-1}a(x)u(x)y(x)$$

is *$\mathcal{M}(I)$-equivalent* to

$$y(qx) = a(x)y(x).$$

The algebraic closure of $\mathbb{Q}_p$ being not complete, we often have to consider its completion $\mathbb{C}_p$. However, it is not sufficient to obtain a p-adic q-analog of the Hahn-Banach theorem. Thus we need to enlarge $\overline{\mathbb{Q}_p}$ not just by completion. This give rise to the following construction. For more details we refer the reader to [Rob00, Chapter 3].

1.2. We will consider an extension of normed fields Ω/K satisfying the following properties:

(i) the field Ω is complete and algebraically closed;
(ii) we have $|\Omega| = \mathbb{R}_{\geqslant 0}$;
(iii) the residue field of Ω is transcendental extension of k;
(iv) for any $\rho \in \mathbb{R}_{\geqslant 0}$ the field Ω contains an element t_ρ such that t_ρ is transcendant over K and $|t_\rho| = \rho$.

The following definition is a tool to measure the radius of convergence of a formal solution in a disc of radius ρ around every point of norm equal to ρ. We can restrict to only look at the radius at the generic point t_ρ because it is invariant at every point of norm ρ.

Definition 1.3. For any $\rho \in I$, we call *generic radius of convergence* of $y(qx) = a(x)y(x)$ at t_ρ the number

$$R_\rho(\sigma_q - a(x)) = \inf\left(\rho, \liminf_{n\to+\infty} \left|\frac{g_n(t_\rho)}{[n]_q^!}\right|^{-1/n}\right).$$

Here are some elementary properties satisfied between solution of a q-difference equation and its radius of convergence.

Proposition 1.4. *([DV04])*

(i) *(**Twisted Taylor expansion** Let $d_q y(x) = g(x)y(x)$ be a q-difference equation with coefficient $g(x) \in \mathcal{M}(I)$ and let $\xi \in \mathcal{C}(I)$. Suppose that $g(x)$ does not have any pole in $q^{\mathbb{N}}\xi$. Then $d_q y(x) = g(x)y(x)$ has an analytic solution in a neighborhood of ξ if an only if*

$$R = \liminf_{n\to\infty} \left| \frac{g_n(\xi)}{[n]_q^!} \right|^{-1/n} > |(q-1)\xi|.$$

In that case the unique analytic solution of $y(qx) = a(x)y(x)$ in the open disc $D(\xi, R^-)$ verifying $y(\xi) = 1$ coincides with

$$\sum_{n\geqslant 0} \frac{g_n(\xi)}{[n]_q^!}(x,\xi)_{n,q} \text{ where } (x,\xi)_{n,q} = \prod_{i=0}^{n-1}(x - q^i\xi).$$

(ii) *Let $b(x) = u(qx)^{-1}a(x)u(x)$ with $u(x) \in \mathcal{M}(I)$. Then $R_\rho(\sigma_q - a(x)) = R_\rho(\sigma_q - b(x))$.*

(iii) *(q-**analog of the Dwork-Robba effective bound**) If $R_\rho > |q-1|\rho$, then*

$$\left| \frac{g_n(t_\rho)}{[n]_q^!} \right| \leqslant \frac{1}{R_\rho^n}, \text{ for any } n \geqslant 1.$$

(iv) *(**Transfert to an ordinary disk**) Let $g(x)$ be analytic over $D(\xi, \rho^-)$, with $\xi \in K$ and $|\xi| \leqslant \rho$, and let $R_\rho > |q-1|\rho$. Then $d_q y(x) = g(x)y(x)$ has an analytic solution over the disc $D(\xi, R_\rho^-)$.*

(v) *(**Transfer to a regular singular disk**) Let $a(x) \in \mathcal{A}(]0,1[)$ and $u(x)$ be a formal power series with coefficients in K such that $u(qx)^{-1}a(x)u(x) \in K$. If $R_\rho = \rho$, the series converges for $|x| < \rho$.*

Corollary 1.5. *Let $a(x) \in \mathcal{A}([0,1[)$ (resp. $\mathcal{A}([0,1[) \cap \mathcal{M}([0,1])$). Then $y(qx) = a(x)y(x)$ has a solution $y(x)$ analytic and bounded over $\mathcal{C}([0,1[)$ if and only if $\lim_{\rho\to 1} R_\rho = 1$ (resp. $R_1 = 1$).*

1.6. As we are studying q-difference equations it seems to be important to define a q-analogue of the exponential

$$e_q(x) = \sum_{n\geqslant 0} \frac{x^n}{[n]_q^!}.$$

We can check that its still verify the usual property of the exponential $d_q e_q = e_q$. Unfortunately, the multiplicative identity of the exponential does not hold in the q-analog setting. After defining the exponential, it make sense to define a q-analog of the logarithm. We know that $\log(1+x)$ converges for $|x| < 1$. For $x \in K$, if $|e_q(x) - 1| < 1$ we consider the analytic function $L_q(x) = \log(e_q(x))$ which by identification (cf. [HL46], [Que04]) has the following expansion

$$L_q(x) = \sum_{n\geqslant 1} \frac{(-1)^{n-1}(q-1)^{n-1}}{[n]_q n} x^n.$$

Proposition 1.7. (i) *The series $L_q(x)$ converges for $|x| < |q-1|^{-1}$.*

(ii) *If $|x| < \frac{|p|^{\frac{1}{p-1}}}{|q-1|} \sup\left(|p|^{\frac{1}{p-1}}, |q-1|\right)$, then $|L_q(x)| < |x|$.*

1.8. In what follows, we assume that K contains the p-th roots of unity. Applying the Krasner lemma ([Rob00]), it then also contains $p-1$ distinct non zero roots of the equation $X^p + pX = 0$. We will pick one of them and denote it by π. We have $|\pi| = |p|^{\frac{1}{p-1}}$.

The following proposition expresses the importance of the q-exponential to construct solutions to q-difference equations.

LEMMA 1.9. *Let $y(qx) = a(x)y(x)$ be a q-difference equation such that $a(x)$ is an analytic function at 0, with $a(0) = 1$. Then write $a(x)$ as an infinite product $\prod_{i\geqslant 1}(1+\mu_i x^i)$. If there exist $\varepsilon > 0$ such that*

$$\sup_{i\geqslant 1} \frac{|\mu_i|}{|q^i - 1|}(1+\varepsilon)^i < |\pi|$$

then the infinite product

$$\prod_{i\geqslant 1} e_{q^i}\left(\frac{\mu_i}{q^i - 1}x^i\right)$$

converges to an overconvergent solution of $y(qx) = a(x)y(x)$ i.e. it has a radius of convergence greater than 1.

COROLLARY 1.10. *Let $y(qx) = a(x)y(x)$ be a q-difference equation such that $a(x) = \prod_{\geqslant 1}(1+\mu_i x^i)$ is an overcongergent analytic function at 0. Then there exists a positive integer M and a positive real number ε such that $y(qx) = a(x)y(x)$ is $\mathcal{M}([0, 1+\varepsilon[)$-equivalent to*

$$y(qx) = \prod_{i=1}^{M}(1+\mu_i x^i)y(x).$$

2. Solvability

2.1. We introduce the following ring

$$\mathcal{E}^\dagger = \cup_{\varepsilon<1}\mathcal{B}(]1-\varepsilon, 1[)$$

endowed with the sup-norm $|\sum a_n x^n|_{\mathcal{E}^\dagger} = \sup |a_n|$. If the valuation of K is discrete, this is a Henselian field with residue field $k((X))$. We will be interested by q-difference equations of the form $y(qx) = a(x)y(x)$ with $a(x) \in \mathcal{E}^\dagger$. The interest of $\mathcal{E}^\dagger$ is that it represents the problems that occur at the boundary which are the kind of problem we are interested in cohomology. Moreover, it's residue field being $k((X))$ we can think of it as a field that lifts the problems that appear in characteristic p.

The next definition yields information on the behaviour of the radius of convergence of a q-difference equation as we get closer to the boundary.

DEFINITION 2.2. The equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, is said to be *solvable* if

$$\lim_{\rho\to 1} R_\rho(\sigma_q - a(x)) = 1.$$

The point of the next lemma is that it makes it easier to determine the solvability of a q-difference equation. The sup-norm on $\mathcal{E}^\dagger$ being easier to manipulate.

LEMMA 2.3. $\lim_{\rho\to 1} R_\rho = \inf\left(1, \liminf_{n\to\infty}\left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger}^{-1/n}\right)$.

2.4. Solvability is invariant under $\mathcal{E}^\dagger$ equivalence. Indeed if $y(x)$ is a solution of $y(qx) = a(x)y(x)$ then $z(x) = \frac{y(x)}{u(x)}$ is a solution of $z(x) = a(x)u(x)u(qx)^{-1}z(x)$ where $u(x) \in \mathcal{E}^\dagger$. Hence, $z(x)$ will have a radius of convergence equals to the radius of convergence of $y(x)$ as $u(x)$ has a radius of convergence equals to 1. It thus makes sense to look at rank 1 q-difference equations under $\mathcal{E}^\dagger$-equivalence.

The following proposition is a q-analog of a result by Dwork and Robba. It gives a characterization of solvability. It is mainly used in the article to prove the proposition 2.8. It also allows us to describe the form of solvable q-difference equations with constant coefficient.

Proposition 2.5. *The following assertions are equivalent:*

(i) *The q-difference equation $y(qx) = a(x)y(x)$, $a(x) \in \mathcal{E}^\dagger$, is solvable.*

(ii) *There exists a sequence $R_n(x) \in \mathcal{E}^\dagger_\Omega$, such that*

$$\lim_{n\to\infty} \left| \frac{R_n(qx)}{R_n(x)} - a(x) \right|_{\mathcal{E}^\dagger_\Omega} = 0.$$

(iii) *There exists a sequence $R_n(x) \in \mathcal{E}^\dagger_\Omega$, such that*

$$\lim_{n\to\infty} \left| \frac{d_q(R_n)(x)}{R_n(x)} - g(x) \right|_{\mathcal{E}^\dagger_\Omega} = 0, \text{ where } g(x) = \frac{a(x)-1}{(q-1)x}.$$

Corollary 2.6. *The q-difference equation $y(qx) = ay(x)$ with constant coefficient $a \in K$, is solvable if and only if $a \in q^{\mathbb{Z}_p}$.*

The following corollary deals with the rank 1 q-difference equation whose coefficient is analytic in $\mathcal{C}(]0,1[)$ and has a pole in 0.

Corollary 2.7. *Consider a solvable q-difference equation $y(qx) = a(x)y(x)$, with $x^N a(x) \in \mathcal{B}_K$ for some positive integer N. Let $a_\infty(x) \in K(x)$ be a rational function such that all the finite zeros and poles of $a_\infty(x)$ are in $\mathcal{C}([0,1[)$, and that $\frac{a(x)}{a_\infty(x)}$ is an invertible analytic function in $\mathcal{B}_K$ having value 1 at 0. Then the q-difference equations $y(qx) = a_\infty(x)y(x)$ and $y(qx) = \frac{a(x)}{a_\infty(x)}y(x)$ are both solvable.*

Proposition 2.8. *Any solvable q-difference equation $y(qx) = a(x)y(x)$ with $a(x) \in \mathcal{E}^\dagger$, is $\mathcal{E}^\dagger$-equivalent to a solvable q-difference equation of the form*

$$y(qx) = q^{l_0} \prod_{i=1}^{M} (1 + \frac{\mu_i}{x^i}) y(x),$$

where $l_0 \in \mathbb{Z}_p$, M is a positive integer, $\mu_i \in K$ and $|\mu_i| \leqslant |q-1|$ for $i = 1, ..., M$.

This proposition may be summarized by saying that we can reduce the study of rank 1 q-difference equations with coefficient in $\mathcal{E}^\dagger$ to rank 1 q-difference equations with polynomial coefficient in $\frac{1}{x}$.

3. Frobenius structure

Let us observe the following situation. If we take a power series $\sum_{n\geqslant 0} a_n x^n$ such that its radius of convergence is equal to $R < 1$. Then, $\sum_{n\geqslant 0} a_n x^{np}$ has a radius of convergence equal to $R^{1/p} > R$. Thus, if we compose such a power series with the Frobenius it allows us to obtain a power series with an enlarged radius of convergence. Because of this kind of heuristic it make sense to study how does the Frobenius interact with q-difference equations. In what follows we will look in more details which q-difference equations are $\mathcal{E}^\dagger$-equivalent to a a deformation of themself under a Frobenius action.

3.1. In the following we will consider a Frobenius automorphism τ of K that lifts the Frobenius of the residue field k. Assume there exists a positive integer s such that q is τ^s-invariant. We consider the endomorphism ϕ of $\mathcal{E}^\dagger$ defined by

$$\phi\left(\sum_{n\in\mathbb{Z}} a_n x^n\right) = \sum_{n\in\mathbb{Z}} \tau^s(a_n) x^{p^s n}.$$

Definition 3.2. We say that a q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, has a *(strong) Frobenius structure* if there exists $u(x) \in (\mathcal{E}^\dagger)^*$ such that

$$\frac{u(qx)}{u(x)} a(x) = \phi(a(x))\phi(a(qx))...\phi(a(q^{p^s-1}x))$$

for a suitable choice of s.

Lemma 3.3. *If q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, has a Frobenius structure then it is solvable.*

3.4. For any $a(x) \in \mathcal{E}^\dagger$ there exist an unique multiplicative decomposition

$$a(x) = \frac{\lambda}{x^N} l(x) m(x),$$

where $\lambda \in K$, $\lambda \neq 0$; $N \in \mathbb{Z}$; $l(x)$ is an invertible function in $1 + x\mathcal{B}$; $m(x)$ is an invertible function in $1 + \frac{1}{x}\left(\cup_{\varepsilon>0}\mathcal{A}(]1-\varepsilon, \infty[)\right)$. This can be found in [Chr11].

The next statements are constructed to prove the theorem 3.5. The theorem aims to give the structure of q-difference equations admitting a Frobenius structure.

Theorem 3.5. *A q-difference equation of rank 1 with coefficient in $\mathcal{E}^\dagger$, i.e.*

$$y(qx) = \frac{\lambda}{x^N} l(x) m(x) y(x),$$

has a Frobenius structure if and only if there exists a positive integer s such that $\lambda^{p^s-1} \in q^{\mathbb{Z}}$ and it is solvable.

From the previous lemma we know that if a q-difference equation has a Frobenius structure then it is solvable. The proposition 2.8 gives us the form of a solvable q-difference equation. Hence, it is enough to prove:

Proposition 3.6. *A q-difference equation*

$$y(qx) = q^{l_0} \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x), \tag{1}$$

with $m_0 \in \mathbb{Z}_p$ and $\mu_1, ..., \mu_M \in K$, has a Frobenius structure if and only if there exists an integer $s \geqslant 0$ such that $l_0 \in \frac{\mathbb{Z}}{p^s-1}$ and it is solvable.

Lemma 3.7. *A q-diffeerence equation $y(qx) = q^{l_0} y(x)$, with $l_0 \in \mathbb{Z}_p$ has a Frobenius structure if and only if $l_0 \in \frac{\mathbb{Z}}{p^s-1}$.*

We can reduce the case of the previous proposition to the following result where we omit the term q^{l_0}. Indeed, suppose that (1) has a Frobenius structure. It implies that it is solvable. Since, $l_0 \in \mathbb{Z}_p$ the lemma 2.6 ensures that $y(qx) = q^{l_0}y(x)$ is solvable. This implies that

$$y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x),$$

is solvable. Reciprocally, if $l_0 \in \frac{\mathbb{Z}}{p^s-1}$ and (1) is solvable than by the previous lemma $y(qx) = q^{l_0}y(x)$ has a Frobenius structure and $y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x)$ is solvable.

Proposition 3.8. *A q-difference equation of the form*

$$y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x),$$

with $\mu_1, ..., \mu_M \in K$, has a Frobenius structure if and only if it is solvable.

This lemma is a q-analog of a particular case of [Mot77, Proposition 1] it is used to prove proposition 3.8.

Lemma 3.9. *Let $u(qx) = v(x)u(x)$ be a q-difference equation such that $v(x)$ is an analytic element over $\mathcal{C}([0,1[)$, without zeros and poles in $\mathcal{C}([0,1[)$ and $u(x)$ is a non zero analytic element over $\mathcal{C}([0,1[)$. Then $u(x)$ is an element over $\mathcal{C}([0,1])$.*

This corollary describes where are located the solution of q-difference equations (with coefficient in $\mathcal{E}^\dagger$) with Frobenius structure.

Corollary 3.10. *Let $y(qx) = a(x)y(x)$ a q-difference equation with Frobenius structure. Then there exists a non negative integer h and a solution $y(x)$ of $y(qx) = a(x)y(x)$ in a finite extension of $\mathcal{E}^\dagger$ such that $y(x)^{p^h} \in \mathcal{E}^\dagger$.*

In definition 3.2 we gave a defintion of the Frobenius structure for q-difference equation of rank 1. In the higher rank we are going to consider a definition of the Frobenius structure better suited for the situation. It appear that in rank 1 both of these definitions are equivalent.

Definition 3.11. We say that $y(qx) = a(x)y(x)$ has *confluent weak Frobenius structure* if there exists a sequence of $q^{p^{sn}}$-difference equations $y(q^{p^{sn}}x) = a_n(x)y(x)$ with $a_0(x) = a(x)$, such that

(i) for any $n \geqslant 1$ the equations

$$y(q^{p^{sn}}x) = a_{n-1}(x)y(x) \text{ and } y(q^{p^{s(n-1)}}x) = a_n(x)^\phi y(x)$$

are $\mathcal{E}^\dagger$-equivalent via $u_n(x) \in (\mathcal{E}^\dagger)^*$;

(ii) the sequences $\frac{a_n(x)-1}{(q^{p^{sn}}-1)x}$ and $u_n(x)$ converge in $\mathcal{E}^\dagger$.

This definition is also useful because it permits to link the Frobenius structure of q-difference equations to a Frobenius structure of classical differential equations.

3.12. In the notation of the previous definition, let $\frac{a_n(x)-1}{(q^{p^{sn}}-1)x} \to g(x)$ and $u_n(x) \to u(x)$. Then we have the following equality

$$g(x) + \frac{u'(x)}{u(x)} = p^s x^{p^s-1} g(x)^\phi.$$

We say that $\frac{dy}{dx}(x) = g(x)y(x)$ has a *strong Frobenius structure* in this case.

PROPOSITION 3.13. *For a q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in (\mathcal{E}^\dagger)^*$, it is equivalent to have a strong Frobenius structure or a confluent weak Frobenius structure.*

The next theorem is an analogue of the classical differential case ([Cre87]). It describes the form of the solution of a rank 1 q-difference equation with coefficient in $\mathcal{E}^\dagger$. In particular, they are located in a finite unramified extension of $\mathcal{E}^\dagger$. Considering unramified extensions mean that we are only considering a lift of an extension of the residue field. Thus, we keep track of the arithmetical problems.

PROPOSITION 3.14. *Let $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, be a solvable q-difference equation (resp. a q-difference equation with a Frobenius structure). Then $y(qx) = a(x)y(x)$ has a solution of the form $x^\alpha u(x)$ (resp. $v(x)$), where $\alpha \in \mathbb{Z}_p$ and $u(x)$ (resp. $v(x)$) is an element of a finite unramified extension of $\mathcal{E}^\dagger$.*

Proof. We know from the proposition 2.8 that $y(qx) = a(x)y(x)$ is $\mathcal{E}^\dagger$-equivalent to an equation of the form

$$y(qx) = q^\alpha \prod_{i=1}^{M} \left(1 + \frac{\mu_i}{x^i}\right) y(x), \tag{2}$$

where $\alpha \in \mathbb{Z}_p$, M is a positive integer, $\mu_i \in K$ and $|\mu_i| \leqslant |q-1|$ for $i = 1, ..., M$. Moreover, using the same proposition we have that

$$y(qx) = \prod_{i=1}^{M} \left(1 + \frac{\mu_i}{x^i}\right) y(x) \tag{3}$$

is solvable. It thus admits a solution $u(x)$ and we clearly have $x^\alpha u(x)$ that is solution of (2). Applying proposition 3.8 we obtain that (3) has a Frobenius structure. To conclude the proof, one can check that $u(x)$ lies in $\mathcal{E}^\dagger[X]/(X - u(x)^{p^h})$ which is an unramified extension of $\mathcal{E}^\dagger$. □

The $\mathcal{E}^\dagger$-equivalence classes of equations of the form $y'(x) = g(x)y(x)$ with a Frobenius structure form a group, which we denote by $d - eq_{\mathcal{E}^\dagger}^{(\phi)}$. Similarly, the $\mathcal{E}^\dagger$-equivalence classes of equations of the form $y(qx) = a(x)y(x)$ with a Frobenius structure form a group, which we denote by $\sigma_q - eq_{\mathcal{E}^\dagger}^{(\phi)}$. We have the following:

THEOREM 3.15. *Let us assume that k is algebraically closed. There are canonical group isomorphisms*

$$\sigma_q - eq_{\mathcal{E}^\dagger}^{(\phi)} \to X_K(G_{k((x))}) \to d - eq_{\mathcal{E}^\dagger}^{(\phi)}$$

the composite being given by confluence.

Thus, it is equivalent to look at q-difference equations of rank 1 with a Frobenius structure and differential equations of rank 1 with a Frobenius structure. The first part of the article of Yves André and Lucia di Vizio gives a good description of what happen in rank 1. The second part focus on the higher rank case. One may wonder what happen if we look at the case in several variables.

References

ADV04 Yves André and Lucia Di Vizio. q-difference equations and p-adic local monodromy. 2004.

Chr11 Gilles Christol. *Le Théoréme de Turrittin p-adique.* 2011.

Cre87 Richard Crew. F-isocrystals and p-adic representations. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 111–138. Amer. Math. Soc., Providence, RI, 1987.

DV04 Lucia Di Vizio. Introduction to p-adic q-difference equations (weak Frobenius structure and transfer theorems). In *Geometric aspects of Dwork theory. Vol. I, II*, pages 615–675. Walter de Gruyter, Berlin, 2004.

HL46 G. H. Hardy and J. E. Littlewood. Notes on the theory of series. XXIV. A curious power-series. *Proc. Cambridge Philos. Soc.*, 42:85–90, 1946.

Mot77 Elhanan Motzkin. La décomposition d'un élément analytique en facteurs singuliers. *Ann. Inst. Fourier (Grenoble)*, 27:ix, 67–82, 1977.

Que04 C. Quesne. Disentangling q-exponentials: A general approach. *International Journal of Theoretical Physics*, 43(2):545–559, 2004.

Rob00 Alain M. Robert. *A course in p-adic analysis*, volume 198 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

Pierre Houedry pierre.houedry@etudiant.univ-rennes1.fr