An introduction to condensed mathematics (*after Dustin Clausen and Peter Scholze*) – Teaser – *(Rennes – 2024)*

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is not a homeomorphism in general. In other words, topological abelian groups do not satisfy AB2 (not abelian).

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In down to earth terms, a condensed abelian group is the data of an abelian group $\mathcal{M}(S)$ for any compact Hausdorff space S and a compatible family of homomorphisms $\mathcal{M}(S) \to \mathcal{M}(S')$ for any continuous map $S' \to S$.

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