Université de Rennes 2023-2024

*An introduction to condensed mathematics*
Tuesday, March 26th
Start: 1:15 pm - Duration: 1 hour.

Write down a complete solution for each of the following exercises (you can use any previous result from the course).

1. (4 points) (Exercise 3.1) Show that, for a set $E$ and a presheaf $T$, we have the adjunction (where $1_{\widehat{\mathcal{C}}}$ denotes a final object)

$$\mathrm{Hom}(E_{\mathcal{C}}, T) \simeq \mathrm{Hom}_{\mathbb{S}\mathrm{et}}(E, \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, T)) \quad (\simeq \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, T)^E).$$

**Solution:** We may assume that $1_{\widehat{\mathcal{C}}}$ is the constant presheaf associated to $1 := \{0\}$ (which is clearly a final object). There exists a natural bijection

$$E \simeq \mathrm{Hom}(1, E) \simeq \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, E_{\mathcal{C}}), \quad e \mapsto e_{\mathcal{C}}.$$

Composition therefore provides us with a natural map

$$\mathrm{Hom}(E_{\mathcal{C}}, T) \to \mathrm{Hom}(E, \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, T)), \quad \varphi \mapsto (e \mapsto \varphi_e := \varphi \circ e_{\mathcal{C}}).$$

By construction, if $X \in \mathcal{C}$, we have $\varphi_{e,X}(0) = \varphi_X(e)$ which implies that the map is injective and provides a candidate for an inverse. It is however necessary to check that $\phi$ will be a morphism of presheaves but if $f : Y \to X$, we have $f^*(\varphi_X(e)) = f^*(\varphi_{e,X}(0)) = \varphi_{e,Y}(0) = \varphi_Y(e)$.

2. (2 points) (Exercise 3.16) Show that, for a set $E$ and a sheaf $\mathcal{F}$ on a site $\mathcal{C}$, we have the adjunction (where $1_{\widetilde{\mathcal{C}}}$ denotes a final object)

$$\mathrm{Hom}(\widetilde{E}_{\mathcal{C}}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathbb{S}\mathrm{et}}(E, \mathrm{Hom}(1_{\widetilde{\mathcal{C}}}, \mathcal{F})) \quad (\simeq \mathrm{Hom}(1_{\widetilde{\mathcal{C}}}, \mathcal{F}))^E).$$

**Solution:** By definition, we have $\mathrm{Hom}(\widetilde{E}_{\mathcal{C}}, \mathcal{F}) \simeq \mathrm{Hom}(E_{\mathcal{C}}, \mathcal{F})$. By left exactness, $1_{\widetilde{\mathcal{C}}}$ is the sheaf associated to $1_{\widehat{\mathcal{C}}}$. It follows that we also have $\mathrm{Hom}(1_{\widetilde{\mathcal{C}}}, \mathcal{F}) \simeq \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, \mathcal{F})$. We may then apply the previous result.

3. (5 points) (Exercise 4.6) Show that the functor

$$\mathbf{S}\mathrm{et} \to \mathbf{C}\mathrm{ond}, \quad E \mapsto \underline{E} := \underline{E^{\mathrm{disc}}}$$

is fully faithful and that

$$\mathrm{Hom}(\underline{E}, X) \simeq X(\bullet)^E.$$

Show that $\underline{E}$ is naturally isomorphic to the constant sheaf $\widetilde{E}$ associated to the set $E$.

**Solution:** The first assertion is obtained by composition. More precisely, the functor $E \mapsto E^{\text{disc}}$ is fully faithful (any map between discrete topological spaces is continuous), $E^{\text{disc}}$ is compactly generated and the functor $X \mapsto \underline{X}$ is fully faithful on compactly generated spaces. Now, we know that (still using the fact that $E^{\text{disc}}$ is compactly generated)

$$\text{Hom}(\underline{E^{\text{disc}}}, X) \simeq \mathcal{C}(E^{\text{disc}}, X(\bullet)) \simeq \mathcal{F}(E, X(\bullet)) \simeq X(\bullet)^E.$$

By left exactness, we know that $\underline{\bullet}$ is a final object of **C**ond. Then, we know from the previous result that there exists a natural isomorphism

$$\text{Hom}(\widetilde{E}, X) \simeq \text{Hom}(\underline{\bullet}, X)^E \simeq X(\bullet)^E.$$

The last assertion therefore follows from Yoneda's lemma.

4. (3 points) (Exercise 5.29) Show that $\mathbb{Z} \cdot X \otimes_{\mathbb{Z}} \mathbb{Z} \cdot Y \simeq \mathbb{Z} \cdot (X \times Y)$.

**Solution:** Also follows from Yoneda's Lemma since

$$\begin{aligned}
\text{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot X \otimes_{\mathbb{Z}} \mathbb{Z} \cdot Y, N) &\simeq \text{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot X, \mathcal{H}\text{om}_{\mathbb{Z}}(\mathbb{Z} \cdot Y, N)) \\
&\simeq \text{Hom}(X, \mathcal{H}\text{om}(Y, N)) \\
&\simeq \text{Hom}(X \times Y, N) \\
&\simeq \text{Hom}(\mathbb{Z} \cdot (X \times Y), N).
\end{aligned}$$

5. (3 points) (Exercise 6.6) Show that if $N$ is a closed subgroup of a locally compact Hausdorff abelian group $M$, then $\underline{M/N} \simeq \underline{M}/\underline{N}$.

**Solution:** There exists an exact sequence of topological abelian groups $0 \to N \to M \to M/N \to 0$. Since $N$ is closed, $M/N$ is Hausdorff. Moreover, $M$ is locally compact. Then, we know that the sequence $0 \to \underline{N} \to \underline{M} \to \underline{M/N} \to 0$ is exact. It means that $\underline{M/N} \simeq \underline{M}/\underline{N}$.

6. (3 points) (Exercise 6.12) Show that if $M$ is a locally compact Hausdorff abelian group, then

$$\underline{M^*} \simeq \mathcal{H}\text{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}) \quad \text{and} \quad M^* \simeq \text{Hom}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}).$$

**Solution:** Since $M$ is locally compact, it is compactly generated and therefore

$$\underline{M^*} = \underline{\mathcal{C}_{\mathbb{Z}}(M, \mathbb{T})} \simeq \mathcal{H}\text{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}).$$

It follows that

$$M^* \simeq \underline{M^*}(\bullet) \simeq \mathcal{H}\text{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}})(\bullet) \simeq \text{Hom}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}).$$