Université de Rennes 2024-2025

An introduction to condensed mathematics
Homework (due March 7th)

A set $\mathcal{F}$ of subsets of a set X is called a *(proper) filter* if

1. $\emptyset \notin \mathcal{F}$,
2. $\forall A, B \in \mathcal{F}, \quad A \cap B \in \mathcal{F}$,
3. $\forall A \subset B \subset X, \quad A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$.

1. Show that any filter is contained in a maximal filter.
2. Show that
   1. if $\mathcal{F}$ is a filter and $A \subset X$, then
   $$A \in \mathcal{F} \Rightarrow (\forall B \in \mathcal{F}, A \cap B \neq \emptyset),$$
   2. if $\mathcal{F}$ is maximal, then conversely
   $$(\forall B \in \mathcal{F}, A \cap B \neq \emptyset) \Rightarrow A \in \mathcal{F}.$$
3. Show that, for a filter $\mathcal{F}$, the following are equivalent
   1. $\mathcal{F}$ is maximal,
   2. $\forall A, B \in X, \quad A \cup B \in \mathcal{F} \Rightarrow A \in \mathcal{F}$ or $B \in \mathcal{F}$,
   3. $\forall A \subset X, \quad A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

We endow the set $F(X)$ of all maximal filters of X with the topology generated by all $U_A := \{\mathcal{F} \in F(X), A \in \mathcal{F}\}$ with $A \subset X$.

4. Show that if $(A_i)_{i=1}^n$ is a *finite* family of subsets of X, then $U_{\cap_i^n A_i} = \cap_i^n U_{A_i}$ and $U_{\cup_i^n A_i} = \cup_i^n U_{A_i}$.
5. Show that $F(X)$ is a compact Hausdorff space.
6. Show that
   1. if $f : X \to Y$ is any map and $\mathcal{F}$ is a filter on X, then
   $$f_*\mathcal{F} := \{B \subset Y, f^{-1}(B) \in \mathcal{F}\}$$
   is also a filter on Y,
   2. if $\mathcal{F}$ is maximal, then $f_*\mathcal{F}$ also is maximal,
   3. the map $f_* : F(X) \to F(Y)$ is continuous.
7. Show that

1. $X \mapsto F(X)$ and $f \mapsto f_*$ define a functor (from sets to topological spaces),
2. if $x \in X$, then $\mathcal{F}_x := \{A \subset X, x \in A\}$ is a maximal filter and the map

$$i_X : X \to F(X), \quad x \mapsto \mathcal{F}_x$$

   is natural (when X has the discrete topology),

3. given any map $f : X \to Y$, there exists a unique continuous map φ making commutative the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\varphi} & F(Y) \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_Y} \\ X & \xrightarrow{f} & Y. \end{array}$$

Let X be topological space, $\mathcal{F}$ a filter on X and $x \in X$. We say that $\mathcal{F}$ *converges* to x or x is a *limit* of $\mathcal{F}$ (notation: $\mathcal{F} \to x$) if $\mathcal{F}$ contains all the neighborhoods of x.

8. For a subset Y of a topological space X and $x \in X$, show that the following are equivalent:

    1. there exists a maximal filter $\mathcal{F}$ converging to x with $Y \in \mathcal{F}$,
    2. there exists a filter $\mathcal{F}$ converging to x with $Y \in \mathcal{F}$,
    3. $x \in \overline{Y}$ (closure of Y in X).

9. For a subset V of a topological space X, show that the following are equivalent:

    1. V is open in X,
    2. if a maximal filter $\mathcal{F}$ converges to a point of V, then $V \in \mathcal{F}$.

10. For a topological space X, show that the following are equivalent:

    1. X is Hausdorff,
    2. any convergent filter on X has a unique limit.

11. For a topological space X, show that the following are equivalent:

    1. X is compact,
    2. any maximal filter on X has a limit.

12. Show that if X is a compact Hausdorff topological space, then the map $\ell_X : F(X) \to X$ that sends a maximal filter $\mathcal{F}$ to its limit $x = \ell_X(\mathcal{F})$ is a continuous section of the canonical map $i_X : X \to F(X)$.

13. Show that if X is a discrete topological space, then $F(X)$ is (homeomorphic to) the Stone-Čech compactification of X.