An introduction to condensed mathematics

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– What I care most about are definitions (Peter Scholze - quoted by Michael Harris).

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Introduction

Teaser

The right environment in order to perform linear algebra is that of an abelian category. For example, vector spaces over $\mathbb{R}$ (or any field) form an abelian category. More interesting, we can consider the category of vector spaces endowed with an operator. This is again an abelian category. Also abelian sheaves on a topological space (or more generally on a site) form an abelian category. This is not true however for Banach (or Hilbert) spaces for example. The presence of a topology makes Noether first isomorphism (or equivalently Grothendieck AB2 axiom) fail.

The queen of abelian categories is the category of abelian groups. Actually, this is even a "Grothendieck category satisfying AB6 and AB4*". Introducing a topology leads to consider the category of topological abelian groups. This is not an abelian category unless we focus on compact Hausdorff abelian groups in which case we miss usual (infinite) abelian groups as well as (non trivial) Banach spaces. Dustin Clausen and Peter Scholze define a condensed abelian group as an abelian sheaf on the site (for the precanonical topology) of all compact Hausdorff spaces[1]. They form a Grothendieck abelian category satisfying AB6 and AB4* that contains all compactly generated abelian groups.

Vector spaces often come with a topology, and linear maps between them are then required to be continuous. In particular, an isomorphism is not only supposed to be linear and bijective, but it also needs to be a homeomorphism. Unfortunately, it is not true anymore, in this situation, that a linear map whose kernel and cokernel

[1]Actually, they consider profinite sets but this is equivalent.

vanish is an isomorphism. The baby example is given by the identity on the real line $\mathbb{R}$ when $\mathbb{R}$ is given the discrete topology on the one side and its usual topology on the other. In particular, all the standard tools from commutative algebra are not anymore available when some topology is involved. Mathematicians have been trying to resolve this issue for some time now, starting maybe with the work of Choquet in the late 40's and first formalized by Johnstone in the late 70's ([Joh79]).

His original idea was to not only consider the points of a given topological space X but also the set of all convergent sequences in it (in order to keep track of the topology). A point of X may be seen – in a very fancy way – as a (continuous) map defined on a one point space $\bullet$ with values in X. Similarly, a convergent sequence (together with the choice of a limit) may be seen as a continuous map defined on the one-point compactification $\overline{\mathbb{N}}$ of the set of all integers $\mathbb{N}$ with values in X. Let us denote by $X(\bullet)$ and $X(\overline{\mathbb{N}})$ the corresponding sets. They are not unrelated because there exists various maps between them obtained for example by composing with the various points of $\overline{\mathbb{N}}$ (meaning maps from $\bullet$ to $\overline{\mathbb{N}}$). One may then consider more generally any couple of sets $T(\bullet)$ and $T(\overline{\mathbb{N}})$, endowed with some family of maps between them and subject to some conditions. In modern language, this is a sheaf on a site. The collection of all these sheaves may be seen as an enlargement of the collection of all topological spaces. Abelian sheaves always form a Grothendieck abelian category satisfying AB5 and AB3*.

Although promising, this is not completely satisfying. For example, there exists the general notion of a (quasi-) compact sheaf but the unit interval $[0, 1]$ is *not* (quasi-) compact in Johnstone's theory. This may be fixed by adding a new test-space, for example the Cantor space $2^{\mathbb{N}}$ which surjects continuously onto $[0, 1]$. The idea of Clausen and Scholze is to actually use the site of all (light) profinite spaces S and not merely $\bullet$ and $\overline{\mathbb{N}}$ or even $2^{\mathbb{N}}$. A *condensed set* is then simply a sheaf (of sets) on this site. And a *condensed abelian group* is an abelian sheaf on this site. It happens that condensed abelian groups have all the properties expected from commutative algebra: much like abelian groups, they form a Grothendieck abelian category satisfying AB6 and AB4*. Moreover, their cohomology provides the expected invariants for locally compact abelian groups.

There also exists the notion of a completion in this theory called *solidification* that matches the properties of usual non-Archimedean completion. Clausen and Scholze are then able to provide very elegant proofs of several theorems from algebraic geometry and even obtain new results. The idea is that many theorems in geometry require strong global assumptions such as properness because of finiteness constraints. Condensed geometry provides a way to remove the global condition and work locally, which is much more natural. Note that there also exists the notion of a *liquid* vector space that matches the properties of Banach spaces in the Archimedean world. Their introduction is necessary in order to recover classical analytic geometry (and functional analysis).

This course is intended for regular students. We shall present some material which is necessary to understand the basics of the theory of condensed sets. We may only reach the definition of a condensed abelian group in the end – and hopefully be able to show that they satisfy the expected properties. We hope however that we can cover some cohomological results. We apologize before the the reader that the limited amount of time for the course does not allow us to do more and send him to further readings.

The first chapter is a short presentation of standard category theory. The second one is a review of topology with special focus on compact Hausdorff spaces. The third chapter is an introduction to topos theory (it is completely independent from the second one). Chapter 4 is devoted to condensed sets. Chapter 5 presents the theory of abelian categories. Chapter 6 is devoted to condensed abelian groups. Chapter 7 deals with cohomology theory. Chapter 8 is devoted to the computation of cohomology of condensed abelian groups.

Here is non exhaustive list of links to yet unpublished documents related to the theory of condensed sets (see also the bibliography at the end of the course):

- Lectures on condensed mathematics – Clausen, Scholze.
- Lectures on analytic geometry – Clausen, Scholze.
- Condensed mathematics and complex geometry – Clausen, Scholze.
- The foundation of condensed mathematics – Àsgeirsson.
- Mathématiques condensées – Le Bras.
- Condensed mathematics – Mathew.
- Crash Course Condensed Mathematics – Barton, Commelin.
- Condensed and locally compact abelian groups – Deglise.
- Condensed Mathematics Seminar – Morgan, Rodrigez-Camargo.
- Condensed Mathematics – Leptien.

Many thanks to the students who took my course - and those who freely attended it - for their feedback that helped me improve on these notes. I am also indepted to Dagur Àsgeirsson and Florian Leptien for making me aware of two mistakes (in proofs, not statements) in the first release.

1. Categories and functors

For those who might worry about set-theoretic issues (see [Shu08] for example), we shall stay in a fixed *universe* (some large set, see definition 1.1.1 of [KS06] for example). We shall then call *set* only those sets that belong ($\in$) to our universe[1] and rename *collection* a set that is only contained ($\subset$) in our universe.

1.1 Category

1.1.1 Definition/Examples

Definition 1.1.1 A *category* consists in the following data:

1. a collection of *objects* $\mathcal{C}$,
2. for all $X, Y \in \mathcal{C}$, a set of *morphisms* $\mathrm{Hom}(X, Y)$,
3. for all $X \in \mathcal{C}$, an *identity* morphism $\mathrm{Id}_X \in \mathrm{End}(X) := \mathrm{Hom}(X, X)$,
4. for all $X, Y, Z \in \mathcal{C}$, a composition rule

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z), \quad (f,g) \mapsto g \circ f$$

   such that
   (a) $f \circ \mathrm{Id}_X = f$ and $\mathrm{Id}_Y \circ f = f$,
   (b) if $h \in \mathrm{Hom}(Z,T)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

The category is said to be *small* if its objects (or equivalently its morphisms) form a set and *finite* if there is a finite number of (objects and) morphisms.

We will usually write $f : X \to Y$ instead of $f \in \mathrm{Hom}(X,Y)$ and call X (resp. Y) the *domain* (resp. le *codomain*) of f. Note that Id_X is uniquely determined by the conditions (4a). In practice, we shall simply say that the set $\mathcal{C}$ *is* a category[2] but we

[1] They are usually called *small sets* but we do not want to keep this epithet everywere.
[2] As one may denote a group by G without explicitly mentioning the multiplication rule.

must not forget that it involves some extra structure: morphisms and composition.

Examples 1. If G is a monoid, one can consider the category $\mathbf{G}$ with a unique object $\bullet$, $\mathrm{End}(\bullet) := G$ and composition given by multiplication on G. This way, we get essentially all categories with a unique object.

2. If $\leq$ is a preorder on a set X, we will then denote by $\mathbf{X}$ the category whose objects are the elements $x \in X$ and morphisms are couples (x, y) for $x \leq y$. This way, we get essentially all the small categories whose Hom have at most one element.
3. As a particular case, one may always endow a set with the relation " $=$ " and consider any set as a category (with only identities). This provides all small categories with only identities as morphisms.
4. The naturel number $n := \{0, \ldots, n-1\}$ is ordered as usual and we will denote by $\mathbf{n}$ the corresponding category. For example, $\mathbf{0}$ is the empty category that has no objects and no morphisms, $\mathbf{1}$ is the category that has exactly one object and one morphism, $\mathbf{2}$ is a category with two distinct objects and a unique morphism between them plus the identities.
5. If X is a topological space, then $\mathrm{Open}(X)$ is ordered by inclusion and we shall denote by $\mathbf{O}\mathrm{pen}(X)$ the corresponding category.
6. We shall denote by $\mathbf{S}\mathrm{et}$ the (large) category whose objects are sets and morphisms are maps between them. We shall write $\mathcal{F}(X, Y)$ for the set of all maps between two sets.
7. In the same way, we shall consider the (large) category $\mathbf{T}\mathrm{op}$ whose objects are topological spaces and morphisms are continuous maps. We shall write $\mathcal{C}(X, Y)$ for the set of continuous maps between two topological spaces.
8. Finally, we will denote by $\mathbf{A}\mathrm{b}$ the (large) category whose objects are abelian groups and morphisms are homomorphisms. We shall write $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ for the set of homomorphisms of abelian groups.

Exercise 1.1 Define the categories $\mathbf{M}\mathrm{on}$, $\mathbf{G}\mathrm{r}$, $\mathbf{R}\mathrm{ng}$, G-$\mathbf{S}\mathrm{et}$, A-$\mathbf{M}\mathrm{od}$ and k-$\mathbf{A}\mathrm{lg}$ of monoids, groups, rings, G-sets, A-modules and k-algebras.

Definition 1.1.2 The *simplex category* $\mathbf{\Delta}$ has *positive* integers

$$[n] := n + 1 := \{0, \ldots, n\}$$

for $n \in \mathbb{N}$ as objects and order preserving maps as morphisms. The injective (resp. surjective) maps

$$\delta_n^i : [n-1] \to [n] \quad (\text{resp. } \sigma_n^i : [n+1] \to [n])$$

that forget (resp. repeats) the i-th term are called the *face* (resp. *degeneracy*) maps.

Exercise 1.2 Show that any morphism in $\mathbf{\Delta}$ is a composition of face and degeneracy maps.

The *product* $\mathcal{C} \times \mathcal{C}'$ of two categories $\mathcal{C}$ and $\mathcal{C}'$ is itself a category (everything is done termwise). The *opposite category* to a category $\mathcal{C}$ is the category $\mathcal{C}^{\mathrm{op}}$ with the same objects as $\mathcal{C}$ but $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$ (and composition going in the reverse direction). We have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

Exercise 1.3 Show that if $\mathcal{C}$ is any category, then there exists a category $\mathbf{Mor}(\mathcal{C})$ defined as follows: an object is a morphism of $\mathcal{C}$ and a morphism between $f : X \to Y$ and $g : X' \to Y'$ is a pair of morphisms in $\mathcal{C}$, $\varphi : X \to X'$ and $\psi : Y \to Y'$, such that $g \circ \varphi = \psi \circ f$.

Exercise 1.4 Show that, if G is a monoid, then the set of objects of $\mathbf{Mor}(\mathbf{G})$ is G and that, if X is a preordered set, then the set of objects of $\mathbf{Mor}(\mathbf{X})$ is the graph of the relation.

Exercise 1.5 Show that if $\mathcal{C}$ is a category and $X \in \mathcal{C}$, then there exists a category ${}_{X\backslash}\mathcal{C}$ (of X-*objects* of $\mathcal{C}$) defined as follows: an object of ${}_{X\backslash}\mathcal{C}$ is a morphism $f : X \to Y$ and a morphism from $f : X \to Y$ to $g : X \to Z$ is a morphism $h : Y \to Z$ such that $h \circ f = g$. Make explicit the category $\mathcal{C}_{/X} := ({}_{X\backslash}\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ (of *objects of* $\mathcal{C}$ *over* X). For example, in $\mathbf{Top}$, an object Y over X is called a *bundle* or a *fibration*.

1.1.2 Isomorphism

Definition 1.1.3 In a category $\mathcal{C}$,

1. a *section* (resp. a *retraction*) of a morphism $f : X \to Y$ is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ (resp. $g \circ f = \mathrm{Id}_X$):
$$\begin{tikzcd} X \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "g"'] \arrow[loop right] \end{tikzcd} \quad \left(\text{resp.} \quad \begin{tikzcd} X \arrow[loop left] \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "g"'] \end{tikzcd}\right).$$
2. an *isomorphism* is a morphism that has at the same time a section and a retraction. When there exists an isomorphism $X \to Y$, one says that X and Y are *isomorphic* and we write $X \simeq Y$. An isomorphism between X and itself is called an *automorphism*.

A section for g is a retraction for f and conversely. A retraction in $\mathcal{C}$ is the same thing as a section in $\mathcal{C}^{\mathrm{op}}$ and conversely. More generally, for any notion, there exists a dual notion obtained by applying the same definition to the opposite category.

Examples
1. A *retract* A of a topological space X is a subspace such that the inclusion map $A \hookrightarrow X$ has a retraction.
2. A *direct factor* M' of a module M is a submodule such that the inclusion map $M' \hookrightarrow M$ has a retraction.

Proposition 1.1.4 If f is an isomorphism, then it has a unique section and a unique retraction and they are the same.

Proof. If $f \circ g = \mathrm{id}_Y$ and $h \circ f = \mathrm{id}_X$, then
$$h = h \circ \mathrm{id}_Y = h \circ f \circ g = \mathrm{id}_X \circ g = g. \qquad \blacksquare$$

The unique section/retraction of an isomorphism f is called its *inverse* and denoted by f^{-1}.

Exercise 1.6 What is an isomorphism in **Set**, in **Top**, in **Ab**, etc. ? In $\mathbf{X}$ if X is a preordered set ? In $\mathbf{G}$ if G is a monoid ?

1.1.3 Subcategory

Definition 1.1.5 A *subcategory* of a category $\mathcal{C}$ is the data of

1. a subcollection $\mathcal{C}' \subset \mathcal{C}$,
2. for all $X, Y \in \mathcal{C}'$, a subset $\mathrm{Hom}_{\mathcal{C}'}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$, such that
   (a) if $X \in \mathcal{C}'$, then $\mathrm{Id}_X \in \mathrm{End}_{\mathcal{C}'}(X) := \mathrm{Hom}_{\mathcal{C}'}(X,X)$,
   (b) if $X,Y,Z \in \mathcal{C}'$, $f \in \mathrm{Hom}_{\mathcal{C}'}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}'}(Y,Z)$, then $g \circ f \in \mathrm{Hom}_{\mathcal{C}'}(X,Z)$.

It is a *full* subcategory if actually

$$\forall X, Y \in \mathcal{C}', \quad \mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

It is a *wide* subcategory if any object of $\mathcal{C}$ is in $\mathcal{C}'$.

A subcategory becomes a category with the induced composition. A full subcategory is uniquely determined by its objects.

Examples 1. **Ab** is a full subcategory of **Gr** which itself is a full subcategory of **Mon** (which itself is a *non-full* subcategory of the category of semigroups or magmas for example).
2. If X is a topological space, then $\mathbf{Open}(X)$ is a subcategory of **Set** which is *not* full.
3. **Top** is *not* a subcategory of **Set** and neither is **Ab** (but see the notion of a forgetful functor below).
4. If X is any topological space, then an *espace étalé* over X is a local homeomorphism $X' \to X$. They form a full subcategory $\mathbf{Et}(X) \subset \mathbf{Top}_{/X}$.
5. We may consider the wide subcategory $\mathbf{\Delta}_{\mathrm{inj}}$ of $\mathbf{\Delta}$ with the same objects but only order preserving *injective* maps as morphism.
6. Conversely, we may consider $\mathbf{\Delta}$ as a full subcategory of $\mathbf{\Delta}^+$ which is defined by adding $[-1] = \emptyset$.

1.2 Functor

1.2.1 Definition/Examples

Definition 1.2.1 1. A *(covariant) functor* $F : \mathcal{C} \to \mathcal{C}'$ between two categories is the data for all $X \in \mathcal{C}$ of $F(X) \in \mathcal{C}'$ and for all $f : X \to Y$ of $F(f) : F(X) \to F(Y)$, in such a way that we always have $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.
2. If $G : \mathcal{C}' \to \mathcal{C}''$ is another functor, then their *composite* is the functor $G \circ F$ given by $(G \circ F)(X) = G(F(X))$ and $(G \circ F)(f) = G(F(f))$.

We will denote by $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$ the set of all functors $\mathcal{C} \to \mathcal{C}'$. We will often describe the functors by their action on the objects and let the reader guess what happens on the morphisms.

There always exists an identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that doesn't change anything. A functor $F : \mathcal{C} \to \mathcal{C}'$ is called an *isomorphism* it there exists a functor G such that $G \circ F = \mathrm{Id}_{\mathcal{C}}$ and $F \circ G = \mathrm{Id}_{\mathcal{C}'}$ (but this is not a very interesting notion).

A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'$ is also called a *contravariant* functor from $\mathcal{C}$ to $\mathcal{C}'$. Any functor $F : \mathcal{C} \to \mathcal{C}'$ provides a functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'^{\mathrm{op}}$ and this construction is "functorial" : $\mathrm{Id}_{\mathcal{C}^{\mathrm{op}}} = \mathrm{Id}_{\mathcal{C}}^{\mathrm{op}}$ and $(G \circ F)^{\mathrm{op}} = G^{\mathrm{op}} \circ F^{\mathrm{op}}$.

Examples

1. There exists a functor that forgets topology (and continuity) $\mathbf{Top} \to \mathbf{Set}$ (underlying set). In the other direction, there exits two functors $X \mapsto X^{\mathrm{disc}}$ and $X \mapsto X^{\mathrm{coarse}}$ that endow a set X with the discrete topology (maximal) or the coarse topology (minimal).
2. There exists a functor that forgets the algebraic structure $\mathbf{Ab} \to \mathbf{Set}$ (underlying set) and, in the other direction, a functor $X \mapsto \mathbb{Z} \cdot X$ (or $\mathbb{Z}^{(X)}$) which sends a set to the free *free abelian group*[3] generated by X:
$$\mathbb{Z} \cdot X := \left\{ \sum_{\text{finite}} n_x x \ : \ n_x \in \mathbb{Z}, x \in X \right\}.$$
3. There exists an inclusion functor $\mathbf{Gr} \hookrightarrow \mathbf{Mon}$ and two functors $G \mapsto G^{\times}$ and
$$G \mapsto G^{\mathrm{gr}} := \langle \{x_g\}_{g \in G} \ / \ \{x_{gh}^{-1} x_g x_h\}_{g,h \in G} \rangle$$
in the other direction.
4. Small categories and functors between them form a category that we shall denote $\mathbf{Cat}$ and there exists a (contravariant) functor
$$\mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}, \quad X \mapsto \mathbf{Ouv}(X), \quad u \mapsto u^{-1}.$$

Exercise 1.7 What is a functor $\mathbf{G} \to \mathbf{H}$ between categories associated to monoïds ? What is a functor $\mathbf{X} \to \mathbf{Y}$ between categories associated to preordered sets ?

Exercise 1.8 What are the analog of the "free abelian group" functor $X \mapsto \mathbb{Z} \cdot X$ for the categories $\mathbf{Mon}$, $\mathbf{Gr}$, $\mathbf{Rng}$, G-$\mathbf{Set}$, A-$\mathbf{Mod}$ and k-$\mathbf{Alg}$.

Solution. For example, the free monoid generated by a set X is $G := \{g : n \to X, n \in \mathbb{N}\}$ endowed with
$$gh : n + m \to X, \quad i \mapsto \begin{cases} g(i) & \text{if } i < n \\ h(i-n) & \text{otherwise.} \end{cases}$$
In other words, X is the alphabet, G is the set of words in this alphabet and the operation is concatenation. ■

Exercise 1.9 Show that, besides the inclusion functor $\mathbf{Ab} \hookrightarrow \mathbf{Gr}$, there exists an *abelianization* functor $G \mapsto G^{\mathrm{ab}} = G/[G,G]$ in the other direction. Show however hat the center of a group is *not* functorial: a group homomorphism $\varphi : G \to H$ does not necessarily induce a morphism of abelian groups $\mathrm{Z}(G) \to \mathrm{Z}(H)$.

[3] Some people write $\mathbb{Z}[X]$ instead of $\mathbb{Z} \cdot X$ but this may be confused with a polynomial ring.

Exercise 1.10 Show that the categories $\mathbb{Z}$-**Mod** and **Ab** are isomorphic. Same thing with the categories $\mathbb{Z}$-**Alg** and **Rng**, and, more generally, with k-**Alg** and a full subcategory of $_{k\backslash}$**Rng** (the image of k must be in the center).

Exercise 1.11 Show that the image of a section (resp. a retraction, resp. an inverse) by a functor is a section (resp. a retraction, resp. an inverse).

If we are given two categories $\mathcal{C}$ and $\mathcal{C}'$, then the projections $\mathcal{C} \times \mathcal{C}'$ on $\mathcal{C}$ and $\mathcal{C}'$ are functorial. The same holds for the obvious partial functors $\mathcal{C}' \hookrightarrow \mathcal{C} \times \mathcal{C}'$ or $\mathcal{C} \hookrightarrow \mathcal{C} \times \mathcal{C}'$ associated to a fixed object $X \in \mathcal{C}$ or $X' \in \mathcal{C}'$.

If $\mathcal{C}$ is any category, then there always exists a (bi-) functor

$$\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}, \quad (X, Y) \mapsto \mathrm{Hom}(X, Y)$$

that sends (f, g) to the map $h \mapsto g \circ h \circ f$. If we compose with the partial functors, we get the (fundamental) functors

$$h^X : \mathcal{C} \to \mathbf{Set}, \quad Y \mapsto \mathrm{Hom}(X, Y)$$

and

$$h_Y : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}, \quad X \mapsto \mathrm{Hom}(X, Y).$$

Exercise 1.12 Define the domain and codomain functors $\mathbf{Mor}(\mathcal{C}) \to \mathcal{C}$ as well as the forgetful functors $_{X\backslash}\mathcal{C} \to \mathcal{C}$ and $\mathcal{C}_{/X} \to \mathcal{C}$.

Exercise 1.13 Let us denote by $\mathbf{Op}(\mathcal{C}) \subset \mathbf{Mor}(\mathcal{C})$ the full subcategory made of morphisms whose codomain is identical to the domain (objects with operator). Show that, if k is commutative ring, then $\mathbf{Op}(k\text{-}\mathbf{Mod}) \simeq k[T]\text{-}\mathbf{Mod}$.

1.2.2 Natural transformation

Definition 1.2.2
1. If $F, G : \mathcal{C} \to \mathcal{C}'$ are two functors, then a *natural transformation* $\alpha : F \Rightarrow G$ is the data for all $X \in \mathcal{C}$ of a morphism $\alpha_X : F(X) \to G(X)$ such that, for all $f : X \to Y$, we have $\alpha_Y \circ F(f) = G(f) \circ \alpha_X$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_Y"] & F(Y) \arrow[d, "\alpha_X"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

We shall say *natural isomorphism* and write $F \simeq G$ if all α_X are isomorphisms.
2. If $\beta : G \Rightarrow H$ is another natural transformation, then their *composition* $\beta \circ \alpha : F \Rightarrow H$ is the natural transformation defined by $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$ for $X \in \mathcal{C}$.

We shall denote by $\mathrm{Hom}(F, G)$ the set of all natural transformations form F to G. If $\mathcal{C}$ is a small category, then the functors $F : \mathcal{C} \to \mathcal{C}'$ make a (large) category

$\mathbf{Hom}(\mathcal{C}, \mathcal{C}')$ with natural transformations as morphisms. One can check that the isomorphisms are exactly the natural isomorphisms defined above.

Examples
1. We obtain a natural transformation by considering $\det_A : \mathrm{GL}_n(A) \to A^\times$ (between functors from commutative rings to groups). This is a natural isomorphism for $n = 1$.
2. If $\mathcal{C}$ is any category, then there exists isomorphisms of categories $\mathbf{Hom}(\mathbf{0}, \mathcal{C}) \simeq \mathbf{1}$, $\mathbf{Hom}(\mathbf{1}, \mathcal{C}) \simeq \mathcal{C}$ and $\mathbf{Hom}(\mathbf{2}, \mathcal{C}) \simeq \mathbf{Mor}(\mathcal{C})$.

Definition 1.2.3 A functor $F : \mathcal{C} \to \mathcal{C}'$ is
1. *faithful* (resp. *full*, resp. *fully faithful*) if for all $X, Y \in \mathcal{C}$, the map
$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y)), \quad f \mapsto F(f)$$
is injective (resp. surjective, resp. bijective).
2. *essentially surjective* if for all $X' \in \mathcal{C}'$, there exists $X \in \mathcal{C}$ such that $X' \simeq F(X)$.
3. an *equivalence of categories* if there exists $G : \mathcal{C}' \to \mathcal{C}$ such that $\mathrm{Id}_{\mathcal{C}'} \simeq F \circ G$ and $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$ (and G is then called a *quasi-inverse*).

One also defines the *essential image* of a functor F as the set of all $X' \in \mathcal{C}'$ such that there exists $X \in \mathcal{C}$ with $X' \simeq F(X)$. The functor F is then essentially surjective when the essential image is equal to $\mathcal{C}'$.

The inclusion of a (full) subcategory is a (fully) faithful functor. An isomorphism of categories is an equivalence (but not conversely). We shall use the notation $\mathcal{C} \simeq \mathcal{C}'$ for the wider notion of *equivalence* (and not merely isomorphism) of categories. One sometimes say that two categories $\mathcal{C}$ and $\mathcal{C}'$ are anti-equivalent if $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C}'$.

Exercise 1.14 Show that the forgetful functors $\mathbf{Top} \to \mathbf{Set}$ and $\mathbf{Ab} \to \mathbf{Set}$ are faithful but not fully faithful.

Exercise 1.15 Show that there exists a fully faithful functor $X \mapsto \mathbf{X}$ from pre-ordered sets to small categories. What is the essential image ? Same questions with a functor $G \mapsto \mathbf{G}$ from monoids to small categories.

Exercise 1.16 Show that if $F \simeq F'$, then F is faithful (resp. full, resp. fully faithful, essentially surjective, an equivalence) if and only if F' is.

Exercise 1.17 Show that a fully faithful functor is essentially injective (if $F(X) \simeq F(Y)$, then $X \simeq Y$).

Theorem 1.2.4 A functor is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Proof. In order to show that the condition is necessary, we first remark that our functor $F : \mathcal{C} \to \mathcal{C}'$ will be essentially surjective since we will always have $X' \simeq F(X)$ with $X := G(X')$ if G is a quasi-inverse for F. Then, we consider the following sequence of maps

$$\mathrm{Hom}(X, Y) \xrightarrow{F} \mathrm{Hom}(F(X), F(Y))$$

$$\xrightarrow{G} \mathrm{Hom}(G(F(X)), G(F(Y))) \xrightarrow{F} \mathrm{Hom}(F(G(F(X))), F(G(F(Y))))$$

Since both $G \circ F$ and $F \circ G$ are fully faithful, all the maps are necessarily bijective and F is therefore fully faithful.

In order to show that the condition is sufficient, we *choose* for all $X' \in \mathcal{C}'$ an object $X \in \mathcal{C}$ and an isomorphism $\alpha_{X'} : X' \simeq F(X)$. We set $G(X') := X$. Since F is fully faithful, there exists for each $f' : X' \to Y'$ a unique $f : G(X') \to G(Y')$ such that $F(f) = \alpha_{Y'} \circ f' \circ \alpha_{X'}^{-1}$ and we set $G(f') = f$. One easily checks that G is a functor and we obtain by construction a natural isomorphism $\alpha : \mathrm{Id}_{\mathcal{C}'} \simeq F \circ G$. In particular, if $X \in \mathcal{C}$, there exists a natural isomorphism $\alpha_{F(X)}^{-1} : F(G(F(X))) \simeq F(X)$ and, since F is fully faithful, there exists a unique isomorphism $\beta_X : (G \circ F)(X) \simeq X$ such that $F(\beta_X) = \alpha_{F(X)}^{-1}$. One easily checks that β is indeed a natural transformation. ■

Exercise 1.18 Show that if X is a preordered set and Y denotes its ordered quotient, then the categories $\mathbf{X}$ and $\mathbf{Y}$ are equivalent.

Exercise 1.19 Show that, if k is a commutative ring, then $\mathbf{Mat}(k) := \mathbb{N}$, endowed with $\mathrm{Hom}(m, n) = \mathrm{M}_{n \times m}(k)$ and multiplication of matrices, is a small category. Show that if k is a field, then $\mathbf{Mat}(k)$ is equivalent, but not isomorphic, to the category of finite dimensional k-vector spaces (which is large).

1.2.3 Representable functor

Definition 1.2.5 An object $X \in \mathcal{C}$, together with an element $s \in F(X)$, is said to be *universal* for a functor $F : \mathcal{C} \to \mathbf{Set}$ if

$$\forall Y \in \mathcal{C}, \forall t \in F(Y), \exists! f : X \to Y, \quad F(f)(s) = t.$$

We shall also say that F is *represented* by X (and s).

We may also say that the couple $(X \in \mathcal{C}, s \in F(X))$ is universal among all couples $(Y \in \mathcal{C}, t \in F(Y))$, or that $s \in F(X)$ is universal for $t \in F(Y)$.

Examples
1. If k is a commutative ring then $k[t]$ represents the forgetful functor $k{-}\mathbf{Alg} \to \mathbf{Set}$. More precisely, $(k[t], t)$ is universal among all (A, a) where A is a k-algebra and $a \in A$: there exists a unique morphism of k-algebras $\varphi : k[t] \to A$ such that $\varphi(t) = a$.
2. $\mathbb{Z}$ is universal among all rings ($F = \emptyset$).
3. The inclusion $Y \hookrightarrow X$ of a subspace into a topological space is universal for continuous maps $f : Z \to X$ such that $f(Z) \subset Y$ (contravariant F).

Exercise 1.20 Show that the other usual forgetful functors with value in $\mathbf{Set}$ are also representable.

Exercise 1.21 Show that the tensor product $(M, N) \mapsto M \otimes_A N$ (resp. $(M, N) \mapsto M \otimes_k N$) is universal for $\mathbb{Z}$-bilinear (resp. k-bilinear) maps.

Exercise 1.22 Let k be a commutative ring and $f_1, \dots, f_r \in k[t_1, \dots, t_n]$. Show that the functor that sends a commutative k-algebra A to the set

$$\mathcal{S}(A) := \{(a_1, \dots, a_n) \in A^n \ / \ f_1(a_1, \dots, a_n) = \cdots = f_r(a_1, \dots, a_n) = 0\}$$

of all solutions with values in A, is representable.

Exercise 1.23 Show that, if F is represented by both X and X', then $X \simeq X'$. More precisely, show that if both (X, s) and (X', s') are universal for F, then there exists a unique isomorphism $f : X \simeq X'$ such that $F(f)(s) = s'$.

Lemma 1.2.6 — Yoneda. If $F : \mathcal{C} \to \mathbf{Set}$ is any functor and $X \in \mathcal{C}$, then there exists a natural bijection

$$\mathrm{Hom}(h^X, F) \simeq F(X), \quad \alpha \mapsto \alpha_X(\mathrm{Id}_X).$$

Proof. Given $s \in F(X)$, if $Y \in \mathcal{C}$ and $f : X \to Y$, then we set $\alpha_Y(f) := F(f)(s)$. This defines a map $\alpha_Y : \mathrm{Hom}(X, Y) \to F(Y)$ and we shall show that this is natural, meaning that

$$\alpha_Z \circ h^X(g) = F(g) \circ \alpha_Y$$

if $g : Y \to Z$. Indeed, we do have

$$\begin{aligned}&(\alpha_Z \circ h^X(g))(f) = \alpha_Z(h^X(g)(f)) = \alpha_Z(g \circ f) = F(g \circ f)(s)\\ &= (F(g) \circ F(f))(s) = (F(g)(F(f)(s)) = F(g)(\alpha_Y(f)) = (F(g) \circ \alpha_Y)(f).\end{aligned}$$

It only remains to check that we did define an inverse (exercise). ■

Proposition 1.2.7 A functor $F : \mathcal{C} \to \mathbf{Set}$ is represented by $X \in \mathcal{C}$ if and only if $h^X \simeq F$.

Proof. We may simply apply Yoneda lemma: the condition means that there exists a natural transformation $\alpha : h^X \to F$ which is an isomorphism. This α corresponds to some $s \in F(X)$ and the condition exactly means that α_Y is always bijective since, necessarily, $F(f)(s) = \alpha_Y(f)$. ■

In other words, F is represented by X if and only if there exists a natural bijection $\mathrm{Hom}(X, Y) \simeq F(Y)$. Note that, for a contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, the condition reads $\mathrm{Hom}(Y, X) \simeq F(Y)$ or, equivalently, $h_X \simeq F$.

Exercise 1.24 Show that if $\mathcal{C}$ is a small category, then the functor

$$\mathcal{C}^{\mathrm{op}} \to \mathbf{Hom}(\mathcal{C}, \mathbf{Set}), \quad X \mapsto h^X$$

is fully faithful. Deduce that $\mathcal{C}^{\mathrm{op}}$ (resp. $\mathcal{C}$) is equivalent to the full subcategory made of representable functors on $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{op}}$).

Solution. If $X, Y \in \mathcal{C}$, then Yoneda's lemma implies that the map

$$\mathrm{Hom}(h^Y, h^X) \simeq h^X(Y) = \mathrm{Hom}(X, Y), \quad \alpha \mapsto \alpha_Y(\mathrm{Id}_Y)$$

is bijective. It is therefore sufficient to notice that, for $f : X \to Y$, we have $h^f_Y(\mathrm{Id}_Y) = \mathrm{Id}_Y \circ f = f$. If we denote by $\mathcal{R}$ the full subcategory of representable functors, then the induced functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{R}$ is fully faithful and essentially surjective. It is therefore an equivalence. The resp. assertion is obtained by duality. ■

1.3 Limit

1.3.1 Diagrams, cones and limits

Definition 1.3.1 Let I be a small category and $\mathcal{C}$ any category. Then, a *commutative diagram* on I in $\mathcal{C}$ is a functor $D : I \to \mathcal{C}$.

A commutative diagram on I in $\mathcal{C}$ is therefore the data of an object X_i for all $i \in I$ and a morphism $f_\alpha : X_i \to X_j$ for all $\alpha : i \to j$ satisfying $f_{\beta\circ\alpha} = f_\beta \circ f_\alpha$ and $f_{\mathrm{Id}_i} = \mathrm{Id}_{X_i}$:

$$\begin{tikzcd} X_i \arrow[rr, "f_\alpha"] \arrow[dr, "f_{\beta\circ\alpha}"'] & & X_j \arrow[dl, "f_\beta"] \\ & X_k & \end{tikzcd}$$

We shall denote such a diagram as $(f_\alpha : X_i \to X_j)$ or (X_i, f_α), and the category of all commutative diagrams on I in $\mathcal{C}$ by $\mathcal{C}^I := \mathbf{Hom}(I, \mathcal{C})$.

Definition 1.3.2 A *simplicial* (resp. *semi-simplicial*) *object* is a diagram on $\mathbf{\Delta}^{\mathrm{op}}$ (resp. $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$). A *(semi-) cosimplicial* object is a (semi-) simplicial object in $\mathcal{C}^{\mathrm{op}}$. It is said to be *augmented* if we actually use $\mathbf{\Delta}^+$ instead of $\mathbf{\Delta}$

Exercise 1.25 Show that giving a *semi-simplicial object* is equivalent to giving a sequence of morphisms

$$\cdots\cdots\; X_{n+1} \overset{d_0^{n+1}}{\underset{d_{n+1}^{n+1}}{\rightrightarrows\!\vdots}} X_n \overset{d_0^{n}}{\underset{d_{n}^{n}}{\rightrightarrows\!\vdots}} X_{n-1} \;\cdots\cdots\; X_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_1 \overset{d_0^1}{\underset{d_1^1}{\rightrightarrows}} X_0$$

such that

$$\forall n \in \mathbb{N}, \forall 0 \le i < j \le n+1, \quad d_i^n \circ d_j^{n+1} = d_{j-1}^n \circ d_i^{n+1}.$$

Analog for simplicial objects ?

Example There exists a cosimplicial object in $\mathbf{Top}$ sending $[n]$ to the standard *topological simplex*

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\ge 0}, \sum_{i=0}^n x_i = 1 \right\}$$

and $u : [n] \to [m]$ to the unique linear map sending e_i to $e_{u(i)}$ if $(e_0, \ldots, e_n)$ denotes the usual basis.

By composition, any functor $\lambda : I \to J$ between small categories will provide a functor $\lambda^* : \mathcal{C}^J \to \mathcal{C}^I$ between diagrams (and this is functorial). As a particular case, the unique functor $I \to \mathbf{1}$ induces the *constant diagram* functor

$$\mathcal{C} \simeq \mathcal{C}^{\mathbf{1}} \to \mathcal{C}^I, \quad X \mapsto \underline{X}.$$

Definition 1.3.3 A *cone* for a diagram D in $\mathcal{C}$ is a morphism $\underline{X} \to D$ with $X \in \mathcal{C}$.

In more down to earth terms, a cone for (X_i, f_α) is a family of morphisms $(p_i : X \to X_i)$ such that for all $\alpha : i \to j$, we have $p_j = f_\alpha \circ p_i$. It is said to be *finite* when I is finite. The dual notion is that of a *cocone*.

Definition 1.3.4 A *limit* X of a commutative diagram D on I in $\mathcal{C}$ is a universal cone.

In other words, X is a limit for D if and only if X is a cone for D and, given any cone Y for D, there exists a unique morphism $f : Y \to X$ making commutative the diagram

$$\begin{tikzcd} \underline{Y} \arrow[rr, "f"] \arrow[dr] & & \underline{X} \arrow[dl] \\ & D. & \end{tikzcd}$$

In down to earth terms, a commutative diagram (X_i, f_α) has X as a limit if and only if we are given for all $i \in I$ a morphism $p_i : X \to X_i$ such that for all $\alpha : i \to j$, we have $p_j = f_\alpha \circ p_i$ with the following universal property: if we are given some $Y \in \mathcal{C}$ endowed for all $i \in I$ with a morphism $g_i : Y \to X_i$ such that for all $u : i \to j$, we have $g_j = f_\alpha \circ g_i$, then there exists a unique morphism $g : Y \to X$ such that for all $i \in I$, we have $g_i = p_i \circ g$:

$$\begin{tikzcd} & & X_i \arrow[d, "f_\alpha"] \\ Y \arrow[r, dashed, "g"] \arrow[urr, bend left, "g_i"] \arrow[drr, bend right, "g_j"'] & X \arrow[ur, "p_i"] \arrow[dr, "p_j"'] & X_j \end{tikzcd}$$

Alternatively, the definition says that the composite functor $h_D \circ \_$ is representable by X in $\mathcal{C}^{\mathrm{op}}$. Equivalently, there exists a natural isomorphism for $Y \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}^I}(\underline{Y}, D) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

We call a limit *finite* when I is finite.

A limit is unique up to a unique isomorphism and we may sometimes say *the* limit and denote it by $X = \varprojlim D$. A limit X of a diagram D in $\mathcal{C}^{\mathrm{op}}$ is also called a *colimit* in $\mathcal{C}$ and denoted by $X = \varinjlim D$. Some authors call a limit (resp. colimit) an *inverse* (resp. a *direct*) limit, a *projective* (resp. an *inductive*) limit or a *left* (resp. a *right*) limit. We have the following formulas:

$$\mathrm{Hom}(\underline{Y}, D) \simeq \mathrm{Hom}(Y, \varprojlim D) \quad \text{et} \quad \mathrm{Hom}(\varinjlim D, Y) \simeq \mathrm{Hom}(D, \underline{Y}).$$

When we write a limit or a colimit, we implicitly assume that it exists.

Exercise 1.26 Make the notion of a colimit explicit.

1.3.2 Specific limits

Definition 1.3.5 A limit of the empty diagram $\mathbf{0} \to \mathcal{C}$ is called a *final* object of $\mathcal{C}$ and denoted by $1_{\mathcal{C}}$. Dually, we get the notion of an *initial* object $0_{\mathcal{C}}$.

If $X \in \mathcal{C}$, there exists a unique morphism $X \to 1_{\mathcal{C}}$ (resp. $0_{\mathcal{C}} \to X$).

Examples
1. In **S**et, the initial object is $\emptyset$ and 1 is a final object (defined up to a unique bijection).
2. Same thing in **T**op.
3. In **A**b, $\{0\}$ is both a final and an initial object.

Definition 1.3.6 A limit of a family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ (with no morphism apart from identites) is called a *product* and denoted by $\prod_{i\in I} X_i$. Dually, a colimit of $(X_i)_{i\in I}$ is called a *coproduct* and and denoted $\coprod_{i\in I} X_i$. When all X_i are equal to the same X, we shall say *power* (resp. *copower*) and we write X^I (resp. $X^{(I)}$ or $I \cdot X$).

Note that a final (resp. initial) object is nothing else but the empty product (resp. coproduct).

Examples
1. In **S**et, the cartesian product is a product and the disjoint union is a coproduct.
2. In **T**op, this is the same thing with the coarser (resp. finer) topolofy making the projections (resp. injections) continuous.
3. In **A**b, the cartesian product is a product and the direct sum is a coproduct[4] (product equals coproduct when I is finite).

Definition 1.3.7 A limit X of a diagram $(X_1 \xrightarrow{f_1} X_0 \xleftarrow{f_2} X_2)$ is called a *fibered product* of X_1 and X_2 over X_0 and denoted by $X = X_1 \times_{X_0} X_2$. We shall then also say that the diagram

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X_1 \arrow[d, "f_1"] \\ X_2 \arrow[r, "f_2"] & X_0 \end{tikzcd}$$

is *cartesian* or that p_2 is the *pullback* of f_1 along f_2 (and p_1 is the *pullback* of f_2 along f_1). Dually, there exists the notions of a *fibered coproduct* denoted by $X_1 \sqcup_{X_0} X_2$, a *cocartesian* square and a *pushout*.

Note that a product (resp. coproduct) of two objects is nothing but a fibered product (resp. fibered coproduct) over a final (resp. initial) object (if it exists).

[4] Be careful that what is called *free product* is a coproduct in the category of (non abelian) groups.

Examples 1. In **Set**, we have

$$X_1 \times_{X_0} X_2 = \{(x_1, x_2) \ / \ f_1(x_1) = f_2(x_2)\} \subset X_1 \times X_2$$

and

$$X_1 \sqcup_{X_0} X_2 = (X_1 \sqcup X_2)/\sim$$

where $\sim$ is the equivalence relation generated by $f_1(x_0) \sim f_2(x_0)$ when $x_0 \in X_0$.

2. In **Top**, this is the same thing with the induced (resp. quotient) topology.
3. In **Ab** this is the kernel (resp. cokernel) of the canonical map from (resp. to) the direct sum.

Definition 1.3.8 A limit X of a pair $Y \rightrightarrows Z$ is called a *kernel* (or *equalizer*) and denoted by $X = \ker(f, g)$. We shall also say that the sequence

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z \end{tikzcd}$$

is *left exact*. Dually, there exists the notion of a *cokernel* (or *coequalizer*) coker (f, g) and *right exact* sequence[a].

[a] We should say *exact sequence* and *coexact sequence*.

Examples 1. In **Set**, we have

$$\ker(f, g) = \{y \in Y \ / \ f(y) = g(y)\} \quad \text{and} \quad \operatorname{coker}(f, g) = Z/\sim$$

where $\sim$ is the equivalence relation generated by $f(y) \sim g(y)$ for $y \in Y$.

2. Same thing in **Top**.
3. In **Ab**, $\ker(f) := \ker(0, f)$ is the usual kernel and we have $\ker(f, g) = \ker(g - f)$, (and dual) .

Definition 1.3.9 When a commutative diagram of the form

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, "i"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

is cartesian, then Y is called a *monomorphism* and we shall write $i : Y \hookrightarrow X$. A morphism $i : Y \to X$ is called a *regular* monomorphism if it is the kernel of some morphism. The dual notion is that of a *(regular) epimorphism* and we shall then write $X \twoheadrightarrow Y$.

Exercise 1.27 1. Write down the definition of an epimorphism.
2. Show that $X \to Y$ is a monomorphism (resp. an epimorphism) if and only if for all $Z \in \mathcal{C}$, the map $\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, X)$ (resp. $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$) induced by composition is injective.
3. Show that a regular monomorphism (resp. epimorphism) is indeed a

monomorphism (resp. an epimorphism).

Examples
1. A morphism in **S**et (resp. **A**b, resp. **T**op) is a monomorphism/epimorphism if and only if it is injective/surjective.
2. A regular monomorphism/epimorphism in **T**op is a homeomorphism with a subspace/a quotient map.
3. The inclusion map $\mathbb{Q} \hookrightarrow \mathbb{R}$ is at the same time a monomorphism and an epimorphism (but not an isomorphism) in the category of *Hausdorff* topological spaces.
4. Same thing for the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in the category of rings.

Definition 1.3.10 If $Y \hookrightarrow X$ is a monomorphism, we shall also write $Y \subset X$ and call Y a *subobject*[a] of X. When $Y, Z \subset X$, their *intersection* is $Y \cap Z := Y \times_X Z \subset X$ (if it exists). If $f : X' \to X$ is any morphism and $Y \subset X$, then its *inverse image* is $f^{-1}(Y) := Y \times_X X' \subset X'$ (if it exists).

[a]More precisely, a subobject is an equivalence class of such.

Exercise 1.28 Make explicit some classical limits and colimits in **M**on, **G**r, **R**ng, G-**S**et, A-**M**od and k-**A**lg.

Exercise 1.29 Show that the fibered coproduct in the category of commutative rings is the tensor product.

Exercise 1.30 Show that in a poset, the limit (resp. colimit) is the greatest lower bound or *meet* (resp. least upper bound or *join*). What about cone and cocone ? Make explicit the case of a finite ordinal as well as the set of open subsets of a topological space.

Exercise 1.31 Does the category **C**at have a final object? an initial object ? finite products ? Make them explicit. Show that if $\mathcal{C}$ is a small category, then **O**p$(\mathcal{C})$ is the kernel of the domain and codomain functors **M**or$(\mathcal{C}) \rightrightarrows \mathcal{C}$ in **C**at.

Exercise 1.32 Show that a morphism $f : Y \to X$ is a monomorphism (resp. an epimorphism) if and only if the induced functor $\mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ (resp. ${}_{X\backslash}\mathcal{C} \to {}_{Y\backslash}\mathcal{C}$) is faithful.

Definition 1.3.11 The *image* $\mathrm{im}(f)$ of a morphism $f : X \to Y$ is a subobject I of Y which is universal (smallest) for factorizations $f : X \to I \hookrightarrow Y$ (if it exists). The dual notion is that of a *coimage* $\mathrm{coim}(f)$. A morphism is said to be *strict* if $\mathrm{im}(f) = \mathrm{coim}(f)$.

Exercise 1.33 Show that any morphism is strict in **S**et or **A**b but not in **T**op.

1.3.3 Constructions of limits

Exercise 1.34 We give ourselves a commutative diagram (X_i, f_α). Assume that both $X' := \prod_i X_i$ and $X'' := \prod_{\alpha:i\to j} X_j$ exist. Denote by p (resp. f) the morphisms $X' \to X''$ induced by the projections onto the codomain (resp. the composition of the projection onto the domain and f_α). Show that $X = \varprojlim(X_i, f_\alpha)$ if and only if there exists a left exact sequence

$$X \longrightarrow X' \overset{p}{\underset{f}{\rightrightarrows}} X''.$$

In particular, $X \hookrightarrow X'$ is a regular monomorphism.

Exercise 1.35 Show that if $(f_\alpha : X_i \to X_j)$ is a commutative diagram of *sets*, then $\varinjlim X_i = \coprod X_i/\sim$ where $\sim$ denotes the equivalence relation generated by $x_i \sim x_j$ whenever $f_\alpha(x_i) = x_j$.

Exercise 1.36 Show that, if we are given two morphisms $f, g : X \to Y$ in a category $\mathcal{C}$, then the commutative diagram

$$\begin{tikzcd} \ker(f,g) \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Y \times Y. \end{tikzcd}$$

is cartesian. More precisely, show that if $Y \times Y$ exists, then there exists such a fibered product if and only if $\ker(f, g)$ exists in which case they coincide.

Exercise 1.37 Let $\mathcal{C}$ be a category.

1. Show that, if all (finite) products and all kernels exist, then all (finite) limits exist.
2. Show that if all fibered products exist and there is a final object, then all finite limits exist.

Analogues for colimits.

When all (finite) limits exist, then $\mathcal{C}$ is said to be *(finitely) complete*. The dual terminology is *cocomplete*.

Exercise 1.38 Show that all limits and colimits do exist in **S**et, **T**op, **A**b, etc.

1.4 Miscelaneous

1.4.1 Filtered colimit

Definition 1.4.1 A small category I is said to be *filtered* if it has all finite cocones. A *filtered diagram* is a diagram $I \to \mathcal{C}$ with I filtered. A *filtered colimit* is a colimit of a filtered diagram.

Exercise 1.39 Show that a poset is directed if and only if it is filtered (as a category).

There exists a partial converse (we may always replace a filtered category with a directed set):

Proposition 1.4.2 If I is a filtered category, then there exists a directed set J and a functor $u : J \to I$ such that, for all diagram $D : I \to \mathcal{C}$, if $\varinjlim(D \circ u)$ exists, then $\varinjlim D$ exists and $\varinjlim(D \circ u) \simeq \varinjlim D$.

Proof. To do. ∎

Exercise 1.40 Show that a small category I is filtered if and only if

1. $I \neq \emptyset$,
2. $\forall i, j \in I, \exists i \to k, j \to k$,
3. $\forall u, v : i \to j, \exists c : j \to k \ / \ c \circ u = c \circ v$.

Exercise 1.41 Show that a category with filtered colimits and finite colimits (resp. finite coproducts) has all colimits (resp. coproducts).

Definition 1.4.3 An *ind-object* "$\varinjlim X_i$" of a category $\mathcal{C}$ is a filtered diagram $(X_i)_{i \in I}$. They form a category $\mathrm{Ind}(\mathcal{C})$ with

$$\mathrm{Hom}(\text{“}\varinjlim X_i\text{”}, \text{“}\varinjlim Y_j\text{”}) = \varprojlim_{j \in J} \varinjlim_{i \in I} \mathrm{Hom}(X_i, Y_j).$$

The dual notion is that of a *pro-object* "$\varprojlim X_i$"and they form a category $\mathrm{Pro}(\mathcal{C}) := \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

The notion of ind-object lives somehow between the notion of a diagram and the notion of a limit. One can show that we get an equivalent category by considering only directed sets instead of filtered categories (difficult).

Example The category of profinite sets (pro-objects of the category of finite sets) is equivalent to the category of compact Hausdorff totally disconnected spaces (as we shall later prove).

Exercise 1.42 Show that the obvious functor $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ is fully faithful.

1.4.2 Preservation of limit

Any functor $F : \mathcal{C} \to \mathcal{D}$ provides by composition a functor

$$F^I : \mathcal{C}^I \to \mathcal{D}^I, \quad D \mapsto F(D) := F \circ D$$

We shall usually simply write F instead of F^I so that $F(X_i, f_\alpha) = (F(X_i), F(f_\alpha))$.

Definition 1.4.4 If D is a commutative diagram in $\mathcal{C}$, then a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve* (or *commute* with) the limit of D, if

$$F(\varprojlim D) \simeq \varprojlim F(D),$$

Of course, it is assumed here that the limit of D exists and it implies that the limit of $F(D)$ also exists. Also, there exists an obvious analogue for colimits. Be careful that a (contravariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ preserves a limit when it turns a *colimit* in $\mathcal{C}$ into limit in $\mathcal{D}$.

Exercise 1.43 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. Show that if F preserves all (finite) products and all kernels, then F preserves all (finite) limits.
2. Show that if F preserves all fibered products and the final object, then F preserves all finite limits.

Analogues for colimits.

Exercise 1.44 Show that a functor that preserves filtered colimits and finite colimits (resp. finite coproducts) preserves all colimits (resp. coproducts).

Exercise 1.45 Show that the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ preserves all limits and colimits and that the functor $\mathbf{Ab} \to \mathbf{Set}$ preserves all limits (but not colimits).

Exercise 1.46 Show that, if F preserves fibered products, then F preserves monomorphisms (and dual).

Proposition 1.4.5 A representable functor $F : \mathcal{C} \to \mathbf{Set}$ preserves all limits.

Proof. We may assume that $F = h^X$ with $X \in \mathcal{C}$. It is then sufficient to check that if D is a commutative diagram in $\mathcal{C}$, we have a sequence of bijections

$$h^X(\varprojlim D) \simeq \mathrm{Hom}(X, \varprojlim D) \simeq \mathrm{Hom}(\underline{X}, D)$$

$$\simeq \mathrm{Hom}(\underline{\{0\}}, h^X(D)) \simeq \mathrm{Hom}(\{0\}, \varprojlim h^X(D)) \simeq \varprojlim h^X(D).$$

Only the middle one needs to be checked by hand: if we write $D =: (X_i, f_\alpha)$, then, giving a morphism $\underline{\{0\}} \to h^X(D)$ is equivalent to give a compatible family of maps $\{0\} \to \mathrm{Hom}(X, X_i)$, or in other words, to give for each $i \in I$, a morphism $g_i : X \to X_i$ such that $f_\alpha \circ g_i = g_j$, or finaly a morphism $\underline{X} \to D$. ■

There is no dual statement here and the notion of a limit plays a special role. When applied to the functor Hom, we have the following fundamental formulas

$$\mathrm{Hom}(X, \varprojlim Y_i) \simeq \varprojlim \mathrm{Hom}(X, Y_i) \quad \text{et} \quad \mathrm{Hom}(\varinjlim X_i, Y) \simeq \varprojlim \mathrm{Hom}(X_i, Y).$$

Definition 1.4.6 A functor is said to be *left exact* (resp. *right exact*) if it preserves all finite limits (resp. colimits). Il is said to be *exact* if it is both left and right exact[a].

[a]We should say exact, coexact and biexact respectively but we will follow the mainstream terminology.

Examples

1. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is exact, as well as the functor $\mathbf{Set} \to \mathbf{Top}$ that endows a set with the discrete topology, but the functor $\mathbf{Set} \to \mathbf{Top}$ that endows a set with the coarse topology is only left exact.
2. The forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is left exact but not right exact, and the free abelian group functor $\mathbf{Set} \to \mathbf{Ab}$ is right exact but not left exact.

Exercise 1.47 Show that the obvious functor $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ is exact and preserves all limits.

1.4.3 Projective/Injective

Definition 1.4.7 An object X of a category $\mathcal{C}$ is said to be *projective* if h^X preserves epimorphisms. The dual notion is that of an *injective* object (h_X sends mono to epi).

It means that, if $Z \twoheadrightarrow Y$ is an epimorphism (resp. $Y \hookrightarrow Z$ is a monomorphism) in $\mathcal{C}$, then following map is surjective:

$$\mathrm{Hom}(X, Z) \twoheadrightarrow \mathrm{Hom}(X, Y) \quad (\text{resp. } \mathrm{Hom}(Z, X) \twoheadrightarrow \mathrm{Hom}(Y, X)).$$

In other words, X is projective (resp. injective) when any diagram

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad (\text{resp.} \quad \begin{tikzcd} Z \arrow[dr, dashed] & \\ Y \arrow[u, hook] \arrow[r] & X \end{tikzcd} \quad)$$

can be completed with the dotted arrow. When X is projective (resp. injective), any epimorphism $Y \twoheadrightarrow X$ (resp. monomorphism $X \hookrightarrow Y$) has a section (resp. a retraction). Also, if $X \to Y$ has a retraction and Y is projective, then X also is projective (analog for injective).

Examples 1. In **Set**, all objects are projective and injective.
2. In **Ab**, projective objects are free abelian groups and injective objects are divisible groups (for example $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$).
3. In $R-$**Mod**, projective objects are direct factors of free R-modules ($\mathbb{Z}/2$ is projective – but not free – over $\mathbb{Z}/6$).
4. We shall show that the projective objects of the category of compact Hausdorff spaces are the Stonean (meaning extremally disconnected) spaces (or equivalently the retracts of free compact Hausdorff spaces).

Exercise 1.48 Show that a coproduct of projectives is projective (analog for injective).

Exercise 1.49 Assume $\mathcal{C}$ admits fibred products and that epimorphisms are universal (see definition 3.2.11 below). Show that X is projective if and only if any epimorphism $Y \twoheadrightarrow X$ has a section.

Solution. Pulling back any epimorphism $Z \twoheadrightarrow Y$ along a morphism $X \to Y$ provides (by universality) an epimorphism $X \times_Y Z \twoheadrightarrow X$ that has a section $X \to X \times_Y Z$ that we may compose with the projection $X \times_Y Z \to Z$ in order to get a lifting $X \to Z$ of the original map $X \to Y$ along the epimorphism $Z \twoheadrightarrow Y$. ■

Definition 1.4.8 A category $\mathcal{C}$ has *enough projectives* if, given any $X \in \mathcal{C}$, there exists a projective Y and an epimorphism $Y \twoheadrightarrow X$ (dually, *enough injective*: a monomorphism $X \hookrightarrow Y$ with Y injective).

Examples 1. The category **Set** has enough projectives and injectives.
2. The category $R-$**Mod** has enough projectives and injectives.
3. We will show that the category of compact Hausdorff spaces has enough projectives.

Definition 1.4.9 An object X of a category $\mathcal{C}$ is said to be *finitely presented* (or *compact*) if h^X preserves filtered colimits.

In other words,

$$\varinjlim \mathrm{Hom}(X, Y_i) \simeq \mathrm{Hom}(X, \varinjlim Y_i)$$

when (Y_i) is filtered. Or, in more down to earth terms, any morphism $X \to \varinjlim Y_i$ factors through some Y_i.

Examples 1. A set is finitely presented if and only if it is finite.
2. A topological space is finitely presented if and only if it is finite discrete.
3. An abelian group is finitely presented (resp. finitely presented projective) if and only if it is finitely generated (resp. free of finite rank).
4. A topological space is compact (in the usual sense) if and only if X is a finitely presented object of $\mathrm{Open}(X)$.

1.4.4 Algebraic structure

Definition 1.4.10 Let $\mathcal{C}$ be a category with finite products (a *cartesian* category). A *monoid* of $\mathcal{C}$ is an object G endowed with a *multiplication map* $\mu : G \times G \to G$ and a *unit map* $\epsilon : 1 \to G$ making commutative the following diagrams:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu\times\mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G\times\mu"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\epsilon\times\mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G\times\epsilon"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G. \end{tikzcd}$$

A *morphism of monoids* $G \to G'$ of $\mathcal{C}$ is a morphism $f : G \to G'$ in $\mathcal{C}$ making the following commutative:

$$\begin{tikzcd} G \times G \arrow[r, "f\times f'"] \arrow[d, "\mu"] & G' \times G' \arrow[d, "\mu'"] \\ G \arrow[r, "f"] & G' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[rr, "f"] & & G' \\ & 1 \arrow[ul, "\epsilon"] \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

It is a *group* if there exists an *inversion map* $\iota : G \to G$ making commutative

$$\begin{tikzcd}
G \arrow[rr, "(\iota,\mathrm{Id}_G)"] \arrow[dd, "(\mathrm{Id}_G,\iota)"] \arrow[dr] & & G \times G \arrow[dd, "\mu"] \\
& 1 \arrow[dr, "\epsilon"] & \\
G \times G \arrow[rr, "\mu"] & & G.
\end{tikzcd}$$

Il is *abelian* if $\mu \circ \tau = \mu$ if τ denotes the map that exchanges factors in $G \times G$.

Monoids (resp. groups, resp. abelian groups) of $\mathcal{C}$ make a category $\mathbf{Mon}(\mathcal{C})$ (resp. $\mathbf{Gr}(\mathcal{C})$, resp. $\mathbf{Ab}(\mathcal{C})$). We shall concentrate on abelian groups.

Example An abelian group of the category **Set** is nothing but a usual abelian group. An abelian group of **Top** is a topological abelian group (with continuous multiplication and continuous inversion).

Exercise 1.50 A *bialgebra* is a monoid of the category opposite to the category of k-algebras. Show that $k[t]$ (resp. $k[t, 1/t]$) endowed with $t \mapsto t \otimes 1 + 1 \otimes t$ (resp. $t \mapsto t \otimes t$) is a bialgebra.

Exercise 1.51 Define the notion of a ring A in a category $\mathcal{C}$ with finite products as well as the notion of a G-object, an A-modules or a k-algebra. We shall denote by $\mathbf{Rng}(\mathcal{C})$, G-$\mathbf{Set}(\mathcal{C})$, A-$\mathbf{Mod}(\mathcal{C})$ and k-$\mathbf{Alg}(\mathcal{C})$ these categories.

Exercise 1.52 Show that, if all limits exist in $\mathcal{C}$, then the same holds in $\mathbf{Ab}(\mathcal{C})$ and they are preserved by the obvious forgetful functor $\mathbf{Ab}(\mathcal{C}) \to \mathcal{C}$.

Exercise 1.53 Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between categories with finite products. Show that

1. if F preserves finite products, then F induces a functor $\mathbf{Ab}(F) : \mathbf{Ab}(\mathcal{C}) \to \mathbf{Ab}(\mathcal{C}')$,
2. if F preserves all limits, so does $\mathbf{Ab}(F)$.

1.4.5 Localization

Definition 1.4.11 The *localization* of a small category $\mathcal{C}$ with respect to a set of morphisms W is a category[a] $\mathrm{ho}(\mathcal{C})$ which is universal for functors $\mathcal{C} \to \mathcal{D}$ sending W to isomorphisms in $\mathcal{D}$.

[a] ho stands for *homotopy category*.

It means that there exists a functor $\gamma : \mathcal{C} \to \mathrm{ho}(\mathcal{C})$ sending W to isomorphisms such that, given any category $\mathcal{D}$, the functor

$$\gamma^* : \mathbf{Hom}(\mathrm{ho}(\mathcal{C}), \mathcal{D}) \to \mathbf{Hom}(\mathcal{C}, \mathcal{D})$$

induces an isomorphism with the full subcategory of functors sending W is isomorphisms.

Proposition 1.4.12 The localization $\mathrm{ho}(\mathcal{C})$ of $\mathcal{C}$ with respect to W always exists.

Proof. (Sketch) We may assume that W contains all isomorphisms in $\mathcal{C}$. Then, the objects of $\mathrm{ho}(\mathcal{C})$ are the objects of $\mathcal{C}$ and morphisms are finite chains

$$X = X_0 \overset{W}{\leftarrow} X_1 \to X_2 \overset{W}{\leftarrow} \cdots \to X_{n-1} \overset{W}{\leftarrow} X_n \to X_{n+1} = Y$$

up to equivalence. ∎

Definition 1.4.13 A category $\mathcal{C}$ admits *right calculus of fractions* with respect to a set of morphisms W if

1. W contains all identities and is stable under composition,
2. given any $f : X \to Y$ in $\mathcal{C}$ and $\varphi : Y' \to Y$ in W, there always exists $f' : X' \to Y'$ and $\varphi' : X' \to X$ in W with $\varphi \circ f' = f \circ \varphi'$,
3. given any $f, g : X \to Y'$ in $\mathcal{C}$ and $\varphi : Y' \to Y$ in W such that $\varphi \circ f = \varphi \circ g$, there exists $\varphi' : X' \to X$ in W such that $f \circ \varphi' = g \circ \varphi'$.

Proposition 1.4.14 If a small category $\mathcal{C}$ admits right calculus of fraction with respect to W, then $\mathrm{ho}(\mathcal{C})$ is the category having the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{\mathrm{ho}(\mathcal{C})}(X, Y) = \varinjlim_{X' \to X \in W} \mathrm{Hom}_{\mathcal{C}}(X', Y).$$

Proof. (sketch) By definition, morphisms and composition are described, up to equivalence, by the following diagram

$$\begin{tikzcd}
X'' \arrow[d, "W"'] \arrow[dr] & & \\
X' \arrow[d, "W"'] \arrow[dr] & Y' \arrow[d, "W"'] \arrow[dr] & \\
X \arrow[r, dashed] & Y \arrow[r, dashed] & Z.
\end{tikzcd}$$

It is then a matter of checking the various properties. ∎

Exercise 1.54 Show that if $\mathcal{C}$ admits right calculus of fraction with respect to W, then the functor $Q : \mathcal{C} \to \mathrm{ho}(\mathcal{C})$ is exact.

1.4.6 Comma category

The following notion is a generalization of many older constructions:

Definition 1.4.15 Assume given two functors $S : \mathcal{S} \to \mathcal{C}$ and $T : \mathcal{T} \to \mathcal{C}$ with same target $\mathcal{C}$. An object in the *comma category* $(S \downarrow T)$ is a triple $(X \in \mathcal{S}, Y \in \mathcal{T}, f : S(X) \to T(Y))$. A morphism $(X, Y, f) \to (X', Y', f')$ is a pair of morphisms $u : X \to X'$, $v : Y \to Y'$ such that $F(v) \circ f = f \circ F(u)$.

Example
1. In case $\mathcal{C} = 1$, we have $(S \downarrow T) \simeq \mathcal{S} \times \mathcal{T}$.
2. We have $(S \downarrow T)^{\mathrm{op}} \simeq (T^{\mathrm{op}} \downarrow S^{\mathrm{op}})$.
3. $(\mathrm{Id}_{\mathcal{C}} \downarrow \mathrm{Id}_{\mathcal{C}}) \simeq \mathbf{Mor}(\mathcal{C}) \simeq \mathbf{Hom}(\mathbf{2}, \mathcal{C})$.

4. $(\mathrm{Id}_{\mathcal{C}} \downarrow 1 \xrightarrow{X} \mathcal{C}) \simeq \mathcal{C}_{/X}$ and $(1 \xrightarrow{X} \mathcal{C} \downarrow \mathrm{Id}_{\mathcal{C}}) \simeq {}_{X\backslash}\mathcal{C}$.
5. One sets more gererally $\mathcal{C}_{/T} := (\mathrm{Id}_{\mathcal{C}} \downarrow T)$ and ${}_{S\backslash}\mathcal{C} := (S \to \mathcal{C} \downarrow \mathrm{Id}_{\mathcal{C}})$.
6. There also exists a characterization of adjointness (see below) using comma categories.

There exists may functorialities in terms of comma categories.

1.5 Adjointness

1.5.1 Definition

Definition 1.5.1 A functor $F : \mathcal{C} \to \mathcal{C}'$ is *adjoint* to a functor $G : \mathcal{C}' \to \mathcal{C}$ if there exists a natural isomorphism

$$\forall X \in \mathcal{C}, X' \in \mathcal{C}', \quad \mathrm{Hom}(F(X), X') \simeq \mathrm{Hom}(X, G(X')).$$

The dual notion is that of a *coadjoint* so that G is coadjoint to F if and only if F is adjoint to G. We may then write $F \dashv G$ or $F : \mathcal{C} \leftrightarrows \mathcal{C}' : G$ but usually simply make explicit the natural isomorphism.

Examples 1. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ has both an adjoint (discrete topology) and a coadjoint (coarse topology):

$$\mathcal{C}(X^{\mathrm{disc}}, Y) \simeq \mathcal{F}(X, Y) \quad \text{et} \quad \mathcal{F}(X, Y) \simeq \mathcal{C}(X, Y^{\mathrm{coarse}}).$$

2. The forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ has an adjoint (free abelian group):

$$\mathrm{Hom}(\mathbb{Z} \cdot X, M) \simeq \mathcal{F}(X, M).$$

Exercise 1.55 Write explicitly what it means for

$$\Phi_{X,X'} : \mathrm{Hom}(F(X), X') \simeq \mathrm{Hom}(X, G(X'))$$

and its inverse to be natural.

Solution. Given $g : Y \to X, g' : X' \to Y'$, then

$$\forall f : F(X) \to X', \quad \Phi_{Y,Y'}(g' \circ f \circ F(g)) = G(g') \circ \Phi_{X,X'}(f) \circ g \quad \text{and}$$

$$\forall f' : X' \to G(X), \quad \Phi^{-1}_{Y,Y'}(G(g') \circ f \circ g) = g' \circ \Phi^{-1}_{X,X'}(f) \circ F(g). \qquad \blacksquare$$

Exercise 1.56 Show that most forgetful and inclusion functors we have already met have an adjoint (and sometimes a coadjoint) and make them explicit.

Exercise 1.57 Show that the functor $X \mapsto X \times Y$ from $\mathbf{Set}$ to itself is adjoint to the functor $Z \mapsto \mathcal{F}(Y, Z)$:

$$\mathcal{F}(X \times Y, Z) \simeq \mathcal{F}(X, \mathcal{F}(Y, Z)).$$

This is called *Currying*. Analogue for $\mathbf{Cat}$, $\mathbf{Ab}$?

Exercise 1.58 Show that if both $F_1 \dashv G$ and $F_2 \dashv G$, then $F_1 \simeq F_2$ (and dual).

Solution. Both $F_1(X)$ and $F_2(X)$ represent the same functor $X' \mapsto \mathrm{Hom}(X, G(X'))$ and there exists therefore a isomorphism $F_1(X) \simeq F_2(X)$ which is easily seen to be natural. ∎

Exercise 1.59 Show that if $F_1 : \mathcal{C} \leftrightarrows \mathcal{D} : G_1$ and $F_2 : \mathcal{D} \leftrightarrows \mathcal{E} : G_2$ then, $F_2 \circ F_1 \dashv G_1 \circ G_2$.

1.5.2 Unit and counit

Proposition 1.5.2 A functor $F : \mathcal{C} \to \mathcal{C}'$ is adjoint to G if and only if there exists $\alpha : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\beta : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{C}'}$ such that $\beta_F \circ F(\alpha) = \mathrm{Id}_F$ and $G(\beta) \circ \alpha_G = \mathrm{id}_G$.

Proof. Assume first that there exists a natural isomorphism

$$\Phi_{X,X'} : \mathrm{Hom}(F(X), X') \simeq \mathrm{Hom}(X, G(X')).$$

We may then set $\alpha_X := \Phi_{X,F(X)}(\mathrm{Id}_{F(X)})$ and, dually, $\beta_{X'} := \Phi^{-1}_{G(X'),X'}(\mathrm{Id}_{G(X')})$. Then, we have

$$\begin{aligned}
\beta_{F(X)} \circ F(\alpha_X) &= \Phi^{-1}_{G(F(X)),F(X)}(\mathrm{Id}_{G(F(X))}) \circ F(\Phi_{X,F(X)}(\mathrm{Id}_{F(X)})) \\
&= \Phi^{-1}_{X,F(X)}(\Phi_{X,F(X)}(\mathrm{Id}_{F(X)})) \\
&= \mathrm{Id}_{F(X)}
\end{aligned}$$

and symetrically. Conversely, given $f : F(X) \to X'$, we set

$$\Phi_{X,X'}(f) : X \xrightarrow{\alpha_X} G(F(X)) \xrightarrow{G(f)} G(X')$$

and define dually for $f' : X \to G(X')$

$$\Psi_{X,X'}(f') : F(X) \xrightarrow{F(f')} F(G(X')) \xrightarrow{\beta_{X'}} X'.$$

Then, we have

$$(\Psi_{X,X'} \circ \Phi_{X,X'})(f) = \beta_{X'} \circ F(G(f)) \circ F(\alpha_X) = f \circ \beta_{F(X)} \circ F(\alpha_X) = f$$

and symetrically. ∎

Definition 1.5.3 The morphisms α and β are called *unit* and *counit* (or else *adjunction morphisms*).

Exercise 1.60 Assume $F \dashv G$ with unit α and counit β. Show that G is faithful (resp. fully faithful) if and only if α_X is always a monomorphism (resp. an isomorphism). Analogue for G ?

Solution. Let us consider composite map

$$\mathrm{Hom}(Y,X) \to \mathrm{Hom}(F(Y),F(X)) \simeq \mathrm{Hom}(Y,G(F(X)))$$

where the first one is $f \mapsto F(f)$ and the second one is the adjunction $\Phi_{Y,F(X)}$. We have

$$\Phi_{Y,F(X)}(F(f)) = \alpha_Y \circ G(F(f)) = f \circ \alpha_X = h_{\alpha_X}(f).$$

Thus we see that G is faithful (resp. fully faithful) if and only h_{α_X} injective (resp. bijective) for all X and all Y. This means that α_X is a monomorphism (resp. an isomorphism) for all X.

Now, we have $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$ and the unit for this adjunction is β^{op}. Moreover, $\beta^{\mathrm{op}}_{X'}$ is a monomorphism (resp. an isomorphism) if and only if $\beta_{X'}$ is an epimorphism (resp. an isomorphism). Therefore, G is faithful (resp. fully faithful) if and only if G^{op} is faithful (resp. fully faithful) if and only if $\beta_{X'}$ is an epimorphism (resp. an isomorphism) for all X'. ■

Exercise 1.61 Describe unit and counit in all the examples studied so far. Deduce in each case faithfulness or full faithfulness of the functors.

Exercise 1.62 Show that, if a small category $\mathcal{C}$ has copowers, then all representable functors F on $\mathcal{C}$ have an adjoint.

Solution. We may assume that $F = h^X$, consider the functor

$$\mathbf{Set} \to \mathcal{C}, \quad I \mapsto X^{(I)},$$

define unit $I \to \mathrm{Hom}(X, X^{(I)})$ and counit $X^{(\mathrm{Hom}(X,Y))} \to Y$ and check the properties. ■

1.5.3 Adjoint and limit

Proposition 1.5.4 In a category $\mathcal{C}$, all limits on I exist if and only if the functor $X \mapsto \underline{X}$ has a coadjoint which is then given by $D \mapsto \varprojlim D$.

Proof. We have indeed a natural isomorphism in X given by

$$\mathrm{Hom}(\underline{X}, D) \simeq \mathrm{Hom}(X, \varprojlim D)$$

and it is sufficient to show that it is also natural in D. This immediately follows from the universal property of limits. ■

Exercise 1.63 Show that any adjunction $F \dashv G$ extends to an adjunction on diagrams on I:

$$\mathrm{Hom}(F(D), E) \simeq \mathrm{Hom}(D, G(E)).$$

Theorem 1.5.5 A functor that admits an adjoint (resp. a coadjoint) preserves all limits (resp. colimits).

Proof. Assume that $F : \mathcal{C} \to \mathcal{D}$ is adjoint to a functor G and that we are given a diagram D in $\mathcal{D}$. If $X \in \mathcal{C}$, then there exists a sequence of natural isomorphisms

$$\mathrm{Hom}(X, G(\varprojlim D)) \simeq \mathrm{Hom}(F(X), \varprojlim D) \simeq \mathrm{Hom}(\underline{F(X)}, D)$$

$$= \mathrm{Hom}(F(\underline{X}), D) \simeq \mathrm{Hom}(\underline{X}, G(D)) \simeq \mathrm{Hom}(X, \varprojlim G(D)).$$

It follows that $G(\varprojlim D)$ and $\varprojlim G(D)$ represent the same functor and are therefore naturally isomorphic. ∎

As a consequence, we see that limits (resp. colimits) commute with limits (resp. colimits).

Example As we already noticed, the forgetful functor **Top** $\to$ **Set** preserves all limits and all colimits and the forgetful functor **Ab** $\to$ **Set** preserves all limits.

There exists a partial converse which is called *Freyd adjunction theorem*:

Theorem 1.5.6 If $\mathcal{D}$ is a small complete category, then any functor $G : \mathcal{D} \to \mathcal{C}$ that preserves limits has an adjoint (and dual).

Proof. (Sketch) It is sufficient to set

$$FX := \varprojlim_{X \to GY} Y.$$ ∎

One can replace the smallness condition on $\mathcal{D}$ by the weaker *solution-set condition*: given any X in $\mathcal{C}$, there exists a *set* of morphisms $X \to GY_i$ such that any morphism $X \to GY$ factors through some GY_i.

Exercise 1.64 Assume F is adjoint to a fully faithful G. Show that if D' is a diagram in $\mathcal{C}'$ and $X = \varprojlim G(D')$, then $X' := F(X) = \varprojlim D'$ and $X \simeq G(X')$.

Exercise 1.65 Show that the forgetful functors on **Mon**, **Gr**, **Rng**, G-**Set**, A-**Mod** and k-**Alg** preserve all limits.

Exercise 1.66 Study the exactness of various forgetful and inclusion functors as well as their adjoints.

Exercise 1.67 Show that if all limits on I exist in $\mathcal{C}$, then all limits on I exist in $\mathcal{C}^J$ and that, if $D \in \mathcal{C}^{I \times J} (\simeq (\mathcal{C}^J)^I)$, then

$$\forall j \in J, \quad \left(\varprojlim_i D\right)_j = \varprojlim_i D_j.$$

Analogue for colimits.

Proposition 1.5.7 Filtered colimits of *sets* are exact.

Proof. The point is to show that if I is a directed set, then the functor

$$\varinjlim : \mathbf{Set}^I \to \mathbf{Set}$$

is exact. It is sufficient to treat the cases of a final object, a product of two objects or a kernel. We shall also use the fact that colimits of diagrams are computed term-wise (see exercice 1.67). Clearly, $\varinjlim \{0\} = \{0\}$. We consider now two families of morphisms $(u_{ij} : X_i \to X_j)$ and $(v_{ij} : Y_i \to Y_j)$ defined for $i < j$ with $u_{jk} \circ u_{ij} = u_{ik}$ and $v_{jk} \circ v_{ij} = v_{ik}$ when $i < j < k$. Recall that one can write $\varinjlim X_i = \coprod X_i / \sim$ and we will denote by $\overline{x}_i$ the class of $x_i \in X_i$ (and we will do the same for other colimits on I). Recall also that, for all $i < j$, we have $\overline{u_{ij}(x_i)} = \overline{x}_i$ and that we may therefore always replace any x_i with some $x_j \in X_j$ when $j > i$ without changing its class. By considering colimits on the projections, we have the following morphisms

$$\varinjlim (X_i \times Y_i) \to \varinjlim X_i \quad \text{and} \quad \varinjlim (X_i \times Y_i) \to \varinjlim Y_i.$$

The universal property of products provides us with a map

$$\varinjlim (X_i \times Y_i) \to \varinjlim X_i \times \varinjlim Y_i, \quad \overline{(x_i, y_i)} \mapsto (\overline{x}_i, \overline{y}_i).$$

and we have to show that it is bijective. Assume that $(\overline{x}_i, \overline{y}_i) = (\overline{x}'_j, \overline{y}'_j)$. Since I is directed, there exists $k \geq i, j$ and we may assume that $i = j = k$ so that $(x_i, y_i) = (x'_i, y'_i)$. This shows injectivity. Assume now given $x_i \in X_i$ and $x_j \in Y_j$. After replacement of i and j by $k \geq i, j$, we may assume that $i = j = k$. Surjectivity follows. This takes care of the product of two sets and we will treat the case of kernels in the same way. We give ourselves two families of maps $(f_i, g_i : X_i \to Y_i)$ compatible with u_{ij}'s and v_{ij}'s. The universal property of kernels (and the explicit description of colimits of sets) provides us with an inclusion

$$\varinjlim \ker(f_i, g_i) \subset \ker(\varinjlim f_i, \varinjlim g_i)$$

We need to show that this is an equality. We give ourselves $\overline{x}_i \in \varinjlim X_i$ such that $\overline{f_i(x_i)} = \overline{g_i(x_i)}$. As usual, there exists $k \geq i, j$ and we may thererefore assume that $i = j = k$. ∎

Concretely, the proposition states that, if I is a *filtered* category, J is a *finite* category and $(X_{i,j})$ is a diagram of *sets* based on $I \times J$, then we have

$$\varinjlim_i \varprojlim_j X_{ij} \simeq \varprojlim_j \varinjlim_i X_{ij}.$$

Exercise 1.68 Show that filtered colimits are exact in $\mathbf{Ab}$, etc. and that they commute with the forgetful functors (one may show the second assertion first).

Exercise 1.69 Show that $\mathbf{Ab}$ satisfies AB6 extra condition which means that filtered colimits commute with products:

$$\prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j} \simeq \varinjlim_{\prod_{j \in J} I_j} \prod_{j \in J} M_{i_j}$$

when each I_j is filtered.

1.5.4 Reflective subcategory

Definition 1.5.8 A full subcategory $\mathcal{C}' \subset \mathcal{C}$ is said to be *reflective* if the inclusion functor has an adjoint, called *reflection*. The dual terminology is *coreflective*.

In other words, a functor $F : \mathcal{C} \to \mathcal{C}'$ is a reflection if there exists a natural bijection

$$\mathrm{Hom}(F(X), X') \simeq \mathrm{Hom}(X, X')$$

when $X' \in \mathcal{C}'$. It means that there exists a natural map $X \to F(X)$ with the following universal property:

$$\begin{tikzcd} X \arrow[r, "\forall f"] \arrow[d] & X'. \\ F(X) \arrow[ur, dashed, "\exists ! \bar{f}"'] & \end{tikzcd}$$

Examples 1. The category of abelian groups is a reflective subcategory of the category of all groups.
2. The category of groups is both a reflective and a corefllective subcategory of the category of all monoids.
3. We shall show that the category of compact Hausdorff spaces is a reflective subcategory of the category of all topological spaces.
4. The category of sets is isomorphic to the reflective (resp. coreflective) subcategory of discrete (resp. chaotic) topological spaces.

Exercise 1.70 Show that if $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$, then any diagram D' in $\mathcal{C}'$ that has a limit (resp. a colimit) in $\mathcal{C}$ has a limit (resp. a colimit) in $\mathcal{C}'$. Show also that $\mathcal{C}'$ is stable under limits that exist in $\mathcal{C}$. Finally, show that, if a colimit in $\mathcal{C}$ of objects of $\mathcal{C}'$ is an object of $\mathcal{C}'$, then this is a colimit in $\mathcal{C}'$.

Exercise 1.71 Show that, if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$, then a functor $F : \mathcal{C} \to \mathcal{C}'$ is a reflection if and only if there exists a natural morphism $\alpha_X : X \to F(X)$ for $X \in \mathcal{C}$ such that α_X is an isomorphism when $X \in \mathcal{C}'$.

Solution. The subcategory $\mathcal{C}'$ is reflective with reflection F if and only if there exists a natural morphism $\alpha_X : X \to F(X)$ for $X \in \mathcal{C}$ and a natural isomorphism $\beta_X : F(X) \simeq X$ when $X \in \mathcal{C}'$ such that $\beta_{F(X)} \circ F(\alpha_X) = \mathrm{Id}_{F(X)}$ when $X \in \mathcal{C}$ and $\beta_X \circ \alpha_X = \mathrm{id}_X$ when $X \in \mathcal{C}'$. This is clearly equivalent to our condition with $\beta_X := \alpha_X^{-1}$. ■

Exercise 1.72 Show that if $\mathcal{C}'$ is a coreflective subcategory of $\mathcal{C}$, then $\mathbf{Mon}(\mathcal{C}')$ is a full subcategory of $\mathbf{Mon}(\mathcal{C})$. Same thing with $\mathbf{Gr}$, $\mathbf{Ab}$, etc.

Exercise 1.73 Show that, if filtered colimits exist in $\mathcal{C}$, then $\mathcal{C}$ is a reflective subcategory of $\mathrm{Ind}(\mathcal{C})$ with adjoint "$\varinjlim X_i$" $\mapsto \varinjlim X_i$.

1.5.5 Kan extension (optional)

Definition 1.5.9 Let $p : \mathcal{C} \to \mathcal{C}'$ be a functor between small categories. The *(left) Kan extension* of a functor $F : \mathcal{C} \to \mathcal{D}$ along p is a functor $p!F$ which is universal for all functors $G : \mathcal{C}' \to \mathcal{D}$ and natural transformations $F \Rightarrow p^{-1}G := G \circ p$.

In other words, $p!F$ represents the functor $G \mapsto \mathrm{Hom}(F, G \circ p)$ on the category $\mathbf{Hom}(\mathcal{C}', \mathcal{D})$. It means that $p!F : \mathcal{C}' \to \mathcal{D}$ is endowed with a natural transformation $\alpha : F \Rightarrow p^{-1}p_!F$ such that, given any natural transformation $\gamma : F \Rightarrow p^{-1}G$, there exists a unique natural transformation $\widetilde{\gamma} : p_!F \Rightarrow G$ such that $\gamma = p^{-1}(\widetilde{\gamma}) \circ \alpha$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "p"'] & \mathcal{D} \\ \mathcal{C}' \arrow[ur, dashed, "p_!F \Rightarrow G"'] & \end{tikzcd} \quad \text{(not commutative)} \quad \text{and} \quad \begin{tikzcd} F \arrow[r, Rightarrow] \arrow[d, Rightarrow] & p^{-1}G. \\ p^{-1}p_!F \arrow[ur, Rightarrow] & \end{tikzcd}$$

There exists the dual notion of a *right Kan extension* p_*F with $\gamma : p^{-1}G \Rightarrow F$ this time.

Example 1. A diagram $D : I \to \mathcal{C}$ has colimit X if and only if the constant functor $\mathbf{1} \to \mathcal{C}, 0 \mapsto X$ is the Kan extenstion of D along the projection $I \to \mathbf{1}$:

$$\begin{tikzcd} I \arrow[r, "D"] \arrow[d] & \mathcal{C} \\ \mathbf{1} \arrow[ur, dashed, "X \to Y"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[r] \arrow[d] & \underline{Y}. \\ \underline{X} \arrow[ur] & \end{tikzcd}$$

2. A functor $F : \mathcal{C} \to \mathcal{D}$ between small categories has a coadjoint G if and only if the Kan extension of $\mathrm{Id}_{\mathcal{C}}$ along F exists and $F \circ F_!\mathrm{Id}_{\mathcal{C}} = F!F$, in which case $G = F_!\mathrm{Id}_{\mathcal{C}}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, equal] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \\ \mathcal{D} \arrow[ur, dashed, "G"] \arrow[urr, dashed, "F \circ G"'] & & \end{tikzcd} \quad \begin{tikzcd} \mathrm{Id}_{\mathcal{C}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & G' \circ F \\ G \circ F \arrow[ur, Rightarrow] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r, Rightarrow] \arrow[d, Rightarrow] & H \circ F. \\ F \circ G \circ F \arrow[ur, Rightarrow] & \end{tikzcd}$$

Proposition 1.5.10 Let $p : \mathcal{C} \to \mathcal{C}'$ be functor between small categories. Then the functor

$$p^{-1} : \mathbf{Hom}(\mathcal{C}', \mathcal{D}) \to \mathbf{Hom}(\mathcal{C}, \mathcal{D}), \quad G \mapsto G \circ p.$$

has an adjoint $p_!$ if and only if all Kan extensions along p with values in $\mathcal{D}$ exist (and dual).

Proof. Follows immediately from the definition. ∎

Proposition 1.5.11 If all colimits exist in $\mathcal{D}$, then the Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along $p : \mathcal{C} \to \mathcal{C}'$ always exists (and dual).

Proof. (Sketch) We set $(p_! F)(X') := \varinjlim_{p(X) \to X'} F(X)$ and check. ■

Exercise 1.74 Show that if $\mathcal{C}$ is small, all colimits exist in $\mathcal{D}$ and $F : \mathcal{C} \hookrightarrow \mathcal{C}'$ is fully faithful, then $F \simeq p!F \circ p$ (and dual).

2. Topology

2.1 Compact Hausdorff space

2.1.1 Compact/Hausdorff space

Definition 2.1.1 A topological space X is said to be
- *Fréchet* if all points are closed,
- *Hausdorff* if any two distinct points have disjoint neighborhoods,
- *normal* if any two disjoint closed subsets have disjoint neighborhoods.

One also say T_1 for Frechet, T_2 for Hausdorff and T_4 for normal Hausdorff (or equivalently normal Frechet).

Examples 1. A discrete space is normal Hausdorff.
2. A finite space is Fréchet if and only if it is discrete (in which case it is normal Hausdorff).
3. The space $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, in which Z is closed if and only if it is finite or contains $+\infty$, is normal Hausdorff.
4. The Sierpiński space $\{s, \eta\}$, in which s is the only closed point is normal but not Hausdorff[1].

Exercise 2.1 Show that a topological space X is Hausdorff it and only if the diagonal $\Delta_X \subset X \times X$ is closed.

Exercise 2.2 Show that any subspace of a Hausdorff space is Hausdorff. Show that a non empty product of topological spaces is Hausdorff if and only if each of them is Hausdorff. Show that any limit (but not colimit) of Hausdorff spaces is Hausdorff.

[1]But it is T_0-space (or Kolmogorov).

Exercise 2.3 Show that the category of Hausdorff topological spaces is a reflective subcategory of the category of all topological spaces (even if there does not exist an explicit reflection).

Solution. Follows from Freyd's adjunction theorem (the solution set is obtained by considering any Hausdorff topology on a quotient $X/\sim$). ■

Exercise 2.4 Let $\pi : X \twoheadrightarrow Y$ be a continuous *closed* surjective map. Show that if X is Fréchet (resp. normal, resp. normal Hausdorff), then Y is Fréchet (resp. normal, resp. normal Hausdorff).

Solution. The Fréchet case is obvious. Let, for $i = 1, 2$, F_i be disjoint closed subsets of Y. Then $\pi^{-1}(F_i)$ are disjoint closed subsets of X. If X is normal, then there exists disjoint open neighborhoods U_i of $\pi^{-1}(F_i)$. It follows that $(\pi(U_i^c))^c$ are disjoint open neighborhoods of F_i. And Y is also normal. Finally, normal Fréchet implies Hausdorff. ■

Exercise 2.5 Show that a normal[a] Hausdorff space X is "locally closed": any neighborhood of $x \in X$ contains a closed neighborhood.

[a]Or even *regular*, meaning that one can separate closed subsets from points.

Lemma 2.1.2 — Urysohn. A topological space X is normal if and only if, given two non-empty closed disjoint subsets A and B, there exists a continuous map $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

Proof. Not trivial but classical. ■

Definition 2.1.3 A topological space S is said to be *compact*[a] if any open covering $S = \bigcup_{i\in I} U_i$ has a refinement $S = \bigcup_{i\in J} U_i$ with J finite.

[a]Some people say *quasi-compact* to insist on the fact that S may not be Hausdorff.

Alternatively, if a family of closed subsets $(F_i)_{i\in I}$ satisfies $\bigcap_{i\in J} F_i \neq \emptyset$ for all finite $J \subset I$, then $\bigcap_{i\in I} F_i \neq \emptyset$.

Examples
1. A finite space (such as the Sierpiński space) is compact (but not Hausdorff in général).
2. A discrete space is compact if and only if it is finite (in which case it is compact Hausdorff).
3. The space $\overline{\mathbb{N}}$ is compact Hausdorff.

Exercise 2.6 Show that a closed subspace of a compact space is compact. Show that the image of a compact space by a continuous map is compact. Show that a finite disjoint union of compact spaces is compact. Show that a compact subspace of a *Hausdorff* space is closed. Show that a continuous (resp. bijective continuous) map from a compact space to a Hausdorff space is closed (resp. a homeomorphism).

Exercise 2.7 Show that a compact Hausdorff space is normal.

Exercise 2.8 Let R be an equivalence relation on a compact topological space S. Show that S/R is compact. Assume now that S is compact Hausdorff. Show that S/R is compact Hausdorff if and only if the subspace $R \subset S \times S$ is closed if and only if the map $S \twoheadrightarrow S/R$ is closed.

Solution. The first assertion follows from the fact that the image of a compact topological space by a continuous map is always compact. We also know that an equivalence relation is closed if and only if the quotient is Hausdorff. Also, if S is compact Hausdorff and $\pi : S \twoheadrightarrow S/R$ is closed, then this is a closed continuous surjective map and S is normal Hausdorff. Then, we know that S/R is Hausdorff. Finally, if S is compact and S/R is Hausdorff, then π is closed because any closed subset of S is compact and any compact subset of S/R is closed. ■

Exercise 2.9 Show that, for a continuous map $\pi : S \to T$ of compact Hausdorff spaces, the following are equivalent

1. π is a quotient map,
2. π is a regular epimorphism,
3. π is an epimorphism,
4. π is surjective.

Solution. This is a straightforwards circular proof. ■

Theorem 2.1.4 — Tykhonov. Any product of compact spaces is compact.

Proof. Not trivial but classical (equivalent to Axiom of Choice). ■

The following result will also follow from proposition 2.1.7 below:

Exercise 2.10 Show that any limit of compact Hausdorff spaces is compact Hausdorff.

Solution. Thanks to Tykhonov theorem, it is sufficient to show that the kernel of two maps $f, g : S \to T$ of compact Hausdorff spaces is compact Hausdorff, or equivalently, closed. But $\ker(f, g)$ is the inverse image of the diagonal $T \subset T \times T$, which is closed since T is Hausdorff, by the continuous map $S \to T \times T, x \mapsto (f(x), g(x))$. ■

We shall need later the following lemma:

Lemma 2.1.5 If $f : S \twoheadrightarrow T$ is a continuous surjection of compact Hausdorff spaces, then there exists a minimal closed subset S' of S such that the restriction of f to S' is surjective.

Proof. We use Zorn's lemma. Thus, we give ourselves a decreasing family S_i of closed subsets of S that for all $i \in I$, the restriction of f to S_i is surjective. If $y \in T$, then $f^{-1}(y)$ is closed and meets any of the S_i which are also closed. Since $f^{-1}(y)$ is a compact subset of S, it also meets necessarily $S' := \bigcap_{i \in I} S_i$. ■

2.1.2 Stone-Čech compactification

Definition 2.1.6 The *Stone-Čech* compactification of a topological space X is the closure βX of the image of the map

$$X \to [0,1]^{\mathcal{C}(X,[0,1])}, \quad x \mapsto (f(x))_{f\in\mathcal{C}(X,[0,1])}.$$

It is hard in general (even when X is discrete infinite) to describe of βX.

Proposition 2.1.7 The category of compact Hausdorff spaces is a reflective subcategory of the the category of all topological spaces with Stone-Čech compactification as reflection.

Proof. We have to prove that Stone-Čech compactification $X \mapsto \beta X$ is adjoint to inclusion. It follows from Tykhonov's theorem that βX is compact Hausdorff. Also the canonical map $X \to \beta X$ is continuous. In order to see this, it is sufficient to show that the map $X \to [0,1]^{\mathcal{C}(X,[0,1])}$ is continuous but this may be checked on some fixed factor $f : X \to [0,1]$ which is continuous by definition. Let us now show that the construction is functorial. Any continuous map $\phi : X \to Y$ will induce a map

$$\mathcal{C}(Y,[0,1]) \to \mathcal{C}(X,[0,1]), \quad g \mapsto \phi^* g := g \circ \phi$$

which, in turn will induce a map

$$\phi_* : [0,1]^{\mathcal{C}(X,[0,1])} \to [0,1]^{\mathcal{C}(Y,[0,1])}, \quad (t_f) \mapsto (t_{\phi^* g}).$$

In order to show that this map is continuous, it is sufficient again to consider its components for various $g \in \mathcal{C}(Y,[0,1])$:

$$[0,1]^{\mathcal{C}(X,[0,1])} \to [0,1], \quad (t_f) \mapsto t_{\phi^*(g)}.$$

But such a component is simply a projection and therefore continuous. Moreover, the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\phi"] & {[0,1]^{\mathcal{C}(X,[0,1])}} \arrow[d, "\phi_*"] \\ Y \arrow[r] & {[0,1]^{\mathcal{C}(Y,[0,1])}} \end{tikzcd}$$

is clearly commutative. It follows that the closure of the image of X is sent into the closure of the image of Y and we do get a continuous map $\beta\phi : \beta X \to \beta Y$. Finally, it follows from Urysohn's lemma that, when S is normal Hausdorff, the natural map $S \to \beta S$ is injective: if $x \neq x' \in S$, then there exists a continuous map $f : S \to [0,1]$ such that $f(x) = 0$ and $f(x') = 1$. Thus, when S is compact Hausdorff, the map $S \to \beta S$ is a homeomorphism : this is a continuous injective map with dense image between compact Hausdorff spaces.

Our assertion therefore follows from exercise 1.71. ■

Be careful that the map $S \to \beta S$ may not be injective when S is not normal Hausdorff.

As a consequence of proposition 2.1.7, any limit (in the category of topological spaces) of compact Hausdorff spaces is automatically compact Hausdorff. In particular, a topological monoid (resp. a group, resp. an abelian group) which is compact Hausdorff is the same thing as a monoid (resp. a group, resp. an abelian group) in the category of compact Hausdorff spaces. There also exists colimits obtained by applying β to the colimit of topological spaces. When we write $\varinjlim X_i$, we always mean the colimit in the category of topological spaces, even if all X_i's are compact Hausdorff. We shall then write $\beta \varinjlim X_i$ if we ever need to consider the colimit in the category of compact Hausdorff spaces.

It is important to remember that there exists a natural bijection

$$\mathcal{C}(\beta X, S) \simeq \mathcal{C}(X, S)$$

when S is compact Hausdorff.

2.2 Projective objects

2.2.1 Free compact Hausdorff space

Definition 2.2.1 A *free compact Hausdorff space* is a topological space F homeomorphic to the Stone-Čech compactification of a discrete space I.

There is a natural bijection $\mathcal{C}(F, S) \simeq S^I$ when S is compact Hausdorff. In other words, the functor $I \mapsto \beta I^{\text{disc}}$ is adjoint to the forgetful functor from compact Hausdorff spaces to sets. One may then call the set I a *basis* for F : it is equivalent to give a continuous map $F \to S$ or a family $(t_i)_{i \in I}$ of elements of S. When $F \twoheadrightarrow S$ is surjective, we may call $(t_i)_{i \in I}$ a set of *generators* for S.

Let us recall that $\mathcal{F} \subset \mathcal{P}(I)$ is a *(proper) filter* on a set I if 0) $\emptyset \notin \mathcal{F}$, 1) $\forall J_1, J_2 \in \mathcal{F}, J_1 \cap J_2 \in \mathcal{F}$ and 2) $\forall J \subset J', J \in \mathcal{F} \Rightarrow J' \in \mathcal{F}$. For example, if $i \in I$, then $\mathcal{F} := \{J, i \in J\}$ is a maximal filter[2].

Exercise 2.11 Show that the free compact Hausdorff space on a set I is (homeomorphic to) the set F of maximal filters on I with the topology generated by the $U_J := \{\mathcal{F} \in F, J \in \mathcal{F}\}$ for $J \subset I$.

Exercise 2.12 Show that, if S, T are two (free) compact Hausdorff spaces, then $S \sqcup T$ is the coproduct in the category of (free) compact Hausdorff spaces.

Proposition 2.2.2
1. Any compact Hausdorff space is a quotient of a free compact Hausdorff space.
2. A free compact Hausdorff space is a projective object in the category of compact Hausdorff spaces.

Proof. 1. If S is a compact Hausdorff space, then there exists a commutative

[2] Also called ultrafilter.

diagram

$$\begin{tikzcd} \beta S^{\mathrm{disc}} \arrow[r] & \beta S \\ S^{\mathrm{disc}} \arrow[u] \arrow[r, two heads] & S \arrow[u, "\simeq"'] \end{tikzcd}$$

and therefore a natural continuous surjection $F := \beta S^{\mathrm{disc}} \twoheadrightarrow S$.

2. Assume given a free compact Hausdorff space F, a continuous surjection of compact Hausdorff spaces $T \twoheadrightarrow S$ and continuous map $F \to S$. We have $F \simeq \beta I$ where I is a discrete space and the composite map $I \to \beta I \simeq F \to S$ lifts to a (automatically) continuous map $I \to T$. Since T is compact Hausdorff, it extends uniquely to $F \simeq \beta I \to T$. The universal property shows that it is indeed a lifting of the original map. ■

In particular, we see that the category of compact Hausdorff spaces has enough projectives.

Definition 2.2.3 A commutative diagram $F' \rightrightarrows F \to S$ of compact Hausdorff spaces is called a *free presentation* if both F, F' are free and both $F \twoheadrightarrow S$ and $F' \twoheadrightarrow R := F \times_S F$ are surjective.

Recall that Noether 1st isomorphism in the category of sets : if $f : X \to Y$ is any map, then $R := X \times_Y X$ is (the graph of) an equivalence relation on X and $X/R \simeq \mathrm{im}(f)$. Conversely, if R is an equivalence relation on X and $Y := X/R$, then $R = X \times_Y X$. In particular, the last condition in the definition means that the image R of F' in $F \times F$ is an equivalence relation and $F/R \simeq S$. Also, automatically $\mathrm{coker}(F' \rightrightarrows F) \simeq S$ but this is a weaker condition in the sense that the image of F' in $F \times F$ might not be an equivalence relation.

Corollary 2.2.4 Any compact Hausdorff space S has a free presentation. Actually, any continuous surjection $F \to S$ from a free compact Hausdorff space extends to a free presentation. ■

2.2.2 Totally/Extremally disconnected space

Definition 2.2.5 Let X be a topological space. A subset of X is said to be *clopen* if it is both open and closed. The space X is said to be *connected* if only $\emptyset$ and X are clopen in X. The *connected components* in X are the maximal non-empty connected subspaces.

Exercise 2.13 Show that the image of a connected space by a continuous map is always connected. Show that the closure of a connected subset is connected. Show that a connected component is closed (but not necessarily open). Show that connected components are disjoints. Show that a clopen subset is a union of connected components.

Exercise 2.14 Show that, if $x \in X$, then the union $C(x)$ of all connected subsets of X containing x is the unique connected component in X containing x (it is called the *connected component of x in X*).

Proposition 2.2.6 If S is compact Hausdorff, then the connected component of $x \in S$ is the intersection of all clopen neighborhoods of x.

Proof. The connected component of x is always contained in the intersection C of all clopen neighborhoods of x. It remains to show that C is connected. Assume that $C = F \cup G$ is the disjoint union of two closed subsets of C. Since C is closed in S, then F and G are also closed in S. Since S is compact Hausdorff, it is normal and there exist disjoint open subsets $U, V \subset S$ with $F \subset U$ and $G \subset V$. The complement K of $U \cup V$ in S is a compact subset and $K \cap C = \emptyset$. Recall now that C is the intersection of all clopen neighborhoods of x. Since a finite intersection of clopen is always clopen and K is compact, there must exist a clopen neighborhood H of x such that $K \cap H = \emptyset$. In other words, we have $H \subset U \cup V$. Since U and V are disjoint open subsets of S, then $H \cap U$ and $H \cap V$ are clopen in H and therefore also in S. We may assume $x \in F$ (or else, it is in G) so that $x \in H \cap U$. Then, $C \subset H \cap U$ so that $C \cap V = \emptyset$ which implies that $G = \emptyset$. ■

Definition 2.2.7 A topological space X is said to be *totally* (resp. *extremally*) *disconnected* if the connected components in X are the points (resp. if the closure of any open subset is clopen). If, moreover, X is compact Hausdorff, then it is called a *Stone* (resp. *Stonean*) space.

Examples 1. $\mathbb{Q}$ is totally disconnected but not extremally disconnected.
2. The Sierpiński space $\{s, \eta\}$ is extremally disconnected but not totally disconnected.
3. The space $\overline{\mathbb{N}}$ is Stone (but not Stonean).
4. The space $\beta\mathbb{N}$ is Stonean.
5. If p is a prime, then

$$\mathbb{Z}_p =: \left\{ \sum_{i=0}^{\infty} a_i p^i, \quad a_i \in \mathbb{Z} \right\} \simeq \varprojlim_n \mathbb{Z}/p^{n+1}\mathbb{Z}$$

is Stone (but not Stonean).
6. A Stonean space is metrizable if and only if it is finite discrete.

2.2.3 Stone space

We shall denote by $\pi_0(X)$ the set of connected components in a topological space X. There exists an obvious surjection $C : X \twoheadrightarrow \pi_0(X)$ that sends a point x to its connected component $C(x)$ and we endow $\pi_0(X)$ with the quotient topology. We obtain a functor $X \mapsto \pi_0(X)$ and a natural transformation $C : X \twoheadrightarrow \pi_0(X)$.

Exercise 2.15 Show that the category of totally disconnected spaces is a reflective subcategory of the category of all topological spaces with reflection π_0.

Solution. The natural map $C : X \to \pi_0(X)$ is an isomorphism if and only if X is totally disconnected. Our assertion therefore follows from exercise 1.71. ■

As a consequence, any limit of totally disconnected spaces is totally disconnected (and there also exists colimits obtained by applying π_0 to the limit of the topological spaces).

Exercise 2.16 Show that a subspace of a totally disconnected space is totally disconnected.

Exercise 2.17 Show that the category of Stone spaces is a reflective subcategory of the category of compact Hausdorff spaces with reflection π_0.

Solution. The point is to show that, if S is compact Hausdorff, then $\pi_0(S)$ is Hausdorff. Let C and C' be two distinct connected components of S. Since C' is compact, and clopen are stable under finite intersection, it follows from proposition 2.2.6 that there exists a clopen U such that $C \subset U$ and $C' \cap U = \emptyset$. Then, U is a union of connected components. In other words, we have $U = \pi_0^{-1}(V)$ where V is a (necessarily) clopen neighborhood of $C \in \pi_0(S)$ and by construction, $C' \notin V$. ■

As a consequence, any limit of Stone spaces is Stone (and there also exists colimits obtained by applying successively β and π_0 to the colimit of topological spaces).

Definition 2.2.8 A *profinite* space is a limit of finite discrete topological spaces.

Exercise 2.18 Show that a profinite space is a directed limit of finite discrete topological spaces with surjective transitions maps.

Solution. If $S = \varprojlim_{i\in I} S_i$ and $J \subset I$ is finite, we set $S_J := \mathrm{im}(S \to \varprojlim_{i\in J} S_i)$ (it also works with diagrams indexed by small categories). Then, S_J is finite discrete, $S \twoheadrightarrow S_J$ is surjective and $S = \varprojlim_J S_J$. The result also follows from the proof of proposition 2.2.9 below. ■

From now on, when we write $S = \varprojlim_{i\in I} S_i$, we shall usually implicitly assume that I is a directed set and the maps $S_i \to S_j$ are surjective when $j \leq i$.

Proposition 2.2.9 A topological space is Stone if and only if it is profinite.

Proof. First of all, a profinite space is a Stone space as a limit of Stone spaces. Assume conversely that S is a Stone space. Let $\mathcal{E}$ be the set of finite families $E \subset \mathbf{Open}(S) \setminus \{\emptyset\}$ such that $S = \coprod_{U\in E} U$. If any $U' \in E'$ is contained is some necessarily unique $U \in E$, then send U' to U and call it restriction. If $E \in \mathcal{E}$, then there exists a continuous surjection $S \to E$, that sends x to U if $x \in U$ and they are compatible with restriction. Therefore, there exists a continuous map $S \to \varprojlim E$. Since S is compact, if $(U_E)_{E\in\mathcal{E}}$ is a system which is compatible with restriction, then $\cap_{E\in\mathcal{E}} U_E \neq \emptyset$: indeed, this is clearly true for a finite family and the U_E's are closed. It follows that the map is surjective. Now, since S is a Stone space, if $x \in S$, then $\{x\}$ it the intersection of all clopen containing x. It follows that the map is injective: if $x \neq y$, there exists U clopen such that $x \in U$ and $y \notin U$ and we may choose $E = \{U, U^c\}$. A bijective continuous map of compact Hausdorff spaces is a homeomorphism. ■

Exercise 2.19 Show that, if S is a profinite space with transition maps $\pi_i : S \to S_i$, then the $U_{x,i} = \pi_i^{-1}(\pi_i(x))$ for various i and $x \in S$ form a basis of clopen neighborhoods stable under intersection.

Solution. By definition of the inverse limit topology, the topology of S is generated by the $\pi_i^{-1}(U)$ when i runs through I and U runs through a basis of open subsets of S_i. Since the points form a basis of open subsets of S_i, we get exactly the $U_{x,i}$'s. Morevoer, we have $U_{x,j} \subset U_{x,i}$ for $i \leq j$, $U_{x,i} = U_{y,i}$ if $\pi_i(x) = \pi_i(y)$ and $U_{x,i} \cap U_{y,i} = \emptyset$ if $\pi_i(x) \neq \pi_i(y)$. ∎

Exercise 2.20 Show that, if S is a Stone space, then any open covering has a finite disjoint clopen refinement.

Solution. It follows from proposition 2.2.9 and exercise 2.18 that $S = \varprojlim_{i\in I} S_k$ is a directed limit of finite discrete spaces and we shall denote by $\pi_i : S \to S_i$ the projection. If $x \in S$, then the clopen subsets $U_{x,i} := \pi_i^{-1}(\pi_i(x))$ with $i \in I$ form a basis of open neighborhoods of x. Since S is compact, our covering has a finite refinement by open subsets of the form U_{x_k,i_k} with $k = 1, \ldots, r$. Since I is directed, there exists $i \geq i_k$ for $i = 1, \ldots, r$. Then, for $a \in S_i$, we set $U_a = \pi_i^{-1}(a)$. This is a finite disjoint clopen covering of S. This is also a refinement of our covering: if $x \in U_a$, then there exists k such that $x \in U_{x_k,i_k}$ and therefore $U_a = U_{x,i} \subset U_{x,i_k} = U_{x_k,i_k}$. ∎

Exercise 2.21 Show that the category of profinite sets (i.e. pro-objects of the category of finite sets) is equivalent to the category of profinite spaces.

Recall that a *Boolean ring* (or algebra) is a commutative ring A such that all $a \in A$ are *idempotent* ($a^2 = a$).

Exercise 2.22 Prove *Stone representation theorem*: the category of Boolean rings is anti-equivalent to the category of Stone spaces.

Solution. One endows $\mathrm{S}(A) := \mathrm{Hom}_{\mathbf{Rng}}(A, \mathbb{F}_2) \subset \mathbb{F}_2^A$ with the induced topology (and $\mathbb{F}_2$ with the discrete topology). A quasi-inverse is given by sending S to the set $\mathrm{A}(S)$ of all clopen subsets of S with the operations $a + b := (a \cup b) \setminus (a \cap b)$ and $ab = a \cap b$. ∎

Incidentally, the category of boolean rings itself is isomorphic to the category of *complemented distributive bounded lattices* (which is a full subcategory of the category of ordered sets).

2.2.4 Stonean space

Exercise 2.23 Show that if X is extremally disconnected and U, V are disjoint open subsets, then $\overline{U}$ and $\overline{V}$ are also disjoint.

Solution. We have $U \subset V^c$ which is closed and therefore $\overline{U} \subset V^c$ so that $V \subset (\overline{U})^c$ which is also closed because $\overline{U}$ is open and therefore $\overline{V} \subset (\overline{U})^c$. ∎

Alternatively, if $T = F \cup G$ is the union of two closed subsets of X, then $T = \mathring{F} \cup \mathring{G}$.

Lemma 2.2.10 An extremally disconnected *Hausdorff* space X is totally disconnected.

Proof. If $x \neq y \in X$, then there exists two disjoint open subsets U and V such that $x \in U$ and $y \in V$. Since X is extremally disconnected, $\overline{U}$ and $\overline{V}$ also are disjoint. Since $\overline{U}$ is a clopen containing x, we have $C(x) \subset \overline{U}$. It follows that $y \notin C(x)$. ∎

Theorem 2.2.11 — Gleason. The projective objects of the category of compact Hausdorff spaces are the Stonean spaces.

Proof. Assume first S is a projective compact Hausdorff space. Let U be an open subset of S and F its complement. Then, the continuous surjection $p : \overline{U} \sqcup F \twoheadrightarrow S$ of compact Hausdorff spaces has a continuous section s. Since $p(F) \cap U = \emptyset$, we have $s(U) \subset \overline{U}$. Since s is continuous, $s(\overline{U}) \subset \overline{U}$. It follows that $\overline{U} = s^{-1}(\overline{U})$ is open.

For the converse, thanks to exercise 1.49, it is sufficient to show that any continuous surjection $f : T \twoheadrightarrow S$ from a compact Hausdorff space to a Stonean space has a section. Thanks to lemma 2.1.5, we may assume that the restriction of f to any closed subset $T' \subsetneq T$ is never surjective. We shall then show that $f : T \simeq S$ is a homeomorphism and it is sufficient to prove that f is injective. Otherwise, there exists $x_1 \neq x_2 \in T$ with $f(x_1) = f(x_2) = y \in S$ and we may pick up disjoint neighborhoods U_i of x_i. Since f is closed, $S_i := f(U_i^c)$ is closed and since f is surjective, $S = S_1 \cup S_2$. Since S is extremally disconnected, we also have $S = \mathring{S}_1 \cup \mathring{S}_2$. We may assume that $y \in \mathring{S}_1$ and we set $T' := (U_1 \cap f^{-1}(\mathring{S}_1))^c$. By construction, T' is a closed subset with $x_1 \notin T'$ and the restriction of f to T' is surjective : we have

$$f(T') = f(U_1^c) \cup f(f^{-1}((\mathring{S}_1)^c)) = S_1 \cup (\mathring{S}_1)^c = S.$$

Contradiction. ∎

Corollary 2.2.12 A topological space is a Stonean space if and only if it is a retract of a free compact Hausdorff space. ∎

In particular, a free compact Hausdorff space is extremally disconnected.

2.3 Compactly generated space

The standard reference is Lewis 78 but you may also consider Strictland.

2.3.1 Definition

Definition 2.3.1 A topological space X is

1. *locally compact Hausdorff* if it is Hausdorff and any point has a compact neighborhood,
2. *compactly (Hausdorff) generated* (also called a *k-space*) if it is a colimit of compact Hausdorff spaces.

Examples 1. A locally compact Hausdorff space is compactly generated (this applies to topological manifolds).

2. A sequential space (a subset is closed if and only if it is stable under convergent sequences) is compactly generated (this applies to metric spaces).
3. If I is uncountable, then $\mathbb{R}^I$ and $\mathbb{Z}^I$ are *not* compactly generated ($\{(x_i)_{i\in I}, \exists J \subset I, |J| \ = n, x_i = 0$ for $i \in J, x_i = n$ for $i \notin J\}$ is k-closed but not closed).

Exercise 2.24 Show that, for a topological space X, the following conditions are equivalent:

1. X is compactly generated.
2. X is a quotient of a disjoint union of compact Hausdorff spaces.
3. X is a quotient of a locally compact Hausdorff space.
4. A subset Y of X is open (resp. closed) when, given any continuous map $f : S \to X$ from a compact Hausdorff space, $f^{-1}(Y)$ is open (resp. closed) in S.
5. A map $X \to Y$ is continuous when, given any continuous map $S \to X$ from a compact Hausdorff space, the composite map $S \to X \to Y$ is also continuous.
6. $X = \varinjlim_{S\to X} S$ when $S \to X$ runs through all continuous maps from a compact Hausdorff space.

Solution. This is a straightforwards circular proof. ■

Theorem 2.3.2 If $X \twoheadrightarrow Y$ is a quotient map and Z is locally compact Hausdorff, then $X \times Z \twoheadrightarrow Y \times Z$ is also a quotient map.

Proof. Classic (but not trivial). ■

Exercise 2.25 Show that, if X is compactly generated and Y locally compact Hausdorff, then $X \times Y$ is compactly generated[a].

[a]But compactly generated spaces are not stable under product in general.

Solution. We assume that $F \subset X \times Y$ is k-closed and we show that it is actually closed. It is sufficient to prove that, given any $(x, y) \notin F$, then there exists some neighborhoods U and V of x and y respectively such that $(U \times V) \cap F = \emptyset$. First of all, $(x, y) \notin (X \times y) \cap F$ which is k-closed, and therefore closed, since $X \times y \simeq X$ is compactly generated. But $X \times y$ is even locally compact Hausdorff and it follows that there exists a compact neighborhood S of x in X such that $(S \times y) \cap F = \emptyset$. After replacing X with S, we may therefore assume that X itself is compact Hausdorff and that $(X \times y) \cap F = \emptyset$. We set $U = X$ and $V := p(F)^c$ where $p : X \times Y \to Y$ denotes the second projection. It only remains to show that $p(F)$ is closed. Since Y is compactly generated, is is sufficient to show that, given any continuous map $f : K \to Y$ with K compact Hausdorff, $f^{-1}(p(F)))$ is closed. After replacing Y with K, we may therefore assume that Y itself is compact Hausdorff and then $p(F)$ is necessarily closed as the image of a closed subset by a continuous map between compact Hausdorff spaces. ■

Definition 2.3.3 A subset Y of a topological space X is said to be *k-open* (resp. *k-closed*) if for all continuous maps $f : S \to X$ from a compact Hausdorff space, $f^{-1}(Y)$ is open (resp. closed) in S. The corresponding topology on X is called the *k-topology* (or *compactly (Hausdorff) generated topology*).

We shall denote by kX the underlying set of X endowed with the k-topology.

Exercise 2.26 Show that the k-topology is indeed a topology and that

$$kX = \varinjlim_{S \to X} S$$

when S runs through all compact Hausdorff spaces.

Proposition 2.3.4 Compactly generated spaces form a coreflective subcategory of the category of topological spaces with coreflection k.

Proof. The assignment $X \mapsto kX$ is functorial, the identity is a natural continuous map $kX \to X$ which is a homeomorphism when X itself is compactly generated. Our assertion therefore follows from exercise 1.71. ■

As a consequence, compactly generated spaces are stable under all colimits (and consequently quotients[3]). We shall use the obvious properties $kkX = kX$ when X is any topological space and $\mathcal{C}(kX, kY) = \mathcal{C}(kX, Y)$ if Y is another topological space.

Exercise 2.27 Show that if X, Y are two topological spaces, then

$$k(kX \times Y) = k(X \times Y).$$

Show that, if Y is locally compact Hausdorff, then $kX \times Y = k(X \times Y)$.

Proof. The second assertion follows from the first one (since then $kX \times Y$ is compactly generated) that itself follows from Yoneda lemma: both spaces are compactly generated and, given any compactly generated T, we have natural bijections

$$\mathcal{C}(T, k(kX \times Y)) \simeq \mathcal{C}(T, X \times Y) \simeq \mathcal{C}(T, k(X \times Y)).$$ ■

2.3.2 Compact-open topology

Definition 2.3.5 Let X and Y be two topological spaces. If $S \to X$ is a continuous map from a compact Hausdorff space and V an open subset of Y, we denote by $W_{S,V}$ the set of all continuous maps $f : X \to Y$ such that the image of the composite map $S \to X \to Y$ is contained in V. The *compact-open topology*[a] on $\mathcal{C}(X, Y)$ is the topology generated by all $W_{S,V}$.

[a] This is compatible with the usual definition when X is (weak) Hausdorff.

We shall systematically endow $\mathcal{C}(X, Y)$ with the compact-open topology.

Exercise 2.28 Show that if X is discrete, then there exists a homeomorphism $\mathcal{C}(X, Y) \simeq Y^X$ (with the product topology).

[3] $X/R = \operatorname{coker}(R \rightrightarrows X)$.

Exercise 2.29 Show that if S is compact Hausdorff and X is (semi-) metric (complete), then the topology of $\mathcal{C}(S,X)$ is (semi-) metric (complete) with $d(f,g) = \sup_{x\in S} d(f(x),g(x))$.

Solution. We start from $f \in W_{K,U}$ with $K \subset S$ compact and $U \subset X$ open. Denote by F the complement of U in X. Since $f(K)$ is compact and $f(K) \cap F = \emptyset$, then $d(F,K) = \epsilon > 0$ and $\mathbb{B}(f,\epsilon^-) \subset W_{K,U}$. Conversely, given $\mathbb{B}(f,\epsilon^-)$, there exists for all $x \in S$ a compact neighborhood of K_x of x such that $f(K_x) \subset \mathbb{B}(f(x),\epsilon^-)$. We can write $S = \bigcup_{i=1}^r K_{x_i}$. Then, if we set $K_i = K_{x_i}$ and $U_i := \mathbb{B}(f(x),\epsilon^-)$, we have $f \in \bigcap_{i=1}^r W_{K_i,U_i} \subset \mathbb{B}(f,\epsilon^-)$. ■

Note in particular that $\mathcal{C}(S,X)$ is compactly generated when S is compact Hausdorff and X semi-metric. It is usually quite hard to tell if $\mathcal{C}(X,Y)$ is compactly generated. Note also that the equality of sets $\mathcal{C}(kX,kY) = \mathcal{C}(kX,Y)$ need not be a homeomorphism a priori, even for the k-topology.

Theorem 2.3.6 If X,Y,Z are three topological spaces, then there exists a homeomorphism

$$k\mathcal{C}(k(X\times Y),kZ) \simeq k\mathcal{C}(kX,k\mathcal{C}(kY,kZ)).$$

Proof. One first shows that the natural bijection (called currying after the mathematician H. Curry)

$$\mathcal{F}(X\times Y,Z) \simeq \mathcal{F}(X,\mathcal{F}(Y,Z))$$

induces a bijection

$$\mathcal{C}(k(X\times Y),Z) \simeq \mathcal{C}(X,\mathcal{C}(Y,Z)) \tag{2.1}$$

for *compactly generated spaces*. This is done by hand (see theorem 5.9.8 in [Bro06] for example). One can then use Yoneda lemma: if T is any compactly generated space, then we will have a sequence of natural bijections

$$\begin{aligned}\mathcal{C}(T,k\mathcal{C}(k(X\times Y),kZ)) &\simeq \mathcal{C}(k(T\times X\times Y),Z)\\ &\simeq \mathcal{C}(k(T\times X),\mathcal{C}(kY,kZ))\\ &\simeq \mathcal{C}(T,k\mathcal{C}(kX,k\mathcal{C}(kY,kZ)).\end{aligned}$$ ■

The category of compactly generated spaces is *cartesian closed* (meaning that 2.1 holds). More generally:

Corollary 2.3.7 The functor $X \mapsto k(X\times Y)$ is adjoint to the functor $Z \mapsto k\mathcal{C}(kY,Z)$ on the category of compactly generated spaces X. ■

As a consequence, the functor $X \mapsto k(X\times Y)$ (resp. $Z \mapsto k\mathcal{C}(kY,Z)$) commutes with colimits (resp. limits) of compactly generated spaces. In particular, colimits are universal (see definition 3.2.11 below) in the category of compactly generated spaces (this is *not* true in the category of all topological spaces).

Exercise 2.30 Show that, if X is compactly generated and Y is locally compact Hausdorff, then

$$\mathcal{C}(X \times Y, Z) \simeq \mathcal{C}(X, \mathcal{C}(Y, Z)) \simeq \mathcal{C}(Y, \mathcal{C}(X, Z)).$$

2.3.3 Weak Hausdorff space

Definition 2.3.8 A topological space X is said to be *weak Hausdorff* (also called a *h-space*) if, for all morphism $f : S \to X$ with S compact Hausdorff, $f(S)$ is closed in X.

Exercise 2.31 Show that any subspace of a weak Hausdorff space is also weak Hausdorff. Show that if X is a weak Hausdorff space, then so is kX.

A compactly generated weak Hausdorff space is also called an *hk-space*.

Exercise 2.32 Show that Hausdorff implies weak Hausdorff implies Fréchet.

Exercise 2.33 Show that, if X is a weak Hausdorff topological space, then, for all morphism $f : S \to X$ with S compact Hausdorff, $f(S)$ is compact Hausdorff.

Solution. If $S' \subset S$, is a closed subset, it is compact Hausdorff and therefore $f(S')$ is closed. The map $S \twoheadrightarrow f(S)$ is closed surjective and S is normal Hausdorff. The assertion therefore follows from exercise 2.4. ■

Exercise 2.34 Show that, if X is weak Hausdorff, then $Y \subset X$ is k-closed (resp. k-open) if and only if for all compact Hausdorff $S \subset X$, $S \cap Y$ is open (resp. closed).

Exercise 2.35
1. Show that if $S \to X$ and $S' \to X$ are two morphisms from a compact Hausdorff space to a weak Hausdorff space, then $S \times_X S'$ is compact Hausdorff.
2. Show that compact Hausdorff subspaces of a weak Hausdorff space are stable under finite unions.

Solution. The second assertion follows from the first one which we now prove. Since X is weak Hausdorff and $S_1 \sqcup S_2$ is compact Hausdorff, then the image T of $S_1 \sqcup S_2$ in X is a compact Hausdorff subset and therefore $S_1 \times_X S_2 = S_1 \times_T S_2$ is compact Hausdorff. ■

Exercise 2.36 Show that, if X is weak Hausdorff, then the diagonal $X \subset X \times X$ is k-closed. Show that the converse holds if X is compactly generated.

Exercise 2.37 Show that if Y is weak Hausdorff and $f, g : X \to Y$ are two continuous maps, then $\ker(f, g) \subset X$ is k-closed.

Solution. This is the inverse image of the diagonal along the continuous map $(f, g) : X \to Y \times Y$. ■

Exercise 2.38 Show that, if Y is weak Hausdorff, then $k\mathcal{C}(X, Y)$ is also weak Hausdorff.

Exercise 2.39 Show that an equivalence relation R on a compactly generated space X is k-closed if and only if X/R is weak Hausdorff.

Exercise 2.40 Show that the category of compactly generated weak Hausdorff spaces is a reflective subcategory of the category of all compactly generated spaces with reflection $X \mapsto hX := X/R$ where R is the smallest closed equivalence relation on X.

It follows that any limit of compactly generated weak Hausdorff spaces, endowed with the k-topology, is weak Hausdorff.

Exercise 2.41 Show that an open (resp. a closed) subspace of a compactly generated weak Hausdorff space is compactly generated weak Hausdorff.

Proposition 2.3.9 A filtered limit $X = \varinjlim_{i \in I} X_i$ under closed inclusion maps of compactly generated weak Hausdorff spaces is compactly generated weak Hausdorff and all X_i's are closed in X.

Proof. We may assume that I is a directed set. If $i, j \in I$, then there exists $k \in I$ such that $i, j \leq k$, and we set

$$R_{ij} := X_i \times_{X_k} X_j = \ker(X_i \times X_j \rightrightarrows X_k)$$

(which does not depend on k). Since X_k is weak Hausdorff, R_{ij} is k-closed. Now, if we set $Y := \coprod_{i \in I} X_i$, then

$$R := \coprod_{i,j} R_{ij} \subset \coprod_{i,j} (X_i \times X_j) = Y \times Y$$

is an equivalence relation on Y and we have $X = Y/R$. Since colimits are universal in the category of compactly generated spaces, we have a homeomorphism $k(Y \times Y) \simeq \coprod_{i,j} k(X_i \times X_j)$. It follows that R is k-closed and this implies that X is weak Hausdorff. Now, we assumed that X_i is closed in X_k and it follows that $X_i \cap X_j$ is closed in X_j. Thus, if we denote by $p : Y \twoheadrightarrow X$ the quotient map, we see that $p^{-1}(X_i)$ is closed in Y, and this means that X_i is closed in X. ■

Exercise 2.42 Let $X = \varinjlim_{n \in \mathbb{N}} S_n$ be a countable directed colimit of compact Hausdorff spaces under inclusion maps. Then, any continuous map $S \to X$ from a compact Hausdorff space factors through some S_n.

Solution. Since X is weak Hausdorff, we can assume that $S_n \subset S_{n+1}$ for all $n \in \mathbb{N}$. We may also replace S with its image in X. Then, we have $S = \varinjlim_{n \in \mathbb{N}} S \cap S_n$ and we may finally assume that $X = S$ and $S = \varinjlim_{n \in \mathbb{N}} S_n$. We have to show that there exists n such that $S = S_n$. Otherwise, we may assume that $S_n \subsetneq S_{n+1}$ for all $n \in \mathbb{N}$ and pick up a point $x_n \in S_{n+1} \setminus S_n$. If $T \subset \{x_n\}_{n \in \mathbb{N}}$, then $T \cap S_n$ is (finite) closed in S_n for all $n \in \mathbb{N}$. Since S has the colimit topology, then T is closed in S. This shows that $\{x_n\}_{n \in \mathbb{N}}$ is discrete in S compact and therefore finite. Contradiction. ■

3. Sites and topos

We shall now give a brief review of Grothendieck's splendid theory of topos.

Recall that we are supposed to work in a fixed universe. Unfortunately, if we are given two categories $\mathcal{C}$ and $\mathcal{D}$, then the collection of all functors $F : \mathcal{C} \to \mathcal{D}$ does *not* make a category (in our sense) in general because the collection of all natural transformations between two of them is not always a set. This is the case however when $\mathcal{C}$ is small, but then, the new category $\mathbf{Hom}(\mathcal{C}, \mathcal{D})$ is *not* small in general, and the process cannot be iterated. The simplest solution is to enlarge our universe as needed. We shall do that informally and not worry too much about set-theoretic issues.

3.1 Presheaf

We fix a category $\mathcal{C}$.

3.1.1 Definition

Definition 3.1.1 A *presheaf (of sets)* on $\mathcal{C}$ is a (contravariant) functor $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. A *morphism of presheaves* is a natural transformation between them.

Examples

1. A presheaf on a category $\mathcal{C}$ is the same thing as a (large) diagram of sets on $\mathcal{C}^{\mathrm{op}}$.
2. A presheaf on a preordered set $(I, \leq)$ is given by a family of sets T_i, together with a compatible family of "restriction" maps $T_j \to T_i$ for $i < j$.
3. If X is a topological space, then a presheaf T on $\mathbf{Open}(X)$ (also called a *presheaf on* X) is the following data :
   (a) a set $T(U)$ for any open subset U of X, and

(b) compatible *restriction maps*

$$\forall U' \subsetneq U, \quad T(U) \to T(U'), \quad s \mapsto s_{|U'}.$$

4. For fixed set E, we can consider the *constant presheaf* $E_{\mathcal{C}}$ on a category $\mathcal{C}$ that sends any X to E and any f to Id_E. When $\mathcal{C} = \mathbf{O}\mathrm{pen}(X)$, we shall write E_X.
5. For fixed $X \in \mathcal{C}$, we can consider the presheaf

$$h_X : Y \mapsto \mathrm{Hom}(Y, X)$$

on $\mathcal{C}$. Recall that a presheaf is said to be *representable* if it is isomorphic to h_X.

If T is a presheaf on $\mathcal{C}$ and $f : Y \to X$ any morphism, we shall then write $f^{-1} := T(f) : T(X) \to T(Y)$. For $s \in T(X)$, we may also write $s_{|Y} := f^{-1}(s)$.
We shall denote by $\widehat{\mathcal{C}} := \mathbf{H}\mathrm{om}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}\mathrm{et})$ the category[1] of all presheaves on $\mathcal{C}$.

Examples 1. We have $\widehat{\mathbf{0}} \simeq \mathbf{1}$ and $\widehat{\mathbf{1}} \simeq \mathbf{S}\mathrm{et}$.
2. If G is a monoid, then $\widehat{\mathbf{G}} \simeq G{-}\mathbf{S}\mathrm{et}$.
3. If X is a topological space, then $\widehat{\mathbf{O}\mathrm{pen}(X)}$ is the « usual » category of presheaves (of sets) on X.

Exercise 3.1 Show that, for a set E and a presheaf T, we have the adjunction (where $1_{\widehat{\mathcal{C}}}$ denotes a final object)

$$\mathrm{Hom}(E_{\mathcal{C}}, T) \simeq \mathrm{Hom}_{\mathrm{Set}}(E, \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, T)) \quad (\simeq \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, T)^E).$$

Solution. We may assume that $1_{\widehat{\mathcal{C}}}$ is the constant presheaf associated to $1 := \{0\}$ (which is clearly a final object). There exists a natural bijection

$$E \simeq \mathrm{Hom}(1, E) \simeq \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, E_{\mathcal{C}}), \quad e \mapsto e_{\mathcal{C}}.$$

Composition therefore provides us with a natural map

$$\mathrm{Hom}(E_{\mathcal{C}}, T) \to \mathrm{Hom}(E, \mathrm{Hom}(1_{\widehat{\mathcal{C}}}, T)), \quad \varphi \mapsto (e \mapsto \varphi_e := \varphi \circ e_{\mathcal{C}}).$$

By construction, if $X \in \mathcal{C}$, we have $\varphi_{e,X}(0) = \varphi_X(e)$ which implies that the map is injective and provides a candidate for an inverse. It is however necessary to check that ϕ will be a morphism of presheaves but if $f : Y \to X$, we have $f^*(\varphi_X(e)) = f^*(\varphi_{e,X}(0)) = \varphi_{e,Y}(0) = \varphi_Y(e)$. ∎

Exercise 3.2 Show that a presheaf on $\mathbf{T}\mathrm{op}$ is equivalent to the following data :
1. a presheaf T_X (its *realization*) on each topological space X, and
2. a compatible family of morphisms $\varphi_f : T_X \to \widehat{f}_* T_Y$ for all continuous map $f : Y \to X$, where $\widehat{f}_* T_Y$ is the presheaf on X defined by

$$\widehat{f}_* T_Y(U) = T_Y(f^{-1}(U)).$$

[1] This is where it is necessary to enlarge our universe and we shall call *small* a set that belongs to the original category.

3.1.2 Yoneda embedding

Theorem 3.1.2 All limits and colimits exist in $\widehat{\mathcal{C}}$ and for fixed $X \in \mathcal{C}$, the functor

$$\widehat{\mathcal{C}} \to \mathbf{Set}, \quad T \mapsto \Gamma(X, T) := T(X)$$

preserves all limits and colimits.

Proof. Follows from exercises 1.67 and 1.38. ■

Examples 1. A morphism $T \to T'$ of presheaves of sets is a monomorphism (resp. an epimorphism) in $\widehat{\mathcal{C}}$ if and only if all $T(X) \to T'(X)$ are injective (resp. surjective).
2. Giving a subobject T' of a presheaf T is equivalent to giving compatible family of subsets $T'(X) \subset T(X)$ for all $X \in \mathcal{C}$. Compatibility means that, if $f : Y \to X$ is any morphism and $s \in T'(X)$, then $f^{-1}(s) \in T'(Y)$.
3. Filtered colimits are exact in $\widehat{\mathcal{C}}$.

Proposition 3.1.3 — Yoneda embedding. The functor

$$よ : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}, \quad X \mapsto h_X$$

is fully faithful and preserves all limits (that exist).

Proof. It follows from Yoneda lemma that, if $X, Y \in \mathcal{C}$, then

$$\mathrm{Hom}(X, Y) = h_Y(X) \simeq \mathrm{Hom}(h_X, h_Y).$$

The other assertion follows from the theorem and the fact that the functor Hom preserves all limits. ■

In other words, the category $\mathcal{C}$ can be identified with a full subcategory of $\widehat{\mathcal{C}}$ but this embedding does *not* preserve colimits in general. However, since it is left exact, it preserves any kind of algebraic structure.

Exercise 3.3 Show that there exists a fully faithful functor

$$\mathrm{Ind}(\mathcal{C}) \hookrightarrow \widehat{\mathcal{C}}, \quad \text{“}\varinjlim X_i\text{”} \mapsto \varinjlim h_{X_i}.$$

A presheaf T isomorphic to $\varinjlim h_{X_i}$ is said to be *ind-representable*.

3.1.3 Equivalence relation

Definition 3.1.4 1. An *equivalence relation* on $T \in \widehat{\mathcal{C}}$ is a diagram $R \rightrightarrows T$ such that, for all $X \in \mathcal{C}$, the map $R(X) \to T(X) \times T(X)$ is a bijection onto (the graph of) an equivalence relation.
2. An equivalence relation on $X \in \mathcal{C}$ is a diagram $R \rightrightarrows X$ in $\mathcal{C}$ such that $h_R \rightrightarrows h_X$ is an equivalence relation on h_X.

3. An equivalence relation R on $X \in \mathcal{C}$ is said to be *effective* if

$$R = X \times_{\overline{X}} X \quad \text{with} \quad \overline{X} = \operatorname{coker}(R \rightrightarrows X).$$

In this case $X/R := \overline{X}$ is called the *quotient* of X by R and $\pi : X \to \overline{X}$ the *quotient* morphism.

In other words, a diagram $R \rightrightarrows X$ is an effective equivalence relation if and only if there exists a morphism $\pi : X \to \overline{X}$ such that the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
X \arrow[r, "\pi"] & \overline{X}
\end{tikzcd}$$

is both cartesian and cocartesian.

Exercise 3.4 Show that, if $X \to Y$ is any morphism (resp. a regular epimorphism), then $R := X \times_Y X$ (if it exists) is an (resp. an effective) equivalence relation on X (resp. and $X/R = Y$).

Examples 1. An equivalence relation on a set or a topological space is a usual equivalence relation. It is always effective.

2. An equivalence relation in $\widehat{\mathcal{C}}$ is always effective (and any epimorphism in $\widehat{\mathcal{C}}$ is always regular).

Exercise 3.5 Show that a quotient morphism[a] is always an epimorphism. Show that when $\mathcal{C}$ has fibered products, then a regular epimorphism is the same thing as a quotient morphism.

[a] Also called *effective epimorphism*.

3.1.4 Slice category

Definition 3.1.5 If T is a presheaf of sets on a category $\mathcal{C}$, then the *slice category* $\mathcal{C}_{/T}$ is defined as follows:

1. an object of $\mathcal{C}_{/T}$ is a pair made of an object X of $\mathcal{C}$ and a section $s \in T(X)$,
2. a morphism in $\mathcal{C}_{/T}$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $T(f)(s') = s$.

There exists an obvious forgetful functor $j_T : \mathcal{C}_{/T} \to \mathcal{C}$. Note that, in terms of *comma* category, we have $\mathcal{C}_{/T} \simeq (\mathcal{C} \hookrightarrow \widehat{\mathcal{C}} \downarrow 1 \xrightarrow{T} \widehat{\mathcal{C}})$.

Exercise 3.6 1. Show that, if $X \in \mathcal{C}$, then there exist an equivalence $\mathcal{C}_{/X} \simeq \mathcal{C}_{/h_X}$.

2. Show that the diagram

$$\begin{tikzcd}
\mathcal{C}_{/T} \arrow[r, hook] \arrow[d] & \widehat{\mathcal{C}}_{/T} \arrow[d] \\
\mathcal{C} \arrow[r, hook, "\text{よ}"] & \widehat{\mathcal{C}}
\end{tikzcd}$$

is cartesian.

Exercise 3.7 1. Prove the *density theorem*: any presheaf T on a category $\mathcal{C}$ is a colimit of representable presheaves. More, precisely:

$$T \simeq \varinjlim_{(X,s)\in\mathcal{C}_{/T}} h_X.$$

2. Show that if T is another presheaf, then

$$\mathrm{Hom}(T,T') \simeq \varprojlim_{(X,s)\in C_{/T}} T'(X).$$

In practice, one may also write $s \in T(X)$ instead of $(X,s) \in \mathcal{C}_{/T}$ but we may also shorten to $X \in \mathcal{C}_{/T}$ (even if s is part of the landscape).

Solution. By definition, giving a morphism

$$\varinjlim_{(X,s)\in\mathcal{C}_{/T}} h_X \to T'$$

amounts to giving a compatible family of morphisms $s' : h_X \to T'$ indexed by $(X,s) \in \mathcal{C}_{/T}$. Thanks to Yoneda's lemma, this is equivalent to giving a compatible family of elements $s' \in T'(X)$ indexed by elements $s \in T(X)$. This is the definition of a morphism $T \to T'$. This shows the first assertion and the second one follows from the fact that the functor Hom preserves all limits. ∎

Exercise 3.8 Show that a presheaf T on a category $\mathcal{C}$ is ind-representable if and only if $\mathcal{C}_{/T}$ is filtered if and only if T is left exact.

3.2 Site

3.2.1 Topology

Definition 3.2.1 If $\mathcal{C}$ is a category and $X \in \mathcal{C}$, then a subobject[a] R of h_X is called a *sieve* of X. The *inverse image* of R under a morphism $f : Y \to X$ is the sieve $f^{-1}(R) := R \times_{h_X} h_Y$ of Y.

[a]Say $T' \subset T$ if $\forall X \in \mathcal{C}, T('(X) \subset T(X)$.

The set $\mathcal{C}_{/R}$ is sometimes also called a sieve (this was actually the original definition).

Exercise 3.9 Show that it is equivalent to give a sieve R of X or a set $\mathcal{R}$ $(= \mathcal{C}_{/R})$ of morphisms $X' \to X$ such that, if $X' \to X$ belongs to $\mathcal{R}$, then any precomposition $X'' \to X' \to X$ is still in $\mathcal{R}$. Show that $f^{-1}(R)$ then corresponds to the set of all morphisms $Y' \to Y$ such that the composition $Y' \to Y \to X$ is in $\mathcal{R}$.

Since the notion of a sieve of X is stable under intersection, there is a notion of sieve *generated* by a family $(X_i \to X)_{i\in I}$.

Exercise 3.10 Show that the sieve R of X generated by a family $(f_i : X_i \to X)_{i\in I}$ is $\bigcup_{i\in I} \operatorname{im} h_{f_i}$. In other words, $Y \to X$ is in $R(Y)$ if and only if it factors through some X_i.

Examples 1. If $(I, \leq)$ is a preordered set and $x \in I$, then it is equivalent to give a sieve of x or a subset J of I such that
   (a) $\forall y \in J, \quad y \leq x$,
   (b) $\forall y \in J, \forall z \in I, \quad z \leq y \Rightarrow z \in J$.
2. If X is a topological space and U an is open subset of X, then a sieve of U corresponds to a family $\mathcal{U}$ open subsets U' of U such that, if $U'' \subset U'$, then also $U'' \in \mathcal{U}$.

Definition 3.2.2 A *topology* on a category $\mathcal{C}$ is the data of a set $J(X)$ of *covering* sieves of X for each $X \in \mathcal{C}$, such that:
1. $\forall X \in \mathcal{C}, \quad h_X \in J(X)$,
2. $\forall R \in J(X), \forall f : Y \to X, \quad f^{-1}(R) \in J(Y)$,
3. $\forall R, R' \subset h_X, (R' \in J(X)$ and $\forall f \in R'(Y), f^{-1}(R) \in J(Y)) \Rightarrow R \in J(X)$.

A *site* is a category $\mathcal{C}$ endowed with a topology[a].

[a]For set-theoretical reasons, we shall also assume that there exists a small *topologically generating set* - see definition 3.2.3 below

We shall simply say that $\mathcal{C}$ is a site without mentioning explicitly the topology.

Exercise 3.11 Show that $J(X)$ is a *filter* (on the ordered set of all sieves of X):
1. $J(X) \neq \emptyset$,
2. if $R, R' \in J(X)$, then $R \cap R' \in J(X)$,
3. if $R \subset R'$ and $R \in J(X)$, then $R' \in J(X)$.

Solution. If $f \in R(Y)$, then in the first case, $f^{-1}(R \cap R') = f^{-1}(R') \in J(Y)$, and in the second case, $f^{-1}(R') = h_Y \in J(Y)$. ■

Exercise 3.12 Show that the last condition in the definition can be replaced by $\forall R \subset R' \subset h_X$,
- $R \in J(X) \Rightarrow R' \in J(X)$ and
- $(R' \in J(X)$ and $\forall f \in R'(Y), f^{-1}(R) \in J(Y)) \Rightarrow R \in J(X)$.

Solution. ■

Definition 3.2.3 A *topologically generating set* for a site $\mathcal{C}$ is a set $\mathcal{G} \subset \mathcal{C}$ such that any $X \in \mathcal{C}$ admits a covering sieve generated by a family $(f_i : X_i \to X)_{i\in I}$ with $X_i \in \mathcal{G}$.

If $\mathcal{C}$ is any category, then the various topologies on $\mathcal{C}$ are ordered by inclusion from *coarse* (only h_X covers X) to *discrete* (any sieve R covers X). Any intersection of topologies is a topology and it follows that any set of sieves generates a topology. As a consequence, any set of families $(X_i \to X)_{i\in I}$ for various X generates a topology.

In practice, it is convenient to rely on the following:

Definition 3.2.4 A *pretopology* on a category $\mathcal{C}$ is the data of sets $\mathrm{Cov}(X)$ of *covering* families $(X_i \to X)_{i \in I}$ for all $X \in \mathcal{C}$ such that

1. any isomorphism $X' \to X$ is in $\mathrm{Cov}(X)$,
2. if $(X_i \to X)_{i \in I} \in \mathrm{Cov}(X)$ and $f : Y \to X$ is any morphism, then $(X_i \times_X Y \to Y)_{i \in I} \in \mathrm{Cov}(Y)$,
3. if $(X_i \to X)_{i \in I} \in \mathrm{Cov}(X)$, and for each $i \in I$, $(X_{ij} \to X_i)_{j \in I_i} \in \mathrm{Cov}(X_i)$, then $(X_{ij} \to X)_{i \in I, j \in I_j} \in \mathrm{Cov}(X)$.

If $\mathcal{C}$ is endowed with a pretopology, we will consider it as a site with respect to the topology generated by the set of $\mathrm{Cov}(X)$ for all X.

Examples
1. If X is a topological space, we turn $\mathbf{Open}(X)$ into a site by calling a family $(U_i \subset U)_{i \in I}$ a covering if $U = \bigcup_{i \in I} U_i$.
2. One can turn $\mathbf{Set}$ into a site by calling a family $(X_i \to X)_{i \in I}$ a covering when it is jointly surjective : $X = \bigcup_{i \in I} f(X_i)$.
3. We turn $\mathbf{Top}$ into a site by calling a a family $(f_i : X_i \to X)_{i \in I}$ a covering if it is jointly surjective and each f_i induces a homéomorphism $X_i \simeq f(X_i)$ with an *open* subset of X.

Exercise 3.13 Show that, if $\mathcal{C}$ is endowed with a pretopology, then a sieve of $X \in \mathcal{C}$ is a covering sieve if and only if it contains a sieve generated by a covering family.

Exercise 3.14 Show that, if $\mathcal{C}$ is a site with fibered products, then the set of all families $(X_i \to X)_{i \in I}$ that generate a covering sieve of X, is a pretopology that generates the topology of $\mathcal{C}$. This is called the *maximal pretopology* (of the site).

Be careful however that a family that generates a covering sieve is not necessarily a covering family for the *given* pretopology.

3.2.2 Sheaf

Definition 3.2.5 A presheaf $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ on a site $\mathcal{C}$ is *separated* (resp. a *sheaf*) if, for all $X \in \mathcal{C}$ and $R \in J(X)$, the restriction map

$$\mathrm{Hom}(h_X, \mathcal{F}) \to \mathrm{Hom}(R, \mathcal{F}).$$

is injective (resp. bijective).

Exercise 3.15 Show that a presheaf $\mathcal{F}$ is a sheaf if and only if for all $X \in \mathcal{C}$ and all $R \in J(X)$, we have

$$\mathcal{F}(X) \simeq \varprojlim_{Y \in C_{/R}} \mathcal{F}(Y).$$

Solution. Follows from density theorem (exercise 3.7). ■

Exercise 3.16 Show that, if $\mathcal{C}$ is endowed with a pretopology, then a presheaf $\mathcal{F}$ is a sheaf if and only if for all $X \in \mathcal{C}$ and all covering families $(X_i \to X)_{i\in I}$

$$\mathcal{F}(X) \simeq \ker\left(\prod_{i\in I} \mathcal{F}(X_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(X_i \times_X X_j)\right).$$

We shall denote by $\widetilde{\mathcal{C}}$ the full subcategory of sheaves of sets on the site $\mathcal{C}$ and by $\mathcal{H} : \widetilde{\mathcal{C}} \hookrightarrow \widehat{\mathcal{C}}$ the inclusion functor.

Examples
1. A sheaf on a topological space X (meaning on $\mathbf{Open}(X)$) is a presheaf $\mathcal{F}$ that satisfies: given an open covering $U = \cup U_i$ of an open subset of X and a family of $s_i \in \mathcal{F}(U_i)$ such that $(s_i)_{|U_j} = (s_j)_{|U_i}$, there exists a unique $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$.
2. A presheaf $\mathcal{F}$ on $\mathbf{Top}$ is a sheaf if and only for any topological space X, the realization $\mathcal{F}_X$ of $\mathcal{F}$ is a sheaf on X.
3. For the *coarse* topology on a category $\mathcal{C}$, any presheaf is a sheaf and consequently $\widetilde{\mathcal{C}} = \widehat{\mathcal{C}}$.
4. The only sheaf for the *discrete* topology on a category $\mathcal{C}$ is the constant presheaf 1 and consequently $\widetilde{\mathcal{C}} \simeq \mathbf{1}$.

If $\mathcal{C}$ is a site, we shall consider the Čech functor $\check{\mathcal{H}}$ defined on $\widehat{\mathcal{C}}$ by

$$\check{\mathcal{H}}(T)(X) = \varinjlim_{R\in J(X)} \mathrm{Hom}(R,T) \tag{3.1}$$

(note that this is a directed colimit). Given any $X \in \mathcal{C}$, since $h_X \in J(X)$, there exists a natural map $T \to \check{\mathcal{H}}(T)$ given by

$$\begin{tikzcd}
T(X) \arrow[r] \arrow[d, "\simeq"] & \check{\mathcal{H}}(T)(X) \arrow[d, "\simeq"] \\
\mathrm{Hom}(h_X,T) \arrow[r] & \varinjlim_{R\in J(X)} \mathrm{Hom}(R,T).
\end{tikzcd}$$

Lemma 3.2.6 If $R \in J(X)$, then any map $R \to T$ gives rise to a morphism $h_X \to \check{\mathcal{H}}(T)$ via

$$\mathrm{Hom}(R,T) \to \check{\mathcal{H}}(T)(X) \simeq \mathrm{Hom}(h_X, \check{\mathcal{H}}(T))$$

and the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d, hook] & T \arrow[d] \\
h_X \arrow[r] & \check{\mathcal{H}}(T)
\end{tikzcd}$$

is commutative.

Proof. We have to show that, for all $Y \in \mathcal{C}$, the diagram

$$\begin{tikzcd} R(Y) \arrow[r] \arrow[d, hook] & T(Y) \arrow[d] \\ h_X(Y) \arrow[r] & \check{\mathcal{H}}(T)(Y) \end{tikzcd}$$

is commutative. It means that, for all $g \in R(Y)$ seen as a map $g : h_Y \to R$, the diagram

$$\begin{tikzcd} h_Y \times_{h_X} R \arrow[r] \arrow[d, "\simeq"] & R \arrow[r] \arrow[d, hook] & T \arrow[d] \\ h_Y \arrow[r] \arrow[ur, "g"] & h_X \arrow[r] & \check{\mathcal{H}}(T) \end{tikzcd}$$

commutes (note that the left arrow is a monomorphism with a section and therefore an isomorphism). We are therefore reduced to the case $R = h_X$ where commutativity follows from the definitions. ■

Proposition 3.2.7 1. The functor $\check{\mathcal{H}}$ is left exact.
2. If T is a presheaf (resp. a separated presheaf), then $\check{\mathcal{H}}(T)$ is separated (resp. a sheaf).
3. A presheaf T is separated (resp. a sheaf) if and only if $T \to \check{\mathcal{H}}(T)$ is a monomorphism (resp. an isomorphism).

Proof. The first assertion follows from the fact that both filtered colimits and Hom are left exact.

In order to show that $\check{\mathcal{H}}(T)$ is separated, we give ourselves two maps $h_X \rightrightarrows \check{\mathcal{H}}(T)$ that coincide on some $R \in J(X)$, and we show that, after possibly refining R, they come from the same map $R \to T$. We may first assume that they come from two maps $R \rightrightarrows T$. Then, thanks to lemma 3.2.6, both compositions $R \rightrightarrows T \to \check{\mathcal{H}}(T)$ are the same. It is now sufficient to prove that $R' := \ker(R \rightrightarrows T) \in J(X)$ and then replace R with R'. Since $R \in J(X)$, it is sufficient to show that for all $f \in R(Y)$, $f^{-1}(R') = \ker(h_Y \rightrightarrows T) \in J(Y)$. But now, by the very definition of $\check{\mathcal{H}}(T)(Y)$, since both compositions $h_Y \rightrightarrows T \to \check{\mathcal{H}}(T)$ are the same, there exists $S \in J(Y)$ such that both compositions $S \hookrightarrow h_Y \rightrightarrows T$ are the same. Necessarily, $S \subset \ker(h_Y \rightrightarrows T) \in J(Y)$.

As an immediate consequence, if $T \hookrightarrow \check{\mathcal{H}}(T)$ is a monomorphism, then T is also separated as a sub-presheaf of a separated presheaf. Conversely, since filtered limits are exact, if T is separated, then we have

$$T(X) = \operatorname{Hom}(h_X, T) \hookrightarrow \varinjlim_{R \in J(X)} \operatorname{Hom}(R, T) = \check{\mathcal{H}}(T)(X).$$

We shall now prove that, if T is separated, then $\check{\mathcal{H}}(T)$ is a sheaf. We give ourselves a map $R \to \check{\mathcal{H}}(T)$ with $R \in J(X)$. Since $T \to \check{\mathcal{H}}(T)$ is a monomorphism, we have $R' := T \times_{\check{\mathcal{H}}(T)} R \subset R$ and we shall show that $R' \in J(X)$. It is sufficient to prove that, for all $f \in R(Y)$, $f^{-1}(R') = T \times_{\check{\mathcal{H}}(T)} h_Y \in J(Y)$. But this is clear since, by definition of $\check{\mathcal{H}}(T)(Y)$, the map $h_Y \to \check{\mathcal{H}}(T)$ comes from some $S \to T$ with $S \in J(Y)$ and then, necessarily, $S \subset T \times_{\check{\mathcal{H}}(T)} h_Y \in J(Y)$. We may therefore replace R with R'. Then our map factors as $R \to T \hookrightarrow \check{\mathcal{H}}(T)$ which provides a map $h_X \to \check{\mathcal{H}}(T)$ by definition.

By definition, if T is sheaf, then $T \simeq \check{\mathcal{H}}(T)$. Conversely, since $\check{\mathcal{H}}(T)$ is separated, such an isomorphism implies that T is separated in which case $\check{\mathcal{H}}(T)$ is a sheaf and the isomorphism shows that T also is a sheaf. ■

Theorem 3.2.8 If $\mathcal{C}$ is a site, then $\widetilde{\mathcal{C}}$ is a reflective subcategory of $\widehat{\mathcal{C}}$ with *exact* reflection $T \mapsto \widetilde{T}$ called *sheafification* and $\mathcal{H}(\widetilde{T}) = \check{\mathcal{H}}(\check{\mathcal{H}}(T))$.

Proof. Immediate consequence of proposition 3.2.7. ■

We have

$$\forall T \in \widehat{\mathcal{C}}, \mathcal{F} \in \widetilde{\mathcal{C}}, \quad \mathrm{Hom}(\widetilde{T}, \mathcal{F}) \simeq \mathrm{Hom}(T, \mathcal{F})$$

(a morphism $T \to \mathcal{F}$ extends uniquely to $\widetilde{T} \to \mathcal{F}$).

Example
1. If $\mathcal{C}$ has the discrete (resp. the coarse) topology, then $\widetilde{T} = T$ (resp. $\widetilde{T} = \emptyset_{\mathcal{C}}$).
2. If E is any set, then we may consider the *constant sheaf* $\widetilde{E}_{\mathcal{C}}$ (sometimes still denoted by E) which is the sheafification of the constant presheaf $E_{\mathcal{C}}$.
3. If X is a topological space, then $\widetilde{\mathbf{O}\mathrm{pen}(X)}$ is the usual category of sheaves (of sets) on X and $T \mapsto \widetilde{T}$ is the usual sheafification functor.

Corollary 3.2.9
1. Limits in $\widetilde{\mathcal{C}}$ are computed in $\widehat{\mathcal{C}}$.
2. The functor $T \mapsto \widetilde{T}$ preserves all colimits and finite limits. ■

It is important to emphasize the fact that an epimorphism of sheaves $\mathcal{F} \to \mathcal{G}$ is usually not an epimorphism of presheaves. Some people say *local epimorphism* in order to remove the ambiguity. More generally, a morphism of presheaves $T \to T'$ is called a *local epimorphism* if $\widetilde{T} \to \widetilde{T}'$ is an epimorphism (of sheaves).

Exercise 3.17 Show that, if R is an equivalence relation on a presheaf T, then $\widetilde{R}$ is an equivalence relation on $\widetilde{T}$ and $\widetilde{T}/\widetilde{R} = \widetilde{T/R}$.

Solution. ■

Exercise 3.18 Show that a morphism of sheaves $u : \mathcal{F} \to \mathcal{G}$ which is both a monomorphism and an epimorphism is automatically an isomorphism.

Solution. ■

Exercise 3.19 Show that a morphism of sheaves $u : \mathcal{F} \to \mathcal{G}$ has a unique epi-mono factorization: it factors uniquely up to an isomorphism as an epimorphism followed by a monomorphism.

Solution. ■

Exercise 3.20 Show that if $\mathcal{C}$ is a site and $T, T' \in \widehat{\mathcal{C}}$, then

$$\operatorname{im}(\widetilde{T} \to \widetilde{T}') = \operatorname{im}(\widetilde{T \to T'}).$$

Solution. ∎

Exercise 3.21 Show that, if $\mathcal{C}$ is endowed with a pretopology, then a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is an epimorphism if and only if, for all $X \in \mathcal{C}$ and all $s \in \mathcal{G}(X)$, there exists a covering $(X_i \to X)$ such that for all $i \in I$, $s_{|X_i}$ belongs to the image of $\mathcal{F}(X_i) \to \mathcal{G}(X_i)$.

Exercise 3.22 Show that, for a set E and a sheaf $\mathcal{F}$ on a site $\mathcal{C}$, we have the adjunction (where $1_{\widetilde{\mathcal{C}}}$ denotes a final object)

$$\operatorname{Hom}(\widetilde{E}_{\mathcal{C}}, \mathcal{F}) \simeq \operatorname{Hom}_{\mathsf{Set}}(E, \operatorname{Hom}(1_{\widetilde{\mathcal{C}}}, \mathcal{F})) \quad (\simeq \operatorname{Hom}(1_{\widetilde{\mathcal{C}}}, \mathcal{F}))^E).$$

Solution. By definition, we have $\operatorname{Hom}(\widetilde{E}_{\mathcal{C}}, \mathcal{F}) \simeq \operatorname{Hom}(E_{\mathcal{C}}, \mathcal{F})$. By left exactness, $1_{\widetilde{\mathcal{C}}}$ is the sheaf associated to $1_{\widehat{\mathcal{C}}}$. It follows that we also have $\operatorname{Hom}(1_{\widetilde{\mathcal{C}}}, \mathcal{F}) \simeq \operatorname{Hom}(1_{\widehat{\mathcal{C}}}, \mathcal{F})$. We may then apply the analog presheaf assertion. ∎

Definition 3.2.10 The sheaf *associated* to an object X of a site $\mathcal{C}$ is $\underline{X} := \widetilde{h_X}$.

The functor $\mathcal{C} \to \widetilde{\mathcal{C}}, X \mapsto \underline{X}$ is left exact but not necessary fully faithful (and does not preserve colimits or infinite limits in general). Also, we have

$$\forall X \in \mathcal{C}, \mathcal{F} \in \widetilde{\mathcal{C}}, \quad \operatorname{Hom}(\underline{X}, \mathcal{F}) \simeq \mathcal{F}(X).$$

Exercise 3.23 Show that if E a set, then

$$\widetilde{E}_{\mathbf{Top}} \simeq \underline{E^{\mathrm{disc}}}$$

(and $\widetilde{E}_{\mathbf{Top}}(X) = E$ if and only if X is connected).

Solution. Notice first that $\underline{1}$ (with $1 = \{0\}$) is the final object of $\widetilde{\mathbf{Top}}$. Now, given any sheaf $\mathcal{F}$, we have on the one hand

$$\operatorname{Hom}(\widetilde{E}_{\mathbf{Top}}, \mathcal{F}) \simeq \operatorname{Hom}(\underline{1}, \mathcal{F})^E \simeq \mathcal{F}(1)^E,$$

and on the other

$$\operatorname{Hom}(\underline{E^{\mathrm{disc}}}, \mathcal{F}) \simeq \mathcal{F}(E^{\mathrm{disc}}) \simeq \prod_{x \in E} \mathcal{F}(\{x\}) \simeq \mathcal{F}(1)^E.$$

Our assertion therefore follows from Yoneda's lemma. ∎

3.2.3 Properties

Definition 3.2.11 In a category $\mathcal{C}$,

1. a colimit $X = \varinjlim_{i\in I} X_i$ is said to be *universal* if for all morphisms $X \to Y$ and $Y' \to Y$, we have $X \times_Y Y' \simeq \varinjlim(X_i \times_Y Y')$,
2. A (regular) epimorphism $f : X \twoheadrightarrow Y$ is said to be *universal* if for all morphism $Y' \to Y$, $f \times_Y Y'$ is a (regular) epimorphism,
3. A coproduct $X = \coprod_{i\in I} X_i$ is said to be *disjoint* if
   (a) for all $i \in I$, $X_i \hookrightarrow X$ is a monomorphism,
   (b) for all $i \neq j \in I$, $X_i \times_X X_j = \emptyset$ is the initial object.

We may also say that an effective equivalence relation is *universal* if the quotient map is a universal regular epimorphism.

Exercise 3.24 Show that if $\mathcal{C}$ is a site, then all limits and colimits exist in $\widetilde{\mathcal{C}}$ and

1. colimits are universal,
2. filtered colimits are exact,
3. epimorphisms are regular and universal,
4. coproducts are disjoint,
5. equivalence relations are effective (and universal).

Solution. The properties hold in **Set**. Therefore, they hold in $\widehat{\mathcal{C}}$. We may then use sheafification which is exact and preserves colimits. As an example, we shall prove the third assertion. We give ourselves an epimorphism $\mathcal{F} \to \mathcal{G}$ in $\widetilde{\mathcal{C}}$. We consider the epi-mono factorization $\mathcal{F} \twoheadrightarrow I \hookrightarrow \mathcal{G}$ in $\widehat{\mathcal{C}}$. We sheafify and get the epi-mono factorization $\mathcal{F} \twoheadrightarrow \widetilde{I} \hookrightarrow \mathcal{G}$ in $\widetilde{\mathcal{C}}$. Since $\mathcal{F} \to \mathcal{G}$ is an epimorphism in $\widetilde{\mathcal{C}}$, we see that the monomorphism $\widetilde{I} \hookrightarrow \mathcal{G}$ is also an epimorphism and therefore un isomorphism. Now, we can write $I = \mathcal{F}/R$ for some equivalence relation R on $\mathcal{F}$ in $\widehat{\mathcal{C}}$ and we know that then $\widetilde{I} = \mathcal{F}/\widetilde{R}$. It follows that $\mathcal{G} = \mathcal{F}/\widetilde{R}$ so that $\mathcal{F} \to \mathcal{G}$ is a regular epimorphism. Now, if we pull our epi-mono factorizations along a morphism of sheaves $\mathcal{G}' \to \mathcal{G}$, then we get diagrams with cartesian squares of epi-mono factorizations in $\widehat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ respectively

$$\begin{tikzcd} \mathcal{F}' \arrow[r, two heads] \arrow[d] & I' \arrow[r, hook] \arrow[d] & \mathcal{G}' \arrow[d] \\ \mathcal{F} \arrow[r, two heads] & I \arrow[r, hook] & \mathcal{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}' \arrow[r, two heads] \arrow[d] & \widetilde{I'} \arrow[r, hook] \arrow[d] & \mathcal{G}' \arrow[d] \\ \mathcal{F} \arrow[r, two heads] & \widetilde{I} \arrow[r, equal] & \mathcal{G}. \end{tikzcd}$$

The last right hand square being cartesian, we must have $\widetilde{I'} = \mathcal{G}'$ and we get an epimorphism $\mathcal{F}' \twoheadrightarrow \mathcal{G}'$. This shows that epimorphisms are universal. ■

Exercise 3.25 Show that if $\mathcal{C}$ is a site and $\mathcal{F}, \mathcal{G} \in \widetilde{\mathcal{C}}$, then

$$\mathrm{im}(\mathcal{F} \to \mathcal{G}) = \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G})$$

in $\widetilde{\mathcal{C}}$ (an dual). Show that any morphism in $\widetilde{\mathcal{C}}$ is strict.

Solution. Note first that, if $\mathcal{F} \subset \mathcal{G}$, then the canonical map $\mathcal{F} \to \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G})$ is an isomorphism. Since sheafification is exact, it is sufficient to consider a category of presheaves $\widehat{\mathcal{C}}$. Now, limits and colimits are computed argument-wise and we are therefore reduced to the analog statement in the category of sets.

Now, let us write $\mathcal{K} := \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G})$. By definition of the fibered coproduct, both composite maps

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G}$$

are the same. By definition of the kernel, f factors as $\mathcal{F} \to \mathcal{K} \hookrightarrow \mathcal{G}$. Assume now that f factors as $\mathcal{F} \to \mathcal{J} \hookrightarrow \mathcal{G}$. Since $\mathcal{J} \subset \mathcal{G}$, we have $\mathcal{J} = \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{J}} \mathcal{G})$. By functoriality of fibered coproduct and kernel, there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{K} \arrow[r, hook] \arrow[d] & \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G} \arrow[d] \\
\mathcal{J} \arrow[r, hook] & \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{G} \sqcup_{\mathcal{J}} \mathcal{G}
\end{tikzcd}$$

which shows that $\mathcal{K} \subset \mathcal{J}$.

The dual case follows exactly the same pattern (but it is not obtained by duality because the dual of $\widetilde{\mathcal{C}}$ is not a category of sheaves).

Now, the commutativity of the diagram

$$\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F} \to \mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G},$$

implies the existence of a natural map

$$\operatorname{coker}(\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F}) \to \ker(\mathcal{G} \rightrightarrows \mathcal{G} \sqcup_{\mathcal{F}} \mathcal{G}).$$

As above, it formally follows from the analog assertion in the category of sets that this is an isomorphism. ■

Exercise 3.26 Show that if T is a presheaf on a site $\mathcal{C}$, then

$$\widetilde{T} \simeq \varinjlim_{X \in \mathcal{C}_{/T}} \underline{X}$$

(in fancy terms, $T \mapsto \widetilde{T}$ is the left Kan exension of $X \mapsto \underline{X}$).

In particular, if $\mathcal{F}$ is a sheaf on $\mathcal{C}$, then there exists an epimorphism $\coprod_{i \in I} \underline{X_i} \twoheadrightarrow \mathcal{F}$ with $X_i \in \mathcal{C}$.

Exercise 3.27 Show that $\mathcal{G} \to \mathcal{F}$ is an epimorphism (resp. an isomorphism) of sheaves if and only if for all $\underline{X} \to \mathcal{F}$, the morphism $\underline{X} \times_{\mathcal{F}} \mathcal{G} \to \underline{X}$ is an epimorphism (resp. isomorphism).

Solution. The condition is necessary since epimorphisms (resp. isomorphisms) are universal. It is sufficient because colimits preserve epimorphisms (resp. isomorphisms) and are universal. ■

Proposition 3.2.12 Let $\mathcal{C}$ be a site $X \in \mathcal{C}$ and R a sieve of X. Then the following are equivalent:

1. $R \in J(X)$,
2. $\widetilde{R} \simeq \underline{X}$,
3. $\underline{X} \simeq \varinjlim_{Y \in \mathcal{C}_{/R}} \underline{Y}$,
4. $\coprod_{Y \in \mathcal{C}_{/R}} \underline{Y} \twoheadrightarrow X$ is an epimorphism.

Proof. The composite map $R \hookrightarrow h_X \to \widetilde{h_X} = \underline{X}$ extends uniquely to $u : \widetilde{R} \to \underline{X}$ and the second assertion should be understood in the sense that this is an isomorphism.

Now, if $R \in J(X)$, then the map $R \to \widetilde{R}$ extends uniquely to $h_X \to \widetilde{R}$ and then to $v : \underline{X} = \widetilde{h_X} \to \widetilde{R}$. We have $v \circ u = \mathrm{Id}_{\widetilde{R}}$ (resp. $u \circ v = \mathrm{Id}_{\underline{X}}$) because this is the unique extension of Id_R (resp. Id_X).

Assume conversely that the above map u is an isomorphism $\widetilde{R} \simeq \underline{X}$. In order to show that $R \in J(X)$, it is sufficient to show that there exists $R' \in J(X)$ such that, given any $f \in R'(Y)$, we have $f^{-1}(R) \in J(Y)$. The composite map $h_X \to \underline{X} \simeq \widetilde{R} = \check{\mathcal{H}}(\check{\mathcal{H}}(R))$ comes from a morphism $\phi : R' \to \check{\mathcal{H}}(R)$ for some $R' \in J(X)$. If $f \in R'(Y) \subset h_X(Y)$, we may then consider its image under ϕ in $\check{\mathcal{H}}(R)(Y)$. It comes from a morphism $S \to R$ with $S \in J(Y)$. By construction, we have $S \subset f^{-1}(R)$ and therefore also $f^{-1}(R) \in J(Y)$. The whole construction may be displayed as follows:

$$\begin{tikzcd}
S \arrow[rr] \arrow[d, hook] & & R \arrow[d] \arrow[dl] \\
h_Y \arrow[r, "f"] & R' \arrow[r] \arrow[d, hook] & \check{\mathcal{H}}(R) \arrow[d, hook] \\
& h_X \arrow[r] \arrow[d] & \check{\mathcal{H}}(\check{\mathcal{H}}(R)) \arrow[d] \\
& \underline{X} & \widetilde{R}. \arrow[l, "\simeq"']
\end{tikzcd}$$

The other equivalences then follow from exercise 3.26. ■

Corollary 3.2.13 A family $(X_i \to X)_{i \in I}$ generates a covering sieve if and only if the morphism $\coprod_{i \in I} \underline{X_i} \twoheadrightarrow \underline{X}$ is an epimorphism (in $\widetilde{C}$).

Proof. The sieve R generated by $(f_i : X_i \to X)_{i \in I}$ is characterized by the fact that $\coprod_{i \in I} h_{X_i} \twoheadrightarrow R \subset h_X$ is an epimorphism of presheaves. It follows that $\coprod_{i \in I} \underline{X_i} \twoheadrightarrow \widetilde{R} \subset \underline{X}$ is an epimorphism of sheaves. Our assertion therefore follows from proposition 3.2.12. ■

Lemma 3.2.14 On any category $\mathcal{C}$, there exists a finest topology such that all representable presheaves are sheaves : a sieve R of $X \in \mathcal{C}$ is a covering (for this topology) if and only if

$$\forall f : Y \to X, \forall Z \in \mathcal{C}, \quad \mathrm{Hom}(Y, Z) \simeq \mathrm{Hom}(f^{-1}(R), h_Z). \tag{3.2}$$

Proof. It is sufficient to check that, if we denote by $J^{\mathrm{can}}(X)$ the set of all sieves R of X satisfying condition (3.2), then this defines a topology on $\mathcal{C}$. Only the last condition of definition 3.2.2 requires a proof and we know that it is sufficient to show that, given $X \in \mathcal{C}$ and $R \subset R' \subset h_X$, we have

- $R \in J^{\mathrm{can}}(X) \Rightarrow R' \in J^{\mathrm{can}}(X)$ and
- $(R' \in J^{\mathrm{can}}(X)$ and $\forall f \in R'(Y), f^{-1}(R) \in J^{\mathrm{can}}(Y)) \Rightarrow R \in J^{\mathrm{can}}(X)$.

In both cases, we will have

$$\mathrm{Hom}(Y, Z) \simeq \mathrm{Hom}(f^{-1}(R), h_Z)$$

whenever $f \in R'(Y)$ and $Z \in \mathcal{C}$. It follows that

$$\varprojlim_{f \in R'(Y)} \mathrm{Hom}(Y, Z) \simeq \varprojlim_{f \in R'(Y)} \mathrm{Hom}(f^{-1}(R), h_Z).$$

and therefore

$$\mathrm{Hom}\left(\varinjlim_{f \in R'(Y)} Y, h_Z\right) \simeq \mathrm{Hom}\left(\varinjlim_{f \in R'(Y)} f^{-1}(R), h_Z\right)$$

We already know that $R' = \varinjlim_{f \in R'(Y)} Y$, and by restriction to $R \subset R'$, we shall also have $R = \varinjlim_{f \in R'(Y)} f^{-1}(R)$ and we get

$$\mathrm{Hom}(R', h_Z) \simeq \mathrm{Hom}(R, h_Z).$$

Therefore, in the first (resp. second) case, we obtain

$$\mathrm{Hom}(X, Z) \simeq \mathrm{Hom}(R', h_Z) \quad (\text{resp. } \mathrm{Hom}(X, Z) \simeq \mathrm{Hom}(R, h_Z))$$

Finally, since our conditions are stable under any pull back $Y \to X$, we see that $R' \in J^{\mathrm{can}}(X)$ (resp. $R \in J^{\mathrm{can}}(X)$). ∎

Definition 3.2.15 This is called the *canonical topology* on $\mathcal{C}$. A topology on $\mathcal{C}$ is said to be *subcanonical* if it is coarsest than the canonical topology.

In other words, a topology on $\mathcal{C}$ is subcanonical if and only if any representable presheaf is a sheaf if and only if for all $X \in \mathcal{C}, \underline{X} = h_X$ if and only if よ factors through $\widetilde{\mathcal{C}}$ if and only if the functor $X \mapsto \underline{X}$ is fully faithful.

Exercise 3.28 Show that if R is a universal effective equivalence relation on $X \in \mathcal{C}$, then $\underline{X/R} = \underline{X}/\underline{R}$ for the canonical topology.

Solution. ∎

Exercise 3.29 Show that $X := \coprod_{i\in I} X_i$ is disjoint universal in a category $\mathcal{C}$ with finite limits if and only if $\underline{X} \simeq \coprod_{i\in I} \underline{X}_i$ for any/some subcanonical topology.

3.3 Topos

3.3.1 Pretopos

Definition 3.3.1 A *pretopos*[a] is a category $\mathcal{C}$ such that

1. finite limits exist,
2. finite coproducts exist and are disjoint and universal,
3. equivalence relations are effective,
4. epimorphisms are regular and universal.

[a] Be careful that *not* all finite colimits exist in general.

Examples 1. If $\mathcal{C}$ is any site, then $\widetilde{\mathcal{C}}$ is a pretopos.
2. The category of finite sets is a pretopos.
3. The category of compact Hausdorff spaces is a pretopos.
4. The category $\mathbf{Top}$ is not a pretopos (epimorphisms are not always regular).

Exercise 3.30 Show that, in a pretopos, any morphism factors uniquely (up to a unique isomorphism) as an epimorphism followed by a monomorphism (an *epi-mono factorization*).

Solution. Let $f : X \to Y$ be a morphism. We let $R := X \times_Y X$ and $\overline{X} := X/R = \mathrm{coker}(R \rightrightarrows X)$. There exists a factorisation $X \overset{\pi}{\twoheadrightarrow} \overline{X} \overset{\iota}{\to} Y$ and we shall prove that ι is a monomorphism. If $g, h : Z \to \overline{X}$, we may consider the cartesian diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[r, two heads, "\widetilde{\pi}"] \arrow[d, "(\widetilde{g},\widetilde{h})"] & Z \arrow[d, "(g,h)"] \\ X \times X \arrow[r, two heads, "\pi\times\pi"] & \overline{X} \times \overline{X}. \end{tikzcd}$$

It is easily checked that if $\iota \circ g = \iota \circ h$, then $f \circ \widetilde{g} = f \circ \widetilde{h}$. It follows that $(\widetilde{g}, \widetilde{h})$ factors through R. This implies that $g \circ \widetilde{\pi} = h \circ \widetilde{\pi}$. Since, by construction, $\widetilde{\pi}$ is a (regular) epimorphism, we obtain $g = h$. Assume now that there exists another epi-mono factorization $X \overset{\pi'}{\twoheadrightarrow} X' \overset{\iota'}{\hookrightarrow} Y$. Then, necessarily, π' factors as $X \overset{\pi}{\to} \overline{X} \overset{s}{\to} X'$ and we must have $\iota = \iota' \circ s$:

$$\begin{tikzcd} X \arrow[r, two heads, "\pi'"] \arrow[d, two heads, "\pi"] & X' \arrow[d, hook, "\iota'"] \\ \overline{X} \arrow[ur, "s"] \arrow[r, hook, "\iota"] & Y. \end{tikzcd}$$

It follows that the diagram $X \times_{X'} X \rightrightarrows X \to \overline{X}$ is commutative. Since π' is regular, we have $X' = \mathrm{coker}(X \times_{X'} X \rightrightarrows X)$ and there exists necessarily a factorization $X \overset{\pi'}{\to} X' \overset{t}{\to} \overline{X}$. On easily checks that t is an inverse for s. ■

Exercise 3.31 Show that a pretopos is *balanced*: A morphism which is at the same time a monomorphism and an epimorphism is automatically an isomorphism.

Exercise 3.32 Show that, in a pretopos, any morphism is strict.

If $X_1, X_2 \subset X$, their *union* is $X_1 \cup X_2 := \operatorname{im}(X_1 \coprod X_2 \to X)$.

Exercise 3.33 Show that the set of subobjects of a given object in a pretopos is a *bounded lattice*[a] and pullback is a *morphism*[b] of bounded lattices.

[a] A partially ordered set with finite inf and finite sup.
[b] It preserves finite inf and finite sup.

3.3.2 Precanonical topology

Definition 3.3.2 The *precanonical pretopology* on a pretopos $\mathcal{C}$ is the pretopology made of *finite* families $(X_i \to X)_{i\in I}$ such that $\coprod_{i\in I} X_i \twoheadrightarrow X$ is an epimorphism.

Thus, almost by definition, a presheaf $\mathcal{F}$ is a sheaf if and only if for all epimorphism $\coprod_{i\in I} X_i \twoheadrightarrow X$ with I finite, the sequence

$$\mathcal{F}(X) \simeq \ker\left(\prod_{i\in I} \mathcal{F}(X_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(X_i \times_X X_j)\right)$$

is exact.

Exercise 3.34 Show that a family $(X_i \to X)_{i\in I}$ generates a covering sieve if and only if there exists a finite $J \subset I$ such such that $\coprod_{i\in J} X_i \twoheadrightarrow X$ is an epimorphism.

Exercise 3.35 Show that the precanonical topology is generated by finite disjoint unions $\coprod_{i\in I} X_i \simeq X$ and epimorphisms $X_1 \twoheadrightarrow X$.

Exercise 3.36 Show that a presheaf $\mathcal{F}$ is a sheaf if and only if:

1. given X_i with $i \in I$ finite,

$$\mathcal{F}\left(\coprod_{i\in I} X_i\right) \simeq \prod_{i\in I} \mathcal{F}(X_i) \quad \text{and} \tag{3.3}$$

2. if R is an equivalence relation on X, then

$$\mathcal{F}(X/R) \simeq \ker\left(\mathcal{F}(X) \rightrightarrows \mathcal{F}(R)\right). \tag{3.4}$$

In particular, a functor that preserves all finite limits (turns finite colimits into limits) is automatically a sheaf - but this is usually a stronger condition (and finite colimits may not exist in general).

Exercise 3.37 Show that the precanonical topology is subcanonical.

Solution. We have to show that, given any $Y \in \mathcal{C}$ and any covering family $(X_i \to X)_{i\in I}$, then

$$h_Y(X) \simeq \ker\left(\prod_{i\in I} h_Y(X_i) \rightrightarrows \prod_{i,j\in I} h_Y(X_i \times_X X_j)\right).$$

It is sufficient to consider a finite disjoint union $X = \coprod_{i\in I} X_i$ or an epimorphism $X' \twoheadrightarrow X$. In the first case, this boils down to $h_Y(\coprod_{i\in I} X_i) = \prod_{i\in I} h_Y(X_i)$ which holds just because h_Y is left exact. In the second case, we have $X = X'/R$ with $R := X' \times_X X'$ and we have to show that

$$\mathrm{Hom}(X'/R, Y) \simeq \ker(\mathrm{Hom}(X', Y)) \rightrightarrows \mathrm{Hom}(R, Y))),$$

which is the very definition of a quotient. ■

Proposition 3.3.3 Let $\mathcal{C}$ be a pretopos endowed with its precanonical topology.

1. If I is finite, then
$$\underline{\coprod_{i\in I} X_i} \simeq \coprod_{i\in I} \underline{X_i}.$$
2. If R is an equivalence relation on X, then
$$\underline{X/R} \simeq \underline{X}/\underline{R}.$$

Proof. This is completely formal but we can give the details.

1. We proceed by induction. Let $\mathcal{F}$ be any sheaf. We have $\underline{\emptyset} = \emptyset$ since $\mathrm{Hom}(\underline{\emptyset}, \mathcal{F}) = \mathcal{F}(\emptyset) = 1$. Moreover, if $X, Y \in \mathcal{C}$, then
$$\mathrm{Hom}(\underline{X \sqcup Y}, \mathcal{F}) = \mathcal{F}(X \sqcup Y) \simeq \mathcal{F}(X) \times X(Y)$$
$$= \mathrm{Hom}(\underline{X}, \mathcal{F}) \times \mathrm{Hom}(\underline{Y}, \mathcal{F}) = \mathrm{Hom}(\underline{X} \sqcup \underline{Y}, \mathcal{F}).$$
2. The same kind of arguments also works for the other assertion. More precisely, the sequence
$$\mathcal{F}(X/R) \to \mathcal{F}(X) \rightrightarrows \mathcal{F}(R)$$
is exact and may be rewritten as
$$\mathrm{Hom}(\underline{X/R}, \mathcal{F}) \to \mathrm{Hom}(\underline{X}, \mathcal{F}) \rightrightarrows \mathrm{Hom}(\underline{R}, \mathcal{F}).$$ ■

Be careful however that the functor $X \mapsto \underline{X}$ does *not* preserve finite colimits (which may not even exist) in general.

Exercise 3.38 Show that, if $\mathcal{C}$ be a pretopos endowed with the precanonical topology, then a morphism $f : X \to Y$ is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if $\underline{f} : \underline{X} \to \underline{Y}$ is a monomorphism (resp. epimorphism, resp. isomorphism).

Solution. The equivalence is true for isomorphisms because the topology is subcanonical. The direct implication holds for monomorphisms since $X \mapsto \underline{X}$ is left exact. If f is an epimorphism, then it is regular and $Y \simeq X/R$ so that $\underline{Y} = \underline{X/R} \simeq \underline{X}/\underline{R}$ which shows that $\underline{f}$ also is an epimorphism. In general, there exists an epi-mono factorization $X \twoheadrightarrow Z \hookrightarrow Y$ providing an epi-mono factorization $\underline{X} \twoheadrightarrow \underline{Z} \hookrightarrow \underline{Y}$. If $\underline{f}$ is a monomorphism then $\underline{X} = \underline{Z}$ so that $X = Z$ and f is a monomorphism. If $\underline{f}$ is an epimorphism, then $\underline{Z} = \underline{Y}$ so that $Z = Y$ and f is an epimorphism. ■

3.3.3 Generator

Definition 3.3.4 A set $S \subset \mathcal{C}$ is a *set of generators* (or *separators*) for a category $\mathcal{C}$ if the functor $\prod_{G \in S} h^G$ is faithful. When $S = \{G\}$, we say that G is a *generator*.

In down to earth terms, it means that if $f_1 \neq g_1 : X \to Y$, then there exists $g : G \to X$ with $G \in S$ such that $f_1 \circ g \neq f_2 \circ g$.

Examples 1. $1 := \{0\}$ is a generator for **Set**.
2. A is a generator for A-**Mod**.
3. If $\mathcal{C}$ is a category, then $\mathcal{C}$ is a set of generators for $\widehat{\mathcal{C}}$.

Exercise 3.39 Show that, if $\mathcal{C}$ is a site and $S \subset \mathcal{C}$ is a topologically generating subset, then $\{\underline{X}, X \in S\}$ is a set of generators for $\widetilde{\mathcal{C}}$.

Exercise 3.40 Show that, if $\mathcal{C}$ has all coproducts, then S is a set of generators if and only if we have for all $X \in \mathcal{C}$ an epimorphism

$$\coprod_{G \in S, f : G \to X} G \twoheadrightarrow X.$$

3.3.4 Topos

Definition 3.3.5 A category $\mathcal{T}$ is a *(Grothendieck) topos* if
1. there exists a small set of generators[a],
2. finite limits exist,
3. coproducts exist and they are disjoint and universal,
4. equivalence relations are effective and universal.

[a]We shall not worry about this set theoretical condition.

Examples 1. The categories **1**, **Set** and G-**Set** are topos.
2. If $\mathcal{C}$ is a category, then $\widehat{\mathcal{C}}$ is a topos.
3. If $\mathcal{C}$ is a site, then $\widetilde{\mathcal{C}}$ is a topos.
4. The category of sheaves of sets on a topological space X is a topos.
5. The category of espaces étalés (local homeomorphisms) over a topological space X is a topos.
6. The category of topological spaces or even compact Hausdorff spaces is not a topos.
7. The category of *condensed sets* is a topos.

Theorem 3.3.6 — Giraud. For a category $\mathcal{T}$, the following are equivalent:
1. $\mathcal{T}$ is a topos,
2. All sheaves for the canonical topology on $\mathcal{T}$ are representable and there exists a small set of generators,
3. there exists a site $\mathcal{C}$ such that $\mathcal{T} \simeq \widetilde{\mathcal{C}}$,
4. there exists a category $\mathcal{C}$ such that $\mathcal{T}$ is a reflective subcategory of $\widehat{\mathcal{C}}$ with exact reflection.

Proof.
- (2) $\Rightarrow$ (3) : we can[2] choose $\mathcal{C} = \mathcal{T}$ with its canonical topology.
- (3) $\Rightarrow$ (4) : shown in theorem 3.2.8.
- (4) $\Rightarrow$ (1) : it is sufficient to notice that $\widehat{C}$ itself is a topos and use the reflection.
- (1) $\Rightarrow$ (2) : we endow $\mathcal{T}$ with the canonical topology and first prove:

  Fact if $\coprod_{i\in I}\mathcal{F}_i \to \mathcal{F}$ is an epimorphism of sheaves on $\mathcal{T}$ with all $\mathcal{F}_i$ and $\mathcal{F}_i \times_{\mathcal{F}} \mathcal{F}_j$ representable, then $\mathcal{F}$ also is representable.

  Proof. We have $\mathcal{F}_i \simeq \underline{X}_i$ and $\mathcal{F}_i \times_{\mathcal{F}} \mathcal{F}_i \simeq \underline{X}_{i,j}$. Then, $R := \coprod_{i,j\in I} X_{i,j}$ defines an equivalence relation on $X := \coprod_{i\in I} X_i$ and it follows from exercises 3.29 and 3.28 that $\mathcal{F} \simeq \underline{X/R}$. ■

  Now, if we are given any sheaf $\mathcal{F}$ on $\mathcal{T}$, then there exists an epimorphism $\coprod_{i\in I}\underline{X}_i \twoheadrightarrow \mathcal{F}$ with $X_i \in \mathcal{C}$. It is therefore sufficient to show that each $\underline{X}_i \times_{\mathcal{F}} \underline{X}_j$ is representable. Assume first that $\mathcal{F} \subset \underline{Y}$ is a subsheaf of a representable sheaf. Then $\underline{X}_i \times_{\mathcal{F}} \underline{X}_j \simeq \underline{X_i \times_Y X_j}$ is representable and we are done. In general, $\underline{X}_i \times_{\mathcal{F}} \underline{X}_j \subset \underline{X_i \times X_j}$ and we are also done.

■

If follows that, in a topos $\mathcal{T}$, all limits and colimits exist, colimits are universal, epimorphisms are regular and universal, filtered colimits are exact and equivalence relations are effective and universal. A topos is balanced and any morphism has a unique epi-mono factorization. Subobjects form a bounded lattice and pulling back is a morphism of bounded lattices.

3.3.5 Internal Hom

Unless otherwise specified, we will always consider a topos $\mathcal{T}$ as a site for its *canonical* topology so that $\mathcal{T} \simeq \widetilde{\mathcal{T}}$. We will then identify $X \in \mathcal{T}$ with h_X and with $\underline{X}$ and write for example $Y(X) = \mathrm{Hom}(X, Y)$.

The topos $\mathcal{T}$ is automatically endowed with the maximal pretopology (a covering is a family that generates a covering sieve).

Exercise 3.41 Show that, in a topos $\mathcal{T}$, a family $(X_i \to X)_{i\in I}$ is a covering if and only if the map $\coprod_{i\in I} X_i \to X$ is an epimorphism.

Example A family of maps $(f_i : X_i \to X)_{i\in I}$ is a covering in the topos **Set** (for the canonical topology) if and only if it is *jointly surjective*:

$$\forall x \in X, \exists i \in I, \exists x_i \in X_i, \quad f_i(x_i) = x.$$

Exercise 3.42 Show that a presheaf T on a topos $\mathcal{T}$ is a sheaf if and only if it preserves all limits (it sends colimits to limits).

Solution. If R is a covering sieve of $X \in \mathcal{T}$, then it follows from proposition 3.2.12 that

$$X \simeq \varinjlim_{X' \in \mathcal{T}_{/R}} X'.$$

[2]For set-theoretic reasons, it is actually necessary to replace $\mathcal{T}$ with the subcategory generated by a small set of generators.

If T preserves limits, then

$$T(X) \simeq \varprojlim_{X' \in \mathcal{T}_{/R}} T(X')$$

and this shows that T is a sheaf. The converse follows from proposition 1.4.5. ■

Proposition 3.3.7 If $\mathcal{T}$ is a topos and $Y, Z \in \mathcal{T}$, then the presheaf

$$X \mapsto \mathrm{Hom}(X \times Y, Z)$$

is representable by an object $\mathcal{H}om(Y, Z) \in \mathcal{T}$.

Proof. It is sufficient to show that it is a sheaf but this follows from the fact that colimits are universal in a topos. ■

As a consequence, a topos is *cartesian closed*: there exists a natural isomorphism (currying)

$$\mathrm{Hom}(X \times Y, Z) \simeq \mathrm{Hom}(X, \mathcal{H}om(Y, Z)) \quad (\simeq \mathrm{Hom}(Y, Z)(X)).$$

In particular, for fixed Y, the functor $X \mapsto X \times Y$ is adjoint to the functor $Z \mapsto \mathcal{H}om(Y, Z)$.

Example If $\mathcal{F}, \mathcal{G}$ are two (pre-) sheaves on a topological space X, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$ (with $\mathcal{F}_{|U}(V) := \mathcal{F}(V)$ for $V \subset U$).

Exercise 3.43 Show that, in a topos,

$$\mathcal{H}om(X \times Y, Z) \simeq \mathcal{H}om(X, \mathcal{H}om(Y, Z)).$$

Solution. It is sufficient to notice that, given any object T, we have a natural isomorphism

$$\begin{aligned}
\mathrm{Hom}(T, \mathcal{H}om(X \times Y, Z)) &\simeq \mathrm{Hom}(T \times X \times Y, Z)) \\
&\simeq \mathrm{Hom}(T \times X, \mathcal{H}om(Y, Z))) \\
&\simeq \mathrm{Hom}(T, \mathcal{H}om(X, \mathcal{H}om(Y, Z))).
\end{aligned}$$ ■

Exercise 3.44 Show that, if 1 denotes the final object of a topos, then

1. $\mathcal{H}om(1, X) \simeq X$,
2. $\mathrm{Hom}(X, Y) = \mathcal{H}om(X, Y)(1)$.

The last assertion means that $\mathcal{T}$ is *enriched* over itself.

Exercise 3.45 Show that, if $\mathcal{C}$ is a site, $T \in \widehat{\mathcal{C}}$ and $\mathcal{F} \in \widetilde{\mathcal{C}}$, then $\mathcal{H}om_{\widetilde{\mathcal{C}}}(\widetilde{T}, \mathcal{F}) = \mathcal{H}om_{\widehat{\mathcal{C}}}(T, \mathcal{F})$.

Solution. For $X \in \mathcal{C}$, we have the sequence of isomorphisms

$$\begin{aligned}\mathcal{H}om_{\widehat{\mathcal{C}}}(\widetilde{T}, \mathcal{F})(X) &= \mathrm{Hom}(\underline{X}, \mathcal{H}om_{\widehat{\mathcal{C}}}(\widetilde{T}, \mathcal{F})) \\ &= \mathrm{Hom}(\underline{X} \times \widetilde{T}, \mathcal{F}) \\ &= \mathrm{Hom}(\widetilde{h_X \times T}, \mathcal{F}) \\ &= \mathrm{Hom}(h_X \times T, \mathcal{F}) \\ &= \mathrm{Hom}(h_X, \mathcal{H}om_{\widehat{\mathcal{C}}}(T, \mathcal{F})) \\ &= \mathcal{H}om_{\widehat{\mathcal{C}}}(T, \mathcal{F})(X).\end{aligned}$$

■

3.3.6 Quasi-compact/quasi-separated

See for example Lurie.

Definition 3.3.8 An object X of a site $\mathcal{C}$ is said to be *quasi-compact* if, given any family $(X_i \to X)_{i \in I}$ that generates a covering sieve, there exists a finite subset J of I such that the sieve generated by $(X_i \to X)_{i \in J}$ is also a covering.

In particular, an object X of a topos $\mathcal{T}$ is quasi-compact if and only if, given an epimorphism $\coprod_{i \in I} X_i \twoheadrightarrow X$, there exists a finite subset J of I such that $\coprod_{i \in J} X_i \twoheadrightarrow X$ is an epimorphism. Actually, quasi-compactness may always be checked in a topos:

Exercise 3.46 Show that an object X of a site $\mathcal{C}$ is quasi-compact if and only if $\underline{X}$ is quasi-compact (for the canonical topology).

Solution. Direct implication is clear. For the converse, consider an epimorphism $\coprod_i \mathcal{F}_i \to \underline{X}$. There exists for each $\mathcal{F}_i$ an epimorphism $\coprod_j \underline{X}_{ij} \to \mathcal{F}_i$. Then, there exists for each $\underline{X}_{ij} \to \underline{X}$ a covering $(X_{ijk} \to X_{ij})_k$ and morphisms $X_{ijk} \to X$ in $\mathcal{C}$ giving rise to $\underline{X}_{ijk} \to \underline{X}_{ij} \to \mathcal{F} \to \underline{X}$. We may then pick up a finite number of i, j, k. ■

Exercise 3.47 Show that if X has a finite covering by quasi-compacts objects, then X is quasi-compact.

Proposition 3.3.9 A sheaf $\mathcal{F}$ on a *pretopos* $\mathcal{C}$ (for the precanonical topology) is quasi-compact if and only if there exists an epimorphism $\underline{X} \twoheadrightarrow \mathcal{F}$ with $X \in \mathcal{C}$.

Proof. There exists an epimorphism $\coprod_{i \in I} \underline{X}_i \twoheadrightarrow \mathcal{F}$ with $X_i \in \mathcal{C}$. If $\mathcal{F}$ is quasi-compact, we may assume that I is finite and set $X := \coprod_{i \in I} X_i$. For the converse, we may assume that $\mathcal{F} = \underline{X}$. It follows from the very definition of the topology that X is quasi-compact. Or, thanks to exercise 3.46, equivalently, that $\underline{X}$ is quasi-compact. ■

Definition 3.3.10 An object X of a topos $\mathcal{T}$ is said to be *quasi-separated* if given any $Y \to X$ and $Z \to X$ with Y and Z quasi-compacts, then $Y \times_X Z$ is also quasi-compact.

Exercise 3.48
1. Show that a subobject of a quasi-separated object is quasi-separated.
2. Show that a coproduct of quasi-separated objects is quasi-separated.

3. Show that a filtered colimit under monomorphisms of quasi-separated objects is quasi-separated.

Solution. 1. Assume X is quasi-separated and $X' \subset X$. We give ourselves $Y \to X$ and $Z \to X$ with Y and Z quasi-compacts. Then, $Y \times_{X'} Z = Y \times_X Z$ is also quasi-compact.

2. Assume that $X = \coprod_{i \in I} X_i$. If $Y \to X$ is any morphism, then we have $Y = \coprod_{i \in I} Y_i$ with $Y_i = Y \times_X X_i$. In other words, the family $(Y_i \hookrightarrow Y)_{i \in I}$ is a covering. Therefore, if Y is quasi-compact, we can then replace I with a finite subset J and we have $Y = \coprod_{i \in J} Y_i$. Moreover, a summand of a quasi-compact is always quasi-compact - as we shall show below - so that each Y_i is quasi-compact. Of course, if $Z \to X$ is another morphism with Z quasi-compact, we can also write $Z = \coprod_{i \in J} Z_i$ and we may assume that this is the same finite J. It is then formal to check that

$$Y \times_X Z \simeq \coprod_{i \in J} Y_i \times_{X_i} Z_i.$$

If we assume that all X_i are quasi-separated, then $(Y_i \times_{X_i} Z_i \to Y \times_X Z)_{i \in J}$ is a finite covering by quasi-compact objects and it follows that $Y \times_X Z$ is also quasi-compact. It remains to show that a summand of a quasi-compact is itself quasi-compact. But if we are given a covering $(X_i \to X)_{i \in I}$ and we know that $X \sqcup Y$ is quasi-compact, we can then consider the covering made of the X_i's and Y of $X \sqcup Y$. It has a finite refinement and we are done.

3. If $X = \varinjlim_{i \in I} X_i$ is any colimit, then the corresponding morphism $\coprod_{i \in I} X_i \twoheadrightarrow X$ is an epimorphism. In other words, the family $(X_i \to X)_{i \in I}$ is a covering. In particular, when X is quasi-compact, there exists a finite subset J of I such that $\coprod_{i \in J} X_i \twoheadrightarrow X$ is an epimorphism and therefore $X = \varinjlim_{i \in J} X_i$. In the case of a filtered colimit, if k is any cocone for J in I, then we will have $X = X_k$. Not assuming X quasi-compact anymore, let $Y \to X$ be a morphism with Y quasi-compact. Then, $Y = \varinjlim_{i \in I} Y_i$ with $Y_i = Y \times_X X_i$. If the colimit is filtered, then there exists k such that $Y = Y_k$. In other words, there exists a factorization $Y \to X_k \to X$ of the original morphism. If $Z \to Y$ is another morphism with Z quasi-compact, then there exists also a factorization $Z \to X_k \to X$ and we may assume that this is the same k since I is filtered. Finally, if we assume that $X_k \subset X$, then we have $Y \times_X Z = Y \times_{X_k} Z$ which is quasi-compact if we assume that X_k is quasi-separated.

■

Theorem 3.3.11 If $\mathcal{C}$ is a pretopos (endowed with the precanonical topology), then the functor $X \mapsto \underline{X}$ induces an equivalence between $\mathcal{C}$ and the category of quasi-compact quasi-separated sheaves on $\mathcal{C}$.

It means that the pretopos is uniquely determined by the topos.

Proof. Since the topology is subcanonical, the functor is fully faithful and it easily follows from proposition 3.3.9 that $\underline{X}$ is always quasi-compact quasi-separated. Considering quasi-separatedness for example, consider two morphisms $\mathcal{F}, \mathcal{G} \to \underline{X}$

with $\mathcal{F}, \mathcal{G}$ quasi-compact. Then, there exists two epimorphisms $\underline{Y} \twoheadrightarrow \mathcal{F}$ and $\underline{Z} \twoheadrightarrow \mathcal{G}$ from which we obtain (by universality) an epimorphism $\underline{Y \times_X Z} \twoheadrightarrow \mathcal{F} \times_{\underline{X}} \mathcal{G}$ which shows that this last sheaf is quasi-compact. Assume conversely that $\mathcal{F}$ is quasi-compact quasi-separated and let $\underline{X} \twoheadrightarrow \mathcal{F}$ (resp. $\underline{X'} \twoheadrightarrow \underline{X} \times_{\mathcal{F}} \underline{X}$) be an epimorphism. The composite morphism

$$\underline{X'} \twoheadrightarrow \underline{X} \times_{\mathcal{F}} \underline{X} \hookrightarrow \underline{X} \times \underline{X} \simeq \underline{X \times X}$$

Is an epi-mono factorization. It comes from a morphism $X' \to X \times X$ in $\mathcal{C}$ that has an epi-mono factorization $X' \twoheadrightarrow R \hookrightarrow X \times X$. It follows that $\underline{X'} \twoheadrightarrow \underline{R} \hookrightarrow \underline{X \times X}$ is also an epi-mono factorization. By uniqueness, $\underline{R} \simeq \underline{X} \times_{\mathcal{F}} \underline{X}$ is an equivalence relation with quotient $\mathcal{F}$. It follows that R also is an equivalence relation and that $\mathcal{F} = \underline{X}/\underline{R} \simeq \underline{X/R}$. ■

3.4 Morphism of topos (optional)

3.4.1 Definition

Definition 3.4.1 A *morphism of topos* $f : \mathcal{T} \longrightarrow \mathcal{T}'$ is a couple of functors

$$\left(f^{-1} : \mathcal{T}' \longrightarrow \mathcal{T}, \quad f_* : \mathcal{T} \longrightarrow \mathcal{T}'\right)$$

with f^{-1} exact and adjoint to f_*.

We say that f is an *embedding of topos* when f_* is fully faithful. Note that if f is a morphism of topos, then both f^{-1} and f_* preserve algebraic structures.

Examples
1. If $\mathcal{C}$ is a site, then there exists an embedding of topos $i : \widetilde{\mathcal{C}} \hookrightarrow \widehat{\mathcal{C}}$ given by $i^{-1}(T) = \widetilde{T}$ and $i_*\mathcal{F} = \mathcal{F}$.
2. There exists a unique morphism of topos $p : \mathcal{T} \to \mathbf{Set}$. We have $p_*\mathcal{F} = \mathcal{F}(1_{\mathcal{T}})$ and $p^{-1}E = \widetilde{E}_{\mathcal{C}}$.

Exercise 3.49 Show that, if $E = \varinjlim_{i \in I} E_i$ in $\mathbb{S}\mathrm{et}$, then $\widetilde{E}_{\mathcal{C}} = \varinjlim_{i \in I} \widetilde{E}_{i\mathcal{C}}$.

Exercise 3.50 Show that if $f : \mathcal{T} \longrightarrow \mathcal{T}'$ is a morphism of topos, then $f^{-1}\widetilde{E}_{\mathcal{T}'} = \widetilde{E}_{\mathcal{T}}$.

Exercise 3.51 Show that if $f : \mathcal{T} \longrightarrow \mathcal{T}'$ is a morphism of topos, then

$$f_*\mathcal{H}om(f^{-1}X', Y) \simeq \mathcal{H}om(X', f_*Y).$$

One defines in the obvious way the composition of two morphisms of topos so that we may consider the topos as the objects of a category.

3.4.2 Presheaves

Theorem 3.4.2 If $g : \mathcal{C} \to \mathcal{C}'$ is any functor, then the functor

$$\widehat{g}^{-1} : \qquad \widehat{\mathcal{C}'} \longrightarrow \widehat{\mathcal{C}}, \qquad T' \longrightarrow T' \circ g.$$

has an adjoint $\widehat{g}_!$ (resp. a coadjoint $\widehat{g}_*$) $: \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$.

Proof. Follows from proposition 1.5.10. ∎

As a consequence, $\widehat{g}_!$ preserves all colimits, $\widehat{g}^{-1}$ preserves all limits and colimits and $\widehat{g}_*$ preserves all limits. In particular, both $\widehat{g}^{-1}$ and $\widehat{g}_*$ preserve algebraic structures. Note also that we can recover the original functor g from the equality $h_{g(X)} = g_! h_X$.

Corollary 3.4.3 The functor $g : \mathcal{C} \to \mathcal{C}'$ induces a morphism of topos $\widehat{g} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ given by $\widehat{g}^{-1}$ and $\widehat{g}_*$

Proof. The existence of $\widehat{g}_!$ makes sure that $\widehat{g}^{-1}$ is left exact. ∎

Exercise 3.52 Any $X \in \mathcal{C}$ may be seen as a functor $X : \mathbf{1} \to \mathcal{C}$ giving rise to a morphism of topos

$$\widehat{X} : \mathbf{Set} \to \widehat{\mathcal{C}}.$$

Make $\widehat{X}_!$, $\widehat{X}^{-1}$ and $\widehat{X}_*$ explicit.

Solution. First of all, we have $\widehat{X}^{-1}(T) = T(X)$. Next, $(\widehat{X}_!(E))(Y) = E$ if there exists $Y \to X$ and $\emptyset$ otherwise. Finally, $(\widehat{X}_*(E))(Y) = E$ if there exists $X \to Y$ and $\{0\}$ otherwise. ∎

Examples 1. If $f : Y \to X$ is a continuous map, it induces a functor

$$(g =)\ f^{-1} : \mathbf{Open}(X) \to \mathbf{Open}(Y),$$

and by composition, a functor

$$(\widehat{g}^{-1} =)\ \widehat{f}_* : \widehat{\mathbf{Open}(Y)} \to \widehat{\mathbf{Open}(X)},$$

that has adjoint and coadjoint

$$(\widehat{g}_! =)\ \widehat{f}^{-1} \quad \text{and} \quad (\widehat{g}_* =)\ \widehat{f}^! : \widehat{\mathbf{Open}(X)} \to \widehat{\mathbf{Open}(Y)}.$$

2. Explicitly, the adjoint functors are given by

$$\widehat{f}^{-1}(T)(V) = \varinjlim_{V \subset f^{-1}(U)} T(U), \quad \widehat{f}_*(T)(U) = T(f^{-1}(U)),$$

$$\text{and} \quad \widehat{f}^!(T)(V) = \varprojlim_{f^{-1}(U) \subset V} T(U).$$

3. There exists a morphism of topos $\widehat{f} : \widehat{\mathbf{O}\mathrm{pen}(Y)} \to \widehat{\mathbf{O}\mathrm{pen}(X)}$.

By analogy, we may also denote a functor as $f^{-1} := g : \mathcal{C} \to \mathcal{C}'$ (even if this is not an inverse image), and consequently write:

$$\widehat{f}^{-1} := \widehat{g}_!, \quad \widehat{f}_* := \widehat{g}^{-1}, \quad \widehat{f}^! := \widehat{g}_*,$$

so that now $h_{f^{-1}(X)} = \widehat{f}^{-1}(h_X)$.

Exercise 3.53 Let X be a topological space. One can see a point $x \in X$ as a continuous map $1 \to X$. Make $\widehat{x}^{-1}$, $\widehat{x}_*$ and $\widehat{x}^!$ explicit. If T is a presheaf, then $T_x := \widehat{x}^{-1}T$ is called the *stalk* of T at x. Show that T is a separated presheaf if and only if for all open subset U of X, the map $T(U) \to \prod_{x \in U} T_x$ is injective.

3.4.3 Morphisms of sites

Definition 3.4.4 If $\mathcal{C}$ and $\mathcal{C}'$ are two sites, then a functor $f^{-1} := g : \mathcal{C} \to \mathcal{C}'$ is said to be *continuous* if the functor

$$\widehat{f}_* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}, \quad T' \mapsto T' \circ f^{-1}$$

preserves sheaves.

We shall then denote by $\widetilde{f}_* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ (or f_* for short) the induced functor.

Exercise 3.54 Show that if $\mathcal{C}$ and $\mathcal{C}'$ are two sites and if $f^{-1} : \mathcal{C} \to \mathcal{C}'$ is a continuous functor, then the functor

$$\widetilde{f}^{-1} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}, \quad \mathcal{F} \mapsto \widetilde{\widehat{f}^{-1}(\mathcal{F})}$$

is adjoint to f_*.

Since we always have $\underline{f^{-1}(X)} = \widetilde{f}^{-1}(\underline{X})$, we may simply write f^{-1} instead of $\widetilde{f}^{-1}$ when there is no ambiguity.

Definition 3.4.5 A *morphism of sites* is a continuous functor $f^{-1} : \mathcal{C} \to \mathcal{C}'$ such that $\widetilde{f}^{-1}$ is (left) exact.

Equivalently, it means that

$$f := (f^{-1}, f_*) : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$$

is a morphism of topos. Note that there exists a commutative diagram of morphisms of topos

$$\begin{tikzcd}
\widehat{\mathcal{C}'} \arrow[r, "\widehat{f}"] & \widehat{\mathcal{C}} \\
\widetilde{\mathcal{C}'} \arrow[u, hook] \arrow[r, "f"] & \widetilde{\mathcal{C}}. \arrow[u, hook]
\end{tikzcd}$$

Examples 1. If we endow two topos $\mathcal{T}$ and $\mathcal{T}'$ with their canonical topology, then a morphism of sites $f : \mathcal{T}' \to \mathcal{T}$ is nothing but a morphism of topos.

2. If $f : Y \to X$ is a continuous map of topological spaces, then the inverse image functor f^{-1} defines a morphism of sites and therefore a morphism of topos

$$f : \widetilde{\mathbf{Open}(Y)} \to \widetilde{\mathbf{Open}(X)}.$$

If $\mathcal{G}$ is a sheaf on Y, we have for any open subset U of X,

$$f_*\mathcal{G}(U) = \mathcal{G}(f^{-1}(U)).$$

And if $\mathcal{F}$ is a sheaf on X, then $f^{-1}\mathcal{F}$ is the *sheafification* of

$$V \mapsto \varinjlim_{f(V)\subset U} \mathcal{F}(U).$$

Exercise 3.55 Make explicit the morphism of topos coming from the final map $p : X \to 1$ when X is a topological space. Show that if $f : Y \to X$ is a continuous map, then $\widetilde{E}_Y = f^{-1}\widetilde{E}_X$.

Solution. We have $p_*\mathcal{F} = \Gamma(X, \mathcal{F})$ and $p^{-1}E = \widetilde{E}_X$. With $q : Y \to 1$, we have $q = p \circ f$ and therefore $\widetilde{E}_Y = q^{-1}E = f^{-1}p^{-1}E = f^{-1}\widetilde{E}_X$. ■

Exercise 3.56 Make explicit the morphism of topos coming from a point $x : 1 \to X$ when X is a topological space.

Solution. We have $(x_*E)(U) = \begin{cases} E \text{ if } x \in U \\ \emptyset \text{ if } x \notin U \end{cases}$ and $x^{-1}\mathcal{F} = \mathcal{F}_x$. ■

Exercise 3.57 Show that if T is a presheaf on a topological space X and $x \in X$, then $\widetilde{T}_x = T_x$.

Exercise 3.58 Show that, if $\mathcal{C}$ has fibered products and $f^{-1} : \mathcal{C} \to \mathcal{C}'$ is a functor between two sites which is left exact and preserves covering families, then f^{-1} is morphism of sites.

Definition 3.4.6 A functor $g : \mathcal{C}' \to \mathcal{C}$ between two sites is said to be *cocontinuous* if $\widehat{g}_*$ preserves sheaves.

Exercise 3.59 Show that if g is cocontinuous, then the induced functor $g_* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ extends uniquely to a morphism of topos $g := (g^{-1}, g_*) : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$.

Exercise 3.60 Show that, if $g : \mathcal{C}' \to \mathcal{C}$ be a functor between two sites and for any $X' \in \mathcal{C}'$, any covering family of $g(X')$ is the image of a covering family of X', then g is cocontinuous.

3.4.4 Induced topology

Definition 3.4.7 If $\mathcal{C}'$ is a site and $f^{-1}:\mathcal{C}\to\mathcal{C}'$ is any functor, then the *induced topology* on $\mathcal{C}$ is the finest topology on $\mathcal{C}$ making $f^{-1}:\mathcal{C}\to\mathcal{C}'$ continuous.

Exercise 3.61 Show that if $\mathcal{C}$ is a site, then the topology of $\mathcal{C}$ is induced by the (canonical) topology of $\widetilde{\mathcal{C}}$.

If $\mathcal{C}$ is a site and $T\in\widehat{\mathcal{C}}$, we can consider the forgetful functor $j_T:\mathcal{C}_{/T}\to\mathcal{C}$ and endow $\mathcal{C}_{/T}$ with the induced topology.

Exercise 3.62 Show that a sieve R of $s\in T(X)$ is a covering in $\mathcal{C}_{/T}$ if and only if $\widehat{j}_{T!}R$ is a covering sieve of X in $\mathcal{C}$.

Unfortunately, the adjoint $j_{T!}$ is *not* left exact in general and we do not get a morphism of sites. However:

Exercise 3.63 Show that, if $\mathcal{C}$ is a site and $T\in\widehat{\mathcal{C}}$, then j_T is also cocontinuous and there exists therefore a morphism of topos

$$j_T:\widetilde{\mathcal{C}_{/T}}\to\widetilde{\mathcal{C}}.$$

Exercise 3.64 Show that, if $\mathcal{C}$ is a site and $T\in\widehat{\mathcal{C}}$, then $\widetilde{\mathcal{C}_{/T}}\simeq\widetilde{\mathcal{C}}_{/\widetilde{T}}$. Show that $j_{T!}$ (resp. j_T^{-1}) corresponds to the forgetful functor $\widetilde{\mathcal{C}}_{/\widetilde{T}}\to\widetilde{\mathcal{C}}$ (resp. to $\mathcal{F}\mapsto(\mathcal{F}\times\widetilde{T}\to\widetilde{T})$.

Exercise 3.65 Show that, in a topos $\mathcal{T}$, we have

$$j_X^{-1}\mathcal{H}\mathrm{om}(X,Y)\simeq\mathcal{H}\mathrm{om}(j_X^{-1}Y,j_X^{-1}Z)\quad\text{and}$$

$$\mathcal{H}\mathrm{om}(j_{X!}X,Y)\simeq j_{X*}\mathcal{H}\mathrm{om}(Y,j_X^{-1}Z).$$

Exercise 3.66 Show that, in a topos $\mathcal{T}$, we have

$$\mathcal{H}\mathrm{om}(X,Y)\simeq j_{X*}j_X^{-1}Y\quad\text{and}$$

$$\mathcal{H}\mathrm{om}(X,Y)(Z)=\mathrm{Hom}(j_Z^{-1}X,j_Z^{-1}Y).$$

3.4.5 Topological spaces

If X is a topological space, we may then consider the site $\mathbf{Top}_{/X}$ of all topological spaces over X. The inclusion map

$$\mathbf{Open}(X)\hookrightarrow\mathbf{Top}_{/X}$$

is continuous, cocontinuous and left exact giving rise to two morphisms of topos

$$\widetilde{\mathbf{Top}_{/X}}\underset{\psi_X}{\overset{\varphi_X}{\rightleftarrows}}\widetilde{\mathbf{Open}(X)}$$

with $\varphi_X\circ\psi_X=\mathrm{Id}$ and $\varphi_{X*}=\psi_X^{-1}$.

Any continuous map $f : Y \to X$ will provide a commutative diagram of topos

$$\begin{tikzcd}
\widetilde{\mathbf{Top}_{/Y}} \arrow[r, "\varphi_Y"] \arrow[d, "j"] & \widetilde{\mathbf{Open}(Y)} \arrow[d, "f"] \\
\widetilde{\mathbf{Top}_{/X}} \arrow[r, "\varphi_X"] & \widetilde{\mathbf{Open}(X)}.
\end{tikzcd}$$

Let X be any topological space. If $\mathcal{F}$ is a sheaf on $\mathbf{Top}_{/X}$ and Y is a topological space over X, then the *realization* of $\mathcal{F}$ on Y is

$$\mathcal{F}_Y := \varphi_{Y*}\mathcal{F}_{/Y}.$$

Any morphism $f : Z \to Y$ over S will induce a morphism

$$\alpha_f : f^{-1}\mathcal{F}_Y \to \mathcal{F}_Z$$

between the realizations.

Exercise 3.67 Show that, giving a sheaf $\mathcal{F}$ on $\mathbf{Top}_{/X}$ is equivalent to giving the set of all $\mathcal{F}_Y$ and compatible morphisms $\alpha_f : f^{-1}\mathcal{F}_Y \to \mathcal{F}_Z$.

We shall call a sheaf $\mathcal{F}$ on $\mathbf{Top}_{/X}$ *crystalline* if all the maps α_f are isomorphisms $f^{-1}\mathcal{F}_Y \simeq \mathcal{F}_Z$.

Exercise 3.68 Show that realization $\mathcal{F} \mapsto \mathcal{F}_X$ induces an equivalence between crystalline sheaves on $\mathbf{Top}_{/X}$ and sheaves on X.

Exercise 3.69 Show that the category $\mathbf{Et}(X)$ is equivalent to the category of crystalline sheaves on $\mathbf{Top}_{/X}$, and consequently to the category of sheaves on X.

This shows that $\mathbf{Et}(X)$ is a topos because $\mathbf{Et}(X) \simeq \widetilde{\mathbf{Open}(X)}$.

Exercise 3.70 Show that a sheaf on a topological space X is quasi-compact (for the canonical topology) if and only if the corresponding espace étalé is compact.

4. Condensed sets

Again, we shall not worry much about set-theoretical issues and work in a fixed universe. Note however that there exists an unconditional theory of condensed sets (that does not depend on the choice of a universe). There also exists a theory of *light* condensed sets.

4.1 Condensed set

4.1.1 Compact Hausdorff spaces

We shall denote by **CHaus** the category of compact Hausdorff spaces and continuous maps. We shall also write $\bullet$ for the one point space (final object).

Proposition 4.1.1 In **CHaus**,

1. all limits and colimits exist,
2. finite coproducts are disjoint and universal,
3. equivalence relations are effective and universal,
4. epimorphisms are regular and universal.

Proof. Recall that **Top** has all limits and colimits and that they are computed in **Set**. Moreover, all the above properties are satisfied in **Set** which is a topos. Also, it follows from proposition 2.1.7 that **CHaus** has all limits and colimits with limits computed in **Top**. Moreover, any colimit in **Top** of compact Hausdorff spaces which is itself compact Hausdorff is also the colimit in **CHaus**. This applies in particular to finite coproducts, epimorphisms and closed equivalence relations. It follows that finite coproducts are disjoint, epimorphisms are universal and equivalence relations are effective. Finally, a continuous surjective map between compact Hausdorff spaces is a quotient map. In other words, an epimorphism is regular. ■

Corollary 4.1.2 The category **CHaus** is a pretopos (see definition 3.3.1). ■

We may also stress out the fact that a continuous map is a monomorphism (resp. an epimorphism, an isomorphism) if and only if it is injective (resp. surjective, resp. bijective). Any monomorphism (resp. epimorphism) is regular: any continuous injective (resp. surjective) map is a homeomorphism onto a closed subspace (resp. a quotient map).

4.1.2 Definition

We shall systematically endow **CHaus** with its precanonical topology: a family $(S_i \to S)_{i\in I}$ of continuous maps of compact Hausdorff spaces is a covering if and only if I is finite and $\coprod_{i\in I} S_i \twoheadrightarrow S$ is surjective. Recall that this topology is subcanonical.

Definition 4.1.3 A *condensed set* is a sheaf of sets on **CHaus**.

We shall denote by **Cond** $:= \widetilde{\mathbf{CHaus}}$ the category of condensed sets.

Examples
1. We shall see that, if X is any topological space, then $S \mapsto \underline{X}(S) := \mathcal{C}(S, X)$ defines a condensed set.
2. If X is a topological space, then $\underline{X}(\overline{\mathbb{N}})$ is the set of convergent sequences $(x_n)_{n\in\mathbb{N}}$, together with a specified limit x_∞.
3. There exists a unique condensed set Q such that $Q(S) = \mathcal{C}(S, \mathbb{R})/\mathcal{C}(S, \mathbb{R}^{\mathrm{disc}})$ when S is Stonean (or even Stone, see exercise 8.3 below).

Proposition 4.1.4 A presheaf of sets X on **CHaus** is a condensed set if and only if:
1. given any compact Hausdorff spaces S_i for $i \in I$ finite, then

$$X\left(\coprod_{i\in I} S_i\right) \simeq \prod_{i\in I} X(S_i) \quad \text{and} \tag{4.1}$$

2. given any closed equivalence relation R on a compact Hausdorff space S,

$$X(\mathrm{coker}(R \rightrightarrows S)) \simeq \ker\left(X(S) \rightrightarrows X(R)\right). \tag{4.2}$$

Note that $\mathrm{coker}(R \rightrightarrows S)$ is usually abbreviated into S/R but the formulation here insists on the symmetry. Note also that it is sufficient (but not necessary) to show that X preserves finite limits for it to be a condensed set.

Proof. This follows from exercise 3.36. ■

The first condition can be reduced to

$$X(\emptyset) \simeq \{0\} \quad \text{and} \quad X(S \sqcup S') = X(S) \times X(S')$$

for compact Hausdorff spaces S and S'. Also, the second one may be rephrased by saying that, if $f : S' \twoheadrightarrow S$ is a continuous surjection, then the sequence

$$X(S) \to X(S') \rightrightarrows X(S' \times_S S')$$

is (left) exact.

Theorem 4.1.5 The category **Cond** is a topos.

Proof. Follows from theorem 3.3.6. ∎

Corollary 4.1.6 In the category **Cond**, all limits and colimits exist and

1. colimits are universal,
2. filtered colimits are exact,
3. epimorphisms are regular and universal,
4. equivalence relations are effective universal. ∎

More generally, any formula that involves colimits and finite limits that holds for sets also holds for condensed sets. Also, any morphism has a unique epi-mono factorization and subobjects form a bounded lattice (and pulling back is a morphism of bounded lattices).

The topos **Cond** will be endowed with its canonical topology (a sheaf is a representable presheaf).

4.1.3 Free compact Hausdorff spaces

Let us consider now the subcategory **FCHaus** of *free* compact Hausdorff spaces and continuous maps. Any compact Hausdorff space is a quotient of a free compact Hausdorff space and any such quotient may be extended to a free presentation. Also, any free compact Hausdorff space is a projective object of the category of compact Hausdorff spaces, or equivalently a Stonean space. Any surjection from a compact Hausdorff space to a Stonean space (and, in particular, to a free compact Hausdorff space) has a section.

Exercise 4.1 Show that finite disjoint unions define a pretopology on **FCHaus**.

We shall systematically endow **FCHaus** with this pretopology: a family $(F_i \to F)_{i\in I}$ of continuous maps of free compact Hausdorff spaces is a covering if and only if I is finite and $\coprod_{i\in I} F_i \simeq F$ is bijective.

Exercise 4.2 Show that a presheaf of sets X on **FCHaus** is a sheaf if and only if it preserves finite products (condition (4.1) above).

Theorem 4.1.7 Inclusion induces an equivalence of categories

$$\mathbf{Cond} := \widetilde{\mathbf{CHaus}} \simeq \widetilde{\mathbf{FCHaus}}.$$

Proof. Clearly, any sheaf X on **CHaus** will restrict to a sheaf on **FCHaus**. Conversely, if X be a sheaf on **FCHaus** and S any compact Hausdorff space, we set

$$X(S) := \varprojlim_{F\to S} X(F)$$

where F runs through all free compact Hausdorff spaces over S.

Let us prove that, if

$$F' \rightrightarrows F \twoheadrightarrow S \tag{4.3}$$

is a free presentation (meaning that F, F' are free and both $F \twoheadrightarrow S$ and $F' \twoheadrightarrow F \times_S F$ are surjective), then

$$X(S) = \varprojlim_{F'' \to S} X(F'') \simeq \ker\left(X(F) \rightrightarrows X(F')\right).$$

By restriction, there exists a natural map from the left hand side to the right one. Conversely, if x lives in the right-hand side and F'' is a free compact Hausdorff space, then any continuous map $F'' \to S$ will lifts to some $f : F'' \to F$ and we may consider $f^{-1}(x) \in X(F'')$. If we are given two such liftings $f_1, f_2 : F'' \to F$, then they induce a map $F'' \to F \times_S F$ that lifts to a map $f' : F'' \to F'$. This implies that $f_1^{-1}(x) = f_2^{-1}(x)$ and our map is well defined. It is straightforward so check that this defines an inverse.

Assume now that $F \twoheadrightarrow S$ is a continuous surjective map with F free. Since it can be extended to a free presentation as above, we have $X(S) \subset X(F)$. In particular, since $F' \twoheadrightarrow F \times_S F$ is surjective, we have $X(F \times_S F) \subset X(F')$ and the sequence

$$X(S) \to X(F) \rightrightarrows X(F \times_S F)$$

is therefore exact (with no reference to F' anymore).

If R is a closed equivalence relation on a compact Hausdorff spaces S, then there exists a free compact Hausdorff space F and a continuous surjection $F \twoheadrightarrow S$. The commutative diagram

$$\begin{tikzcd}
F \times_{S/R} F \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F \arrow[r, two heads] \arrow[d] & S/R \arrow[d, equal] \\
R \arrow[r, shift left] \arrow[r, shift right] & S \arrow[r, two heads] & S/R
\end{tikzcd}$$

provides a commutative diagram

$$\begin{tikzcd}
X(S/R) \arrow[r] \arrow[d, equal] & X(F) \arrow[r, shift left] \arrow[r, shift right] & X(F \times_{S/R} F) \\
X(S/R) \arrow[r] & X(S) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, hook] & X(R). \arrow[u, hook]
\end{tikzcd}$$

Since F is free and the composite map $F \twoheadrightarrow S \twoheadrightarrow S/R$ is surjective, the upper sequence is left exact. Since the vertical maps are inclusions, the bottom sequence also is left exact.

This shows that our presheaf X is a sheaf: condition (4.1) follows from the fact that a disjoint union of free presentations is a free presentation and we just checked condition (4.2). ■

Our assertion means that a sheaf on **FCH**aus extends uniquely to a condensed set. We will say for short that the sheaf *is* a condensed set.

There exists analogous statements with many intermediate categories such as Stone or Stonean spaces. More precisely, the category of Stone spaces is endowed with the pretopology made of jointly surjective maps (like **CH**aus) and the category of Stonean spaces with the pretopology made of disjoint unions (like **FCH**aus). This was the original approach to the theory and the comparison theorems are left as (easy) exercises.

Proposition 4.1.8 A morphism $X \to Y$ of condensed sets is an epimorphism if and only if, for all free compact Hausdorff space F, the map $X(F) \to Y(F)$ is surjective.

Proof. It is equivalent to prove this property in $\widetilde{\mathbf{FCH}\mathrm{aus}}$. Let $p : X \twoheadrightarrow Y$ be an epimorphism, F a free compact Hausdorff space and $y \in Y(F)$. Then there exists a disjoint covering $F = \coprod_{i=1}^n F_i$ and, for all $i = 1, \dots, n$, an $x_i \in X(F_i)$ such that $p(x_i) = y_{|F_i}$. If we set $x = (x_i)_{i=1}^n \in \prod X(F_i) = X(F)$, we will have $p(x) = y$. ■

4.2 Topological space and condensed set

4.2.1 Associated condensed set

Lemma 4.2.1 If X is any topological space, then the presheaf

$$\underline{X} : \mathbf{CHaus}^{\mathrm{op}} \hookrightarrow \mathbf{Top}^{\mathrm{op}} \xrightarrow{h_X} \mathbf{Set}, \quad S \mapsto \mathcal{C}(S, X)$$

is a condensed set.

Proof. This functor preserves all limits. ■

Proposition 4.2.2 1. If S_i are compact Hausdorff spaces for $i \in I$ finite, then

$$\underline{\coprod_{i \in I} S_i} \simeq \coprod_{i \in I} \underline{S_i}.$$

2. If R is a closed equivalence relation on a compact Hausdorff space S, then

$$\underline{S/R} \simeq \underline{S}/\underline{R}.$$

Proof. This is completely formal but was shown in proposition 3.3.3. ■

Be careful that the functor $S \mapsto \underline{S}$ does not preserve finite colimits of compact Hausdorff spaces in general. However, a morphism $f : S \to T$ of compact Hausdorff spaces is injective (resp. surjective, resp. bijective) if and only if $\underline{f} : \underline{S} \to \underline{T}$ is a monomorphism (resp. an epimorphism, resp. an isomorphism).

Exercise 4.3 Show that, if F is a free compact Hausdorff space, then $\underline{F}$ is a projective condensed set. Show that the category $\mathbf{Cond}$ has enough projectives.

Solution. We know from proposition 4.1.8 that, if $Y \twoheadrightarrow X$ is an epimorphism of condensed sets, then $Y(F) \to X(F)$ is surjective. Thanks to Yoneda lemma, it exactly means that $\mathrm{Hom}(\underline{F}, Y) \to \mathrm{Hom}(\underline{F}, X)$ is surjective. This shows that $\underline{F}$ is projective. The second assertion follows from the fact that a coproduct of projectives is projective. ■

Let X be a condensed set. If S is a compact Hausdorff space, then there exists a natural (Yoneda) bijection

$$\mathrm{Hom}(\underline{S}, X) \simeq X(S).$$

In other words, any $f \in X(S)$ may be seen as a morphism $f : \underline{S} \to X$. It provides us with a map

$$f_\bullet : S \simeq \underline{S}(\bullet) \to X(\bullet),$$

Definition 4.2.3 If X is a condensed set, then its *underlying topological space* is $X(\bullet)$ endowed with the finest topology such that

- Given any (free) compact Hausdorff space S and any $f \in X(S)$, the map $f_\bullet$ is continuous.

Exercise 4.4 Show that if X is a condensed set, then a subset Y of $X(\bullet)$ is open (resp. closed) if and only if given any (free) compact Hausdorff S and any $f \in X(S)$, $f_\bullet^{-1}(Y)$ is open (resp. closed) in S.

Let us denote by $k\mathbf{Top}$ the full subcategory of compactly generated topological spaces. We know that it is a coreflective subcategory of $\mathbf{Top}$ with coadjoint $X \mapsto kX$.

Theorem 4.2.4 The functor

$$\mathbf{Top} \to \mathbf{Cond}, \quad (\text{resp. } k\mathbf{Top} \to \mathbf{Cond}) \quad X \mapsto \underline{X}$$

is faithful (resp. fully faithful) with adjoint $X \mapsto X(\bullet)$.

Proof. Let us first show that, if X is a topological space, then $\underline{X}(\bullet) = kX$. First of all, as sets, we have $\underline{X}(\bullet) = \mathcal{C}(\bullet, X) \simeq X$. Moreover, if S is a compact Hausdorff space, then $\underline{X}(S) \simeq \mathcal{C}(S, X)$. Under this bijection, an $f \in \underline{X}(S)$ corresponds to the composite map

$$S \xrightarrow{f_\bullet} \underline{X}(\bullet) \simeq X.$$

Our claim then follows from the very definitions of the topology of kX and $\underline{X}(\bullet)$ respectively and we can move to the main statement.

Our previous computation will imply that the functor will be fully faithful in the compactly generated case. Therefore, since coreflection $X \mapsto kX$ is trivially faithful, we are left to show that, if we are given a condensed set X and a topological space Y, then the natural map

$$\mathrm{Hom}(X, \underline{Y}) \to \mathcal{C}(X(\bullet), Y), \quad \phi \mapsto \phi_\bullet \tag{4.4}$$

is bijective. Assume given a continuous map $\varphi : X(\bullet) \to Y$. Then, if S is a compact Hausdorff space and $g \in X(S)$, the composite map

$$\phi_S(g) : S \simeq S(\bullet) \xrightarrow{g_\bullet} X(\bullet) \xrightarrow{\varphi} Y$$

is continuous. It provides a compatible family of maps $\phi_S : X(S) \to \mathcal{C}(S, Y)$, or equivalently a morphism $\phi : X \to \underline{Y}$. One easily checks that this is an inverse to the the map in (4.4). ∎

It follows that the functor $X \mapsto \underline{X}$ preserves all limits. Actually, we can identify the category of compactly generated spaces with a reflective subcategory of the category of all condensed sets (with all the pleasant consequences). Let us also mention that there are factorizations

$$\mathbf{Top} \xrightarrow{k} k\mathbf{Top} \hookrightarrow \mathbf{Cond}, \quad \text{and} \quad \mathbf{Cond} \to k\mathbf{Top} \hookrightarrow \mathbf{Top}$$

and in particular, $\underline{kX} = \underline{X}$ if X is any topological space.

Exercise 4.5 Show that, for a condensed set X, the following are equivalent:

1. X is *discrete*: there exists a discrete topological space E such that $X \simeq \underline{E}$,
2. X is *constant*: there exists a set E such that $X \simeq \widetilde{E}$,
3. for any compact Hausdorff space S,

$$X(S) \simeq \varinjlim_{S \twoheadrightarrow S'} X(S')$$

when S' runs through all *discrete* quotients of S.

Solution. ■

Exercise 4.6 Show that if $X = \bigcup_{i\in I} X_i$ is an open covering, then $(\underline{X}_i \to \underline{X})_{i\in I}$ is a covering.

Solution. In the case $X = S$ is compact Hausdorff, it follows from exercise 2.5 that we can replace our open covering with a finite compact covering and we are done. In general, any morphism $\underline{S} \to \underline{X}$ where S is a compact Hausdorff space has the form $\underline{f}$ and we can apply the first case to the covering $S = \bigcup_{i\in I} f^{-1}(X_i)$. It means that

$$(\underline{S} \times_{\underline{X}} \underline{X}_i \to \underline{S})_{i\in I}$$

is a covering and we can use exercise 3.27. ■

Exercise 4.7 Show that, if X is a topological space and $Y \to \underline{X}$ a morphism from a condensed set, then the presheaf

$$U \mapsto Y(U) := \mathrm{Hom}_{/\underline{X}}(\underline{U}, Y)$$

(the Hom in $\mathbf{Cond}_{/\underline{X}}$) is a sheaf on X.

Exercise 4.8 Show that if $X = \coprod_{i\in I} X_i$, then $\underline{X} = \coprod_{i\in I} \underline{X}_i$.

Solution. Using exercise 3.27, we may assume that X is compact. ■

Proposition 4.2.5 Let $\pi : X \to Y$ be a continuous map. Assume Y is weak Hausdorff and any compact Hausdorff subset of Y is equal to (or equivalently contained in) some $\pi(S)$ with S compact Hausdorff. Then, $\underline{\pi} : \underline{X} \to \underline{Y}$ is an epimorphism.

Proof. It is sufficient to show that if F is a free compact Hausdorff space, then the map $\underline{X}(F) \to \underline{Y}(F)$, or equivalently $\mathcal{C}(F, X) \to \mathcal{C}(F, Y)$, is surjective. But if $g : F \to Y$ is a continuous map, then $g(F)$ is compact Hausdorff (because we assumed that Y is weak Hausdorff) and there exists therefore a compact Hausdorff subset $S \subset X$ such that $g(F) = \pi(S)$. Since F is a projective object of the category of compact Hausdorff spaces, the map $F \to \pi(S)$ induced by g lifts to some continuous map $F \to S$ and we can compose with the inclusion map $S \hookrightarrow X$. ■

Exercise 4.9 Show that the hypothesis is satisfied when X locally compact Hausdorff, Y is weak Hausdorff and π is open surjective.

Solution. Pick up for each $x \in X$ a compact neighborhood S_x of x. Since π is open, $\pi(S_x)$ is a neighborhood of $\pi(x)$. Since π is surjective, $Y = \bigcup_{x \in X} \pi(S_x)$. Any compact subset of Y is contained in a finite union $\bigcup_{i=1}^r \pi(S_{x_i})$ and we set $S := \bigcup_{i=1}^r S_{x_i}$. ■

4.2.2 Internal Hom

Since **Cond** is a topos, it is *cartesian closed*: there exists an internal Hom and a natural isomorphism

$$\forall X, Y, Z \in \mathbf{Cond}, \quad \mathcal{H}om(X \times Y, Z) \simeq \mathcal{H}om(X, \mathcal{H}om(Y, Z)).$$

Recall from theorem 2.3.6 that the category of compactly generated spaces is also cartesian closed. Actually, if X, Y, Z are three topological spaces, then there exists a homeomorphism

$$k\mathcal{C}(k(X \times Y), kZ) \simeq k\mathcal{C}(kX, k\mathcal{C}(kY, kZ)).$$

The adjunction in theorem 4.2.4 can be enriched as follows:

Proposition 4.2.6 If X is a condensed set and Y a compactly generated topological space, then there exists a natural isomorphism of condensed sets

$$\underline{C(X(\bullet), Y)} \simeq \mathcal{H}om(X, \underline{Y}).$$

Proof. We shall first do the case $X = \underline{S}$ with S compact Hausdorff so that $X(\bullet) = S$. If T is compact Hausdorff, then

$$\begin{aligned}
\mathcal{H}om(\underline{S}, \underline{Y})(T) &\simeq \mathrm{Hom}(\underline{T}, \mathcal{H}om(\underline{S}, \underline{Y})) \\
&\simeq \mathrm{Hom}(\underline{T} \times \underline{S}, \underline{Y}) \\
&\simeq \mathrm{Hom}(\underline{T \times S}, \underline{Y}) \\
&\simeq \mathcal{C}(T \times S, Y) \\
&\simeq \mathcal{C}(T, \mathcal{C}(S, Y)) \\
&\simeq \underline{\mathcal{C}(S, kY)}(T).
\end{aligned}$$

In general, we obtain for S compact Hausdorff

$$\begin{aligned}
\mathcal{H}om(X, \underline{Y})(S) &\simeq \mathrm{Hom}(\underline{S}, \mathcal{H}om(X, \underline{Y})) \\
&\simeq \mathrm{Hom}(X, \mathcal{H}om(\underline{S}, \underline{Y})) \\
&\simeq \mathrm{Hom}(X, \underline{\mathcal{C}(S, Y)}) \\
&\simeq \mathcal{C}(X(\bullet), \mathcal{C}(S, Y)) \\
&\simeq \mathcal{C}(S, \mathcal{C}(X(\bullet), Y)) \\
&\simeq \underline{\mathcal{C}(X(\bullet), Y)}(S).
\end{aligned}$$

■

If X and Y are two condensed sets, then

$$\mathrm{Hom}(X,Y) = \mathcal{H}\mathrm{om}(X,Y)(\bullet)$$

inherits the structure of a compactly generated space. Then, if X is a condensed set and Y is a compactly generated topological space, then there exists a *homeomorphism*

$$kC(X(\bullet),Y) \simeq \mathrm{Hom}(X,\underline{Y}).$$

In particular, if X and Y are two compactly generated topological spaces, then there exists a *homeomorphism*

$$kC(X,Y) \simeq \mathrm{Hom}(\underline{X},\underline{Y}).$$

Exercise 4.10 Show that if E is any set and X a compactly generated topological space, then

$$\mathcal{H}\mathrm{om}(\underline{E},\underline{X}) \simeq \underline{X^E}.$$

Corollary 4.2.7 The functor

$$\mathbf{Cond} \to k\mathbf{Top}, \quad X \mapsto X(\bullet)$$

preserves finite products:

$$\forall X,Y \in \mathbf{Cond}, \quad (X\times Y)(\bullet) \simeq k(X(\bullet)\times Y(\bullet)).$$

Proof. We use uniqueness of the adjoint. If X,Y are two condensed sets and Z is any compactly generated topological space, then

$$\begin{aligned}
\mathcal{C}(k(X(\bullet)\times Y(\bullet)),Z) &\simeq \mathcal{C}(X(\bullet),k\mathcal{C}(Y(\bullet),Z)) \\
&\simeq \mathrm{Hom}(X,\underline{\mathcal{C}(Y(\bullet),Z)}) \\
&\simeq \mathrm{Hom}(X,\mathcal{H}\mathrm{om}(Y,\underline{Z})) \\
&\simeq \mathrm{Hom}(X\times Y,\underline{Z}) \\
&= \mathcal{C}((X\times Y)(\bullet),Z).
\end{aligned}$$

■

It is important to notice that the result does not hold anymore if we replace $k\mathbf{Top}$ with $\mathbf{Top}$.

Exercise 4.11 Show that there exists a *projection formula*

$$S\times X(\bullet) \simeq (\underline{S}\times X)(\bullet)$$

when X is any condensed set and S (locally) compact Hausdorff.

4.2.3 Quasi-compact/quasi-separated condensed set

Theorem 4.2.8
1. A condensed set X is quasi-compact if and only if there exists an epimorphism $\underline{F} \twoheadrightarrow X$ with F free compact Hausdorff.
2. The functor $S \mapsto \underline{S}$ induces an equivalence between compact Hausdorff spaces and quasi-compact quasi-separated (qcqs for short) condensed sets.

Proof. This was shown in proposition 3.3.9 and theorem 3.3.11 (use also proposition 2.2.2). ■

Lemma 4.2.9 If X is a weak Hausdorff topological space, then $\underline{X}$ is quasi-separated.

Proof. We may assume that X is compactly generated. For $i = 1, 2$, let $X_i \to \underline{X}$ be a morphism with X_i quasi-compact. There exists a compact Hausdorff space S_i and an epimorphism $\underline{S_i} \twoheadrightarrow X_i$. We may then consider the epimorphism $\underline{S_1 \times_X S_2} = \underline{S_1} \times_{\underline{X}} \underline{S_2} \twoheadrightarrow X_1 \times_{\underline{X}} X_2$. Since X is weak Hausdorff, then $S_1 \times_X S_2$ is compact Hausdorff. ■

Lemma 4.2.10 A condensed set X is quasi-separated if and only if $X \simeq \varinjlim_{i \in I} \underline{S_i}$ as a filtered colimit of compact Hausdorff spaces under inclusion maps.

Proof. Thanks to exercise 3.48, only the direct implication needs a proof. There always exists an isomorphism $X \simeq \varinjlim_{i\in I} \underline{T_i}$ with T_i compact Hausdorff. Thus, if we denote by X_i the image of $\underline{T_i}$ in X, we have $X = \varinjlim_{i\in I} X_i$. As an image of $\underline{T_i}$, X_i is quasi-compact. But it is also quasi-separated because X is assumed to be quasi-separated. It follows that $X_i \simeq \underline{S_i}$ with S_i compact Hausdorff. We may then replace the family (S_i) with the family of $S_J = \cup_{i\in J} S_i$ with J finite to get a directed set. ■

In other words, there exists an equivalence between the subcategory of Ind($\mathbf{C}$Haus) of ind-objects "$\varinjlim S_i$" with injective transition maps and the category of quasi-compact quasi-separated condensed sets.

Exercise 4.12 Show that if $X = \varinjlim S_n$ is a *countable* filtered colimit of compact Hausdorff spaces under inclusion maps, then $\underline{X}$ is quasi-separated and $\underline{X} = \varinjlim \underline{S_n}$.

Solution. This follows from exercise 2.42. More precisely, if we are given a compact Hausdorff space T, then an element of $\underline{X}(T)$ is a continuous map $T \to X$ and it comes from a morphism $T \to S_n$ which is an element of $\underline{S_n}(T)$. ■

Lemma 4.2.11 If X is a quasi-separated condensed set, then $X(\bullet)$ is weak Hausdorff.

Proof. We can write $X = \varinjlim_{i\in I} \underline{S_i}$ as a filtered colimit of inclusions of compact Hausdorff spaces and it follows that $X(\bullet) = \varinjlim_{i\in I} S_i$. Our assertion therefore follows from proposition 2.3.9. ■

Proposition 4.2.12 A compactly generated space X is weak Hausdorff if and only if $\underline{X}$ is quasi-separated.

Proof. Follows from lemmas 4.2.9 and 4.2.11. ∎

Exercise 4.13 Show that if X is a quasi-separated condensed set, then we have a monomorphism $X \hookrightarrow \underline{X(\bullet)}$

5. Commutative algebra

5.1 Additive category

5.1.1 Pre-additive category

Definition 5.1.1 A *pre-additive category* (also called $\mathbf{A}$b-*category*)[a] is a category $\mathcal{C}$ endowed with a factorization of the Hom functor:

$$\begin{tikzcd} \mathrm{Hom}: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[rr] \arrow[drr, dashed] & & \mathbf{S}\mathrm{et} \\ & & \mathbf{A}\mathrm{b}. \arrow[u] \end{tikzcd}$$

[a] This is a particular instance of the notion of an *enriched* category.

In other words, we require that for all $M, N \in \mathcal{C}$, $\mathrm{Hom}(M, N)$ is endowed with the structure of an abelian group and that for all $M, N, P \in \mathcal{C}$, composition

$$\begin{aligned} \mathrm{Hom}(M,N) \times \mathrm{Hom}(N,P) &\longrightarrow \mathrm{Hom}(M,P) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

is bilinear. In particular, $\mathrm{End}(M)$ becomes a ring. We shall simply say that the category *is* pre-additive (and not mention the factorization through the category of abelian groups).

Examples
1. The categories $\mathbf{A}$b and $A-\mathbf{M}$od are pre-additive.
2. If $\mathcal{C}$ is any category with finite products, then $\mathbf{A}\mathrm{b}(\mathcal{C})$ is pre-additive. This applies in particular to the category $\mathbf{A}\mathrm{b}\mathbf{T}\mathrm{op}$ of topological abelian groups.
3. The category $\mathbf{Mat}_A$ is pre-additive.

4. The category $\mathbf{A}$ associated to the multiplicative monoid of a ring A, is a pre-additive category. Any pre-additive category with exactly one object has this form.
5. The categories $\mathbf{Set}$, $\mathbf{Mon}$, $\mathbf{Gr}$ or $\mathbf{Rng}$ *cannot* be endowed with the structure of a pre-additive category.

If $\mathcal{C}$ is a pre-additive category, we will actually consider the functor Hom as a functor with values in $\mathbf{Ab}$. Consequently, if $M \in \mathcal{C}$, we will write

$$h^M : \mathcal{C} \to \mathbf{Ab} \quad \text{and} \quad h_M : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}.$$

If $\mathcal{C}$ is a pre-additive category, then $\mathcal{C}^{\mathrm{op}}$ also. Same for $\mathcal{C}^I$ if I is any small category.

Definition 5.1.2 A functor $F : \mathcal{C} \to \mathcal{D}$ between two pre-additive categories is *additive* if for any $M, N \in \mathcal{C}$, the map

$$\mathrm{Hom}(M, N) \to \mathrm{Hom}(F(M), F(N))$$

is a group homomorphism.

The composite of two additive functors is additive. If F is an additive functor, so is F^{op}. And so is F^I if I is a category.

Examples 1. If $\mathcal{C}$ is any pre-additive category, then the functors h^M and h_M are additive.
2. If A is a ring, then the functors Hom_A and $\otimes_A$ are additive.

5.1.2 Additive category

Definition 5.1.3 Let $\mathcal{C}$ be a pre-additive category.
1. An $M \in \mathcal{C}$ is a *zero object* if $\mathrm{End}(M) = 1$ (meaning $\mathrm{Id}_M = 0_M$).
2. An $M \in \mathcal{C}$ is a *direct sum* of two objects M_1 and M_2 (in which case we write $M = M_1 \oplus M_2$) if there exists

$$p_k : M \to M_k, i_k : M_k \to M, k = 1, 2 \quad \text{such that}$$

$$p_1 \circ i_1 = \mathrm{Id}_{M_1}, \quad p_2 \circ i_2 = \mathrm{Id}_{M_2} \quad \text{and} \quad i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{Id}_M.$$

Both notions are autodual in the sense that the property is satisfied in $\mathcal{C}$ if and only if it is satisfied in $\mathcal{C}^{\mathrm{op}}$.

Exercise 5.1 Let $\mathcal{C}$ be a pre-additive category.
1. Show that an $M \in \mathcal{C}$ is a zero object if and only if it is a final object (and dual).
2. Show that an $M \in \mathcal{C}$ is a direct sum of M_1 and M_2 if and only if it is a product of M_1 and M_2 with projections p_1 and p_2 (and dual).

More generally, if the *coproduct* of a family $(M_i)_{i \in I}$ exists in a pre-additive category $\mathcal{C}$, it is denoted by $\oplus_{i \in I} M_i$ and called a *direct sum*.

Definition 5.1.4 An *additive* category is a pre-additive category with a zero object and all direct sums.

Equivalently, it means that all finite products *or* all finite sums exist (and then they both exist and are equal).

If $\mathcal{C}$ is an additive category then $\mathcal{C}^{\mathrm{op}}$ is also an additive category, and so is $\mathcal{C}^I$ if I is a small category.

Examples 1. The categories $\mathbf{A}$b, $A-\mathbf{M}$od and $\mathbf{M}$at$_A$ are additive.
2. If $\mathcal{C}$ is any category with finite products, then $\mathbf{A}\mathrm{b}(\mathcal{C})$ is additive.
3. If A is a non-zero ring, then the category $\mathbf{A}$ is *not* an additive category.

Exercise 5.2 Show that if $\mathcal{C}$ is an additive category, then the factorization of Hom through $\mathbf{A}$b is unique.

Exercise 5.3 Show that a functor between two additive categories is additive if and only if it preserves all direct sums and the zero object - or equivalently all finite products (and dual).

Exercise 5.4 Show that if a functor F between two additive categories is adjoint to a functor G, then both functors are additive and there exists a natural isomorphism of abelian groups

$$\mathrm{Hom}(FM, N) \simeq \mathrm{Hom}(M, GN)).$$

5.1.3 Exact sequence

Let $\mathcal{C}$ be an additive category.

Definition 5.1.5 1. The *kernel* of a morphism $f : M \to N$ is $\ker f := \ker(f, 0)$ if it exists. The *cokernel* $\operatorname{coker} f$ of f is the kernel of f in $\mathcal{C}^{\mathrm{op}}$ (if it exists).
2. A sequence

$$0 \to M' \to M \xrightarrow{f} M''$$

is said to be *left exact* if the sequence

$$\begin{tikzcd} M' \arrow[r] & M \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & M'' \end{tikzcd}$$

is left exact ($M' \simeq \ker f$). A sequence $M' \to M \to M'' \to 0$ is *right exact* if it is left exact in $\mathcal{C}^{\mathrm{op}}$ ($M'' \simeq \operatorname{coker} f$).
3. A *short exact sequence* is a sequence

$$0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$$

which is exact both on the left and on the right. We shall also say that M is an *extension* of M'' by M'.

Exercise 5.5 In the the category $\mathbf{AbHaus}$ of *Hausdorff* topological abelian groups, show that

1. if $f : M \to N$ is any continuous homomorphism, then $\ker f = f^{-1}(0)$ with the induced topology and $\operatorname{coker} f = N/\overline{f(M)}$ with the quotient topology.
2. a continuous map is a kernel (resp. cokernel) if and only if it is closed injective (resp. open surjective).
3. a short sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ is exact[a] if and only if it is exact as a sequence of abelian groups with ι closed and π open.

[a]Sometimes called *strict exact* in order to insist on the fact that this is not just exact as a sequence of abelian groups.

Exercise 5.6 Show that left (resp. right, resp. short) exact sequences in $\mathcal{C}$ form an additive subcategory of $\mathcal{C}^{\mathbf{3}}$.

Definition 5.1.6 A short exact sequence is said to *split* if it is isomorphic to

$$0 \longrightarrow M_1 \xrightarrow{i_1} M_1 \oplus M_2 \xrightarrow{p_2} M_2 \longrightarrow 0\,.$$

An extension is said to be *trivial* if it splits.

Exercise 5.7 Show that a short exact sequence

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

splits if and only if p has a section (and dual).

5.2 Abelian category

5.2.1 Definition

Definition 5.2.1 A *pre-abelian* category is an additive category $\mathcal{C}$ where any morphism has both a kernel and a cokernel.

Equivalently, a pre-additive category $\mathcal{C}$ is pre-abelian if and only if all finite limits and all finite colimits exist in $\mathcal{C}$.

Examples

1. The categories $\mathbf{A}$b, $A-\mathbf{M}$od and $\mathbf{Mat}_A$ are pre-abelian.
2. The categories $\mathbf{A}$b$\mathbf{T}$op and $\mathbf{A}$b$\mathbf{H}$aus are pre-abelian.

If $\mathcal{C}$ is pre-abelian, so are $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{C}^{I}$ if I is a small category.

Definition 5.2.2 An *abelian category* is a pre-abelian category satisfying one of the following equivalent properties:

1. Every monomorphism is regular and dual.
2. If $i : N \hookrightarrow M$ is a monomorphism, then $N \simeq \ker(M \twoheadrightarrow \operatorname{coker}(i))$ and dual.
3. Any morphism $f : M \to N$ factors uniquely up to an isomorphism as an epimorphism followed by a monomorphism.

4. If $f : M \to N$ is a morphism, then

$$\ker(N \to \operatorname{coker} f) \simeq \operatorname{coker}(\ker f \to M).$$

In an abelian category, it is common to say injection and surjection instead of monomorphism and epimorphism.

Exercise 5.8 Show that all the above conditions are equivalent.

Exercise 5.9 Show that, in an abelian category $\mathcal{C}$, we have for any morphism $f : M \to N$,

$$\operatorname{im} f := \ker(N \to \operatorname{coker} f) \quad \text{(and dual)}.$$

Show that any morphism is strict.

Examples

1. The category $A-\mathbf{Mod}$ is abelian.
2. The category $\mathbf{Op}(k\text{-}\mathbf{Mod})$ of k-modules endowed with an operator is abelian.
3. If $\mathcal{T}$ is a topos, then $\mathbf{Ab}(\mathcal{T})$ is abelian (see theorem 5.3.2 below).
4. The category $\mathbf{AbCHaus}$ is abelian.
5. The category $\mathbf{AbTop}$ is not abelian however because the identity $\mathbb{Z}^{\mathrm{disc}} \to \mathbb{Z}^{\mathrm{coarse}}$ is not strict.
6. The category $\mathbf{AbHaus}$ is not abelian either because the map $\ell^1(\mathbb{R}) \to \ell^2(\mathbb{R})$ is not strict.
7. The category $\mathbf{Mat}_{\mathbb{Z}}$ is not abelian because 2 is a monomorphism which is not regular.

Definition 5.2.3 A sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be *exact (in M)* if $\operatorname{Im}(f) = \ker(g)$.

Exercise 5.10 Show that a sequence

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

is a short exact sequence (resp. left exact, resp. right exact) if and only if it is exact in M', M, M'' (resp. M', M, resp. M, M'').

If $\mathcal{C}$ is an abelian category, then $\mathcal{C}^{\mathrm{op}}$ is also an abelian category, as well as $\mathcal{C}^I$ if I is any small category.

Exercise 5.11 Show that a functor $F : \mathcal{C} \to \mathcal{D}$ between two abelian categories is left exact if and only if it is additive and preserves left exact sequences (and dual).

Exercise 5.12 Let $\mathcal{D}$ be an additive (resp. abelian) category. Show that, if a fully faithful (resp. and exact) functor $\mathcal{C} \hookrightarrow \mathcal{D}$ has an adjoint or a coadjoint, then $\mathcal{C}$ also is additive (resp. abelian).

We shall not prove the next result which is useful to reduce many general statements on abelian categories to the case of a category of A-modules:

Theorem 5.2.4 — Freyd-Mitchell. If $\mathcal{C}$ is a small abelian category, then there exists a ring A and a fully faithful exact functor $\mathcal{C} \hookrightarrow A-\mathrm{Mod}$. ■

5.2.2 Grothendieck category

We now introduce Grothendieck axioms:

Definition 5.2.5 A category $\mathcal{C}$ is:

1. AB1 : pre-abelian.
2. AB2 : abelian.
3. AB3 : AB2 with all colimits (AB3* : dual).
4. AB4 : AB3 and coproducts are exact (AB4* : dual).
5. AB5 : AB4 and filtered colimits are exact (AB5* : dual).
6. AB6 : AB5 and filtered colimits commute with products (AB5* : dual).

Examples
1. The category $\mathbf{Ab}$ satisfies AB6 and AB4*.
2. The category $\mathbf{AbTop}$ satisfies AB1.
3. The category $\mathbf{AbCHaus} \simeq \mathbf{Ab}^{\mathrm{op}}$ satisfies AB4 and AB6* (Pontryagin duality).
4. If $\mathcal{C}$ is any category, then $\mathbf{Ab}\widehat{\mathcal{C}}$ satisfies AB6 and AB4*.
5. If $\mathcal{T}$ is a topos, then $\mathbf{Ab}\mathcal{T}$ satisfies AB5 and AB3* (theorem 5.3.2).
6. It is not true however that $\mathbf{Ab}\mathcal{T}$ satisfies AB6 or AB4* in general.
7. There is no category satisfying AB5 and AB5* besides $\{0\}$.

Definition 5.2.6 A Grothendieck category is an AB5 category that has a generator.

Examples
1. A-$\mathbf{Mod}$ is a Grothendieck category.
2. One can show that if $\mathcal{C}$ is an abelian category, then $\mathrm{Ind}(\mathcal{C})$ is a Grothendieck category.
3. If $\mathcal{T}$ is a topos, then $\mathbf{Ab}(\mathcal{T})$ is a Grothendieck category (theorem 5.3.2).

Exercise 5.13 Show that if $\mathcal{C}$ is an AB5 category, then G is a generator if and only if, for all $M \in \mathcal{C}$, there exists an epimorphism $G^{(I)} \twoheadrightarrow M$.

Proposition 5.2.7 A Grothendieck category is automatically AB3*.

Proof. To do. ■

Proposition 5.2.8 A Grothendieck category has enough injectives.

Proof. To do. ■

5.3 Abelian sheaf

5.3.1 Defintiion/Properties

Definition 5.3.1
1. A *presheaf* on a category $\mathcal{C}$ with values in a category $\mathcal{D}$ is a (contravariant) functor $T : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. A *morphism of presheaves* is a natural transformation.
2. If $\mathcal{C}$ is a site, then a *sheaf* $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a presheaf such that, for all $Y \in \mathcal{D}$, the presheaf of sets

$$X \mapsto \mathrm{Hom}(Y, \mathcal{F}(X))$$

is a sheaf.

One can extend many former results from sheaf theory to this situation but we shall concentrate on the case of sheaves with values in $\mathbf{A}\mathrm{b}$ and say *sheaf of abelian groups* or *abelian sheaf* or even *abelian group* on $\mathcal{C}$. We shall denote by $\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b}) := \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, (\mathbf{A}\mathrm{b})$ (resp. $\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b}) \subset \widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})$) the category of presheaves (resp. the full subcategory of sheaves) of abelian groups on $\mathcal{C}$. Note that a presheaf is a particular case of a sheaf where we endow $\mathcal{C}$ with the coarse topology. When there is no risk of ambiguity, we shall write

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathcal{N}) := \mathrm{Hom}_{\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})}(\mathcal{M}, \mathcal{N}).$$

By composition, the forgetful functor $\mathbf{A}\mathrm{b} \to \mathbf{S}\mathrm{et}$ provides a forgetful functor $\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b}) \to \widehat{\mathcal{C}}$ that sends a presheaf of abelian groups to the *underlying* presheaf of sets.

Exercise 5.14 Show that a presheaf of abelian groups $\mathcal{M}$ on a site $\mathcal{C}$ is a sheaf if and only if the underlying presheaf of sets is a sheaf.

Solution. By definition, the presheaf $\mathcal{M}$ is a sheaf if and only if, for all covering sieves R of $X \in \mathcal{C}$ and for all $N \in A$, we have

$$\mathrm{Hom}_{\mathbb{Z}}(N, \mathcal{M}(X)) \simeq \varprojlim_{Y \in C_{/R}} \mathrm{Hom}_{\mathbb{Z}}(N, \mathcal{M}(Y)) \simeq \mathrm{Hom}_{\mathbb{Z}}\left(N, \varprojlim_{Y \in C_{/R}} \mathcal{M}(Y)\right)$$

By Yoneda lemma, this is equivalent to require that

$$\mathcal{M}(X) \simeq \varprojlim_{Y \in C_{/R}} \mathcal{M}(Y)$$

(in $\mathbf{A}\mathrm{b}$) and we know that the forgetful functor preserves all limits. ■

As a consequence, there also exists a forgetful functor $\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b}) \to \widetilde{\mathcal{C}}$ that sends a sheaf of abelian groups to the *underlying sheaf* of sets.

Exercise 5.15 Show that if $\mathcal{C}$ is a site, then there exists an equivalence of categories $\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b}) \simeq \mathbf{A}\mathrm{b}(\widetilde{\mathcal{C}})$.

Solution. It follows from exercise 5.17 that the global section functor preserves products. Therefore, if $\mathcal{M} \in \mathbf{A}\mathrm{b}(\widetilde{\mathcal{C}})$ and $X \in \mathcal{C}$, then $\mathcal{M}(X)$ is a usual abelian group and this is clearly functorial. Conversely, if $\mathcal{M} \in \widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})$, then the obvious family of maps

$$\mu_X : \mathcal{M}(X) \times \mathcal{M}(X) \to \mathcal{M}(X), \quad \epsilon_X : \{0\} \to \mathcal{M}(X) \quad \text{and} \quad \iota_X : \mathcal{M}(X) \to \mathcal{M}(X)$$

define a structure of abelian group on the underlying sheaf of sets of $\mathcal{M}$. ■

We shall identify these categories: a sheaf of abelian groups on $\mathcal{C}$ is the same thing as an abelian group in the topos $\widetilde{\mathcal{C}}$. In particular, it only depends on the topos and not on the site itself.

Exercise 5.16 Show that if $\mathcal{C}$ is a site, then $\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})$ is a reflective subcategory of $\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})$ with exact (additive) reflection $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ which is compatible with underlying sheaves and presheaves of sets.

Solution. Since sheafification is (left) exact, this easily follows from theorem 3.2.8. ■

Exercise 5.17 Show that if $\mathcal{C}$ is a site and $X \in \mathcal{C}$ then the functor

$$\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b}) \to \mathbf{A}\mathrm{b}, \quad (\text{resp. } \widehat{\mathcal{C}}(\mathbf{A}\mathrm{b}) \to \mathbf{A}\mathrm{b}) \quad \mathcal{M} \mapsto \Gamma(X, \mathcal{M}) := \mathcal{M}(X).$$

is additive and preserves all limits (resp. limits and colimits).

Solution. In the presheaf case, this is shown as in theorem 3.1.2 and we may then use exercise 5.16. ■

5.3.2 Abelian group

Exercise 5.18 Show that if $\mathcal{T}$ is a topos, then the forgetful functor $\mathcal{T}(\mathbf{A}\mathrm{b}) \to \mathcal{T}$ has an adjoint[a] $X \mapsto \mathbb{Z} \cdot X$.

[a] One also sometimes write $\mathbb{Z}[X]$.

Solution. We can write $\mathcal{T} = \widetilde{\mathcal{C}}$. We know that the forgetful functor $\mathbf{A}\mathrm{b} \to \mathbf{S}\mathrm{et}$ as an adjoint $E \mapsto \mathbb{Z} \cdot E := \mathbb{Z}^{(E)}$. It follows from exercise 1.63 that the forgetful functor $\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b}) \to \widehat{\mathcal{C}}$ as an adjoint $T \mapsto \mathbb{Z} \overset{p}{\cdot} T$ (p for presheaf). If $\mathcal{F}$ is a sheaf, we may then define $\mathbb{Z} \cdot \mathcal{F}$ as the sheafification of $\mathbb{Z} \overset{p}{\cdot} \mathcal{F}$. ■

It means that there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot X, M) \simeq \mathrm{Hom}(X, M) = M(X)$$

for $X \in \mathcal{T}$ and $M \in \mathcal{T}(\mathbf{A}\mathrm{b})$. In particular, if $\mathcal{C}$ is a site, $X \in \mathcal{C}$ and $\mathcal{M} \in \widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})$, then

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot \underline{X}, \mathcal{M}) \simeq \mathcal{M}(X).$$

Example If E is a set, then $\mathbb{Z} \cdot \widetilde{E} \simeq \widetilde{\mathbb{Z}^{(E)}}$. In particular, $\mathbb{Z} \cdot \widetilde{1} \simeq \widetilde{\mathbb{Z}}$.

Exercise 5.19 Show that, if $\mathcal{C}$ is any category, then $\mathbb{Z} \cdot h_X$ is projective in $\widehat{\mathcal{C}}$.

Solution. The functor $\mathcal{M} \mapsto \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot h_X, \mathcal{M}) \simeq \mathcal{M}(X)$ is exact on presheaves. ■

Exercise 5.20 Show that is S is a set of generators of a topos $\mathcal{T}$, then $\{\mathbb{Z}\cdot X, X \in S\}$ is a set of generators for $\mathcal{T}(\mathbf{A}\mathrm{b})$.

Theorem 5.3.2 If $\mathcal{T}$ is a topos, then $\mathcal{T}(\mathbf{A}\mathrm{b})$ is a Grothendieck category.

Proof. In the case $\mathcal{T} = \widehat{\mathcal{C}}$, this is may be checked component by component (details left to the reader). In general, we already know that $\mathcal{T}(\mathbf{A}\mathrm{b}) \simeq \mathbf{A}\mathrm{b}(\mathcal{T})$ is pre-abelian and we shall now write $\mathcal{T} = \widetilde{\mathcal{C}}$. In order to see that $\mathcal{T}(\mathbf{A}\mathrm{b})$ is abelian, it remains to check that if $u : M \to N$ is any morphism, then $\mathrm{coker}(\ker(u)) \simeq \ker(\mathrm{coker}(u))$. Denote by $\iota : \widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b}) \hookrightarrow \widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})$ the inclusion functor and $\pi : \widehat{\mathcal{C}}(\mathbf{A}\mathrm{b}) \twoheadrightarrow \widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})$ the exact reflection. We have $\mathrm{coker}(\ker(\iota(u))) \simeq \ker(\mathrm{coker}(\iota(u)))$ in $\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})$ and we apply π which is exact and satisfies $\pi \circ i = \mathrm{Id}$. The same kind of argument shows that filtered direct limits are exact. The fact that all limits and colimits exist follows directly from the fact that $\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})$ is a reflexive subcategory of $\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})$. Finally, since $\mathcal{T}$ has a small set of generators, it follows from exercise 5.20 that $\mathcal{T}(\mathbf{A}\mathrm{b})$ has a small set of generators S. Then $\oplus_{M \in S} M$ is a generator. ■

5.3.3 Internal Hom and tensor product

Let us first remark that, if $\mathcal{T}$ is a topos, $X \in \mathcal{T}$ and $M \in \mathcal{T}(\mathbf{A}\mathrm{b})$, then the sheaf

$$Y \mapsto \mathcal{H}\mathrm{om}(X, M)(Y) \simeq \mathrm{Hom}(X \times Y, M) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot (X \times Y), M)$$

is a sheaf of abelian groups.

Exercise 5.21 Show that if $\mathcal{T}$ is a topos and $M, N \in \mathcal{T}(\mathbf{A}\mathrm{b})$, then the presheaf

$$X \mapsto \mathrm{Hom}_{\mathbb{Z}}(M, \mathcal{H}\mathrm{om}(X, N))$$

on $\mathcal{T}$ is representable by some $\mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, N) \in \mathcal{T}(\mathbf{A}\mathrm{b})$.

Solution. Thanks to exercise 3.42, it is sufficient to notice that the presheaf commutes will all limits. ■

As a consequence, there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathcal{H}\mathrm{om}(X, N)) \simeq \mathrm{Hom}(X,\ \mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, N)) \simeq \mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, N)(X).$$

Exercise 5.22 Show that, if $\mathcal{C}$ is a site and $\mathcal{M}, \mathcal{N} \in \widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})$, then $\mathcal{H}\mathrm{om}_{\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})}(\mathcal{M}, \mathcal{N}) = \mathcal{H}\mathrm{om}_{\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})}(\mathcal{M}, \mathcal{N})$.

Solution. For $X \in \mathcal{C}$, we have

$$\begin{aligned}
\mathcal{H}\mathrm{om}_{\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b})}(\mathcal{M}, \mathcal{N})(X) &= \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathcal{H}\mathrm{om}_{\widetilde{\mathcal{C}}}(\underline{X}, \mathcal{N})) \\
&= \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathcal{H}\mathrm{om}_{\widehat{\mathcal{C}}}(h_X, \mathcal{N})) \\
&= \mathcal{H}\mathrm{om}_{\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})}(\mathcal{M}, \mathcal{N})(X).
\end{aligned}$$

■

Exercise 5.23 Show that, if $\mathcal{T}$ is a topos and $M, N \in \mathcal{T}(\mathbf{A}\mathrm{b})$, then $\mathrm{Hom}_{\mathbb{Z}}(M, N) = \mathcal{H}om_{\mathbb{Z}}(M, N)(1)$.

It means that $\mathcal{T}(\mathbf{A}\mathrm{b})$ is *enriched* over itself.

Exercise 5.24 Show that if $\mathcal{T}$ is a topos, $X \in \mathcal{T}$ and $M \in \mathcal{T}(\mathbf{A}\mathrm{b})$, then

$$\mathcal{H}om_{\mathbb{Z}}(\mathbb{Z} \cdot X, M) \simeq \mathcal{H}om(X, M).$$

Exercise 5.25 Show that if $\mathcal{T}$ is a topos and $M, N \in \mathcal{T}(\mathbf{A}\mathrm{b})$, then the functor

$$P \mapsto \mathrm{Hom}_{\mathbb{Z}}(M, \mathcal{H}om_{\mathbb{Z}}(N, P))$$

is representable by an $M \otimes_{\mathbb{Z}} N \in \mathcal{T}(\mathbf{A}\mathrm{b})$.

Solution. It is sufficient to consider the case $\mathcal{T} = \widehat{\mathcal{C}}$ and then apply sheafification. This in turn blows down to the analog assertion in the category of usual abelian groups. ∎

As a consequence, $\mathcal{T}(\mathbf{A}\mathrm{b})$ is a *closed symmetric monoidal category*: there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}} N, P) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \ \mathcal{H}om_{\mathbb{Z}}(N, P)).$$

In particular, for fixed N, the functor $M \mapsto M \otimes_{\mathbb{Z}} N$ is adjoint to the functor $P \mapsto \ \mathcal{H}om_{\mathbb{Z}}(N, P)$.

Exercise 5.26 Show that, if $\mathcal{M}, \mathcal{N}$ are two abelian sheaves on a site $\mathcal{C}$, then $\mathcal{M} \otimes_{\mathbb{Z}} \mathcal{N}$ is the sheafification of the *presheaf*

$$\mathcal{M} \overset{p}{\otimes}_{\mathbb{Z}} \mathcal{N} : X \mapsto \mathcal{M}(X) \otimes_{\mathbb{Z}} \mathcal{N}(X).$$

Exercise 5.27 Show that
1. $\forall M, N \in \mathcal{T}(\mathbf{A}\mathrm{b}), \quad M \otimes_{\mathbb{Z}} N \simeq N \otimes_{\mathbb{Z}} M$,
2. $\forall M, N, P \in \mathcal{T}(\mathbf{A}\mathrm{b}), \quad (M \otimes_{\mathbb{Z}} N) \otimes_{\mathbb{Z}} P \simeq M \otimes_{\mathbb{Z}} (N \otimes_{\mathbb{Z}} P)$,
3. $\forall M \in \mathcal{T}(\mathbf{A}\mathrm{b}), \quad M \otimes_{\mathbb{Z}} \mathbb{Z} \cdot 1 \simeq M$.

Exercise 5.28 Show that there exists a natural isomorphism

$$\mathcal{H}om_{\mathbb{Z}}(M \otimes_{\mathbb{Z}} N, P) \simeq \mathcal{H}om_{\mathbb{Z}}(M, \ \mathcal{H}om_{\mathbb{Z}}(N, P)).$$

Exercise 5.29 Show that $\mathbb{Z} \cdot X \otimes_{\mathbb{Z}} \mathbb{Z} \cdot Y \simeq \mathbb{Z} \cdot (X \times Y)$.

Solution. Follows from Yoneda's Lemma since

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot X \otimes_{\mathbb{Z}} \mathbb{Z} \cdot Y, N) &\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot X, \mathcal{H}om_{\mathbb{Z}}(\mathbb{Z} \cdot Y, N)) \\
&\simeq \mathrm{Hom}(X, \mathcal{H}om(Y, N)) \\
&\simeq \mathrm{Hom}(X \times Y, N) \\
&\simeq \mathrm{Hom}(\mathbb{Z} \cdot (X \times Y), N).
\end{aligned}$$ ∎

Exercise 5.30 Show that, if we set $M \cdot X := M \otimes_{\mathbb{Z}} \mathbb{Z} \cdot X$, then

$$\mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, N)(X) \simeq \mathrm{Hom}_{\mathbb{Z}}(M \cdot X, N)$$

and

$$\mathcal{H}\mathrm{om}_{\mathbb{Z}}(M \cdot X, N) \simeq \mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, \mathcal{H}\mathrm{om}(X, N)) = \mathcal{H}\mathrm{om}(X, \mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, N)).$$

Definition 5.3.3 An abelian group P of a topos $\mathcal{T}$ is said to be *flat* if the functor $M \mapsto P \otimes_{\mathbb{Z}} M$ is exact.

Example A usual abelian group is flat if and only if it is torsion free.

Exercise 5.31 Show that if $\mathcal{T}$ is a topos and $X \in \mathcal{T}$, then $\mathbb{Z} \cdot X$ is flat.

Solution. It is sufficient to consider the case $\mathcal{T} = \widehat{\mathcal{C}}$ and then apply sheafification. This reduces to the case of ordinary abelian groups and a free abelian group is torsion-free. ∎

6. Condensed abelian groups

6.1 Condensed abelian group

6.1.1 Definition

Definition 6.1.1 A *condensed abelian group* is an abelian group in the category of condensed sets.

They form a category $\mathbf{Ab}\mathbf{Cond}$ which maybe also denoted $\mathbf{Cond}\mathbf{Ab}$ thanks to:

Proposition 6.1.2 The following are equivalent:
1. The category of condensed abelian groups.
2. The category of sheaves of abelian groups on the site $\mathbf{Cond}$ of condensed sets.
3. The category of sheaves of abelian groups on the site $\mathbf{CHaus}$ of compact Hausdorff spaces.
4. The category of sheaves of abelian groups on the site $\mathbf{FCHaus}$ of free compact Hausdorff spaces.

Proof. This follows from the definition and proposition 4.1.7 thanks to exercise 5.15. ■

The category $\mathbf{Cond}\mathbf{Ab}$ is also equivalent to the category of sheaves of abelian groups on the site of Stone or Stonean spaces. We shall identify all these categories. The category $\mathbf{Cond}\mathbf{Ab}$ is automatically a Grothendiek category but we shall do a lot better (theorem 6.1.4 below).

Exercise 6.1 Show that a presheaf M of abelian groups on $\mathbf{FCHaus}$ is a condensed abelian group if and only if it preserves finite products: for all free compact

Hausdorff spaces $F_1, \ldots, F_r$, we have

$$M\left(\coprod_{j=1}^{r} F_j\right) \simeq \bigoplus_{j=1}^{r} M\left(F_j\right).$$

Solution. Follows from exercise 4.2. ■

Equivalently: $M(\emptyset) = 0$ and $M(F \sqcup F') \simeq M(F) \oplus M(F')$ for $F, F' \in \mathbf{FCHaus}$.

Lemma 6.1.3 If F is a free compact Hausdorff space (more generally a Stonean space), then the functor

$$\mathbf{Ab}\mathbf{Cond} \to \mathbf{Ab}, \quad M \mapsto \Gamma(F, M)$$

preserves all limits and colimits.

Proof. Thanks to exercise 5.17, it suffices to show that if $(M_i)_{i\in I}$ is diagram of sheaves of abelian groups on $\mathbf{FCHaus}$, then its colimit $\varinjlim^{\wedge}_{i\in I} M_i$ in the category of *presheaves* of abelian groups is automatically a sheaf. We give ourselves free compact Hausdorff spaces $F_1, \ldots, F_r$ and we compute

$$\begin{aligned}
\left(\varinjlim_{i\in I}^{\wedge} M_i\right)\left(\coprod_{j=1}^{r} F_j\right) &= \varinjlim_{i\in I} M_i\left(\coprod_{j=1}^{r} F_j\right) \\
&= \varinjlim_{i\in I} \bigoplus_{j=1}^{r} M_i\left(F_j\right) \\
&= \bigoplus_{j=1}^{r} \varinjlim_{i\in I} M_i\left(F_j\right) \\
&= \bigoplus_{j=1}^{r} \left(\varinjlim_{i\in I}^{\wedge} M_i\right)\left(F_j\right).
\end{aligned}$$

■

6.1.2 Grothendieck category

Theorem 6.1.4 The category $\mathbf{Cond}\mathbf{Ab}$ is a Grothendieck category satisfying axioms AB6 and AB4*.

Proof. Thanks to lemma 6.1.3, this follows from the analog assertion in $\mathbf{Ab}$. ■

Corollary 6.1.5 The category $\mathbf{Cond}\mathbf{Ab}$ is an abelian category and

1. there exists a a generator,
2. all limits and colimits exist,
3. all products and coproducts are exact,
4. filtered colimits are exact and commute with products. ■

Lemma 6.1.6 If F is a free compact Hausdorff space (more generally Stonean), then $\mathbb{Z} \cdot \underline{F}$ is a finitely presented[a] projective condensed abelian group.

[a] Also called compact.

Proof. It is means that the functor[1]

$$\mathbf{CondAb} \to \mathbf{Ab}, \quad M \mapsto \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z} \cdot \underline{F}, M)$$

preserves epimorphisms and filtered colimits. This follows from proosition 6.1.3 since

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z} \cdot \underline{F}, M) \simeq \mathrm{Hom}(\underline{F}, M) \simeq M(F) = \Gamma(F, M)$$

and the functor actually commutes with all colimits. ■

Proposition 6.1.7 The category $\mathbf{CondAb}$ is generated by finitely presented projective condensed abelian groups. In particular, it has enough projectives.

Proof. Since $\mathbf{Cond}$ is generated the family of $\underline{F}$ where F is a free compact Hausdorff space, we know from exercise 5.20 that $\mathbf{CondAb}$ is generated by all $\mathbb{Z} \cdot \underline{F}$. ■

The category $\mathbf{CondAb}$ is a *closed symmetric monoidal* category (in particular, it is *enriched* over itself):

Proposition 6.1.8 There exists two bifunctors $\mathcal{H}\mathrm{om}_{\mathbb{Z}}$ and $\otimes_{\mathbb{Z}}$ on $\mathbf{CondAb}$ with natural isomorphisms of abelian groups

$$\mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, N)(\bullet) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, N)$$

and

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}} N, P) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathcal{H}\mathrm{om}_{\mathbb{Z}}(N, P)).$$

Proof. This is exercises 5.23 and 5.25. ■

Note that $\mathrm{Hom}_{\mathbb{Z}}(M, N) = \mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, N)(\bullet)$ is naturally a compactly generated topological space. Be careful however that this is *not* a topological abelian group because compactly generated spaces are not closed under products of topological spaces.

6.2 Topological abelian groups

6.2.1 Associated condensed group

Proposition 6.2.1 There exists a faithful functor that preserves all limits:

$$\mathbf{AbTop} \to \mathbf{AbCond}, \quad M \mapsto \underline{M}$$

It is even fully faithful on compactly generated abelian groups.

Proof. Formally follows from theorem 4.2.4. ■

[1] We may use $\mathbf{Ab}$ or $\mathbf{Ens}$ indifferently.

Be careful however that there is no obvious adjoint because the functor $X \mapsto X(\bullet)$ does not preserve products. In the same way, if M is a topological abelian group, then kM does not get the structure of a topological abelian group in general.

If M, N are two topological abelian groups, then we denote by $\mathcal{C}_{\mathbb{Z}}(M,N) \subset \mathcal{C}(M,N)$ the subspace of continuous homomorphisms. Note that $\mathcal{C}_{\mathbb{Z}}(\underline{M},\underline{N}) = \mathrm{Hom}_{\mathbb{Z}}(M,N)$ when M is compactly generated.

Proposition 6.2.2 If M, N are two topological abelian groups with M compactly generated, then there exists a natural isomorphism of condensed abelian groups

$$\underline{\mathcal{C}_{\mathbb{Z}}(M,N)} \simeq \mathcal{H}om_{\mathbb{Z}}(\underline{M},\underline{N}).$$

Proof. Let S be a compact Hausdorff space. It follows from proposition 6.2.1 that

$$\begin{aligned}
\mathcal{H}om_{\mathbb{Z}}(\underline{M},\underline{N})(S) &\simeq \mathrm{Hom}(\underline{S},\ \mathcal{H}om_{\mathbb{Z}}(\underline{M},\underline{N}))\\
&\simeq \mathrm{Hom}_{\mathbb{Z}}(\underline{M},\mathcal{H}om(\underline{S},\underline{N}))\\
&\simeq \mathrm{Hom}_{\mathbb{Z}}(\underline{M},\underline{\mathcal{C}(S,N)})\\
&\simeq \mathcal{C}_{\mathbb{Z}}(M,\mathcal{C}(S,N)).
\end{aligned}$$

On the other hand, we have

$$\underline{\mathcal{C}_{\mathbb{Z}}(M,N)}(S) \simeq \mathrm{Hom}(\underline{S},\ \underline{\mathcal{C}_{\mathbb{Z}}(M,N)}) \simeq \mathcal{C}(S,\ \mathcal{C}_{\mathbb{Z}}(M,N)).$$

Our isomorphism is then induced by the natural bijection

$$\mathcal{C}(M,\mathcal{C}(S,N)) \simeq \mathcal{C}(M\times S,N) \simeq \mathcal{C}(S,\mathcal{C}(M,N))$$

coming from theorem 2.3.6. ■

Proposition 6.2.3 Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ be an exact sequence of topological abelian groups with M'' weak Hausdorff. Assume that for all compact Hausdorff $K'' \subset M''$, there exists a compact Hausdorff $K \subset M$ such that $K'' \subset \pi(K)$. Then, the sequence of condensed abelian groups $0 \to \underline{M'} \to \underline{M} \to \underline{M''} \to 0$ is also exact.

Proof. Since left exactness is automatic, this follows from proposition 4.2.5. ■

Exercise 6.2 Show that the hypothesis is satisfied if M is locally compact.

Exercise 6.3 Show that if $(M_i)_{i\in I}$ is a family of abelian groups, then

$$\underline{\left(\bigoplus_{i\in I} M_i\right)^{\mathrm{disc}}} \simeq \bigoplus_{i\in I} \underline{M_i^{\mathrm{disc}}}.$$

6.2.2 Locally compact abelian groups

The standard reference is Morris 79.

Exercise 6.4 Show that the category of locally compact Hausdorff abelian groups is pre-abelian but *not* abelian.

Solution. The kernel of $f : M \to N$ is the usual kernel with the induced topology. The cokernel is the quotient $N/\overline{f(M)}$ with the quotient topology. The category is not abelian because the identity $\mathbb{R}^{\text{disc}} \to \mathbb{R}$ is not strict. ■

Exercise 6.5 Show that if N is a closed subgroup of a locally compact Hausdorff abelian group M, then $\underline{M/N} \simeq \underline{M}/\underline{N}$.

We shall denote by $\mathbb{T} := \{z \in \mathbb{C}^\times, \|z\| = 1\}$ and

$$M^* := \mathcal{C}_{\mathbb{Z}}(M, \mathbb{T})$$

the *Pontryagin dual* of a topological abelian group M.

Exercise 6.6 Show that Fourier transform

$$\mathbb{R} \times \mathbb{R} \to \mathbb{T}, \quad (x, y) \mapsto e^{2i\pi xy}$$

induces an isomorphism $\mathbb{R} \simeq \mathbb{R}^*$. Show that $\mathbb{Z} \simeq \mathbb{T}^*$ and $\mathbb{T} \simeq \mathbb{Z}^*$.

Solution. If $f \in \mathbb{R}^*$, then f is continuous and $f(0) = 1$. It follows that there exists $\epsilon > 0$ such that $\delta := \int_0^\epsilon f(t)\mathrm{d}t \neq 0$. Since f is a homomorphism, we have

$$\int_y^{y+\epsilon} f(t)\mathrm{d}t = \int_0^\epsilon f(y+t)\mathrm{d}t = \int_0^\epsilon f(y)f(t)\mathrm{d}t = \delta f(y).$$

It follows that $f(y) = \delta^{-1} \int_y^{y+\epsilon} f(t)\mathrm{d}t$, and in particular, f is differentiable. Since f is a homomorphism, we have $f(y+h) - f(y) = f(y)(f(h) - f(0))$. Dividing by h and taking limit provides $f'(y) = f'(0)f(y)$ and therefore $f(y) = Ce^{f'(0)y}$. Since $f(0) = 1$, we have $C = 1$ and since $|f(1)| = 1$, we have $f'(0) \in \mathbb{R}i$. We can therefore write $f(y) = e^{2i\pi xy}$ for a unique $x \in \mathbb{R}$. This implies that $\mathbb{R} \simeq \mathbb{R}^*$. The isomorphism $\mathbb{Z} \simeq \mathbb{T}^*$ follows immediately (by composition with the covering $\mathbb{R} \twoheadrightarrow \mathbb{T}$) and $\mathbb{T} \simeq \mathbb{Z}^*$ is trivial. ■

Theorem 6.2.4 — Pontryagin-van Kampen. The functor $M \mapsto M^* := \mathcal{C}_{\mathbb{Z}}(M, \mathbb{T})$ is a self-equivalence (an equivalence which is self-adjoint) on the category of locally compact Hausdorff abelian groups.

Proof. To do. ■

It means that, if M is a locally compact Hausdorff abelian group, then M^* also and $(M^*)^* = M$.

Exercise 6.7 Show that if $0 \to M' \to M \to M'' \to 0$ is a (strict) exact sequence of locally compact abelian groups, then the sequence $0 \to M''^* \to M^* \to M'^* \to 0$

is also exact.

Solution. There exists an exact sequence of topological abelian groups $0 \to N \to M \to M/N \to 0$. Since N is closed, M/N is Hausdorff. Moreover, M is locally compact. Then, we know that the sequence $0 \to \underline{N} \to \underline{M} \to \underline{M/N} \to 0$ is exact. It means that $\underline{M/N} \simeq \underline{M}/\underline{N}$. ■

Exercise 6.8 Show that Pontryagin duality reduces to usual duality on finite dimensional real Banach spaces[a].

[a] A real Banach space is locally compact if and only if it is finite dimensinal.

Exercise 6.9 Show that Pontryagin duality induces an equivalence between discrete abelian groups and compact Hausdorff abelian groups. Show that torsion corresponds to Stone[a] and torsion free corresponds to connected.

[a] Profinite.

Proof. If M is a discrete abelian group, then there exists a surjective homomorphism $\mathbb{Z}^{(I)} \twoheadrightarrow M$ and, by duality, a closed embedding $M^* \hookrightarrow \mathbb{T}^I$ which shows that M^* is compact. Conversely, if $U \subset \mathbb{T}$ is defined by $\mathrm{Im}(z) > 0$, then $\mathcal{C}_{\mathbb{Z}}(M, \mathbb{T}) \cap \mathcal{C}(M, U) = \{0\}$ and it easily follows that M^* is discrete. As a consequence, we see that the dual of a finite abelian group is also a finite abelian group. Now, M is torsion if and only if $M = \varinjlim M_i$ with M_i finite if and only if $M^* = \varprojlim N_i$ with N_i finite and only if M^* is Stone. Also, M is not torsion free if and only if there an injection $M' \hookrightarrow M$ with M' finite not trivial if and only if there exists a surjection $M^* \twoheadrightarrow N'$ with N' finite not trivial if and only if M^* is not connected. ■

Exercise 6.10 Show that any compact Hausdorff abelian group M has a two terms resolution (a short exact sequence)

$$0 \to M \to M_0 \to M_1 \to 0$$

with M_0, M_1 connected.

Exercise 6.11 Show that if M is a locally compact Hausdorff abelian group, then

$$\underline{M^*} \simeq \mathcal{H}om_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}) \quad \text{and} \quad M^* \simeq \mathrm{Hom}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}).$$

Solution. Since M is locally compact, it is compactly generated and therefore

$$\underline{M^*} = \underline{\mathcal{C}_{\mathbb{Z}}(M, \mathbb{T})} \simeq \mathcal{H}om_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}).$$

It follows that

$$M^* \simeq \underline{M^*}(\bullet) \simeq \mathcal{H}om_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}})(\bullet) \simeq \mathrm{Hom}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}).$$ ■

Theorem 6.2.5 Any locally compact Hausdorff abelian group M has an open subgroup of the form $V \times K$ where V is a finite dimensional real Banach space and K is a compact Hausdorff abelian group.

Proof. To do. ■

In other words, there exists an exact sequence

$$0 \to V \times K \to M \to D \to 0$$

with D discrete.

7. Cohomology (optional)

7.1 Complex

7.1.1 Definition

Let $\mathcal{C}$ be an additive category.

Definition 7.1.1 1. A *(long) sequence* in $\mathcal{C}$ is a commutative diagram on the ordered set $(\mathbb{Z}, \leq)$:

$$\cdots \longrightarrow K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \longrightarrow \cdots$$

2. If $d^n \circ d^{n-1} = 0$ for each $n \in \mathbb{Z}$, we call $K^\bullet$ a *(cochain) complex*. The dual notion is that of *chain complex*. Then, one usually writes $K_n := K^{-n}$ and $d_n := d^{1-n}$.
3. A complex $K^\bullet$ is said to be *bounded below* (resp. *bounded above*, resp. *bounded*) if $K^n = 0$ for $n << 0$ (resp. it is bounded below in $\mathcal{C}^{\mathrm{op}}$, resp. if it is bounded both above and below).

Exercise 7.1 Show that the following hold:
1. The complexes of $\mathcal{C}$ form an additive subcategory $\mathbf{C}(\mathcal{C})$ of $\mathcal{C}^{(\mathbb{Z},\leq)}$.
2. Limits and colimits in $\mathbf{C}(\mathcal{C})$ are computed argument by argument.
3. The equivalence $(\mathbb{Z}, \leq) \simeq (\mathbb{Z}, \geq)$ induces an equivalence $\mathbf{C}(\mathcal{C}^{\mathrm{op}}) \simeq \mathbf{C}(\mathcal{C})$ between chain complexes and cochain complexes.

We shall denote by $\mathbf{C}^+(\mathcal{C})$, $\mathbf{C}^-(\mathcal{C})$ and $\mathbf{C}^b(\mathcal{C})$ the categories of complexes that are respectively bounded below, bounded above and bounded. All the coming development has an equivalent with $+$, $-$ and b.

Exercise 7.2 Show that if $K_\bullet$ is a (semi-) simplicial object and we set $d^n := \sum_{i=0}^{n-1}(-1)^i d_i^n$, then $K_\bullet$ becomes a chain complex.

Example If X is any topological space, we can consider the simplicial set $S_\bullet(X) = h_X \circ \Delta^\bullet$ so that $S_n(X) = \mathrm{Hom}_{\mathrm{cont}}(\Delta^n, X)$. We may then consider the simplicial (resp. cosimplicial) group

$$C_\bullet(X) := \mathbb{Z} \cdot S_\bullet(X) \quad (\text{resp. } C^\bullet(X) := \mathbb{Z}^{S_\bullet(X)})$$

and see it as a chain (resp. cochain) complex.

Exercise 7.3 1. Show that the inclusion $\mathbf{1} = \{0\} \hookrightarrow (\mathbb{Z}, \leq)$ induces a functor

$$\mathcal{C} \hookrightarrow \mathbf{C}(\mathcal{C}), \quad M \mapsto [M]$$

(so that $[M]^0 = M$ and $[M]^n = 0$ otherwise) which is fully faithful and commutes with all limits and colimits.
2. Show that $\mathbf{2} \hookrightarrow (\mathbb{Z}, \leq)$ induces a functor

$$\mathbf{Mor}(\mathcal{C}) \hookrightarrow \mathbf{C}(\mathcal{C}), \quad (M \xrightarrow{f} N) \mapsto [M \xrightarrow{f} N].$$

3. Same thing with left, right or short exact sequences.

7.1.2 Homotopy

Definition 7.1.2 Two morphisms $f, g : K^\bullet \to L^\bullet$ are said to be *homotopic* if there exists a family of $s^n : K^n \to L^{n-1}$:

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r, "d^{n-1}"] \arrow[dl, "s^{n-1}"'] & K^n \arrow[r, "d^n"] \arrow[dl, "s^n"'] & K^{n+1} \arrow[r] \arrow[dl, "s^{n+1}"'] & \cdots \\
\cdots \arrow[r] & L^{n-1} \arrow[r, "d^{n-1}"] & L^n \arrow[r, "d^n"] & L^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that for all n, we have

$$f^n - g^n = s^{n+1} \circ d^n + d^{n-1} \circ s^n.$$

We shall then write $s : f \sim g$.

Exercise 7.4 Show that morphisms that are homotopic to 0 form a subgroup of $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(K^\bullet, L^\bullet)$. Show that composition on both sides with a morphism homotopic to 0 always gives a morphism homotopic to 0.

Exercise 7.5 Show that homotopy is an equivalence relation compatible with composition on both sides.

Definition 7.1.3 The category $\mathbf{K}(\mathcal{C})$ of *complexes up to homotopy* is the category that has the same objects as $\mathbf{C}(\mathcal{C})$ and

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(K^\bullet, L^\bullet)/\{\text{homotopic to } 0\}$$

with the induced composition. A morphism of complexes that becomes an isomorphism in $\mathbf{K}(\mathcal{C})$ is called a *homotopy equivalence*. A complex that becomes 0 in $\mathbf{K}(\mathcal{C})$ is said to be *homotopically trivial*.

In other words, a diagram

$$\begin{tikzcd} K^\bullet \arrow[rr, "h"] \arrow[dr, "f"] & & M^\bullet \\ & L^\bullet \arrow[ur, "g"] & \end{tikzcd}$$

in $\mathbf{C}(\mathcal{C})$ is commutative in $\mathbf{K}(\mathcal{C})$ if it *commutes up to homotopy*: there exists a homotopy $s : h \sim g \circ f$:

Exercise 7.6
1. Show that a morphism of complexes $f : K^\bullet \to L^\bullet$ is a homotopy equivalence if and only if there exists a morphism $g : L^\bullet \to K^\bullet$ such that $g \circ f \sim \mathrm{Id}$ and $f \circ g \sim \mathrm{Id}$.
2. Show that a complex $K^\bullet$ is a homotopy trivial if and only if $\mathrm{Id}_{K^\bullet} \sim 0_{K^\bullet}$.

Exercise 7.7
1. Show that $\mathbf{K}(\mathcal{C})$ is an additive category and that the obvious functor $\mathbf{C}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$ is additive.
2. Show that any additive functor $F : \mathcal{C} \to \mathcal{C}'$ induces a functor

$$F : \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{C}').$$

On defines $\mathbf{K}^+(\mathcal{C})$, $\mathbf{K}^-(\mathcal{C})$ and $\mathbf{K}^{\mathrm{b}}(\mathcal{C})$ in the same way (may also be seen as full subcategories).

7.1.3 Mapping cone

Definition 7.1.4 The *mapping cone* of a morphism of complexes $f : K^\bullet \to L^\bullet$ is the complex $M(f)^\bullet$ with

$$M(f)^n := K^{n+1} \oplus L^n \quad \text{and} \quad d^n := \begin{bmatrix} -d^{n+1} & 0 \\ f^n & d^n \end{bmatrix}.$$

The *kth shift* $K^\bullet[k]$ of a complex $K^\bullet$ is defined by $K^\bullet[k]^n := K^{n+k}$ endowed with $(-1)^k d^{n+k}$.

Exercise 7.8 Show that there exists a short exact sequence of complexes

$$0 \to L^\bullet \xrightarrow{g} M(f)^\bullet \xrightarrow{h} K^\bullet[1] \to 0$$

with $g := \begin{bmatrix} \mathrm{Id} \\ 0 \end{bmatrix}$ and $h = \begin{bmatrix} 0 & \mathrm{Id} \end{bmatrix}$.

Definition 7.1.5 A *triangle* in $\mathbf{K}(\mathcal{C})$ is a diagram

$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1] \quad (\text{or } K^\bullet \to L^\bullet \to M^\bullet \xrightarrow{+} \text{for short}).$$

A *morphism of triangles* is a commutative diagram (in $\mathbf{K}(\mathcal{C})$)

$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d, "u"] & L^\bullet \arrow[r] \arrow[d, "v"] & M^\bullet \arrow[r] \arrow[d, "w"] & K^\bullet[1] \arrow[d, "u[1]"] \\ K'^\bullet \arrow[r] & L'^\bullet \arrow[r] & M'^\bullet \arrow[r] & K'^\bullet[1]. \end{tikzcd}$$

The triangle is said to be *distinguished* if it is isomorphic (in $\mathbf{K}(\mathcal{C})$) to

$$K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M(f)^\bullet \xrightarrow{h} K^\bullet[1].$$

Exercise 7.9 Show that if F is an additive functor and $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ is distinguished, then $FK^\bullet \to FL^\bullet \to FM^\bullet \to FK^\bullet[1]$ is also distinguished.

Proposition 7.1.6 A triangle

$$K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \xrightarrow{h} K^\bullet[1]$$

is distinguished if and only if the triangle

$$L^\bullet \xrightarrow{g} M^\bullet \xrightarrow{h} K^\bullet[1] \xrightarrow{-f[1]} L^\bullet[1]$$

is distinguished.

Proof. In order to prove the direct implication, it is sufficient to show that there exists a commutative diagram in $\mathbf{K}(\mathcal{C})$:

$$\begin{tikzcd} L^\bullet \arrow[r, "g"] \arrow[d, equal] & M(f)^\bullet \arrow[r, "h"] \arrow[d, equal] & K^\bullet[1] \arrow[r, "-f[1]"] \arrow[d, "\simeq"] & L^\bullet[1] \arrow[d, equal] \\ L^\bullet \arrow[r, "g"] & M(f)^\bullet \arrow[r] & M(g)^\bullet \arrow[r] & L^\bullet[1]. \end{tikzcd}$$

We can then simply set

$$\phi^n := \begin{bmatrix} -f^n \\ \mathrm{Id} \\ 0 \end{bmatrix}, \quad \psi^n := \begin{bmatrix} 0 & \mathrm{Id} & 0 \end{bmatrix} \quad \text{and} \quad s^n := \begin{bmatrix} 0 & 0 & \mathrm{Id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The diagram is clearly commutative, $\psi \circ \varphi = \mathrm{Id}$ and $s : \varphi \circ \psi \sim \mathrm{Id}$ since $\mathrm{Id} - \phi^n \circ \psi^n = s^{n+1} \circ d^n + d^{n-1} \circ s^n$. For the converse implication, it is equivalent to show that $K^\bullet[1] \to L^\bullet[1] \to M^\bullet[1] \to K^\bullet[2]$ is distinguished and we may apply twice the previous result. ■

Exercise 7.10 Show that $K^\bullet = K^\bullet \to [0] \to K^\bullet[1]$ is distinguished.

Proposition 7.1.7 1. Any morphism $f : K^\bullet \to L^\bullet$ in $\mathbf{K}(\mathcal{C})$ can be extended to a (not unique) distinguished triangle.

2. Any commutative diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "f"] \arrow[d, "u"] & L^\bullet \arrow[d, "v"] \\ K'^\bullet \arrow[r, "f'"] & L'^\bullet \end{tikzcd}$$

in $\mathbf{K}(\mathcal{C})$ can be extended to a (not unique) morphism of distinguished triangles.

Proof. The first assertion is clear. For the second one, we can write $v^n \circ f^n - f'^n \circ u^n = s^{n+1} \circ d^n + d^{n-1} \circ s^n$ and set

$$\begin{bmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{bmatrix} = M(f)^n \to M(f')^n. \qquad \blacksquare$$

In other words, the mapping cone is "almost" functorial in $\mathbf{K}(\mathcal{C})$.

Lemma 7.1.8 If we are given a morphism of distinguished triangles

$$\begin{tikzcd} K^\bullet \arrow[r, "f"] \arrow[d, "0"] & L^\bullet \arrow[r, "g"] \arrow[d, "0"] & M^\bullet \arrow[r, "h"] \arrow[d, "w"] & K^\bullet[1] \arrow[d, "0"] \\ K^\bullet \arrow[r, "f"] & L^\bullet \arrow[r, "g"] & M^\bullet \arrow[r, "h"] & K^\bullet[1] \end{tikzcd}$$

in $\mathbf{K}(\mathcal{C})$, then $w^2 = 0$ (in $\mathbf{K}(\mathcal{C})$).

Proof. We may assume that $M^\bullet = M(f)^\bullet$ and write

$$w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}.$$

Since $\begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} \sim 0$ and $\begin{bmatrix} w_{21} & w_{22} \end{bmatrix} \sim 0$, we have

$$w \sim w' := \begin{bmatrix} 0 & w_{12} \\ 0 & w_{22} \end{bmatrix} \quad \text{and} \quad w \sim w'' := \begin{bmatrix} w_{11} & w_{12} \\ 0 & 0 \end{bmatrix}.$$

It follows that $w^2 \sim w'' \circ w' = 0$. $\blacksquare$

Proposition 7.1.9 In a morphism of distinguished triangles (in $\mathbf{K}(\mathcal{C})$)

$$\begin{tikzcd} K^\bullet \arrow[r, "f"] \arrow[d, "u"] & L^\bullet \arrow[r] \arrow[d, "v"] & M^\bullet \arrow[r] \arrow[d, "w"] & K^\bullet[1] \arrow[d, "u[1]"] \\ K'^\bullet \arrow[r, "f'"] & L'^\bullet \arrow[r] & M'^\bullet \arrow[r] & K'^\bullet[1], \end{tikzcd}$$

if u and v are homotopy equivalences, so is w.

Proof. We may assume that $M^\bullet = M(f)^\bullet$ and that $M'^\bullet = M(f')^\bullet$. After composing both u and v with an inverse in $\mathbf{K}(\mathcal{C})$ and extending to a morphism of distinguished triangles, we are reduced to the case $u = \mathrm{Id}$ and $v = \mathrm{Id}$. Then, we have a morphism of distinguished triangles

$$\begin{tikzcd} K^\bullet \arrow[r, "f"] \arrow[d, "0"] & L^\bullet \arrow[r] \arrow[d, "0"] & M^\bullet \arrow[r] \arrow[d, "w-\mathrm{Id}"] & K^\bullet[1] \arrow[d, "0"] \\ K^\bullet \arrow[r, "f"] & L^\bullet \arrow[r] & M^\bullet \arrow[r] & K^\bullet[1]. \end{tikzcd}$$

If follows from lemma 7.1.8 that $(w-\mathrm{Id})^2 = 0$ in $\mathbf{K}(\mathcal{C})$. Thus, $2\mathrm{Id} - w$ is an inverse for w. ∎

As a consequence, the mapping cone is “almost” unique (up to a homotopy equivalence) in $\mathbf{K}(\mathcal{C})$.

Corollary 7.1.10 In a distinguished triangle

$$K^\bullet \xrightarrow{f} L^\bullet \to M^\bullet \xrightarrow{+},$$

f is a homotopy equivalence if and only if $M^\bullet$ is homotopically trivial.

Proof. It is sufficient to contemplate the following diagram:

$$\begin{tikzcd} K^\bullet \arrow[r, "f"] \arrow[d, "f"] & L^\bullet \arrow[r] \arrow[d, equal] & M^\bullet \arrow[r] \arrow[d] & K^\bullet[1] \arrow[d, "f[1]"] \\ L^\bullet \arrow[r, equal] & L^\bullet \arrow[r] & [0] \arrow[r] & L^\bullet[1]. \end{tikzcd}$$ ∎

7.1.4 Cohomology

Let $\mathcal{A}$ be an *abelian* category.

Exercise 7.11 Show that $\mathbf{C}(\mathcal{A})$ also is abelian.

Definition 7.1.11 The n-th *cohomology* of a cochain complex $K^\bullet$ of $\mathcal{A}$ is

$$\mathrm{H}^n(K^\bullet) := \mathrm{Z}^n(K^\bullet)/\mathrm{B}^n(K^\bullet)$$

with $\mathrm{Z}^n(K^\bullet) := \ker d^n$ and $\mathrm{B}^n(K^\bullet) := \mathrm{im}\, d^{n-1}$. The n-th *homology* $\mathrm{H}_n(K_\bullet)$ of a chain complex $K_\bullet$ is its $(1-n)$-th cohomology in $\mathcal{A}^{\mathrm{op}}$.

Examples 1. We have $\mathrm{H}^n(K^\bullet[k]) = \mathrm{H}^{n+k}(K^\bullet)$.

2. If $M \in \mathcal{A}$, then $\mathrm{H}^0([M]) = M$ and $\mathrm{H}^n([M]) = 0$ otherwise.
3. We have $\mathrm{H}^0([M \xrightarrow{f} N]) = \ker f$ and $\mathrm{H}^1([M \xrightarrow{f} N]) = \mathrm{coker}\, f$ (and 0 otherwise).
4. A short exact sequence $0 \to M' \to M \to M'' \to 0$ has 0 cohomology everywhere.
5. If X is a topological space, then

$$\mathrm{H}_n^{\mathrm{sing}}(X) := \mathrm{H}_n(C_\bullet(X)) \quad \text{and} \quad \mathrm{H}^n_{\mathrm{sing}}(X) := \mathrm{H}^n(C^\bullet(X)).$$

Exercise 7.12 Show that H^n is functorial in $K^\bullet$.

Exercise 7.13 Show that if $K^\bullet$ is a complex, then the sequence

$$0 \to \mathrm{H}^k \to \operatorname{coker} d^{k-1} \to \mathrm{Z}^{k+1} \to \mathrm{H}^{k+1} \to 0$$

est exact (everywhere).

Exercise 7.14 Show that the functor

$$\mathrm{H}^n : \mathbf{C}(\mathcal{A}) \to \mathcal{A}$$

is additive but *not* exact in general.

Definition 7.1.12 If $\mathcal{C}$ is an additive category, then an additive functor $F : \mathbf{K}(\mathcal{C}) \to \mathcal{A}$ is said to be *cohomological* if, whenever $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ is distinguished, then $FK^\bullet \to FL^\bullet \to FM^\bullet$ is exact in the middle.

Exercise 7.15 Show that, if $F : \mathbf{K}(\mathcal{C}) \to \mathcal{A}$ is a *cohomological* functor and $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ a distinguished triangle, then there exists a long exact sequence

$$\cdots \to FK^\bullet[n] \to FL^\bullet[n] \to FM^\bullet[n] \to FK^\bullet[n+1] \to \cdots$$

Proposition 7.1.13 For fixed $K^\bullet$, the functor $L^\bullet \mapsto \operatorname{Hom}_{\mathbf{K}(\mathcal{C})}(K^\bullet, L^\bullet)$ is cohomological (and dual).

Proof. Let $L'^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} L''^\bullet \to L'^\bullet[1]$ be a distinguished triangle. If $K^\bullet \to L'^\bullet$ is any morphism, then there exists a morphism of triangles

$$\begin{tikzcd}
K^\bullet \arrow[r, equal] \arrow[d] & K^\bullet \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K^\bullet[1] \arrow[d] \\
L'^\bullet \arrow[r] & L^\bullet \arrow[r] & L''^\bullet \arrow[r] & L'^\bullet[1].
\end{tikzcd}$$

This shows that $K^\bullet \to L''^\bullet$ is homotopic to zero. Conversely, if we are given a morphism $K^\bullet \to L^\bullet$ such that the composite $K^\bullet \to L^\bullet \to L''^\bullet$ is homotopic to zero, then there exists such a morphism of triangles which provides a suitable morphism $K^\bullet \to L'^\bullet$. ∎

Proposition 7.1.14 — Snake lemma. If

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, "f'"] & M \arrow[r, "p"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "j"] & N \arrow[r, "q"] & N'' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram with exact lines, then there exists a natural (long) exact sequence

$$0 \to \ker f' \to \ker f \to \ker f'' \to \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f'' \to 0.$$

Proof. This is a tedious diagram chasing[1] . ■

Theorem 7.1.15 Any short exact sequence $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ of cochain complexes gives rise to a natural long exact sequence

$$\cdots \to \mathrm{H}^n(K^\bullet) \to \mathrm{H}^n(L^\bullet) \to \mathrm{H}^n(M^\bullet) \to \mathrm{H}^{n+1}(K^\bullet) \to \cdots$$

Proof. This is a consequence of the *snake lemma*. ■

Corollary 7.1.16 The functor $H^n : \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ is cohomological.

Proof. If we are given a distinguished triangle $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$, we may assume that $M^\bullet = M(f)^\bullet$ and the sequence $0 \to L^\bullet \to M^\bullet \to K^\bullet[1] \to 0$ is therefore exact. It follows from the theorem that $\mathrm{H}^n(K^\bullet) \to \mathrm{H}^n(L^\bullet) \to \mathrm{H}^n(M^\bullet)$ is exact in the middle. ■

7.1.5 Quasi-isomorphism

Definition 7.1.17 A *quasi-isomorphism* of complexes $f : K^\bullet \to L^\bullet$ is a morphism such that $\mathrm{H}^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$. An *acylic complex* $K^\bullet$ is a complex such that $\mathrm{H}^n(K^\bullet) = 0$ for all $n \in \mathbb{Z}$.

Note that being quasi-isomorphic is *not* a symmetric relation. Also, a complex is acyclic if and only if $[0]$ is quasi-isomorphic to $K^\bullet$ if and only if $K^\bullet$ is quasi-isomorphic to $[0]$.

Exercise 7.16 Show that if $K^\bullet \to L^\bullet \to M^\bullet \xrightarrow{+}$ is a distinguished triangle, then f is a quasi-isomorphism if and only if $M^\bullet$ is acyclic.

Exercise 7.17 Show that, if

$$\begin{tikzcd}
K^\bullet \arrow[r] \arrow[d, "u"] & L^\bullet \arrow[r] \arrow[d, "v"] & M^\bullet \arrow[r, "+"] \arrow[d, "w"] & {} \\
K'^\bullet \arrow[r] & L'^\bullet \arrow[r] & M'^\bullet \arrow[r, "+"] & {}
\end{tikzcd}$$

is a morphism of distinguished triangles and u, v are quasi-isomorphism, then w is also a quasi-isomorphism.

Exercise 7.18
1. Show that $I \in \mathcal{A}$ is injective if and only if any monomorphism $I \hookrightarrow M$ has a retraction if and only if h_I is exact if and only if any extension by I is trivial (and dual).
2. Show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence with M' injective, then M is injective if and only if M'' is (and dual).

[1]With a lot of fun - assume $\mathcal{A} = A\!-\!\mathbf{Mod}$ to make it easier.

3. Show that, if a functor G has an exact adjoint, then G preserves injective objects (and dual).

We denote by $\mathcal{I}$ (resp. $\mathcal{P}$) the full subcategory of injective (resp. projective) objects.

Lemma 7.1.18 If $K^\bullet$ is acyclic and $I^\bullet \in \mathbf{C}^+(\mathcal{I})$ then any morphism $f : K^\bullet \to I^\bullet$ is homotopic to zero (and dual).

Proof. [2] Since $I^\bullet$ is bounded below, we may assume that we have built a homotopy s_k up to $k = n$. In particular, we have $f^{n-1} = s^n \circ d^{n-1} + d^{n-2} \circ s^{n-1}$. Thus, if we set $g^n := f^n - d^{n-1} \circ s^n$, we have

$$\begin{aligned} g^n \circ d^{n-1} &= f^n \circ d^{n-1} - d^{n-1} \circ s^n \circ d^{n-1} \\ &= d^{n-1} \circ f^{n-1} - d^{n-1} \circ s^n \circ d^{n-1} \\ &= d^{n-1} \circ (f^{n-1} - s^n \circ d^{n-1}) \\ &= d^{n-1} \circ d^{n-2} \circ s^{n-1} \\ &= 0. \end{aligned}$$

In other words g_n is zero on $Z^n(K^\bullet)$. Since $K^\bullet$ is acyclic, its means that g^n factors through $Z^{n+1}(K^\bullet)$. Since I^n is injective, it can be extended to $s^{n+1} : K^{n+1} \to I^n$ and we have $g^n = s^{n+1} \circ d^n$ so that

$$f^n = s^{n+1} \circ d^n + d^{n-1} \circ s^n.$$

■

Proposition 7.1.19 If $K^\bullet \to L^\bullet$ is a quasi-isomorphism and $I^\bullet \in \mathbf{C}^+(\mathcal{I})$, then

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K^\bullet, I^\bullet) \quad \text{(and dual)}.$$

Proof. There exists a distinguished triangle $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$. Applying the cohomological functor $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(-, I^\bullet)$ provides an exact sequence

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M^\bullet[-1], I^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(L^\bullet, I^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K^\bullet, I^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M^\bullet, I^\bullet).$$

Since $M^\bullet$ is acyclic, we can apply lemma 7.1.18. ■

Exercise 7.19 Show that a complex $I^\bullet \in \mathbf{C}^+(\mathcal{I})$ is acyclic if and only if it is homotopically trivial (and dual).

Exercise 7.20 Show that $I^\bullet, J^\bullet \in \mathbf{C}^+(\mathcal{I})$ are quasi-isomorphic if and only if they are homotopically equivalent (and dual).

Definition 7.1.20 If $f : K^\bullet \to L^\bullet$ is a quasi-isomorphism, we also say that $L^\bullet$ is a *right resolution*[a] of $K^\bullet$. It is called an *injective resolution* if each L^n is injective.

[a] Or *replacement*.

A *left* (resp. a *projective*) *resolution* is a right (resp. an injective) resolution in $\mathcal{A}^{\mathrm{op}}$.

[2] The proof in [Sta19, Tag 013R] is not correct.

Examples 1. A sequence $0 \to M \to K^0 \to K^1 \to \cdots$ is exact if and only if $[M] \to K^\bullet$ is a right resolution.
2. Any abelian group M has a free (projective) resolution of length 2: a short exact sequence $0 \to L_1 \to L_0 \to M \to 0$ with L_0 and L_1 free.

Exercise 7.21 Show that if $K^\bullet \to I^\bullet$ and $L^\bullet \to J^\bullet$ are two injective resolutions in $\mathbf{C}^+(\mathcal{A})$, then any morphism $f : K^\bullet \to L^\bullet$ extends to $I^\bullet \to J^\bullet$ (and dual).

Proposition 7.1.21 If $\mathcal{A}$ has enough injectives and $K^\bullet \in \mathbf{C}^+(\mathcal{A})$, then there exists an injective resolution $K^\bullet \to I^\bullet$ made of injective maps (and dual).

Proof. We may assume that $K^n = 0$ for $n < 0$ and, by induction, that there exists an injective morphism of complexes with right exact lines

$$\begin{tikzcd}
I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^{n-2} \arrow[r] & I^{n-1} \arrow[r] & D^n \arrow[r] & 0 \\
K^0 \arrow[r] \arrow[u, hook] & K^1 \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & K^{n-2} \arrow[r] \arrow[u, hook] & K^{n-1} \arrow[r] \arrow[u, hook] & C^n \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

such that $\mathrm{H}^m(K^\bullet) \simeq \mathrm{H}^m(I^\bullet)$ is an isomorphism for $m < n-1$. Then, there exists an injection $E^n := (K^n \oplus D^n)/C^n \hookrightarrow I^n$ into an injective. We obtain injective maps $K^n \hookrightarrow I^n$ and $C^{n+1} \hookrightarrow D^{n+1}$ on the cokernels and a morphism $I^{n-1} \to I^n$ such that $\mathrm{H}^{n-1}(K^\bullet) \simeq \mathrm{H}^{n-1}(I^\bullet)$. This may be shown[3] by playing around with the following diagram:

$$\begin{tikzcd}
I^{n-2} \arrow[r] & I^{n-1} \arrow[r, two heads] & D^n \arrow[rr] \arrow[dr] & & I^n \arrow[r, two heads] & D^{n+1} \\
& & & E^n \arrow[ur, hook] & & \\
K^{n-2} \arrow[r] \arrow[uu, hook] & K^{n-1} \arrow[r, two heads] \arrow[uu, hook] & C^n \arrow[rr] \arrow[uu, hook] \arrow[ur] & & K^n \arrow[r, two heads] \arrow[uu, hook] \arrow[ul] & C^{n+1} \arrow[uu]
\end{tikzcd}$$

■

7.2 Derived functor

7.2.1 Derived category

Let $\mathcal{A}$ be an abelian category.

Proposition 7.2.1 The category $\mathbf{K}(\mathcal{A})$ admits both left and right calculus of fractions with respect to quasi-isomorphisms.

Proof. An identity is a quasi-isomorphism and quasi-isomorphisms are clearly stable under composition. Assume now that we are given a morphism $K^\bullet \to L^\bullet$ and a quasi-isomorphism $L'^\bullet \to L^\bullet$. Then, there exists a morphism of distinguished

[3] With a lot of fun - assume $\mathcal{A} = A{-}\mathbf{Mod}$ to make it easier.

triangles (build $N^\bullet$ first and then $K'^\bullet$)

$$\begin{tikzcd}
K'[-1] \arrow[r] \arrow[d] & K^\bullet \arrow[r] \arrow[d] & N^\bullet \arrow[r] \arrow[d, equal] & K'^\bullet \arrow[d] \\
L'^\bullet \arrow[r] & L^\bullet \arrow[r] & N^\bullet \arrow[r] & L'^\bullet[1].
\end{tikzcd}$$

Since $L'^\bullet \to L^\bullet$ is a quasi-isomorphism, $N^\bullet$ is acyclic and therefore $K'[-1] \to K^\bullet$ is also a quasi-isomorphism. The last condition is proved in the same way. The left property also is shown with the same method. ■

Definition 7.2.2 The *derived category* $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ is the localization of $\mathbf{K}(\mathcal{A})$ at quasi-isomorphisms.

Thus, $\mathbf{D}(\mathcal{A})$ has the same objects as $\mathbf{K}(\mathcal{A})$ (or equivalently $\mathbf{C}(\mathcal{A})$) and a morphism $K^\bullet$ to $L^\bullet$ is a diagram

$$\begin{tikzcd}
K'^\bullet \arrow[d] \arrow[dr] & \\
K^\bullet & L^\bullet
\end{tikzcd}$$

where the vertical map is a quasi-isomorphism. A morphism of complexes $K^\bullet \to L^\bullet$ is a quasi-isomorphism if and only if it becomes an isomorphism in $\mathbf{D}(\mathcal{A})$. Actually, $\mathbf{D}(\mathcal{A})$ is also the localization of $\mathbf{C}(\mathcal{A})$ at quasi-isomorphisms (even it this last category does not admits right or left calculus of fraction).

One can also define $\mathbf{D}^+(\mathcal{A})$, $\mathbf{D}^-(\mathcal{A})$ and $\mathbf{D}^b(\mathcal{A})$ along the same lines. Moreover, there exists a canonical embedding $\mathbf{D}^+(\mathcal{A}) \hookrightarrow \mathbf{D}(\mathcal{A})$ whose essential image is made of complexes such that $\mathrm{H}^n(K^\bullet) = 0$ for $n << 0$ (and analogous statements with $-$ and b).

Definition 7.2.3 A *triangle* in $\mathbf{D}(\mathcal{A})$ is a diagram

$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1].$$

A *morphism of triangles* is a commutative diagram (in $\mathbf{D}(\mathcal{A})$)

$$\begin{tikzcd}
K^\bullet \arrow[r] \arrow[d, "u"] & L^\bullet \arrow[r] \arrow[d, "v"] & M^\bullet \arrow[r] \arrow[d, "w"] & K^\bullet[1] \arrow[d, "u[1]"] \\
K'^\bullet \arrow[r] & L'^\bullet \arrow[r] & M'^\bullet \arrow[r] & K'^\bullet[1].
\end{tikzcd}$$

The triangle is said to be *distinguished* if it is isomorphic (in $\mathbf{D}(\mathcal{A})$) to

$$K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M(f)^\bullet \xrightarrow{h} K^\bullet[1].$$

Proposition 7.2.4 If $0 \to K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to 0$ is an exact sequence of complexes, then there exists a distinguished triangle $K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to K^\bullet[1]$ in $\mathbf{D}(\mathcal{A})$.

Proof. The composite maps $M(f)^n \twoheadrightarrow L^n \twoheadrightarrow M^n$ defines a morphism of complexes and there is an exact sequence

$$0 \to M(\mathrm{Id}_{K^\bullet}) \to M^\bullet(f) \to M^\bullet \to 0.$$

It follows from corollary 7.1.16 that $M(\mathrm{Id}_{K^\bullet})$ is acyclic and then from theorem 7.1.15 that $M^\bullet(f) \to M^\bullet$ is a quasi-isomorphism. ■

Proposition 7.2.5 If $K^\bullet \in \mathbf{C}(\mathcal{A})$ and $I^\bullet \in \mathbf{C}^+(\mathcal{I})$, then

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K^\bullet, I^\bullet) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(K^\bullet, I^\bullet) \quad \text{(and dual)}.$$

Proof. Formal consequence of proposition 7.1.19. ■

Theorem 7.2.6 Assume that $\mathcal{A}$ has enough injectives. Then there exists an equivalence

$$\mathbf{K}^+(\mathcal{I}) \simeq \mathbf{D}^+(\mathcal{A}) \quad \text{(and dual)}.$$

Proof. Follows from propositions 7.1.21 and 7.2.5. ■

7.2.2 Derived functor

Let $F : \mathcal{A} \to \mathcal{A}'$ be a functor between two abelian categories with $\mathcal{A}$ having enough injectives (and dual).

Definition 7.2.7 The *right derived functor*[a] of F is the unique (up to isomorphism) functor $\mathrm{R}F$ making commutative the following diagram:

$$\begin{tikzcd}
\mathbf{K}^+(\mathcal{A}) \arrow[r, "F"] & \mathbf{K}^+(\mathcal{A}') \arrow[dd] \\
\mathbf{K}^+(\mathcal{I}) \arrow[u, hook] \arrow[d, "\simeq"] & \\
\mathbf{D}^+(\mathcal{A}) \arrow[r, "\mathrm{R}F"] & \mathbf{D}^+(\mathcal{A}').
\end{tikzcd}$$

[a]This definition is usually only applied to left exact functors.

In practice, if $K^\bullet \to I^\bullet$ is an injective resolution, then $RFK^\bullet \simeq FI^\bullet$ and this can be made functorial. We will then set

$$\forall n \in \mathbb{N}, \quad \mathrm{R}^n F K^\bullet := \mathrm{H}^n(\mathrm{R}FK^\bullet).$$

If $M \in \mathcal{A}$, we may consider $\mathrm{R}FM := \mathrm{R}F[M] \in \mathbf{D}^+(\mathcal{A}')$ and for all $n \in \mathbb{Z}$, $\mathrm{R}^n FM \in \mathcal{A}'$ (and dual). One defines dually $\mathrm{L}F := \mathrm{R}F^{\mathrm{op}}$ and $\mathrm{L}_n FK_\bullet := \mathrm{H}_n(\mathrm{L}FK_\bullet)$ when $\mathcal{A}$ has enough projectives.

Exercise 7.22 Show that if F is left exact, then $\mathrm{R}F$ is the (left) Kan extension of the composite map $\mathbf{K}^+(\mathcal{A}) \xrightarrow{F} \mathbf{K}^+(\mathcal{A}') \to \mathbf{D}^+(\mathcal{A}')$ along $\mathbf{K}^+(\mathcal{A}) \to \mathbf{D}^+(\mathcal{A})$ (and dual).

Theorem 7.2.8 Any short exact sequence

$$0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$$

in $\mathbf{C}^+(\mathcal{A})$ gives rise to a long exact sequence

$$\cdots \to \mathrm{R}^n FK^\bullet \to \mathrm{R}^n FL^\bullet \to \mathrm{R}^n FM^\bullet \to \mathrm{R}^{n+1} FK^\bullet \to \cdots \quad \text{(and dual).}$$

Proof. We know from propositon 7.2.4 that there exists a distinguished triangle $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ in $\mathbf{D}(\mathcal{A})$. Thanks to theorem 7.2.6, we may assume that all complexes are composed of injective objects and maps defined in $\mathcal{C}(A)$. We can then apply F and then use corollary 7.1.16 ■

Exercise 7.23 Show that F is left exact (resp. exact) if and only if $\forall M \in \mathcal{A}, \mathrm{R}^0 FM \simeq FM$ (resp. $\mathrm{R}FM \simeq \mathrm{F}M$) (and dual).

Solution. Let $M \to I^\bullet$ be an injective resolution. Assume first that F is left exact. Then, since $M \simeq \mathrm{H}^0(I^\bullet) \simeq \ker(I^0 \to I^1)$, we will have $FM \simeq \ker(FI^0 \to FI^1) \simeq \mathrm{H}^0(FI^\bullet) = \mathrm{R}^0 FM$. If F is exact, then $FM \simeq FI^\bullet \simeq \mathrm{R}FM$. The converse follows from theorem 7.2.8. ■

Definition 7.2.9 The *n-th extension group* of $L^\bullet$ by $K^\bullet$ is

$$\mathrm{Ext}^n(K^\bullet, L^\bullet) := \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(K^\bullet, L^\bullet[n]).$$

Example
1. We have $\mathrm{Ext}^n(M,N) = 0$ for $n < 0$,
2. we have $\mathrm{Ext}^0(M,N) = \mathrm{Hom}(M,N)$,
3. $\mathrm{Ext}^1(M,N) = \mathrm{Ext}(M,N)$ classifies extensions of N by M up to isomorphism,
4. we have $\mathrm{Ext}^n(M,N) = 0$ for $n > 1$ in $\mathbf{A}\mathrm{b}$,
5. $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$ with $d = m \wedge n$.

Proposition 7.2.10 If $M \in \mathcal{A}$ and $K^\bullet \in \mathbf{C}^+(\mathcal{A})$, then

$$\mathrm{R}^n\mathrm{Hom}(M, K^\bullet) = \mathrm{Ext}^n(M, K^\bullet) \quad \text{(and dual).}$$

Proof. It is sufficient to prove the assertion when $K^\bullet[n] = I^\bullet \in \mathbf{C}^+(\mathcal{I})$. Thanks to proposition 7.2.5, we have to show that

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M, I^\bullet) = \mathrm{H}^0(\mathrm{Hom}(M, I^\bullet)).$$

A quick inspection shows that

$$\mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(M, I^\bullet) = \mathrm{Z}^0(\mathrm{Hom}(M, I^\bullet))$$

and $f \in \mathrm{B}^0(\mathrm{Hom}(M, I^\bullet))$ if and only if the corresponding map of complexes is homotopic to zero. ■

Note in particular, that if $\mathcal{A}$ has enough injectives *and* projectives, then we can use an injective resolution of N or a projective resolution of M to compute $\mathrm{Ext}^n(M,N)$.

Exercise 7.24 Show that (even if $\mathcal{A}$ does not have enough projectives), then $\mathrm{Ext}^n(P,M) = 0$ for $n \neq 0$ when P is projective.

Solution. If $M \to I^\bullet$ is an injective resolution, then $\mathrm{Hom}(P,M) \to \mathrm{Hom}(P,I^\bullet)$ also and therefore $\mathrm{Hom}(P,M) \simeq \mathrm{RHom}(P,M)$. ■

Exercise 7.25 Show that if M,N are two abelian groups, then $\mathrm{Ext}^n(M,N) = 0$ for $n \neq 0,1$ and $\mathrm{Ext}^0(M,N) = \mathrm{Hom}(M,N)$. Show that $\mathrm{Ext}^1(\mathbb{Z}/k\mathbb{Z},N) \simeq N/kN$.

Definition 7.2.11 An object $M \in \mathcal{A}$ is said to be (right) *F-acyclic* if $FM = \mathrm{R}FM$ (i.e. $FM = R^0FM$ and $\mathrm{R}^nFM = 0$ for $n \neq 0$).

Proposition 7.2.12 — Leray acyclicity. Show that if $K^\bullet \in \mathbf{C}^+(\mathcal{A})$ and each K^n is F-acyclic, then $\mathrm{R}FK^\bullet = FK^\bullet$.

Proof. We may assume that $K^n = 0$ for $n < 0$. Assume first that $K^\bullet$ is bounded and denote by $K'^\bullet := K^1 \to K^2 \to \cdots$. Then, there exists an exact sequence $0 \to K'^\bullet \to K^\bullet \to [K^0] \to 0$. It follows that there exists a morphism of distinguished triangles

$$\begin{tikzcd}
FK'^\bullet \arrow[r] \arrow[d] & FK^\bullet \arrow[r] \arrow[d] & FK^0 \arrow[r] \arrow[d] & FK'^\bullet[1] \arrow[d] \\
\mathrm{R}FK'^\bullet \arrow[r] & \mathrm{R}FK^\bullet \arrow[r] & \mathrm{R}FK^0 \arrow[r] & \mathrm{R}FK'^\bullet[1].
\end{tikzcd}$$

in $\mathbf{D}(\mathcal{A})$. By induction, the middle arrow is also an isomorphism. In général, if we set $K'^\bullet = K^{n+2} \to K^{n+3} \to \cdots$ and $K''^\bullet = K^0 \to \cdots \to K^{n+1}$ (bounded), we get an exact sequence $0 \to K'^\bullet \to K^\bullet \to K''^\bullet \to 0$. It follows that there exists a morphism of long exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H^nFK^\bullet \arrow[r, "\simeq"] \arrow[d] & H^nFK''^\bullet \arrow[r] \arrow[d, "\simeq"] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{R}^nFK^\bullet \arrow[r, "\simeq"] & \mathrm{R}^nFK''^\bullet \arrow[r] & 0.
\end{tikzcd}$$

■

As a consequence, if $M \to K^\bullet$ is a right resolution with each K^n is F-acyclic, then $\mathrm{R}FM = FK^\bullet$ and therefore

$$\forall n \in \mathbb{N}, \quad \mathrm{R}^nFM = H^n(FK^\bullet).$$

Exercise 7.26 Show that an object is injective if and only if it is F-acyclic for all left exact functor F if and only if this is the case when $F = \mathrm{Hom}(M,-)$ for all $M \in \mathcal{A}$ (and dual).

Corollary 7.2.13 Assume $\mathcal{A}'$ also has enough injective, $\mathcal{A}''$ is another abelian category and $G : \mathcal{A}' \to \mathcal{A}''$ an additive functor. Assume that FI is G-acyclic whenever I is injective. Then $R(G \circ F) = RG \circ RF$.

Proof. Follows from Leray acyclicity. ∎

Note that the condition is automatic if F has an exact adjoint.

7.2.3 Spectral sequence

Let $\mathcal{A}$ be an abelian category (with exact countable direct sums[4]).

Definition 7.2.14 A *(decreasing) filtration* on an object $M \in \mathcal{A}$ is diagram on $(\mathbb{Z}, \geq)$ of subobjects of M:

$$M \supset \cdots \supset \mathrm{F}^n M \supset \mathrm{F}^{n+1} M \supset \cdots \supset 0.$$

We shall always assume that the filtration is *exhaustive* : $\bigcup_{n\in\mathbb{Z}} F^n M = M$ and *separated* : $\bigcap_{n\in\mathbb{Z}} F^n M = 0$. It is said to be *finite* if $F^n M = M$ for $n << 0$ and $F^n M = 0$ for $n >> 0$

We shall denote this category by $\mathbf{F}(\mathcal{A})$ and write for each $n \in \mathbb{Z}$, $\mathrm{Gr}^n M = \mathrm{F}^n M / \mathrm{F}^{n+1} M$.

Exercise 7.27 Show that $\mathbf{F}(\mathbf{C}(\mathcal{A})) \simeq \mathbf{C}(\mathbf{F}(\mathcal{A}))$.

Definition 7.2.15 A spectral sequence

$$E_{r_0}^{p,q} \Rightarrow H^{p+q}$$

is

1. a family of complexes with $p, q, r \in \mathbb{Z}$ and $r \geq r_0$

$$\cdots \longrightarrow E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r,q-r+1} \longrightarrow \cdots$$

such that

$$E_{r+1}^{p,q} = H^{p,q}(E_r) := \ker(d_r^{p,q}) / \mathrm{Im}(d_r^{p-r,q+r-1}),$$

2. a family of filtered objects H^n for $n \in \mathbb{Z}$ such that

$$\forall p, q \in \mathbb{N}, \quad \mathrm{Gr}^p H^{p+q} = E_r^{p,q} \quad \text{for } r >> 0.$$

It is called a *first quadrant* spectral sequence if $E_r^{p,q} = 0$ unless $p, q \in \mathbb{N}$.

Exercise 7.28 Represent E_0, E_1 and E_2 with p, q as coordinates.

Exercise 7.29 Show that, if $r \geq 1$, then $\forall p \neq 0, E_r^{p,q} = 0 \Rightarrow \forall q \in \mathbb{Z}, E_r^{0,q} \simeq H^q$ and if $r \geq 2$, then $\forall q \neq 0, E_r^{p,q} = 0 \Rightarrow \forall p \in \mathbb{Z}, E_r^{p,0} \simeq H^p$.

[4]For example a Grothendieck category.

Exercise 7.30 Show that, in a first quadrant spectral sequence, the sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0}$$

is exact.

Theorem 7.2.16 Let $K^\bullet$ is a filtered complex. Assume $K^\bullet$ is bounded below. Then there exists a spectral sequence

$$E_0^{p,q} = \mathrm{Gr}^p K^{p+q} \Rightarrow \mathrm{H}^{p+q} K^\bullet.$$

Proof. To do. ■

Exercise 7.31 Show that $E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}^p K^\bullet)$.

Definition 7.2.17 A *bicomplex* (in an additive category) is a complex of complexes.

In other words, a bicomplex is a diagram $(K^{p,q}, d^{p,q}, d'^{p,q})$ on $(\mathbb{Z}, \leq)^2$ such that for all $p, q \in \mathbb{Z}$,

$$d^{p+1,q} \circ d^{p,q} = 0, \quad d'^{p,q+1} \circ d'^{p,q} = 0 \quad \text{and} \quad d'^{p+1,q} \circ d^{p,q} = d^{p,q+1} \circ d'^{p,q}.$$

Exercise 7.32 Show that if $K^{\bullet,\bullet}$ is a bicomplex, and we endow $K^n := \bigoplus_{p+q=n} K^{p,q}$ with $d^n := \oplus_{p+q=n}(d^{p,q} + (-1)^p d'^{p,q})$, then $K^\bullet$ is a complex.

It is called the *simple complex* associated to the bicomplex or the *total complex* of the bicomplex.

Proposition 7.2.18 Let $K^{\bullet,\bullet}$ be a bicomplex. Assume that it is bounded below (in both variables). Then there exists a spectral sequence

$$E_0^{p,q} = K^{p,q} \Rightarrow \mathrm{H}^{p+q}(K^\bullet).$$

Proof. Apply theorem 7.2.16 with $\mathrm{F}^p K^n := \bigoplus_{i+j=n, i \geq p} K^{p,q}$. ■

Exercise 7.33 Show that $E_1^{p,q} = \mathrm{H}^q(K^{p,\bullet})$ and $E_2^{p,q} = \mathrm{H}^p(\mathrm{H}^q(K^{\bullet,\bullet}))$.

Lemma 7.2.19 If $K^\bullet$ is bounded below, then there exists a bounded below bicomplex $I^{\bullet,\bullet}$ with $I^{\bullet,q} = 0$ for $q < 0$ and a morphism of complexes $K^\bullet \to I^{\bullet,0}$ such that each $K^p \to I^{p,\bullet}$ and $H^p(K^\bullet) \to H^p(I^{\bullet,\bullet})$ are injective resolutions.

Proof. To do. ■

This is called a *Cartan-Eilenberg* resolution.

Exercise 7.34 Show that if $I^\bullet$ denotes the associated simple complex, then $K^\bullet \to I^\bullet$ is also an injective resolution.

Proposition 7.2.20 Let $\mathcal{A} \xrightarrow{F} \mathcal{A}'$ be an additive functor. Assume $\mathcal{A}$ has enough injectives. If $K^\bullet$ is bounded below, then there exists two spectral sequences

$$'E_1^{p,q} = R^qFK^p \Rightarrow R^{p+q}FK^\bullet \quad \text{and} \quad ''E_2^{p,q} = R^pF(H^q(K^\bullet)) \Rightarrow R^{p+q}FK^\bullet.$$

Proof. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $K^\bullet$. Then, there exists a spectral sequence

$$'E_1^{p,q} = \mathrm{H}^q(FI^{p,\bullet}) \Rightarrow \mathrm{H}^{p+q}(FI^\bullet).$$

But we may also exchange the rôle of p and q and consider the spectral sequence

$$''E_2^{p,q} = \mathrm{H}^q(F\mathrm{H}^p(I^{\bullet,\bullet})) \Rightarrow \mathrm{H}^{p+q}(FI^\bullet).$$ ∎

Corollary 7.2.21 Let $\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{G} \mathcal{A}''$ be a sequence of additive functors. Assume $\mathcal{A}$ and $\mathcal{A}'$ have enough injectives and FI is G-acyclic whenever I is injective. If $K^\bullet$ is bounded below, then there exists a spectral sequence

$$E_2^{p,q} = R^pG(R^qF(K^\bullet)) \Rightarrow R^{p+q}(G \circ F)(K^\bullet).$$

Proof. We may assume that $K^\bullet \in \mathbf{C}^+(\mathcal{I})$ and apply the proposition to $FK^\bullet$. ∎

Exercise 7.35 Show that, in the situation of the corollary, when F (resp. G) is exact, there is an isomorphism

$$R^nG(F(K^\bullet)) \simeq R^n(G \circ F)(K^\bullet) \quad (\text{resp. } G(R^nF(K^\bullet)) \simeq R^n(G \circ F)(K^\bullet)).$$

7.3 Sheaf cohomology

7.3.1 Definition

Recall that, if $\mathcal{C}$ is a site and $X \in \mathcal{C}$, then there exists a (left exact additive) functor

$$\widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b}) \to \mathbf{A}\mathrm{b}, \quad \mathcal{M} \to \Gamma(X, \mathcal{M}) := \mathcal{M}(X).$$

Definition 7.3.1 If $\mathcal{M}^\bullet$ is a complex of abelian sheaves on a site $\mathcal{C}$, then its nth cohomology group on $X \in \mathcal{C}$ is

$$\mathrm{H}^n(X, \mathcal{M}^\bullet) := \mathrm{R}^n\Gamma(X, \mathcal{M}^\bullet).$$

Examples The above definition applies in particular to an abelian complex on a topological space. We give (without details) a list of examples from geometry that show that sheaf cohomology agrees with classical cohomology:

1. If X is a locally contractible topological space, then

$$\mathrm{H}^n(X, \mathbb{Z}) \simeq \mathrm{H}^n_{\mathrm{sing}}(X).$$

2. If X is a differentiable manifold (and there exists a complex analog), then

$$\mathrm{H}^n(X, \mathbb{R}) \simeq \mathrm{H}^n(X, \Omega^\bullet_{X/\mathbb{R}}) \simeq \mathrm{H}^n(\Omega^\bullet(X)) =: \mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{R}).$$

3. If X is a smooth algebraic variety over $\mathbb{C}$, we have (GAGA)

$$\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}) := \mathrm{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}) \simeq \mathrm{H}^n(X^{\mathrm{an}}, \mathbb{C})$$

Note that, by definition, if $f : Y \to X$ is a morphism in $\mathcal{C}$, there exists a canonical maps

$$\mathrm{R}\Gamma(X, \mathcal{M}^\bullet) \to \mathrm{R}\Gamma(Y, \mathcal{M}^\bullet) \quad \text{and} \quad \mathrm{H}^n(X, \mathcal{M}^\bullet) \to \mathrm{H}^n(Y, \mathcal{M}^\bullet).$$

Cohomology can be computed in the topos:

Exercise 7.36 Show that $\mathrm{R}\Gamma(X, \mathcal{M}^\bullet) = \mathrm{R}\Gamma(\underline{X}, \mathcal{M}^\bullet)$ and therefore $\mathrm{H}^n(\underline{X}, \mathcal{M}^\bullet) = \mathrm{H}^n(X, \mathcal{M}^\bullet)$.

Solution. Choosing an injective resolution $\mathcal{I}^\bullet$, it is sufficient to show that $\Gamma(X, \mathcal{I}^\bullet) = \Gamma(\underline{X}, \mathcal{I}^\bullet)$. But we know that for any sheaf $\mathcal{M}$, we have $\Gamma(X, \mathcal{M}) = \Gamma(\underline{X}, \mathcal{M})$ ■

Exercise 7.37 Show that in a topos $\mathcal{T}$, we have $\mathrm{R}\Gamma(X, M^\bullet) \simeq \mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z} \cdot X, M^\bullet)$ and therefore $\mathrm{H}^n(X, M^\bullet) \simeq \mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z} \cdot X, M^\bullet)$.

Exercise 7.38 Show that in a topos $\mathcal{T}$, we have $\mathrm{H}^n\left(\coprod_{i\in I} X_i, M^\bullet\right) \simeq \prod_{i\in I} \mathrm{H}^n(X_i, M^\bullet)$.

Solution. Choose an injective resolution, use left exactness of global section and the fact that products are exact in abelian groups. ■

On a topos $\mathcal{T}$ (with enough projectives - but this not really necessary), we can derive our functors $\mathcal{H}\mathrm{om}_{\mathbb{Z}}$ and $\otimes_{\mathbb{Z}}$ (on one side or the other) and obtain

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, N) \quad \text{and} \quad M \otimes^{\mathrm{L}}_{\mathbb{Z}} N.$$

Cohomology is then usually denoted respectively by

$$\mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(M, N) \quad \text{and} \quad \mathcal{T}\mathrm{or}^{\mathbb{Z}}_n(M, N).$$

Exercise 7.39 Show that, in a topos, if P is a flat abelian group and M is a complex of abelian groups which is bounded below and acyclic, then $P \otimes_{\mathbb{Z}} M$ is also acyclic.

7.3.2 Simplicial method

If $f : X_0 \to X$ is a morphism in a topos $\mathcal{T}$, we may then consider the (semi-) simplicial object

$$[n] \mapsto X_n := \underbrace{Y \times_X \cdots \times_X Y}_{n+1}$$

so that

$$X_\bullet : \quad \begin{smallmatrix}\to\\ \to\\ \to\\ \to\end{smallmatrix} X \times_X \times Y \times_X Y \;\begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix}\; Y \times_X Y \;\begin{smallmatrix}\to\\ \to\end{smallmatrix}\; Y.$$

We shall also consider the augmented (semi-) simplicial object

$$X_\bullet^+ : \qquad \begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} X \times_X \times Y \times_X Y \;\begin{smallmatrix}\to\\\to\\\to\end{smallmatrix}\; Y \times_X Y \rightrightarrows Y \longrightarrow X.$$

More generally, if $\mathcal{X} := (X_i \to X)_{i\in I}$ is a family of morphisms, we shall simply write

$$\mathcal{X}_\bullet = \left(\coprod_{i\in I} X_i\right)_\bullet \quad \text{and} \quad \mathcal{X}_\bullet^+ = \left(\coprod_{i\in I} X_i\right)_\bullet^+.$$

Lemma 7.3.2 If $\mathcal{X} := (X_i \to X)_{i\in I}$ is a covering in $\mathcal{T}$, then $\mathbb{Z}\cdot\mathcal{X}_\bullet^+$ is acyclic.

Proof. It is sufficient to consider the case of an epimorphism $X_0 \twoheadrightarrow X$. When $\mathcal{T} = \mathbf{Set}$, it reduces to $X = \{x\}$ in which case, this is clear. It follows that this is also true when $\mathcal{T} := \widehat{\mathcal{C}}$ for any category $\mathcal{C}$. Finally, if $\mathcal{C}$ is a site, then any epimorphism in $\mathcal{T} := \widetilde{\mathcal{C}}$ is the sheafification of an epimorphism in $\widehat{\mathcal{C}}$. ∎

If $\mathcal{X} := (X_i \to X)_{i\in I}$ is a family of morphisms and M is an abelian group of $\mathcal{T}$, then we set

$$\check{\mathcal{C}}^\bullet(\mathcal{X}, M) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\cdot\mathcal{X}_\bullet, M), \quad \check{\mathcal{C}}^\bullet(\mathcal{X}^+, M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\cdot\mathcal{X}_\bullet^+, M)$$

and

$$\forall n \in \mathbb{N}, \quad \check{\mathrm{H}}^n(\mathcal{X}, M) := \mathrm{H}^n(\check{\mathcal{C}}^\bullet(\mathcal{X}, M)).$$

When $\mathcal{X} := (X_i \to X)_{i\in I}$ is a family of morphisms in a site $\mathcal{C}$ and $\underline{\mathcal{X}} := (\underline{X}_i \to \underline{X})_{i\in I}$, we shall write

$$\check{\mathcal{C}}^\bullet(\mathcal{X}, M) := \check{\mathcal{C}}^\bullet(\underline{\mathcal{X}}, M), \quad \check{\mathcal{C}}^\bullet(\mathcal{X}^+, M) := \check{\mathcal{C}}^\bullet(\underline{\mathcal{X}}^+, M), \quad \check{\mathrm{H}}^n(\mathcal{X}, M) := \check{\mathrm{H}}^n(\underline{\mathcal{X}}, M).$$

Exercise 7.40 Show that if $\mathcal{X} := (X_i \to X)_{i\in I}$ is a family of morphisms in a site $\mathcal{C}$ with fibered products and M a sheaf of abelian groups on $\mathcal{C}$, then for $n \geq 0$,

$$\check{\mathcal{C}}^n(\mathcal{X}, M) = \check{\mathcal{C}}^n(\mathcal{X}^+, M) = \prod_{i_0,\ldots,i_n} M(X_{i_0} \times_X \cdots \times_X X_{i_n}).$$

Proposition 7.3.3 If a morphism $X_0 \twoheadrightarrow X$ in $\mathcal{T}$ has a section s and M is an abelian group of $\mathcal{T}$, then $\check{\mathcal{C}}^\bullet((X_0 \to X)^+, M)$ is homotopically trivial. The homotopy has the form $f \mapsto f \circ s_n$ with $s_n : X_n \to X_{n+1}$.

Proof. We consider for all $n \geq -1$ the morphism

$$s_n : s \times_X \mathrm{Id} : X_n \to X_{n+1}$$

(so that $s_{-1} = s$). It induces a morphism

$$s_n : \mathbb{Z}\cdot X_n \to \mathbb{Z}\cdot X_{n+1}$$

and we have for all $n \geq -1$, $\mathrm{Id}^n = s^{n+1} \circ d^n + d^{n-1} \circ s^n$. In other words, $\mathbb{Z}\cdot X_\bullet^+$ is homotopically trivial and so is $\check{\mathcal{C}}^\bullet((X_0 \to X)^+, M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\cdot X_\bullet^+, M)$. ∎

If $\mathcal{C}$ is any category and we endow $\widehat{\mathcal{C}}$ with its canonical topology, we shall consider $\mathrm{R}\Gamma(T, M)$ for any presheaf of sets T and any presheaf of abelian group M.

Exercise 7.41 Show that if $X \in \mathcal{C}$, then $\mathrm{H}^n(h_X, M) = 0$ for $n \neq 0$.

Proposition 7.3.4 If $\mathcal{C}$ is any category and R is the sieve generated by a family $\mathcal{X} := (X_i \to X)_{i \in I}$, then

$$\mathrm{R}\Gamma(R, \mathcal{M}) \simeq \check{\mathcal{C}}^\bullet(\mathcal{X}, \mathcal{M})$$

for all abelian presheaf $\mathcal{M}$ on $\mathcal{C}$ (and therefore $\mathrm{R}^n\Gamma(R, \mathcal{M}) \simeq \check{\mathrm{H}}^n(\mathcal{X}, \mathcal{M})$).

Proof. The family $\mathcal{X} := (h_{X_i} \to R)_{i \in I}$ is a covering in $\widehat{\mathcal{C}}$. It therefore follows from lemma 7.3.2 that $\mathbb{Z} \cdot h_{\mathcal{X}_\bullet} \simeq \mathbb{Z} \cdot R$. This is a projective resolution and therefore

$$\begin{aligned} \mathrm{R}\Gamma(R, \mathcal{M}) &\simeq \mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z} \cdot R, \mathcal{M}) \\ &\simeq \mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z} \cdot h_{\mathcal{X}_\bullet}, \mathcal{M}) \\ &\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot h_{\mathcal{X}_\bullet}, \mathcal{M}) \\ &= \check{\mathcal{C}}^\bullet(\mathcal{X}, \mathcal{M}). \end{aligned}$$

■

7.3.3 Čech cohomology

Let $\mathcal{C}$ be a site with fibered products.

We consider the inclusion functor

$$\mathcal{H} : \widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b}) \hookrightarrow \widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})$$

(so that $\mathcal{H}(\mathcal{M})$ denotes the sheaf $\mathcal{M}$ seen as a presheaf) and write for all $n \in \mathbb{Z}$ and all complex $\mathcal{M}^\bullet$ of abelian sheaves, $\mathcal{H}^n(\mathcal{M}^\bullet) = \mathrm{R}^n\mathcal{H}(\mathcal{M}^\bullet)$.

Exercise 7.42 Show that if $X \in \mathcal{C}$, then $\Gamma(X, \mathrm{R}\mathcal{H}(\mathcal{M}^\bullet)) = \mathrm{R}\Gamma(X, \mathcal{M}^\bullet)$ and therefore $\Gamma(X, \mathcal{H}^n(\mathcal{M}^\bullet)) = \mathrm{H}^n(X, \mathcal{M}^\bullet)$.

Solution. The functor $\mathcal{H} : \widetilde{\mathcal{C}}(\mathbf{A}\mathrm{b}) \hookrightarrow \widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})$ has an right adjoint. Therefore, it preserves injectives. Moreover, the functor $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ is exact on presheaves. We can apply corollary 7.2.13. ■

Exercise 7.43 Show that if $\mathcal{M}$ is an abelian sheaf, then $\widetilde{\mathcal{H}^n(\mathcal{M})} = 0$ for $n \neq 0$.

Exercise 7.44 Show that if $\mathcal{X}$ is family of morphisms of $\mathcal{C}$ and $\mathcal{M}$ is a sheaf on $\mathcal{C}$, then $\check{\mathrm{H}}^n(\mathcal{X}, \mathcal{H}(M)) = \check{\mathrm{H}}^n(\mathcal{X}, M)$.

Recall no that there exists a left exact functor (see (3.1))

$$\check{\mathcal{H}} : \widehat{\mathcal{C}}(\mathbf{A}\mathrm{b}) \to \widehat{\mathcal{C}}(\mathbf{A}\mathrm{b}), \quad \forall X \in \mathcal{C}, \quad \check{\mathcal{H}}(\mathcal{M})(X) = \varinjlim_{R \in J(X)} \mathrm{Hom}(R, \mathcal{M})$$

such that $\check{\mathcal{H}}(\check{\mathcal{H}}(\mathcal{M})) = \mathcal{H}(\widetilde{\mathcal{M}})$. We shall write $\check{\mathcal{H}}^n(\mathcal{M}^\bullet) = \mathrm{R}^n\check{\mathcal{H}}(\mathcal{M}^\bullet)$.

Exercise 7.45 Show that $\check{\mathcal{H}}(\mathcal{H}^n(\mathcal{M})) = 0$ for $n \neq 0$.

Definition 7.3.5 If $\mathcal{M}^\bullet$ is a complex of abelian presheaves on $\mathcal{C}$, its nth *Čech cohomology group* on X is

$$\check{\mathrm{H}}^n(X, \mathcal{M}^\bullet) := \Gamma(X, \check{\mathcal{H}}^n(\mathcal{M}^\bullet)).$$

Exercise 7.46 Show that if $X \in \mathcal{C}$ and $\mathcal{M}$ is an abelian presheaf, then

$$\check{\mathrm{H}}^n(X, \mathcal{M}) = \varinjlim_{R \in J(X)} \mathrm{H}^n(R, \mathcal{M}) = \varinjlim_{\mathcal{X} \in \mathrm{Cov}(X)} \check{\mathrm{H}}^n(\mathcal{X}, \mathcal{M}).$$

Exercise 7.47 Show that $\check{\mathrm{H}}^n([0,1], M) = 0$ for $n \neq 0$ if M is a constant abelian group.

Solution. Any open covering of $[0,1]$ has a refinement of the form $[0,1] = \bigcup_{k=0}^r I_k$ where I_k is an interval and $I_k \cap I_{k-1} = J_k$ is also an interval for $k = 1, \dots r$ (make a pictutre). Then, the augmented Čech complex

$$\begin{array}{ccccc} M & \longrightarrow & M^{r+1} & \longrightarrow & M^r \\ & & (s_k) & \longmapsto & (s_{k+1} - s_k) \end{array}$$

is acyclic. ■

We shall apply the above definition when $\mathcal{M}^\bullet$ is a complex of abelian *sheaves* so that

$$\check{\mathcal{H}}^n(\mathcal{M}^\bullet) := \check{\mathcal{H}}^n(\mathcal{H}(\mathcal{M}^\bullet)) \quad \text{and} \quad \check{\mathrm{H}}^n(X, \mathcal{M}^\bullet) := \check{\mathrm{H}}^n(X, \mathcal{H}(\mathcal{M}^\bullet)).$$

Theorem 7.3.6 — Cartan-Leray. If $\mathcal{X}$ is a covering of X and $\mathcal{M}$ is an abelian sheaf, then there exists a spectral sequence

$$E_2^{p,q} := \check{\mathrm{H}}^p(\mathcal{X}, \mathcal{H}^q(\mathcal{M})) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{M}).$$

Proof. If R denote the sieve generated by $\mathcal{X}$, then $\Gamma(X, \mathcal{M}) = \Gamma(R, \mathcal{H}(\mathcal{M}))$. Since $\mathcal{H}$ preserves injectives, the spectral sequence is obtained from corollary 7.2.21 and proposition 7.3.4. ■

Corollary 7.3.7 Assume $\mathrm{H}^q(\mathcal{X}_k, \mathcal{M}) = 0$ for all $q \neq 0$ and $k \in \mathbb{N}$. Then, there exists an isomorphism

$$\forall n \in \mathbb{N}, \quad \check{\mathrm{H}}^n(\mathcal{X}, M) \simeq \mathrm{H}^n(X, \mathcal{M}).$$

Proof. Assume $q \neq 0$. For all $k \in \mathbb{N}$, we have $\Gamma(\mathcal{X}_k, \mathcal{H}^q(\mathcal{M})) = \mathrm{H}^n(X, \mathcal{M}) = 0$ and therefore $\check{\mathcal{C}}(\mathcal{X}, \mathcal{H}^q(\mathcal{M})) = 0$ so that $E_2^{p,q} = 0$. ■

Corollary 7.3.8 If $X \in \mathcal{C}$ and $\mathcal{M}$ is an abelian sheaf, then there exists a spectral sequence

$$E_2^{p,q} := \check{\mathrm{H}}^p(X, \mathcal{H}^q(\mathcal{M})) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{M}).$$

Proof. Filtered direct limits are exact and therefore preserve spectral sequences. ■

Exercise 7.48 Show that $\check{\mathrm{H}}^1(X, \mathcal{M}) = \mathrm{H}^1(X, \mathcal{M})$.

Solution. We consider the spectral sequence from corollary 7.3.8. It follows from exercise 7.45 that $E_2^{0,1} = 0$. Our assertion therefore follows from exercise 7.30 ■

Definition 7.3.9 An abelian sheaf $\mathcal{M}$ on $\mathcal{C}$ is said to be *acyclic* if $\mathcal{H}^n(\mathcal{M}) = 0$ for $n \neq 0$.

Be careful that this definition depends on the category $\mathcal{C}$ and not only on the topos $\widetilde{\mathcal{C}}$.

Proposition 7.3.10 An abelian sheaf $\mathcal{M}$ on $\mathcal{C}$ is acyclic if and only if $\check{\mathcal{H}}^n(\mathcal{M}) = 0$ for $n \neq 0$.

Proof. It follows from corollary 7.3.8 and exercise 7.42 that there exists a spectral sequence of presheaves

$$E_2^{p,q} := \check{\mathcal{H}}^p(\mathcal{H}^q(\mathcal{M})) \Rightarrow \mathcal{H}^{p+q}(\mathcal{M}).$$

The implication follows immediately. Conversely, it is sufficient to prove by induction on $n > 0$ that for all $p, q \in \mathbb{N}$ such that $0 < p + q \leq n$, we have $E_2^{p,q} = 0$. If this is the case and $0 < q \leq n$, then $\mathcal{H}^q(\mathcal{M}) = 0$, but then also $E_2^{p,q} = 0$ for all $p \in \mathbb{N}$ (and $0 < q \leq n$). Now, it follows from exercise 7.45 that $E_2^{0,q} = 0$ for $q \neq 0$ and from our hypothesis that $E_2^{p,0} = 0$ for $p \neq 0$. Therefore, our assertion is satisfied for $n = 1$ ($E_2^{0,1} = E_2^{1,0} = 0$) and extends from n to $n + 1$ ($E_2^{0,n+1} = 0$). ■

Exercise 7.49 Show that the following are equivalent:

1. $\mathcal{M}$ is acyclic on $\mathcal{C}$,
2. If $X \in \mathcal{C}$, then $\mathrm{H}^n(X, \mathcal{M}) = 0$ for $n \neq 0$,
3. If $\mathcal{X}$ is a covering family in $\mathcal{C}$, then $\check{\mathrm{H}}^n(\mathcal{X}, \mathcal{M}) = 0$ for $n \neq 0$ (equivalently $\check{\mathcal{C}}(\mathcal{X}^+, \mathcal{M})$ is an acyclic complex),
4. Iff $X \in \mathcal{C}$, then $\check{\mathrm{H}}^n(X, \mathcal{M}) = 0$ for $n \neq 0$.

Solution. (1) ⇔ (2) follows from exercise 7.42. Then, (1) ⇒ (3) follows from theorem 7.3.6. And (3) ⇒ (4) is obtained by taking the limit. Finally, (1) ⇔ (4) follows from the proposition. ■

Note that this is also equivalent to $\check{\mathcal{C}}(\mathcal{X}^+, \mathcal{M})$ being an acyclic complex.

If $\mathcal{M}$ is an abelian sheaf on a topological space X, we shall denote its cohomological groups by $\mathrm{H}^n_{\mathrm{sheaf}}(X, \mathcal{M})$ or $\mathrm{H}^n(X, \mathcal{M})$ when there is no ambiguity.

Proposition 7.3.11 If X is a compact Hausdorff space, then

$$\check{\mathrm{H}}^n(X,\mathcal{M}) \simeq \mathrm{H}^n(X,\mathcal{M}).$$

Proof. Classic. ∎

Exercise 7.50 Show that if S is a Stone space, then $\mathrm{H}^n(S,\mathcal{M}) = 0$ for $n \neq 0$.

Solution. Since S is a compact Hausdorff, we can use Čech cohomology. We saw in exercise 2.20 that any covering has a finite disjoint clopen refinement $\mathcal{S}$. It is therefore sufficient to show that $\check{\mathrm{H}}^n(\mathcal{S},\mathcal{M}) = 0$. But then, $\check{\mathcal{C}}^\bullet(\mathcal{S}, M)$ is concentrated in degree 0. ∎

Proposition 7.3.12 If $X = \varprojlim_{i\in I} X_i$ is a filetered limit of compact Hausdorff spaces and M is a constant abelian group, then

$$\mathrm{H}^n(X,M) \simeq \varinjlim_{i\in I} \mathrm{H}^n(X_i, M).$$

Proof. We can use Čech cohomology. If we denote by $\pi_i : X \to X_i$ the projection, then any covering of X has a refinement of the form $\pi_i^{-1}(\mathcal{X})$ for some covering $\mathcal{X}$ of some X_i. Since M is constant, we have

$$\check{\mathcal{C}}(\pi_i^{-1}(\mathcal{X}), M) \simeq \check{\mathcal{C}}(\mathcal{X}, M) \quad \text{so that} \quad \check{\mathrm{H}}^n(\pi_i^{-1}(\mathcal{X}), M) \simeq \check{\mathrm{H}}^n(\mathcal{X}, M).$$

We conclude with exercise 7.46. ∎

7.4 Morphisms of topos (optional)

7.4.1 Morphism

Recall that a morphism of topos $f = \mathcal{T} \to \mathcal{T}'$ is a couple of adjoint functors $f^{-1} : \mathcal{T}' \to \mathcal{T}$ (inverse image) and $f_* : \mathcal{T} \to \mathcal{T}'$ (direct image) with f^{-1} exact.

Exercise 7.51 Show that a morphism of topos induces an adjunction on abelian groups on both sides with exact inverse image.

Exercise 7.52 Show that

$$f_*\mathcal{H}om_{\mathbb{Z}}(f^{-1}M, N) = \mathcal{H}om_{\mathbb{Z}}(M, f_*N) \quad \text{and} \quad f^{-1}(M \otimes_{\mathbb{Z}} N) \simeq f^{-1}M \otimes_{\mathbb{Z}} f^{-1}N.$$

Exercise 7.53 Show that if $f : \mathcal{T} \to \mathcal{T}'$ is a morphism of topos, then f_* preserves injectives.

Solution. Follows from exercise 7.18. ∎

Exercise 7.54 Show that if $f : \mathcal{T} \to \mathcal{T}'$ is a morphism of topos, there exists canonical maps

$$\mathrm{H}^n(X', M^\bullet) \to \mathrm{H}^n(f^{-1}(X'), f^{-1}M^\bullet).$$

Solution. There exists a canonical map

$$\Gamma(X', M) \to \Gamma(X', f_*f^{-1}M) \simeq \Gamma(f^{-1}(X'), f^{-1}M).$$

Let $M^\bullet \to I^\bullet$ and $f^{-1}I^\bullet \to J^\bullet$ be two injective resolutions. Since f^{-1} is exact, the composite map $f^{-1}M^\bullet \to J^\bullet$ is also an injective resolution and therefore

$$\mathrm{R}\Gamma(X', M^\bullet) = \Gamma(X', I^\bullet) \to \Gamma(f^{-1}(X'), f^{-1}I^\bullet) \to \Gamma(f^{-1}(X'), J^\bullet) = \mathrm{R}\Gamma(f^{-1}(X'), f^{-1}M^\bullet)$$

■

Exercise 7.55 Show that if $\mathcal{T} \xrightarrow{f} \mathcal{T}' \xrightarrow{f'} \mathcal{T}'$ is a sequence of morphisms of topos, then $Rf'_* \circ \mathrm{R}f_* = \mathrm{R}(f' \circ f)_*$ and there is a spectral sequence

$$E_2^{p,q} = R^p f'_*(R^q f_*(M^\bullet)) \Rightarrow R^{p+q}(f' \circ f)_*(M^\bullet).$$

Solution. Our assertion follows from proposition 7.2.13 and corollary 7.2.21. ■

Exercise 7.56 Show that if $f : \mathcal{T} \to \mathcal{T}'$ is a morphism of topos and $X' \in \mathcal{T}'$, then there is a spectral sequence

$$E_2^{p,q} = H^p(X', R^q f_*(M^\bullet)) \Rightarrow \mathrm{H}^{p+q}(f^{-1}(X'), M^\bullet).$$

Proof. By adjunction, we have $\Gamma(X', f_*M) = \Gamma(f^{-1}(X'), M)$. ■

Proposition 7.4.1 If $f : \mathcal{T} \to \mathcal{T}'$ is a morphism of topos, then $\mathrm{R}^n f_*M$ is the sheaf associated to $X' \mapsto \mathrm{H}^n(f^{-1}(X'), M)$.

Proof. The morphism $\widehat{f} : \widehat{\mathcal{T}} \to \widehat{\mathcal{T}}$ induced on presheaves as well as sheafification are both exact. From $f_*M = \widehat{f}_*\mathcal{H}(M)$, we obtain $\mathrm{R}f_*M = \widehat{f}_*\mathrm{R}\mathcal{H}(M)$. It follows that $\mathrm{R}^n f_*M$ is the sheaf associated to $\widehat{f}_*\mathcal{H}^n(M)$. By adjunction, we have

$$\Gamma(X', \widehat{f}_*\mathcal{H}^n(M)) = \Gamma(f^{-1}(X'), \mathcal{H}^n(M)) = \mathrm{H}^n(f^{-1}(X'), M).$$ ■

7.4.2 Localization

Exercise 7.57 Show that if $\mathcal{T}$ is a topos and $X \in \mathcal{T}$, then the functor $j_X^{-1} : \mathbf{Ab}(\mathcal{T}) \to \mathbf{Ab}(\mathcal{T}_{/X})$ has an exact left adjoint $j_{X!} : \mathbf{Ab}(\mathcal{T}_{/X}) \to \mathbf{Ab}(\mathcal{T})$.

Solution. In the case $\mathcal{T} = \widehat{\mathcal{C}}$, the functor is given by the explicit formula $j_{S!}M(X) = \bigoplus_{s:X\to S} M(s)$. In general, one uses sheafification. ■

Be careful that the functors $j_{X!}$ on abelian groups and set are not compatible.

Exercise 7.58 Show that $j_{X!}\mathbb{Z} = \mathbb{Z}\cdot X$.

Solution. It follows from exercise 3.64 that $j_{X!}\mathbf{1}_X = X$ where $\mathbf{1}_X = \mathrm{Id}_X$ denotes the final object of $\mathcal{T}_{/X}$. Therefore, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(j_{X!}\mathbb{Z}, M) &\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, j_X^{-1}M)\\ &= \mathrm{Hom}(\mathbf{1}_X, j_X^{-1}M)\\ &= \mathrm{Hom}(j_{X!}\mathbf{1}_X, M)\\ &= \mathrm{Hom}(X, M)\\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\cdot X, M).\end{aligned}$$

■

Exercise 7.59 Show that $\mathrm{H}^n(\mathbf{1}_X, j_X^{-1}M^\bullet) = \mathrm{H}^n(X, M^\bullet)$.

Solution. We have

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, j_X^{-1}M) = \mathrm{Hom}_{\mathbb{Z}}(j_{X!}\mathbb{Z}, M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\cdot X, M).$$

Since j_X^{-1} has an exact adjoint, it preserves injectives and therefore

$$\mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}, j_X^{-1}M^\bullet) = \mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}\cdot X, M^\bullet)$$

It is then sufficient to take cohomology on both sides. ■

Exercise 7.60 Show that

$$\mathcal{H}om_{\mathbb{Z}}(M, N)(X) = \mathrm{Hom}_{\mathbb{Z}}(j_{X!}j_X^{-1}M, N)$$

$$= \mathrm{Hom}_{\mathbb{Z}}(j_X^{-1}M, j_X^{-1}N) = \mathrm{Hom}_{\mathbb{Z}}(M, j_{X*}j_X^{-1}N).$$

Exercise 7.61 Show that

$$j_{X!}(M \otimes_{\mathbb{Z}} j_X^{-1}N) = j_{X!}M \otimes_{\mathbb{Z}} N, \quad j_{X!}(j_X^{-1}M \otimes_{\mathbb{Z}} N) = M \otimes_{\mathbb{Z}} j_{X!}N$$

$$\text{and} \quad M\cdot X := M \otimes_{\mathbb{Z}} \mathbb{Z}\cdot X = j_{X!}j_X^{-1}M.$$

7.4.3 Topological spaces

Recall that, if X is a topological space, then there exists various morphisms of topos

$$\widetilde{\mathbf{Top}} \xleftarrow[j_X]{} \widetilde{\mathbf{Top}_{/X}} \underset{\psi_X}{\overset{\varphi_X}{\rightleftarrows}} \widetilde{\mathbf{Open}(X)}.$$

We shall call $M_X := \varphi_{X*}j_X^{-1}M$ the *realization* of an abelian sheaf M on $\mathbf{Top}$. Then we have a natural isomorphism

$$\mathrm{H}^n(X, M^\bullet) \simeq \mathrm{H}^n(X, M_X^\bullet).$$

Actually, both functors j_X^{-1} and $\phi_{X*} = \psi_X^{-1}$ induce equivalences on constant abelian groups and we shall not make any difference in this case.

Proposition 7.4.2 If $f \sim g : X \to Y$ are two homotopic continuous maps and M is a *constant* abelian group, then the induced maps

$$\mathrm{H}^n(Y, M) \to \mathrm{H}^n(X, M).$$

coincide.

Proof. Classic (see for example Schapira's course on Algebra and Topology). ■

Corollary 7.4.3 If $f : X \to Y$ is a homotopy equivalence and M is a *constant* abelian group, then

$$\mathrm{H}^n(Y, M) \simeq \mathrm{H}^n(X, M).$$

This applies in particular to a projection $p : X \times Y \to Y$ when X is contractile.

8. Condensed cohomology (optional)

8.1 Cohomology

8.1.1 On Stonean spaces

If M is a condensed abelian group and X is a condensed set, we can consider the cohomology groups $\mathrm{H}^n(X, M)$. We may write $\mathrm{H}^n_{\mathrm{cond}}(X, M)$ in order to remove any ambiguity. In the case X is a topological space (and M is a condensed abelian group), then we may write $\mathrm{H}^n(X, M) = \mathrm{H}^n(\underline{X}, M)$. Also, if M is a topological abelian group (and X a condensed set), we may simply write $\mathrm{H}^n(X, M) := \mathrm{H}^n(X, \underline{M})$. Finally, we may still denote by M the constant condensed abelian group associated to a usual abelian group M.

Proposition 8.1.1 Condensed abelian groups are acyclic on Stonean spaces.

Proof. We saw in lemma 6.1.3 that the functor $M \mapsto \Gamma(S, M)$ is exact on Stonean spaces (and has therefore no higher cohomlogy). ■

In other words, we always have $\mathrm{H}^n_{\mathrm{cond}}(S, M) = 0$ for $n \neq 0$ when S is Stonean. Using exercise exercise 7.49, this last result may also be deduced from the following:

Exercise 8.1 Show that, if M is a condensed abelian group and $\mathcal{S} := (S_i \hookrightarrow S)_{i=1}^r$ is a finite disjoint covering in **CH**aus, then $\check{\mathrm{H}}^n(\mathcal{S}, M) = 0$ for $n \neq 0$.

Solution. The canonical map $\coprod_{i=1} S_i \to S$, being an isomorphism, has a section. We can then use proposition 7.3.3. ■

Unfortunately, the category of Stonean spaces does not have fibered products and there is no Čech cohomology[1] on this site. It will be necessary to rely on Stone spaces.

[1]There exists a workaround through the theory of *hypercoverings*.

Exercise 8.2 Show that condensed abelian groups are acyclic on discrete spaces, and more generally on products $E \times S$ of a discrete space and a Stonean space.

Proof. Using exercises 7.38 and 4.5 and 3.49, we are reduced to the case of a Stonean. ■

8.1.2 On Stone spaces

Lemma 8.1.2 *Constant* condensed abelian groups are acyclic on Stone spaces.

Proof. As shown in exercise 7.49, it is sufficient to prove that if M is an abelian group and $f : S_0 \twoheadrightarrow S$ is a surjective map of Stone spaces, then the Čech complex

$$\check{\mathcal{C}}^\bullet((S_0 \to S)^+, \underline{M})$$

is acyclic (note that the case of a finite disjoint covering is taken care of by exercise 8.1). If S and S_0 are finite, this follows from proposition 7.3.3. In general, we can write $S_0 = \varprojlim_k S_{0k}$ with S_{0k} finite and we shall denote by S_{-1k} the image S_{0k} in S. Since $\underline{M}$ is discrete,

$$\check{\mathcal{C}}^\bullet((S_0 \to S)^+, \underline{M}) = \varinjlim \check{\mathcal{C}}^\bullet((S_{0k} \to S_{-1k})^+, \underline{M}).$$

Since filtered direct limits are exact, we are done. ■

In other words, we always have $\mathrm{H}^n_{\mathrm{cond}}(S, M) = 0$ for $n \neq 0$ when S is Stone and M is a discrete abelian group.

Exercise 8.3 Show that if $Q = \underline{\mathbb{R}}/\underline{\mathbb{R}^{\mathrm{disc}}}$, then $Q(S) := \mathcal{C}(S, \mathbb{R})/\mathcal{C}(S, \mathbb{R}^{\mathrm{disc}})$ when S is Stone.

Solution. There exists an exact sequence

$$0 \to \Gamma(S, \mathbb{R}^{\mathrm{disc}}) \to \Gamma(S, \mathbb{R}) \to \Gamma(S, Q) \to \mathrm{H}^1_{\mathrm{cond}}(S, \mathbb{R}^{\mathrm{disc}}) = 0.$$ ■

8.1.3 On locally compact spaces

We shall freely use here the notion of a morphism of topos.

Proposition 8.1.3 If X is a topological space, there exists a morphism of topos

$$c_X : \mathbf{Cond}_{/\underline{X}} \to \widetilde{\mathbf{Open}(X)}$$

given by

$$Y \mapsto \big(U \mapsto Y(U) := \mathrm{Hom}_{/\underline{X}}(\underline{U}, Y)\big) \quad \text{and} \quad \mathcal{F} \mapsto \underline{\mathcal{F}} := \varinjlim_{U \subset X, s \in \mathcal{F}(U)} \underline{U}.$$

Proof. It follows from exercise 4.6 that the functors are well defined. Moreover, inverse image is left exact because direct colimits are exact. Now, we have

$$\mathrm{Hom}_{/\underline{X}}(\underline{\mathcal{F}}, Y) = \varprojlim_{U \subset X, s \in \mathcal{F}(U)} \mathrm{Hom}_{/\underline{X}}(\underline{U}, Y).$$

In other words, a morphism $\underline{\mathcal{F}} \to Y$ is a compatible family of morphisms $\underline{s} : \underline{U} \to Y$ over $\underline{X}$ for all open subset U of X and $s \in \mathcal{F}(U)$. Conversely, a morphism $\mathcal{F} \to Y$ is a compatible family of maps $\mathcal{F}(U) \to \mathrm{Hom}_{/\underline{X}}(\underline{U}, Y), s \mapsto f_s$. This is the same thing and it follows that there exists an adjunction

$$\mathrm{Hom}_{/\underline{X}}(\underline{\mathcal{F}}, Y) \simeq \mathrm{Hom}(\mathcal{F}, Y). \qquad \blacksquare$$

Proposition 8.1.4 — Dyckhoff. If X is a locally compact Hausdorff space and M is a *discrete* abelian group, then

$$\mathrm{H}^n_{\mathrm{cond}}(X, \underline{M}) = \mathrm{H}^n_{\mathrm{sheaf}}(X, M).$$

Proof. Since $c_X^{-1}X = \underline{X}$, there exists a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{sheaf}}(X, \mathrm{R}^q c_{X*}\underline{M}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{cond}}(X, \underline{M}).$$

Since $c_{X*}\underline{M} = M$, it is sufficient to show that $\mathrm{R}^n c_{X*}\underline{M} = 0$ for $n \neq 0$. This is the sheaf associated to $U \mapsto \mathrm{H}^n_{\mathrm{cond}}(U, \underline{M})$. Since X is locally compact, the stalk of this sheaf at $x \in X$ is

$$\varinjlim_{x \in U} \mathrm{H}^n_{\mathrm{cond}}(U, \underline{M}) = \varinjlim_{x \in S} \mathrm{H}^n_{\mathrm{cond}}(S, \underline{M})$$

when U (resp. S) runs trough the open (resp. compact) neighbohoods of x. Thanks to exercise 3.53, it is sufficient to show that this stalk is zero. Fix some compact neighborhood K of x in X. Let $f : K_0 \to K$ be a surjective map with K_0 Stone. Let $S \subset K$ be a compact subset and $S_0 := f^{-1}(K_0)$. Then, S_n is a Stone space for all $n \in \mathbb{N}$ because a product of Stone, as well as a subspace of Stone is automatically Stone. It follows from corollary 7.3.7 and lemma 8.1.2 that

$$\mathrm{H}^n_{\mathrm{cond}}(S, \underline{M}) = \check{\mathrm{H}}^n(S_\bullet, \underline{M}).$$

Since filtered colimits are exact, we are reduced to the case $S = \{x\}$ in which case we can apply proposition 7.3.3. $\blacksquare$

8.2 Banach abelian groups

8.2.1 K-exactness

We recall that a *semi-norm*[2] on an abelian group M is a map $M \to \mathbb{R}_{\geq 0}, s \mapsto \|s\|$ satisfying $\|0\| = 0$, $\|s_1 + s_2\| \leq \|s_1\| + \|s_2\|$ and $\|-s\| = \|s\|$. The topology on M is defined via the semi-distance $\delta(s_1, s_2) = \|s_2 - s_1\|$. Semi-normed abelian groups form a subcategory of the category of all topological abelian groups. Continuous homomorphisms $u : M \to N$ form a semi-normed abelian group for $\|u\| := \sup_{s \in M} \|u(s)\|$. A semi-normed abelian group M is Hausdorff if and only if the semi-norm is a *norm* : $\|s\| = 0 \Leftrightarrow s = 0$. A *Banach* abelian group is a complete normed abelian group. Banach abelian groups form a reflexive subcategory of semi-normed abelian groups with reflection $M \mapsto \widehat{M}$.

[2]It is actually sufficient to require that $\|-\|\ : M \to \mathbb{R}$ satisfies $\|0\| \leq 0$, $\|s_1 + s_2\| \leq \|s_1\| + \|s_2\|$ and $\|-s\| \leq \|s\|$.

Definition 8.2.1 A complex $M^\bullet$ of semi-normed abelian groups is said to be *K-bounded exact* at M^n for some $K \in \mathbb{R}$ if

$$\forall s \in M^n, \forall \epsilon > 0, \exists s' \in M^{n-1}, \quad \|s - d^{n-1}s'\| \leq K\|d^n s\| + \epsilon.$$

The complex is said to be *K-bounded acyclic* if it is K-bounded exact at all M^n.

In practice, we shall not use superscripts unless necessary and simply write d for d^n.

Exercise 8.4 Let $M^\bullet = \varinjlim_k M_k^\bullet$ a direct colimit of complexes of semi-normed abelian groups with isometric transitions maps. Show that, if each $M_k^\bullet$ is K-bounded exact in M_k^n, then $M^\bullet$ is K-bounded exact in M^n.

Lemma 8.2.2 A complex $M^\bullet$ of semi-normed abelian groups is K-bounded exact at M^n if and only if $\widehat{M}^\bullet$ is K-bounded exact at $\widehat{M}^n$.

Proof. We may clearly replace $M^\bullet$ with its Hausdorff quotient and assume that $M^\bullet \subset \widehat{M}^\bullet$. For the implication, fix $\epsilon > 0$ and $s \in \widehat{M}^n$. There exists $t \in M^n$ such that

$$\|s - t\| \leq \frac{\epsilon}{2(1 + K\|d^n\|)}$$

Now, there exists $s' \in M^{n-1}$ such that $\|t - ds'\| \leq K\|dt\| + \epsilon/2$. It follows that

$$\begin{aligned}
\|s - ds'\| &\leq \|s - t\| + \|t - ds'\| \\
&\leq \|s - t\| + K\|dt\| + \epsilon/2 \\
&\leq \|s - t\| + K\|ds\| + K\|ds - dt\| + \epsilon/2 \\
&\leq K\|ds\| + \|s - t\| + K\|d^n\|\|s - t\| + \epsilon/2 \\
&\leq \epsilon.
\end{aligned}$$

Conversely, if $s \in M^n$ then there exists $t' \in \widehat{M}^{n-1}$ such that $\|s - dt'\| \leq K\|ds\| + \epsilon/2$. Then, there exists $s' \in M^{n-1}$ such that $\|s' - t'\| \leq \epsilon/(2\|d^{n-1}\|)$ and therefore

$$\|s - ds'\| \leq \|s - dt'\| + \|d(s' - t')\| \ \leq K\|ds\| + \epsilon/2 + \|d^{n-1}\|\|s' - t'\| \leq \epsilon. \quad \blacksquare$$

Proposition 8.2.3 If a complex $M^\bullet$ of Banach abelian groups is K-bounded exact at M^{n-1} and M^n, then it is exact at M^n.

Proof. Let $s \in M^n$ such that $ds = 0$. Then, there exists $s'_i \in M^{n-1}$ and $s''_i \in M^{n-2}$ such that

$$\|s - ds'_i\| \leq \frac{1}{2^{i+2}K} \quad \text{and} \quad \|s'_{i+1} - s'_i - ds''_i\| \leq K\|d(s'_{i+1} - s'_i)\| + \frac{1}{2^{i+1}}.$$

We have

$$\|d(s'_i - s'_{i+1})\| \leq \|s - ds'_{i+1}\| + \|s - ds'_i\| \leq \frac{1}{2^{i+1}K}.$$

Therefore, if we set $t'_i = s'_i - \sum_{j<i} ds''_j$, we see that

$$\begin{aligned}\|t'_{i+1} - t'_i\| &= \|s'_{i+1} - s'_i - ds''_i\| \\ &\leq K\|d(s'_{i+1} - s'_i)\| + \frac{1}{2^{i+1}}. \\ &=\leq \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}}. \\ &\leq \frac{1}{2^i}.\end{aligned}$$

It follows that $t'_i \to t' \in M^{n-1}$. Now we have $ds'_i \to s$ (by definition) and $dt'_i \to dt'$ bu continuity. Since $ds'_i = dt'_i$, it follows that $dt' = s$. ■

Corollary 8.2.4 A K-acylcic complex $M^\bullet$ of Banach abelian groups is acyclic. ■

8.2.2 On Stone spaces

Exercise 8.5 Show that if S is compact Hausdorff and M is a (semi-) normed (resp. Banach) abelian group, then so is $\mathcal{C}(S, M)$ and $\|f\| = \sup_{x\in S} \|f(x)\|$. Show that if S is a Stone space, then (the image of) [a] $\mathcal{C}(S, M^{\mathrm{disc}})$ is dense in $\mathcal{C}(S, M)$.

[a] The induced topology is not the compact-open topology.

Solution. The first assertion follows from exercise 8.5. Now, if $f : S \to M$ is a continuous map and $\epsilon > 0$, then the open covering $S = \bigcup_{s\in M} f^{-1}(\mathbb{B}(s, \epsilon^-))$ has a finite disjoint clopen refinement $S = \coprod_{i=1}^r S_i$. For each $i = 1, \dots, r$, there exists $s_i \in M$ such that $S_i \subset f^{-1}(\mathbb{B}(s_i, \epsilon^-))$ and we set $g(x) = s_i$ for $x \in S_i$. Then, $\|f - g\| < \epsilon$. ■

Lemma 8.2.5 If M is a semi-normed abelian group and $f : S_0 \twoheadrightarrow S$ is a continuous surjective map of Stone spaces, then the augmented Čech complex

$$\check{\mathcal{C}}^\bullet((S_0 \to S)^+, \underline{M})$$

is 1-bounded acyclic.

Proof. We can write $S_0 = \varprojlim_k S_{0k}$ with S_{0k} finite and we shall denote by S_{-1k} the image S_{0k} in S. Then, we consider

$$\varinjlim \check{\mathcal{C}}^\bullet((S_{0k} \to S_{-1k})^+, \underline{M}) \simeq \check{\mathcal{C}}^\bullet((S_0 \to S)^+, \underline{M}^{\mathrm{disc}}) \to \check{\mathcal{C}}^\bullet((S_0 \to S)^+, \underline{M}).$$

The direct limit topology on $\check{\mathcal{C}}^\bullet((S_0 \to S)^+, \underline{M}^{\mathrm{disc}})$ coincides with the induced topology (this is the sup-norm topology). Using exercise 8.5 and 8.2.2, we see that it is sufficient to show that $\check{\mathcal{C}}^\bullet((S_0 \to S)^+, \underline{M}^{\mathrm{disc}})$ is 1-bounded acyclic (for this topology). Now, thanks to lemma 8.4, we are reduced to the case S, S_0 finite. In this case, we know from proposition 7.3.3 that the complex $\check{\mathcal{C}}^\bullet((S0 \to S)^+, \underline{M})$ is homotopically trivial. Moreover, the homotopy has the form $h_n : f \mapsto f \circ k_n$ for some $k_n : S_n \to S_{n+1}$ and therefore $\|h_n\| \leq 1$. Since $\mathrm{Id} = h \circ d + d \circ h$, we finally obtain that, if $s \in M^n$, then $\|s - dhs\| \leq \|ds\|$. ■

Proposition 8.2.6 A Banach abelian group is acyclic on Stone spaces.

Proof. According to exercise 7.49, we have to show that if M is a Banach abelian group and $f : S_0 \twoheadrightarrow S$ is a surjective map of Stone spaces, then the augmented Čech complex

$$\check{\mathcal{C}}^\bullet((S_0 \to S)^+, \underline{M})$$

is acyclic. This follows from corollary 8.2.4 and lemma 8.2.5. ■

8.2.3 Banach spaces

Theorem 8.2.7 — Tietze. Let X is a normal topological space, $K \subset X$ a compact subset and V a real Banach space, then $\mathcal{C}(X, V) \twoheadrightarrow \mathcal{C}(K, V)$ is surjective.

Proof. Sketch[3]. When V is finite dimensional or $V = \ell_\infty(\mathbb{R})$, this reduces to the case $V = \mathbb{R}$ where this is a classical result in the spirit of Urysohn's lemma. In general, since a compact subset of a Banach space is separable, one may assume that V is separable. Since all infinite dimensional separable Banach spaces are isomorphic, we may assume that $V = c_0(\mathbb{R})$ (space of null sequences). This is a Lipschitz retract to $\ell_\infty(\mathbb{R})$ where we know that the result holds. ■

Theorem 8.2.8 Real Banach spaces are acyclic on compact Hausdorff spaces.

Proof. Let V be a Banach space. Thanks to propositions 7.3.7 and 8.2.6, and corollary 8.2.4, it is sufficient to show that, if $S_0 \to S$ is a continuous surjection from a Stone space, then the augmented Čech complex $\check{\mathcal{C}}^\bullet((S_0 \to S)^+, V)$ is 1-acyclic. We fix some $\epsilon > 0$. We denote by $\pi_n : S_n \to S$ the canonical map and apply for all $x \in S$, lemma 8.2.5 to $\pi_0^{-1}(x) \to x$. If $f : S_n \to V$ is a continuous map and f_x denote its restriction to $\pi_n^{-1}(x)$, then there exists $g_x : \pi_{n-1}^{-1}(x) \to V$ such that (use exercise 8.6)

$$\|f_x - dg_x\| \le \|df_x\| + \epsilon/2 \le \|df\| + \epsilon/2.$$

By Tietze's theorem, g_x extends to $\widetilde{g}_x$ on S_{n-1}. By continuity, there exists a neighborhood V_x of x such that

$$\|f - d\widetilde{g}_x\|_{\pi_n^{-1}(V_x)} < \|df\| + \epsilon.$$

We can pick-up $V_i := V_{x_i}$ for $i = 1, \ldots, r$ that cover S and we write $g_i := \widetilde{g}_{x_i}$. Choose a subordinated partition of unity $\{\phi_i\}_{i=1}^r$ and set $g = \sum_{i=1}^r (\phi_i \circ \pi_{n-1}) g_i$. Then,

$$f = \sum_{i=1}^r (\phi_i \circ \pi_n) f \quad \text{and} \quad dg = \sum_{i=1}^r (\phi_i \circ \pi_{n-1} \circ d) dg_i = \sum_{i=1}^r (\phi_i \circ \pi_n) dg_i$$

[3]This is supposed to be a result of Dugundji but I have not been able to provide a reference.

It follows that

$$\begin{aligned}\|f - dg\| &= \left\| \sum_{i=1}^{r} (\phi_i \circ \pi_n)(f - dg_i) \right\| \\ &\leq \sum_{i=1}^{r} \phi_i \|(f - dg_i)\|_{\pi_n^{-1}(V_i)} \\ &\leq \|df\| + \epsilon.\end{aligned}$$

■

Exercise 8.6 Let $\pi : Y \to X$ be a closed continuous map and $A \subset X$. Then the subsets of the form $\pi^{-1}(U)$ when U is a neighborhood of A in X form a basis of neighborhoods of $\pi^{-1}(A)$ in Y.

Solution. Let V be an open neighborhood of $\pi^{-1}(A)$ and $U := X \setminus \pi(Y \setminus V)$. Then, $\pi^{-1}(A) \subset \pi^{-1}(U) \subset V$. ■

We used above the notion of *subordinated partition of unity* for an open cover $\{V_i\}_{i\in I}$ of a topological space S: this is a family of continuous maps $\phi_i : S \to [0,1]$ with support in V_i such that $1_S = \sum_{i\in I} \phi_i$.

Exercise 8.7 Show that a topological space is paracompact Hausdorff (resp. normal) if and only if any open cover (resp. locally finite open cover) admits a subordinated partition of unity.

8.3 Extensions of abelian groups

8.3.1 First computations

Lemma 8.3.1 If M, N are two condensed abelian groups and S is a Stonean space, then

$$\mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(M,N)(S) \simeq \mathrm{Ext}^n_{\mathbb{Z}}(M \cdot \underline{S}, N).$$

Proof. Since S is Stonean, then $\mathbb{Z} \cdot \underline{S}$ is projective, which implies that the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot \underline{S}, -)$ is exact. We apply exercise 7.35 to the natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(M \cdot S, N) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot \underline{S}, \mathcal{H}\mathrm{om}_{\mathbb{Z}}(M,N))$$

and obtain

$$\mathrm{Ext}^n_{\mathbb{Z}}(M \cdot \underline{S}, N) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot \underline{S}, \mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(M,N)) \simeq \mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(M,N)(S).$$

■

Exercise 8.8 Show that, if X is a condensed set, N a condensed abelian group and S is a Stonean space, then

$$\mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(\mathbb{Z} \cdot X, N)(S) \simeq \mathrm{H}^n(X \times \underline{S}, N).$$

Exercise 8.9 Show that if S is a Stonean space and N a condensed abelian group, then

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\mathbb{Z} \cdot \underline{S}, N) \simeq N(S).$$

Solution. Follows from the fact that $\mathbb{Z} \cdot \underline{S}$ is projective. ■

Exercise 8.10 Show that if E is a discrete topological space and N a condensed abelian group, then

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\mathbb{Z} \cdot \underline{E}, N) \simeq N^E.$$

Solution. Exercises 8.8 and 8.2 show that, if S is a Stonean space, then

$$\mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(M, N)(S) \simeq \mathrm{H}^n(E \times \underline{S}, N) = 0$$

for $n \neq 0$. ■

Exercise 8.11 Show that if M, N are two discrete abelian groups, then

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{N}) = \underline{\mathrm{RHom}_{\mathbb{Z}}(M, N)} \quad \text{and } \underline{M} \otimes^L_{\mathbb{Z}} \underline{N} \simeq \underline{M \otimes^L_{\mathbb{Z}} N}$$

Solution. Since $\underline{M}$ is the constant sheaf associated to M, the functor $M \mapsto \underline{M}$ is exact on discrete abelian groups. Moreover, it follows from exercise 8.10 that constant free abelian groups are acyclic for the functor $\mathcal{H}\mathrm{om}_{\mathbb{Z}}(-, \underline{N})$. We also know that free abelian groups are flat. Our assertion therefore follows from proposition 6.2.2 (resp. exercise 7.52) and corollary 7.35. ■

Exercise 8.12 Show that if S is a compact Hausdorff space and N a real Banach space, then

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\mathbb{Z} \cdot \underline{S}, \underline{N}) \simeq \underline{\mathcal{C}(S, N)}.$$

Solution. It follows from exercise 8.8 and theorem 8.2.8 that, if T is Stonean, we have

$$\mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(\mathbb{Z} \cdot \underline{S}, \underline{N})(T) \simeq \mathrm{H}^n(\underline{S} \times \underline{T}, \underline{N}) = 0.$$ ■

8.3.2 Breen-Deligne resolutions

The following was attributed to Deligne but no full proof was available before Scholze gave one in the appendix to the fourth lecture on condensed mathematics:

Theorem 8.3.2 — Breen-Deligne. If M is an abelian group (of a topos), then there exists a natural left resolution $F(M)_\bullet \to M$ with

$$F(M)_n = \bigoplus_{i=1}^{r_n} \mathbb{Z} \cdot M^{s_{n,i}}.$$

Proof. This is difficult and will not be proved here. ■

It is worth describing the lower part of the (augmented) complex:

$$\cdots \to \mathbb{Z}\cdot M^2 \oplus \mathbb{Z}\cdot M^3 \to \mathbb{Z}\cdot M^2 \to \mathbb{Z}\cdot M \to M$$

We have

1. $d_0 : [s] \mapsto s$,
2. $d_1 : [s_1, s_2] \mapsto -[s_1] + [s_1 + s_2] - [s_2]$,
3. $d_2 : \begin{cases} [s_1, s_2] \mapsto [s_1, s_2] \quad - [s_2, s_1] \\ [s_1, s_2, s_3] \mapsto -[s_2, s_3] \quad + [s_1 + s_2, s_3] - [s_1, s_2 + s_3] \quad + [s_1, s_2]. \end{cases}$

Exercise 8.13 Show that this is indeed the lower terms of a resolution.

Exercise 8.14 Show that, if M, N are two abelian groups of a topos, then there exists a natural spectral sequence

$$E_1^{p,q} = \bigoplus_{i=1}^{r_n} \mathrm{H}^q(M^{s_{p,i}}, N) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbb{Z}}(M, N)$$

Solution. Apply proposition 7.2.20 to $\mathrm{Hom}(-, N)$ and the Breen-Deling resolution. We have $\mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}\cdot M^{s_{p,i}}, N) = \mathrm{H}^q(M^{s_{p,i}}, N)$ ■

We shall need later the following (whose proof is in the same sprit as the proof of Breen-Deligne theorem):

Lemma 8.3.3 If M is an abelian group (in a topos), then multiplication by $p \in \mathbb{Z}$ on the Breen-Deligne resolution $F(M)_\bullet$ and the map $[p]$ induced by multiplication by p on M are naturally homotopic.

Proof. Assume first that $F(M)_\bullet$ is a *projective* resolution of M. Since $p = [p]$ on M, then $[p] = p$ in the derived category on $F(M)_\bullet$ and it follows from proposition 7.2.5 (dual version) that $[p] \sim p$ on $F(M)_\bullet$. Consider now the topos $\mathcal{T} = \widehat{\mathbf{A}\mathrm{b}^{\mathrm{op}}}$ and the abelian group $M := h^{\mathbb{Z}}$ of $\mathcal{T}$. If $s \in \mathbb{N}$ then $M^s = h^{\mathbb{Z}^s}$. If N is a presheaf of abelian groups on $\mathbf{A}\mathrm{b}^{\mathrm{op}}$, then

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\cdot M^s, N) = \mathrm{Hom}(M^s, N) = \mathrm{Hom}(h^{\mathbb{Z}^s}, N) = N(\mathbb{Z}^s).$$

Since limits and colimits are computed argument by argument on presheaves, $\mathbb{Z}\cdot M^s$ is a projective abelian group of $\mathcal{T}$. It follows that the assertion is true in this case. Now, if M is any (usual) abelian group, we can specialize to $M = h^{\mathbb{Z}}(M)$. This extends to any category of presheaves and then to any category of sheaves by sheafification. ■

Proposition 8.3.4 If M, N are two condensed abelian groups and S is a Stonean space, then there exists a natural spectral sequence

$$E_1^{p,q} = \bigoplus_{i=1}^{r_n} \mathrm{H}^q(M^{s_{p,i}} \times \underline{S}, N) \Rightarrow \mathcal{E}\mathrm{xt}^{p+q}_{\mathbb{Z}}(M, N)(S)$$

Proof. Let $F(M)_\bullet$ be the Breen-Deligne resolution of M. Proposition 7.2.20 applied to $\mathrm{Hom}_{\mathbb{Z}}(-,N)$ and and $F(M)_\bullet \cdot \underline{S}$ provides a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}}(F(M)_p \cdot \underline{S}, N) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbb{Z}}(F(M)_\bullet \cdot \underline{S}, N).$$

On the one hand, we have

$$F(M)_p \cdot \underline{S} = \bigoplus_{i=1}^{r_n} \mathbb{Z} \cdot M^{s_{p,i}} \otimes_{\mathbb{Z}} \mathbb{Z} \cdot \underline{S} = \bigoplus_{i=1}^{r_n} \mathbb{Z} \cdot (M^{s_{p,i}} \times \underline{S})$$

and

$$\mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z} \cdot (M^{s_{p,i}} \times \underline{S}), N) = \mathrm{H}^q(M^{s_{p,i}} \times \underline{S}, N).$$

On the other hand, since $\mathbb{Z} \cdot S$ is flat, the morphism

$$F(M)_\bullet \cdot S = F(M)_\bullet \otimes_{\mathbb{Z}} \mathbb{Z} \cdot \underline{S} \to M \otimes_{\mathbb{Z}} \mathbb{Z} \cdot \underline{S} \simeq M \cdot \underline{S}$$

is a quasi-isomorphism and it therefore follows from lemma 8.3.1 that

$$\mathrm{Ext}^n_{\mathbb{Z}}(F(M)_\bullet \cdot \underline{S}, N) \simeq \mathrm{Ext}^n_{\mathbb{Z}}(M \cdot \underline{S}, N) \simeq \mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(M, N)(S).$$ ■

8.3.3 Applications

Proposition 8.3.5 If M is a finite dimensional real Banach space and N a discrete abelian group, then

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{N}) = 0.$$

Proof. Recall first from corollary 8.3.4 that, if S is a Stonean space, then there exists a natural spectral sequence

$$E_1^{p,q} = \bigoplus_{i=1}^{r_n} \mathrm{H}^q(\underline{M}^{s_{p,i}} \times \underline{S}, \underline{N}) \Rightarrow \mathcal{E}\mathrm{xt}^{p+q}_{\mathbb{Z}}(M, \underline{N})(S)$$

and we want to show that the abutment is zero. Now, let $q, s \in \mathbb{N}$. Thanks to corollary 7.4.3 and proposition 8.1.4, since M^s is contractible and $M^s \times S$ is locally compact Hausdorff, we have a natural isomorphism

$$\mathrm{H}^n(\underline{M}^s \times \underline{S}, \underline{N}) \simeq \mathrm{H}^n(\underline{S}, \underline{N}).$$

We may therefore assume that $M = 0$ and we are done. ■

More generally, if V is an $\underline{\mathbb{R}}$-module[4] and N a discrete abelian group, then

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(V, \underline{N}) = \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{R}}(V, \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\mathbb{R}, \underline{N})) = \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{R}}(V, 0) = 0.$$

[4]We haven't discussed this matter.

Proposition 8.3.6 If M is a compact Hausdorff abelian group and N a real Banach space, then

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{N}) = 0.$$

Proof. We consider again the spectral sequence

$$E_1^{p,q} = \bigoplus_{i=1}^{r_n} \mathrm{H}^q(\underline{M}^{s_{p,i}} \times \underline{S}, \underline{N}) \Rightarrow \mathcal{E}\mathrm{xt}_{\mathbb{Z}}^{p+q}(M, \underline{N})(S).$$

It follows from theorem 8.2.8 that $E_1^{p,q} = 0$ for $q \neq 0$ and we shall show that $E_2^{p,0} = 0$, or equivalently, that the complex of Banach spaces $K^\bullet$ with

$$K^n := \bigoplus_{i=1}^{r_n} \mathcal{C}(\underline{M}^{s_{n,i}} \times \underline{S}, N)$$

is acyclic. Now, we proved lemma 8.3.3 that the maps 2 and [2] induced by multiplication by 2 on N and M respectively are homotopic on the Breen-Deligne resolution: $2 - [2] = dh + hd$. Let $f \in K^n$ such that $df = 0$. Then, $2f - [2]^*f = dh_{n-1}^* f$ and therefore $f = \frac{1}{2}[2]^*f + d(\frac{1}{2}h_{n-1}^* f)$. By induction, we get

$$f = \frac{1}{2^n}[2^n]^*f + d\left(\sum_{k=1}^{n} \frac{1}{2^k} h_{n-1}^*([2^{k-1}]^*f)\right).$$

Since $\|[2]^*\| \leq 1$ and $\|h_{n-1}^*([2^{k-1}]^*f)\| \leq \|h_{n-1}^*\|\|f\|$ (so that the series below converges), we finally obtain

$$f = d\left(\sum_{k=1}^{\infty} \frac{1}{2^k} h_{n-1}^*([2^{k-1}]^*f)\right). \qquad \blacksquare$$

Exercise 8.15 Show that, if M and N are finite dimensional Banach spaces, then

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{N}) = \underline{\mathrm{Hom}_{\mathbb{R}}(M, N)}$$

Solution. We may assume that $M = N = \mathbb{R}$. Since $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{\mathbb{T}}, \underline{\mathbb{R}}) = 0$, we have

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{\mathbb{R}}, \underline{\mathbb{R}}) = \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{\mathbb{Z}}, \underline{\mathbb{R}}) \simeq \underline{\mathbb{R}}. \qquad \blacksquare$$

We shall need below the following elementary fact:

Exercise 8.16 In any topos, there exists a natural map

$$M \otimes_{\mathbb{Z}}^{L} \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(P, N) \to \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, P), N).$$

Solution. By adjunction, the identity

$$\mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, P) = \mathcal{H}\mathrm{om}_{\mathbb{Z}}(M, P) \quad (\text{resp. } \mathcal{H}\mathrm{om}_{\mathbb{Z}}(P, N) = \mathcal{H}\mathrm{om}_{\mathbb{Z}}(P, N))$$

provides a map

$$M \otimes_{\mathbb{Z}} \mathcal{H}om_{\mathbb{Z}}(M,P) \to P(\text{resp. } P \otimes_{\mathbb{Z}} \mathcal{H}om_{\mathbb{Z}}(P,N) \to N).$$

From this, we deduce

$$M \otimes_{\mathbb{Z}} \mathcal{H}om_{\mathbb{Z}}(M,P) \otimes_{\mathbb{Z}} \mathcal{H}om_{\mathbb{Z}}(P,N) \to P \otimes_{\mathbb{Z}} \mathcal{H}om_{\mathbb{Z}}(P,N) \to N,$$

and by adjunction again

$$M \otimes_{\mathbb{Z}} \mathcal{H}om_{\mathbb{Z}}(P,N) \to \mathcal{H}om_{\mathbb{Z}}(\mathcal{H}om_{\mathbb{Z}}(M,P),N)$$

We can then derive. ■

Recall that we denote by $\mathbb{T}$ the circle (the one dimensional torus).

Proposition 8.3.7 If M, N are two discrete abelian groups, then

$$\mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{M},\underline{\mathbb{T}}),\underline{N}) \simeq \underline{M \otimes^L_{\mathbb{Z}} N}[-1].$$

Proof. We first define the map

$$\begin{aligned}\underline{M \otimes^L_{\mathbb{Z}} N}[-1] &\simeq \underline{M} \otimes^L_{\mathbb{Z}} \mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\mathbb{Z}[-1],\underline{N}) \\ &\to \underline{M} \otimes^L_{\mathbb{Z}} \mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{\mathbb{T}},\underline{N}) \\ &\to \mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{M},\underline{\mathbb{T}}),\underline{N}).\end{aligned}$$

The first isomorphism comes from exercise 8.11, the second from the exact sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{\mathbb{T}} \to 0$$

(use exercise 6.5) and the last one from exercise 8.16. In order to prove that this is an isomorphism, we may assume that $M = \mathbb{Z} \cdot E$ is free. Then, it follows from exercises 8.10 and 4.10 that

$$\mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{M},\underline{\mathbb{T}}) = \mathcal{H}om(\underline{E},\underline{\mathbb{T}}) = \underline{\mathbb{T}^E}.$$

On the other hand, we have $M \otimes^L_{\mathbb{Z}} N \simeq (N \cdot E)[-1]$ and we are therefore reduced to showing that

$$\mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{\mathbb{T}^E},\underline{N}) \simeq \underline{N \cdot E}[-1].$$

The distinguished triangle

$$\mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{\mathbb{Z}},N) \to \mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{\mathbb{R}},N) \to \mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{\mathbb{T}},N) \xrightarrow{+}$$

provides thanks to proposition 8.3.5 an isomorphism

$$\mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{\mathbb{I}},N) \simeq \mathrm{R}\mathcal{H}om_{\mathbb{Z}}(\underline{\mathbb{Z}},N)[-1] \simeq N[-1].$$

The case where E is finite follows and we write now $E = \varinjlim E'$ when E' runs through the finite subsets. Thanks to the natural spectral sequence of corollary 8.3.4 (and the fact that filtered coimits are exact), we are reduced to show that for all $s \in \mathbb{N}$,

$$\mathrm{H}^q((\underline{\mathbb{T}}^s)^E \times \underline{S}, N) = \varinjlim_{E'} \mathrm{H}^q((\underline{\mathbb{T}}^s)^J \times \underline{S}, N)$$

This follows from propositions 8.1.4 and 7.3.12. ■

Exercise 8.17 Show that $\mathcal{RH}om_{\mathbb{Z}}(\underline{M}, \underline{N})$ is given by

$M \backslash N$	$\mathbb{Z}$	$\mathbb{R}$	$\mathbb{T}$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{R}$	$\mathbb{T}$
$\mathbb{R}$	0	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{T}$	$\mathbb{Z}[-1]$	0	$\mathbb{Z}$

8.3.4 Locally compact abelian groups

Proposition 8.3.8 If M is a locally compact Hausdorff abelian group, then

$$\underline{M^*} \simeq \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}).$$

We also have

$$\underline{M^*} \simeq \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{Z}})[1] \quad (\text{resp. } \underline{M^*} \simeq \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{R}}))$$

if M is compact (resp. Banach).

Proof. Since M is an extension of a discrete abelian group by the sum of a finite dimensional Banach space and a (connected) compact Hausdorff abelian group, it is sufficient to consider the case where it is of one of these types: discrete abelian group, finite dimensional Banach space or compact Hausdorff abelian group. Assume first M is discrete. Then, there exists an exact sequence $0 \to F' \to F \to M \to 0$ with F, F' free. The corresponding long exact sequence reads (use exercise 8.10)

$$0 \to \underline{M}^* \to \underline{F}^* \to \underline{F}'^* \to \mathcal{E}\mathrm{xt}^1_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}) \to 0$$

which implies that $\mathcal{E}\mathrm{xt}^1_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{T}}) = 0$. In the case M is compact, we can first apply the previous result to M^* so that

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M^*}, \underline{\mathbb{T}}) \simeq \underline{M}.$$

Then propositions 8.3.7 (applied to M^* and $\mathbb{Z}$) and 8.3.6 (applied to M and $\mathbb{R}$) provide

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{Z}}) \simeq \underline{M^*}[-1] \quad \text{and} \quad \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{R}}) = 0$$

which allows us to conclude. Finally, if M is a finite dimensional Banach space, we have know from proposition 8.3.5 and exercise 8.15 that

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{Z}}) = 0 \quad \text{and} \quad \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M}, \underline{\mathbb{R}}) = M^*.$$ ■

Exercise 8.18 Show that if M is a connected locally compact Hausdorff abelian group and N a discrete abelian group, then

$$\forall n \neq 1, \quad \mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(\underline{M}, \underline{N}) = 0.$$

Solution. Any connected locally compact Haudorff abelian group is the direct sum of a connected compact Haudorff abelian group and a finite dimensional Banach space.

Using proposition 8.3.5, we may therefore assume that M is compact. Since M is connected, its Pontryagin dual M^* is then torsion free and proposition 8.3.7 provides

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{M},\underline{N}) \simeq \underline{M^* \otimes_{\mathbb{Z}} N}[-1].$$ ■

Theorem 8.3.9 If M, N are two locally compact Hausdorff abelian groups, then

$$\forall n \neq 0, 1, \quad \mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(\underline{M},\underline{N}) = 0.$$

Proof. Since both M and N are an extension of a discrete abelian group by the sum of a finite dimensional Banach space and a connected compact Hausdorff abelian group, it is sufficient to consider the case where they are of one of these types: discrete abelian group, finite dimensional Banach space or connected compact Hausdorff abelian group. If M is discrete, there exists a two terms free resolution and the result follows from exercise 8.10. If N is discrete, then this follows from exercise 8.18. The case N is Banach is taken care of by propositions 8.3.6 and 8.15. So, we may assume now that N is connected compact Hausdorff. There exists a two terms resolution of the form $0 \to N \to \mathbb{T}^E \to \mathbb{T}^{E'} \to 0$. It is therefore sufficient to show that

$$\forall n \neq 0, \quad \mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(\underline{M},\underline{\mathbb{T}}^E) = \mathcal{E}\mathrm{xt}^n_{\mathbb{Z}}(\underline{M},\underline{\mathbb{T}})^E = 0.$$

But this follows from proposition 8.3.8. ■

It is also possible to treat non locally compact groups:

Exercise 8.19 Show that

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{\mathbb{R}^I},\underline{\mathbb{Z}}) = 0 \quad \text{and} \quad \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{Z}}(\underline{\mathbb{Z}^I},\underline{\mathbb{Z}}) = \underline{\mathbb{Z}\cdot I}.$$

Solution. The first assertion follows from the fact that $\underline{\mathbb{R}^I}$ is an $\underline{\mathbb{R}}$-module but we shall give a direct proof. Considering the first assertion, one can use Breen-Deligne resolution for some Stonean space S and it is sufficient to show that

$$\mathrm{H}^p((\underline{\mathbb{R}^I})^s \times \underline{S},\underline{\mathbb{Z}}) = \mathrm{H}^p(\underline{S},\underline{\mathbb{Z}})$$

when $s \in \mathbb{N}$. We may clearly assume $s = 1$. One can write $\mathbb{R}^I = \varinjlim \prod_{i\in I}[-n_i, n_i]$ (for the compact-open topology) and therefore

$$\mathrm{R}\Gamma(\underline{\mathbb{R}^I} \times \underline{S},\underline{\mathbb{Z}}) = \mathrm{R}\varprojlim \mathrm{R}\Gamma\left(\prod_{i\in I}\underline{[-n_i,n_i]} \times \underline{S},\underline{\mathbb{Z}}\right)$$

Now, we can use sheaf cohomology since $\prod_{i\in I}[-n_i, n_i]\times S$ is compact. But $\prod_{i\in I}[-n_i, n_i]$ is acyclic and therefore

$$\mathrm{R}\varprojlim \mathrm{R}\Gamma\left(\prod_{i\in I}[-n_i,n_i] \times S,\mathbb{Z}\right) = \mathrm{R}\varprojlim \mathrm{R}\Gamma\left(S,\mathbb{Z}\right) = \mathrm{R}\Gamma\left(S,\mathbb{Z}\right).$$

The second equality is then obtained from

$$\mathrm{R}\mathcal{H}\mathrm{om}(\underline{\mathbb{T}^I},\underline{\mathbb{Z}}) = \underline{\mathbb{Z}\cdot I}[-1].$$ ■

Note that

$$\mathcal{H}\mathrm{om}(\mathbb{R}^I,\mathbb{Z}) = k\mathcal{C}_{\mathbb{Z}}(k\mathbb{R}^I,\mathbb{Z}) \quad \text{and} \quad \mathcal{H}\mathrm{om}(\mathbb{Z}^I,\mathbb{Z}) = k\mathcal{C}_{\mathbb{Z}}(k\mathbb{Z}^I,\mathbb{Z})$$

so that $\mathcal{C}_{\mathbb{Z}}(k\mathbb{R}^I,\mathbb{Z}) = 0$ and $k\mathcal{C}_{\mathbb{Z}}(k\mathbb{Z}^I,\mathbb{Z}) = \mathbb{Z}\cdot I$.

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