Université de Rennes 1 2023-2024

An introduction to condensed mathematics
Homework (due March 11th)

Write down a complete solution for each of the following exercises (you can use any previous result from the course).

1. **Exercise 1.23** Show that if $\mathcal{C}$ is a small category, then the functor

$$\mathcal{C}^{\mathrm{op}} \to \mathbf{H}\mathrm{om}(\mathcal{C}, \mathbf{S}\mathrm{et}), \quad X \mapsto h^X$$

is fully faithful. Deduce that $\mathcal{C}^{\mathrm{op}}$ (resp. $\mathcal{C}$) is equivalent to the full subcategory made of representable functors on $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{op}}$).

2. **Exercise 1.59** Assume $F \dashv G$ with unit α and counit β. Show that F is faithful (resp. fully faithful) if and only if α_X is always a monomorphism (resp. an isomorphism). Analogue for G ?

3. **Exercise 2.7** Let R be an equivalence relation on a compact topological space S. Show that S/R is compact. Assume now that S is also Hausdorff. Show that S/R is compact Hausdorff if and only if $R \subset S \times S$ is closed if and only if $S \twoheadrightarrow S/R$ is a closed map.

4. **Exercise 2.28** Assume X is compactly generated and Y is locally compact Hausdorff. Show that $X \times Y$ is compactly generated. Show that, if Z is any topological space, then

$$\mathcal{C}(X \times Y, Z) \simeq \mathcal{C}(X, \mathcal{C}(Y, Z)) \simeq \mathcal{C}(Y, \mathcal{C}(X, Z)).$$

5. **Exercise 3.19** Show that if $\mathcal{C}$ is a site and $\mathcal{F}, \mathcal{G} \in \widetilde{\mathcal{C}}$, then

$$\mathrm{im}(\mathcal{F} \to \mathcal{G}) = \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G})$$

in $\widetilde{\mathcal{C}}$ (an dual). Show that any morphism in $\widetilde{\mathcal{C}}$ is strict.

6. **Exercise 3.36** Show that, in a topos,

$$\mathcal{H}\mathrm{om}(X \times Y, Z) \simeq \mathcal{H}\mathrm{om}(X, \mathcal{H}\mathrm{om}(Y, Z)).$$

7. **Exercise 3.41**

   1. Show that a subobject of a quasi-separated object is quasi-separated.
   2. Show that a coproduct of quasi-separated objects is quasi-separated.
   3. Show that a filtered colimit under monomorphisms of quasi-separated objects is quasi-separated.