An introduction to condensed mathematics (after Dustin Clausen and Peter Scholze) – Teaser – (Rennes – 2024)

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Abelian groups

Let *M* and *N* be two abelian groups. If we are given a homomorphism $f : M \to N$, we can then consider its kernel

$$\ker f := \{s \in M \ / \ f(s) = 0\} \subset M$$

and image

$$\operatorname{im} f := \{f(s) : s \in M\} \subset N.$$

This is because abelian groups satisfy AB1 (pre-abelian category). A celebrated theorem of Emmy Noether states that f induces an isomorphism

$$\overline{f}: M/\ker(f) \simeq \operatorname{im}(f).$$

This is because abelian groups satisfy AB2 (abelian category). Actually, they even satisfy up to AB6 (and dually AB4*) but we will not go through their description here.

Topological abelian groups

Let X be a topological space. Then, any subset Y of X inherits a topology (the induced topology). Also, if we are given an equivalence relation R on X, then X/R inherits a topology (the quotient topology).

A topological abelian group is a topological space M endowed with a commutative group law which is continuous as well as the inverse mapping. A morphism of topological abelian groups is a continuous homomorphism $f: M \to N$. Both kerf and imf inherit the structure of a topological abelian group. In other words, topological abelian groups satisfy AB1 (pre-abelian). However,

$$\overline{f}$$
: $M/\ker(f) \simeq \operatorname{im}(f)$

is not a homeomorphism in general. In other words, topological abelian groups do not satisfy AB2 (not abelian).

Condensed abelian groups

This is however the case for compact Hausdorff abelian groups. Unfortunately, an infinite discrete abelian group like \mathbb{Z} or a non-trivial Banach space like \mathbb{R} are not compact.

The trick consists in first considering the category of all compact Hausdorff spaces. They form what is called a pretopos (a very stable category). Then, a condensed abelian set is a sheaf \mathcal{M} on this pretopos (for the subcanonical topology). They form a *topos*. A condensed abelian group is an abelian group in this topos.

In down to earth terms, a condensed abelian group is the data of an abelian group $\mathcal{M}(S)$ for any compact Hausdorff space S and a compatible family of homomorphisms $\mathcal{M}(S) \to \mathcal{M}(S')$ for any continuous map $S' \to S$.

It is subject to the following conditions:

1. $\mathcal{M}(\emptyset) = 0$, 2. If $S \cap S' = \emptyset$, then $\mathcal{M}(S \cup S') = \mathcal{M}(S) \oplus \mathcal{M}(S')$, 3. If *R* is a closed equivalence relation on *S*, then

$$\mathcal{M}(S/R) \simeq \ker(\mathcal{M}(S) \to \mathcal{M}(R)).$$

As an example, if M is topological abelian group, then setting $\underline{M}(S) := \mathcal{C}(S, M)$ defines a condensed abelian group. Actually it is equivalent to give M or \underline{M} as long as M is compactly (Hausdorff) generated. This is the case for example if M is locally compact Hausdorff or metrizable (and in particular if M is a normed vector space).

Condensed abelian groups satisfy AB2 (abelian). Actually, they even satisfy up to AB6 and AB4* exactly like usual abelian groups do (Clausen/Scholze).

Program

- 1. Some category theory (quick review),
- 2. Some topology related to compact Hausdorff spaces,
- 3. The notions of sheaf and topos,
- 4. Condensed sets,
- 5. Abelian categories (quick review),
- 6. Condensed abelian groups,
- 7. Homological algebra (hopefully),
- 8. Cohomology of condensed abelian groups (hopefully).

The students should be comfortable with the basics of category theory from the first semester course of Matthieu.

– Thank you –