An introduction to condensed mathematics
Homework (due March 11th)

Write down a complete solution for each of the following exercises (you can use any previous result from the course).

1. **Exercise 1.23** Show that if $\mathcal{C}$ is a small category, then the functor

$$\mathcal{C}^{\mathrm{op}} \to \mathbf{H}\mathrm{om}(\mathcal{C}, \mathbf{S}\mathrm{et}), \quad X \mapsto h^X$$

is fully faithful. Deduce that $\mathcal{C}^{\mathrm{op}}$ (resp. $\mathcal{C}$) is equivalent to the full subcategory made of representable functors on $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{op}}$).

Solution: If $X, Y \in \mathcal{C}$, then Yoneda's lemma implies that the map

$$\mathrm{Hom}(h^Y, h^X) \simeq h^X(Y) = \mathrm{Hom}(X, Y), \quad \alpha \mapsto \alpha_Y(\mathrm{Id}_Y)$$

is bijective. It is therefore sufficient to notice that, for $f : X \to Y$, we have $h^f_Y(\mathrm{Id}_Y) = \mathrm{Id}_Y \circ f = f$. If we denote by $\mathcal{R}$ the full subcategory of representable functors, then the induced functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{R}$ is fully faithful and essentially surjective. It is therefore an equivalence. The resp. assertion is obtained by duality.

2. **Exercise 1.59** Assume $F \dashv G$ with unit α and counit β. Show that F is faithful (resp. fully faithful) if and only if α_X is always a monomorphism (resp. an isomorphism). Analogue for G ?

Solution: Let us consider composite map

$$\mathrm{Hom}(Y, X) \to \mathrm{Hom}(F(Y), F(X)) \simeq \mathrm{Hom}(Y, G(F(X)))$$

where the first one is $f \mapsto F(f)$ and the second one is the adjunction $\Phi_{Y,F(X)}$. We have

$$\Phi_{Y,F(X)}(F(f)) = \alpha_Y \circ G(F(f)) = f \circ \alpha_X = h_{\alpha_X}(f).$$

Thus we see that G is faithful (resp. fully faithful) if and only h_{α_X} injective (resp. bijective) for all X and all Y. This means that α_X is a monomorphism (resp. an isomorphism) for all X.

Now, we have $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$ and the unit for this adjunction is β^{op}. Moreover, $\beta^{\mathrm{op}}_{X'}$ is a monomorphism (resp. an isomorphism) if and only if $\beta_{X'}$ is an epimorphism (resp. an isomorphism). Therefore, G is faithful (resp. fully faithful) if and only if G^{op} is faithful (resp. fully faithful) if and only if $\beta_{X'}$ is an epimorphism (resp. an isomorphism) for all X'.

3. **Exercise 2.7** Let R be an equivalence relation on a compact topological space S. Show that S/R is compact. Assume now that S is also Hausdorff. Show that S/R is compact Hausdorff if and only if $R \subset S \times S$ is closed if and only if $S \twoheadrightarrow S/R$ is a closed map.

> **Solution:** The first assertion follows from the fact that the image of a compact topological space by a continuous map is always compact. We also know that an equivalence relation is closed if and only if the quotient is Hausdorff. Also, if S is compact Hausdorff and $\pi : S \twoheadrightarrow S/R$ is closed, then this is a closed continuous surjective map and S is normal. Then, we know that S/R is normal and therefore Hausdorff. Finally, if S is compact and S/R is Hausdorff, then π is closed because any closed subset of S is compact and any compact subset of S/R is closed.

4. **Exercise 2.28** Assume X is compactly generated and Y is locally compact Hausdorff. Show that $X \times Y$ is compactly generated. Show that, if Z is any topological space, then

$$\mathcal{C}(X \times Y, Z) \simeq \mathcal{C}(X, \mathcal{C}(Y, Z)) \simeq \mathcal{C}(Y, \mathcal{C}(X, Z)).$$

> **Solution:** The last assertion is a formal consequence of the first one on which we shall focus. We assume that $F \subset X \times Y$ is k-closed and we show that it is actually closed. It is sufficient to prove that, given any $(x, y) \notin F$, then there exists some neighborhoods U and V of x and y respectively such that $(U \times V) \cap F = \emptyset$. First of all, $(x, y) \notin (X \times y) \cap F$ which is k-closed, and therefore closed, since $X \times y \simeq X$ is compactly generated. But $X \times y$ is even locally compact Hausdorff and it follows that there exists a compact neighborhood S of x in X such that $(S \times y) \cap F = \emptyset$. After replacing X with S, we may therefore assume that X itself is compact Hausdorff and that $(X \times y) \cap F = \emptyset$. We set $U = X$ and $V := p(F)^c$ where $p : X \times Y \to Y$ denotes the second projection. It only remains to show that $p(F)$ is closed. Since Y is compactly generated, is is sufficient to show that, given any continuous map $f : K \to Y$ with K compact Hausdorff, $f^{-1}(p(F)))$ is closed. After replacing Y with K, we may therefore assume that Y itself is compact Hausdorff and then $p(F)$ is necessarily closed as the image of a closed subset by a continuous map between compact Hausdorff spaces.

5. **Exercise 3.19** Show that if $\mathcal{C}$ is a site and $\mathcal{F}, \mathcal{G} \in \widetilde{\mathcal{C}}$, then

$$\mathrm{im}(\mathcal{F} \to \mathcal{G}) = \ker (\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G})$$

in $\widetilde{\mathcal{C}}$ (and dual). Show that any morphism in $\widetilde{\mathcal{C}}$ is strict.

Solution: Note first that, if $\mathcal{F} \subset \mathcal{G}$, then the canonical map $\mathcal{F} \to \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G})$ is an isomorphism. Since sheafification is exact, it is sufficient to consider a category of presheaves $\widehat{\mathcal{C}}$. Now, limits and colimits are computed argument-wise and we are therefore reduced to the analog statement in the category of sets.

Now, let us write $\mathcal{K} := \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G})$. By definition of the fibered coproduct, both composite maps

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G}$$

are the same. By definition of the kernel, f factors as $\mathcal{F} \to \mathcal{K} \hookrightarrow \mathcal{G}$. Assume now that f factors as $\mathcal{F} \to \mathcal{J} \hookrightarrow \mathcal{G}$. Since $\mathcal{J} \subset \mathcal{G}$, we have $\mathcal{J} = \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{J}} \mathcal{G})$. By functoriality of fibered coproduct and kernel, there exists a commutative diagram

$$\begin{array}{ccccc} \mathcal{K} & \hookrightarrow & \mathcal{G} & \rightrightarrows & \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G} \\ \downarrow & & \| & & \downarrow \\ \mathcal{J} & \hookrightarrow & \mathcal{G} & \rightrightarrows & \mathcal{G} \sqcup_{\mathcal{J}} \mathcal{G} \end{array}$$

which shows that $\mathcal{K} \subset \mathcal{J}$.

The dual case follows exactly the same pattern (but it is not obtained by duality because the dual of $\widetilde{\mathcal{C}}$ is not a category of sheaves).

Now, the commutativity of the diagram

$$\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F} \to \mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G},$$

implies the existence of a natural map

$$\operatorname{coker}(\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F}) \to \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G}).$$

As above, it formally follows from the analog assertion in the category of sets that this is an isomorphism.

6. **Exercise 3.36** Show that, in a topos,

$$\mathcal{H}\mathrm{om}(X \times Y, Z) \simeq \mathcal{H}\mathrm{om}(X, \mathcal{H}\mathrm{om}(Y, Z)).$$

Solution: It is sufficient to notice that, given any object T, we have a natural isomorphism

$$\begin{aligned} \mathrm{Hom}(T, \mathcal{H}\mathrm{om}(X \times Y, Z)) &\simeq \mathrm{Hom}(T \times X \times Y, Z)) \\ &\simeq \mathrm{Hom}(T \times X, \mathcal{H}\mathrm{om}(Y, Z))) \\ &\simeq \mathrm{Hom}(T, \mathcal{H}\mathrm{om}(X, \mathcal{H}\mathrm{om}(Y, Z))). \end{aligned}$$

7. **Exercise 3.41**

   1. Show that a subobject of a quasi-separated object is quasi-separated.
   2. Show that a coproduct of quasi-separated objects is quasi-separated.
   3. Show that a filtered colimit under monomorphisms of quasi-separated objects is quasi-separated.

Solution:

1. Assume X is quasi-separated and $X' \subset X$. We give ourselves $Y \to X$ and $Z \to X$ with Y and Z quasi-compacts. Then, $Y \times_{X'} Z = Y \times_X Z$ is also quasi-compact.

2. Assume that $X = \coprod_{i \in I} X_i$. If $Y \to X$ is any morphism, then we have $Y = \coprod_{i \in I} Y_i$ with $Y_i = Y \times_X X_i$. In other words, the family $(Y_i \hookrightarrow Y)_{i \in I}$ is a covering. Therefore, if Y is quasi-compact, we can then replace I with a finite subset J and we have $Y = \coprod_{i \in J} Y_i$. Moreover, a summand of a quasi-compact is always quasi-compact - as we shall show below - so that each Y_i is quasi-compact. Of course, if $Z \to X$ is another morphism with Z quasi-compact, we can also write $Z = \coprod_{i \in J} Z_i$ and we may assume that this is the same finite J. It is then formal to check that

$$Y \times_X Z \simeq \coprod_{i \in J} Y_i \times_{X_i} Z_i.$$

   If we assume that all X_i are quasi-separated, then $(Y_i \times_{X_i} Z_i \to Y \times_X Z)_{i \in J}$ is a finite covering by quasi-compact objects and it follows that $Y \times_X Z$ is also quasi-compact.

   It remains to show that a summand of a quasi-compact is itself quasi-compact. But if we are given a covering $(X_i \to X)_{i \in I}$ and we know that $X \sqcup Y$ is quasi-compact, we can then consider the covering made of the X_i's and Y of $X \sqcup Y$. It has a finite refinement and we are done.

3. If $X = \varinjlim_{i \in I} X_i$ is any colimit, then the corresponding morphism $\coprod_{i \in I} X_i \twoheadrightarrow X$ is an epimorphism (as usual, this follows formally from the analogous assertion in **S**et). In other words, the family $(X_i \to X)_{i \in I}$ is a covering. In particular, when X is quasi-compact, there exists a finite subset J of I such that $\coprod_{i \in J} X_i \twoheadrightarrow X$ is an epimorphism and therefore $X = \varinjlim_{i \in J} X_i$. In the case of a filtered colimit, if k is any cocone for J in I, then we will have $X = X_k$.

   Not assuming X quasi-compact anymore, let $Y \to X$ be a morphism with Y quasi-compact. Then, $Y = \varinjlim_{i \in I} Y_i$ with $Y_i = Y \times_X X_i$. If the colimit is filtered, then there exists k such that $Y = Y_k$. In other words, there exists a factorization $Y \to X_k \to X$ of the original morphism. If $Z \to Y$ is another morphism with Z quasi-compact, then there exists also a factorization $Z \to X_k \to X$ and we may assume that this is the same k since I is filtered. Finally, if we assume that $X_k \subset X$, then we have $Y \times_X Z = Y \times_{X_k} Z$ which is quasi-compact if we assume that X_k is quasi-separated.