The overconvergent site

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Abstract

We prove that rigid cohomology can be computed as the cohomology of a site analogous to the crystalline site. Berthelot designed rigid cohomology as a common generalization of crystalline and Monsky-Washnitzer cohomology. Unfortunately, unlike the former, the functoriality of the theory is not built-in. We define the "overconvergent site" which is functorially attached to an algebraic variety. We prove that the category of modules of finite presentation on this ringed site is equivalent to the category of overconvergent isocrystals on the variety. We also prove that their cohomology coincides.

Contents

Chapter 0

Introduction

In order to give an algebraic description of Betti cohomology, one can use de Rham cohomology which can then be interpreted as the cohomology of the infinitesimal site ([21]). The category of coefficients, locally trivial families of finite dimensional vector spaces, is replaced successively by coherent modules with integrable connections, and then, by modules of finite presentation. In the positive characteristic situation, there is no exact equivalent to Betti cohomology and de Rham cohomology has to be replaced by rigid cohomology (and modules with integrable connections by overconvergent isocrystals). We will define here the overconvergent site which plays in positive characteristic the role that the infinitesimal site plays in characteristic zero. A first hint at this approach is already in Berthelot's fundamental article ([10], 2.3.2. ii)) and this is actually the way I liked to define overconvergent isocrystals in my Ph.D. Thesis (see also [18], section 1.1 or [23], definition 7.1.1). Note that Arthur Ogus introduced the convergent site in [25] which is a satisfying solution as long as we are only interested in proper varieties. Of course, for proper smooth varieties, we can also use the crystalline site of Pierre Berthelot ([7]).

Beside its intrinsic interest, there are many reasons to look for such an interpretation of rigid cohomology. For example, we will get for free a Leray spectral sequence giving the overconvergence of the Gauss-Manin connection. Also, our setting should be well-suited to describe Besser's integration ([11]), Chiarellotto-Tsuzuki's descent theory ([16]) or the results of Atsushi Shiho on relative rigid cohomology. Finally, replacing schemes by log-schemes should give a comparison theorem with log-crystalline cohomology. Note that David Brown, who is a student of Bjorn Poonen, makes an essential use of the overconvergent site in his study of rigid cohomology of algebraic stacks ([14]). In order to avoid technical complications, we will not consider étale cohomology nor log-schemes (or algebraic stacks) here.

In our presentation, we will systematically replace rigid geometry with analytic geometry in the sense of Berkovich. I understand that it is unpleasant for those who are accustomed to Tate's theory and feel uncomfortable with Berkovich's. But there are several reasons for this choice. First of all, I really think that classical Tate's theory should be seen as a part of Berkovich theory (using rigid points and Grothendieck topology). Moreover, most young mathematicians start directly with Berkovich's approach. There is also a specific reason here: in the construction of rigid geometry, strict neighborhoods play an essential role; in Berkovich theory those are just usual neighborhoods. Finally, the notion of generic point that is central in Dwork's theory has a very natural interpretation using Berkovich theory ([15]).

Of course, this article owes much to Berthelot's previous work on rigid cohomology ([9], [10], [8]). We only want to rewrite his theory with a slightly different approach. The reader should note however that we do not make any use of Berthelot's results and that, for this purpose, the article is totally self-contained. In particular, the reader needs not know anything about rigid cohomology. However, since the main results are comparison theorems, I should briefly recall how it works.

Rigid cohomology is a cohomological theory for algebraic varieties over a field k of positive characteristic p with values in vector spaces over a p-adic field K whose residue field is k. The idea is to embed the given variety X into a proper variety Y and then Y into a smooth formal scheme P over the valuation ring $\mathcal{V}$ of K. Then, one considers the limit de Rham cohomology on (strict) neighborhoods of the tube of X inside the generic fiber of P. The hard part in the theory is to show that the cohomological spaces obtained this way are independent of the choices (and that they glue when there exists no embedding as above). There is a relative theory and one may also add coefficients. The coefficients are those of the de Rham theory, namely modules with integrable connections, on a neighborhood of the tube of X, with the extra condition that the connection must be *overconvergent* (overconvergent isocrystals). Here again, this is a local definition and one must show that it does not depend on the choices. It is also important to remark that glueing is the only solution when there is no global embedding as described above. And this is unfortunately the case in general even if it can be avoided in practice.

I want however to emphasize the fact that rigid cohomology was designed as a functorial theory from the beginning and that the purpose of the present article is *not* to fill a possible gap in the original theory. Since this is not completely understood in the mathematical community, it might be necessary to recall how this works. For simplicity, I will only consider the case of absolute cohomology without coefficient of "realizable" varieties (say quasi-projective, if you wish). The general case is more technical but works exactly the same. First of all, rigid cohomology is functorial in triples $X \subset Y \subset P$ made of an open embedding into a variety and a closed embedding into a formal scheme. Thus we can define $\mathrm{H}^i_{\mathrm{rig}}(X \subset Y \subset P)$ and if we are given a morphism $u : P' \to P$ that induces $g : Y' \to Y$ and $f : X' \to X$, we can define $\mathrm{H}^i_{\mathrm{rig}}(u)$. This should be clear. The fundamental theorem of the theory tells us that if f is an isomorphism, g is proper and u is smooth, we obtain an isomorphism on cohomology. This being said, here is how you define your functor on the category of (realizable) varieties: given X, you choose an open embedding of X into Y proper as well as a closed embedding of Y into P smooth and define $\mathrm{H}^i_{\mathrm{rig}}(X)$ as the cohomology of *this* triple. You do the same thing with another variety X' and introduce Y' and P'. Now, if you are given a morphism $f : X' \to X$, you consider the graph $X' \subset Y'' \subset P' \times P$, where Y'' denotes the algebraic closure of X' into $Y \times Y'$. Then, you consider the maps of triples defined by the projections p_1 and p_2 and define $\mathrm{H}^i_{\mathrm{rig}}(f)$ as the composite

$$\mathrm{H}^i_{\mathrm{rig}}(X' \subset Y' \subset P') \xleftarrow{\simeq} \mathrm{H}^i_{\mathrm{rig}}(X' \subset Y'' \subset P' \times P) \longrightarrow \mathrm{H}^i_{\mathrm{rig}}(X \subset Y \subset P)$$

(the first map is an isomorphism thanks to the fundamental theorem). It is an exercise to check that this actually defines a functor.

Let us now present more precisely the content of this article. There are three chapters (and an appendix). In the first one, we define and study the overconvergent site. In the second one, we show that finitely presented modules correspond to overconvergent isocrystals. In the third one, we prove that cohomology coincides.

Chapter 1 (Geometry):

If we want to do some analytic geometry over a field k of characteristic zero, the first step consists in embedding k into a complete valued field K (the field $\mathbf{C}$ of complex numbers for example). Then, we may consider all analytic varieties V that appear as an open subset of some X_K^{an} where X is an algebraic variety over k. In other words, an object should be a pair (X, V) made of an algebraic variety X over k, an analytic variety V over K and an open immersion $\lambda : V \to X_K^{\mathrm{an}}$. In order to get more flexibility, it is simpler to allow any such map λ and not only open immersions (big site).

We can mimic this in characteristic $p > 0$. We cannot embed k into a complete ultrametric field of characteristic zero but we can find a complete ultrametric field K of characteristic zero with residue field k. We then use formal schemes over the valuation ring $\mathcal{V}$ of K as a bridge between algebraic varieties over k and analytic varieties over K. More precisely, we may embed an algebraic variety X into a formal $\mathcal{V}$-scheme P and consider open subsets V of the generic fiber P_K of P (which is an analytic variety over K). In other words, an *overconvergent variety* will be a diagram $(X \hookrightarrow P \leftarrow P_K \leftarrow V)$ with a locally closed embedding $X \hookrightarrow P$ and an open immersion $\lambda : V \to P_K$. Again, we will actually allow any map λ but this does not matter at this point. Actually, the formal scheme P should play a secondary role and we will usually write (X, V) as above.

Morphisms of overconvergent varieties are defined in two steps. We first consider formal morphisms as compatible triples of morphisms (algebraic level, formal level and analytic level). This way, we obtain a category $\mathrm{An}(\mathcal{V})$ which is way too rigid. In order to define "true" morphisms, we need to introduce the tube $]X[_V$ of X in V which is simply the inverse image of X in V (through specialization and λ). We will then allow the replacement of the formal scheme by another one as long as the tube does not change. We will also allow the replacement of the analytic variety by any neighborhood of the tube. This is done through the fancy language of category of fraction. Then, we obtain the category $\mathrm{An}^\dagger(\mathcal{V})$ and we endow it with the topology that comes from the topology of analytic varieties (completely discarding the topology on the algebraic side).

The main result of this first chapter is the local section theorem (theorem 1.5.11) that I will try to explain now. Assume that we are given an overconvergent variety $(X \hookrightarrow P \leftarrow V)$ as above, an embedding of X in another formal scheme P' and and a morphism $P' \to P$ of formal schemes that is proper and smooth (at X). Then, we may consider the overconvergent variety (X, V') obtained by base extension. The theorem says that the morphism $(X, V') \to (X, V)$ has locally a a section in $\mathrm{An}^\dagger(\mathcal{V})$. This invariance result will be the key tool in the proof of the main theorem of the second chapter, namely the equivalence of category between overconvergent modules of finite presentation and overconvergent isocrystals. However, it will not be powerful enough to prove compatibility at the cohomology level: in the third chapter, we will need to prove a fibration theorem that is only local for the Zariski topology on X and the Grothendieck topology on V (as this is the case in rigid cohomology).

At this point, I must mention a complication due to the use of Berkovich theory instead of Tate's. In Berkovich original theory ([2]), analytic varieties were locally affinoid. Unfortunately, this definition happens to be too restrictive because the generic fiber of a formal scheme, for example, does not always fulfill this condition. Consequently, the category had to be enlarged in [3] and the original analytic varieties are henceforth called *good*. Note that his condition should be simply seen as a separation condition and it appears in many situations. For example, for bad varieties, the notion of coherent sheaf is not compatible with the analog notion on the rigid analytic underlying variety, therefore there is not much hope to obtain a comparison theorem between our theory and Berthelot's if we do not restrict to good analytic varieties.

Chapter 2 (Coefficients):

Let me first recall the original definition of the category of overconvergent isocrystals ([10]). We fix a formal scheme S and an algebraic variety X over S_k. We assume that there exists some open embedding of X into a proper algebraic variety Y over S_k and a closed embedding of Y into a formal scheme P over S which is smooth in the neighborhood of X. We consider the generic fiber of P as a rigid analytic variety, as well as the admissible open subsets $]X[$ and $]Y[$. It is then necessary to introduce the functor $j^\dagger$ of sections defined on a strict neighborhood of $]X[$ on $]Y[$. An *overconvergent isocrystal* on X is a coherent $j^\dagger\mathcal{O}_{]Y[}$-module E with an integrable connection which is "overconvergent". The point is to show that this definition does not depend on the choices.

In this article (as we actually did in [23]), we work in the opposite direction: we define a category of coefficients and show that they can be interpreted as modules with integrable connection in nice geometric situations. Recall that we introduced above the overconvergent site $\mathrm{An}^\dagger(\mathcal{V})$. We turn it into a ringed site with the following definition

$$\Gamma((X,V),\mathcal{O}^\dagger) = \Gamma(]X[_V, i_X^{-1}\mathcal{O}_V)$$

where $i_X :]X[_V \hookrightarrow V$ denotes the embedding. Recall that, since we work with Berkovich theory, the tube is an analytic domain which is not open in general and the ring on the right hand side consist of functions that are defined on some open neighborhood.

Actually, if we fix an overconvergent variety $(C \hookrightarrow S \leftarrow S_K \leftarrow O)$ and a morphism of algebraic varieties $X \to C$, we may consider all overconvergent varieties (U,V) over (C,O) with a factorization of $U \to X \to C$. We obtain a ringed site $\mathrm{An}^\dagger(X/O)$. Our main result in the second chapter of this article (theorem 2.5.9) states that, if we are given a proper smooth morphism $P \to S$ (at X) and let V denote the inverse image of O, there is an equivalence $E \mapsto (E_{X,V}, \nabla)$ between $\mathcal{O}^\dagger_{X/O}$-modules of finite presentation and coherent $i_X^{-1}\mathcal{O}_V$-modules with an overconvergent integrable connection. If we apply this to the particular case $C = S_k$ and $O = S_K$, we easily obtain that the category of $\mathcal{O}^\dagger_{X/S_K}$-modules of finite presentation is equivalent to the category of overconvergent isocrystals on X/S.

We can be a little more precise. With the same notations, we may consider the ringed site $\mathrm{An}^\dagger(X_V/O)$ of all overconvergent varieties (X',V') over (C,O) with a fixed factorization of $X' \to X$ that extends to *some* factorization $(X',V') \to (X,V)$. It follows from the main theorem of the first chapter that this site is equivalent to $\mathrm{An}^\dagger(X/O)$. Then we are reduced to showing that $\mathcal{O}^\dagger_{X_V/O}$-modules of finite presentation are equivalent to coherent $i_X^{-1}\mathcal{O}_V$-modules with an overconvergent integrable connection. And we can use, as we usually do in such a situation, the bridge provided by modules with stratification.

Note that in this chapter also, it is necessary to work only with good analytic varieties at some point.

Chapter 3 (Cohomology):

What is the original definition of rigid cohomology? As before, we fix a formal scheme S and a morphism of algebraic varieties $p : X \to S_k$. We assume that there exists an open embedding of X into a proper algebraic variety Y over S_k and a closed embedding of Y into a formal scheme P over S which is smooth in the neighborhood of X. The rigid cohomology of an overconvergent isocrystal E on X/S is its de Rham cohomology:

$$\mathrm{R}p_{\mathrm{rig}}E = \mathrm{R}p_{]Y[/S_K*}(E \otimes_{\mathcal{O}_{]Y[}} \Omega^{\bullet}_{]Y[/S_K}).$$

Again, one has to show that this is independent on the choices and it is even harder than the analog problem for coefficients - not to mention the glueing question (see [16] and private notes from Berthelot in [8]).

How does it work on the overconvergent site? If $(C \hookrightarrow S \leftarrow S_K \leftarrow O)$ is an overconvergent variety and $p : X \to C$ a morphism of algebraic varieties, there is a canonical morphism of toposes

$$\begin{array}{ccc} (X/O)_{\mathrm{An}^\dagger} & \xrightarrow{p_{X/O}} & (]C[_O)_{\mathrm{an}} \\ X/O' & \longleftarrow &]C[_{O'}. \end{array}$$

The cohomology of an $\mathcal{O}^\dagger_{X/O}$-module E is simply $\mathrm{R}p_{X/O*}E$. Assume now that p extends to a morphism of formal schemes $P \to S$ that is proper and smooth (at X) and let V be the inverse image of O. The main result of this third chapter (theorem 3.5.3) states there is a canonical isomorphism

$$\mathrm{R}p_{X/O*}E \simeq \mathrm{R}p_{]X[_V*}(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^{\bullet}_{V/O})$$

on $]C[_O$. As a consequence, we see that if E is an overconvergent isocrystal on X/S, then

$$(\mathrm{R}p_{\mathrm{rig}}E)^{\mathrm{an}} = \mathrm{R}p_{X/S_K*}E$$

(the exponent “an” denote the move from rigid to Berkovich topology). For example, in the simplest geometric situation, we obtain that if X is any separated algebraic variety of finite type over k, we have for all $i \in \mathbf{N}$,

$$\mathrm{H}^i_{\mathrm{rig}}(X/K) = \mathrm{H}^i((X/\mathcal{V})_{\mathrm{An}^\dagger}, \mathcal{O}^\dagger).$$

In order to prove our main theorem, we will use linearization of differential operators as in Grothendieck's original article [21]. If one tries to mimic the classical arguments, one keeps stumbling. There is no such thing as a projection morphism. The linearization does not give rise to crystals. Linearization is not even local on X. The coefficients do not get out of the de Rham complex. Some important functors are not exact anymore. Nevertheless, the method works and we get what we want in the end. The main ideas are to use *derived linearization* and the Grothendieck topology. Again, we need to restrict to good analytic varieties in order to obtain the last results.

We finish this chapter with a section on Zariski localization that allow us to state the main comparison theorem 3.6.7.

Appendix:

For the convenience or the reader, we added two sections at the end of the article. The first one recalls the basic results on topos theory that are used here and the second one is a brief introduction to Berkovich theory. It gives us the opportunity to fix the notations and some vocabulary.

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Conventions

Finite presentation for a formal scheme will be a little weaker than the usual convention (in the non-noetherian case) because we allow *usual* schemes of finite presentation on the reduction as well.

Locally of finite type means that there exists a *locally finite* open covering by object of finite type, and not just any open covering. This will apply to schemes and formal schemes.

Algebraic varieties are just assumed to be locally of finite type and not necessarily separated, quasi-compact or reduced for example.

Analytic varieties are meant in Berkovich sense and denote *strictly* analytic spaces (locally defined by strictly affinoid algebras as in rigid geometry) defined on the base field (and not just after some isometric extension).

Formally smooth is used instead of *rig-smooth* or *quasi-smooth* in Berkovich theory. And the same convention applies to étale morphisms.

Boundaryless is used for *closed* in the sense of Berkovich.

Locally compact spaces are not assumed to be Hausdorff (they are only *locally* Hausdorff).

Complexes are always assumed to be *bounded below.*

Notations

Throughout this paper, K is a *non trivial* complete ultrametric field with valuation ring $\mathcal{V}$, maximal ideal $\mathfrak{m}$ and residue field k.

We will usually use the letters $X, Y, Z, U, C, D, \ldots$ to denote algebraic varieties, $P, Q, R, S, \ldots$ for formal schemes and $V, W, \ldots$ for analytic varieties.

As usual, we will write $\mathbf{A}^n$ and $\mathbf{P}^n$ for the affine and projective spaces of dimension n. We will also use $\mathbf{B}^n(0, \lambda^{\pm})$ for the open or closed polydisc of dimension n and radius λ. When $n = 1$, we will write $\mathbf{D}(0, \lambda^{\pm})$.

We will denote by $\mathcal{K}(x)$ the complete residue field at a point of an analytic variety over K (which is usually written $\mathcal{H}(x)$).

Chapter 1

Geometry

In section 1, we fix some notations and vocabulary that will be used throughout the paper. In particular, we introduce the notion of formal embedding. We recall the relation between algebraic varieties over k, formal schemes over $\mathcal{V}$ and analytic varieties over K.

The objects of the overconvergent site, the overconvergent varieties, are introduced in section 2. We also define formal morphisms between them leading to an intermediate category $\mathrm{An}(\mathcal{V})$. We endow this category with a topology that comes from the analytic side.

The category $\mathrm{An}^\dagger(\mathcal{V})$ of overconvergent varieties is introduced in section 3 as the category of fractions of $\mathrm{An}(\mathcal{V})$ with respect to strict neighborhoods that we define there. We give a more down-to-earth description of this category as well as some standard properties.

In section 4, we endow $\mathrm{An}^\dagger(\mathcal{V})$ with the topology coming from $\mathrm{An}(\mathcal{V})$ and derive some of its properties. Then we use the general notion of restriction of a site to a presheaf (usually called *localization*) in order to introduce relative overconvergent sites.

The hard work starts in section 5. The point is to prove invariance theorems under proper smooth maps and derive some consequences for the overconvergent sites. In order to do so, one studies successively how some properties of formal schemes in the "neighborhood" of a subvariety translate into properties of the generic fiber in the neighborhood of the tube.

1.1 Formal embeddings

We mostly fix the notations and vocabulary here.

Formal schemes

Recall that we fixed a non trivial complete ultrametric field K with valuation ring $\mathcal{V}$, maximal ideal $\mathfrak{m}$ and residue field k.

We denote by

$$\mathcal{V}\{T_1, \ldots, T_n\} := \{\sum_{\underline{i}\geq 0} a_{\underline{i}} T^{\underline{i}}, a_{\underline{i}} \in \mathcal{V}, \quad |a_{\underline{i}}| \to 0\}$$

the ring of convergent power series over $\mathcal{V}$. A *formal $\mathcal{V}$-scheme* will always be assumed to have a locally finite open covering by formal affine schemes $\mathrm{Spf}(A)$ where A is a quotient of $\mathcal{V}\{T_1, \dots, T_n\}$ by an ideal of the form $I + \mathfrak{a}\mathcal{V}\{T_1, \dots, T_n\}$, where I is an ideal of finite type and $\mathfrak{a}$ an ideal in $\mathcal{V}$. We will denote by $\mathrm{FSch}(\mathcal{V})$ the category of formal $\mathcal{V}$-schemes. Traditionally, a formal scheme is said to be *admissible* if it is $\mathcal{V}$-flat or, in other words, if it has no torsion.

The introduction of the ideal $\mathfrak{a}$ above is only necessary when the valuation is not discrete in order to recover algebraic varieties over k as particular cases of formal schemes over $\mathcal{V}$. Also, the locally finite condition is necessary in order to define the generic fiber in Berkovich sense.

We will make an extensive use of the notion *restriction* of a category (usually called localization). Namely, if T is a object in a category C, we will consider the *restricted category* $C_{/T}$ whose objects are morphisms $X \to T$ in C and morphisms are simply morphism $Y \to X$ in C compatible with the structural morphisms $Y \to T$ and $X \to T$.

For example, if S is a formal $\mathcal{V}$-scheme, we will consider the category $\mathrm{FSch}(S)$ of formal schemes over S, which is just the restricted category

$$\mathrm{FSch}(S) := \mathrm{FSch}(\mathcal{V})_{/S}.$$

In section 1.5, we will need some geometric properties of formal schemes that we introduce now. If P is a formal $\mathcal{V}$-scheme and X is a subset of P, we will denote by $\overline{X}^P$ or simply $\overline{X}$, the Zariski closure of X in P with its reduced structure.

Definition 1.1.1 *Let $v : P' \to P$ be a morphism of formal schemes and $x' \in P'$. Then, v is said to be*

1. flat *(resp.* smooth*, resp.* étale*) at x' if u is flat (resp. smooth, resp. étale) in the neighborhood of x'.*
2. (relatively) separated *(resp.* (relatively) proper*, resp.* (relatively) finite*) at $x' \in P'_k$ if the induced map $\overline{\{x'\}} \to P_k$ is separated (resp. proper, resp. finite).*

Finally, note that if $K \hookrightarrow K'$ is an isometric embedding of complete ultrametric fields, and $\mathcal{V}'$ denotes the valuation ring of K', there exists an extension functor

$$\begin{array}{rcl} \mathrm{FSch}(\mathcal{V}) & \longrightarrow & \mathrm{FSch}(\mathcal{V}') \\ P & \longmapsto & P_{\mathcal{V}'} \end{array}$$

with $P_{\mathcal{V}'} = \mathrm{Spf}(\mathcal{V}'\widehat{\otimes}_{\mathcal{V}} A)$ when $P = \mathrm{Spf}(A)$.

Formal embeddings

If X is any scheme, we denote by $\mathrm{Sch}(X)$ the category of schemes over X that have a locally finite open covering by schemes of finite presentation over X. In the case $X = \mathrm{Spec}R$, we will write $\mathrm{Sch}(R)$. If k denotes the residue field of K, the category $\mathrm{Sch}(k)$ may (and will) be seen as a full subcategory of $\mathrm{FSch}(\mathcal{V})$ and its objects will be called *algebraic varieties* over k. Moreover, the embedding $\mathrm{Sch}(k) \hookrightarrow \mathrm{FSch}(\mathcal{V})$ has a right adjoint

$$\begin{array}{rcl} \mathrm{FSch}(\mathcal{V}) & \longrightarrow & \mathrm{Sch}(k) \\ P & \longmapsto & P_k \end{array}$$

sending a formal scheme to its special fiber. Recall that the adjunction map $P_k \to P$ is a homeomorphism and we will use it to identify the underlying topological spaces.

We recall the following from section 2.2 of [23]:

Definition 1.1.2 *A* formal embedding $X \hookrightarrow P$ *(or $X \subset P$ for short) is a (locally closed) immersion over $\mathcal{V}$ of a k-variety into a formal $\mathcal{V}$-scheme. A* morphism of formal embeddings

$$(f \subset v) : (X' \subset P') \to (X \subset P)$$

is a pair of morphisms (over k and $\mathcal{V}$ respectively)

$$(f : X' \to X, v : P' \to P),$$

such that the diagram

$$\begin{array}{ccc} X' & \hookrightarrow & P' \\ \downarrow f & & \downarrow v \\ X & \hookrightarrow & P \end{array}$$

is commutative. When $X' = X$ and f is the identity of X, we will just say that v is a morphism of formal embeddings of X*.*

Note that the morphism f is uniquely determined by v.

Example: In order to connect our construction to Monsky-Washnitzer's, we may consider the following situation: we let A be a $\mathcal{V}$-algebra of finite type; the choice of a presentation of A defines an embedding

$$\mathrm{Spec}(A) \hookrightarrow \mathbf{A}^N_{\mathcal{V}} \subset \mathbf{P}^N_{\mathcal{V}}$$

which can be used to embed $X := \mathrm{Spec}(A_k)$ into the formal completion $P := \widehat{\mathbf{P}^N_{\mathcal{V}}}$ of $\mathbf{P}^N_{\mathcal{V}}$. This is a formal embedding.

Proposition 1.1.3 *We have the following results:*

1. *With obvious composition, formal embeddings $(X \subset P)$ form a category $\mathrm{Fmb}(\mathcal{V})$ with finite inverse limits.*
2. *The forgetful functor*
$$\begin{array}{rcl} \mathrm{Fmb}(\mathcal{V}) & \longrightarrow & \mathrm{FSch}(\mathcal{V}), \\ (X \subset P) & \longmapsto & P \end{array}$$
is exact and has an adjoint on the right
$$\begin{array}{rcl} \mathrm{FSch}(\mathcal{V}) & \longrightarrow & \mathrm{Fmb}(\mathcal{V}) \\ P & \longmapsto & (P_k \subset P). \end{array}$$
3. *The forgetful functor*
$$\begin{array}{rcl} \mathrm{Fmb}(\mathcal{V}) & \longrightarrow & \mathrm{Sch}(k) \\ (X \subset P) & \longmapsto & X \end{array}$$
(is left exact and) has an adjoint on the left
$$\begin{array}{rcl} \mathrm{Sch}(k) & \longrightarrow & \mathrm{Fmb}(\mathcal{V}) \\ X & \longmapsto & (X \subset X). \end{array}$$

Proof: It should be clear that $\mathrm{Fmb}(\mathcal{V})$ is indeed a category. The first adjointness assertion simply says that, if $X \subset P$ is a formal embedding, any morphism of formal $\mathcal{V}$-schemes $P \to Q$ sends X into Q_k. The other adjointness assertion is trivial. Finally, we have to check that, if we are given a finite diagram $(X_i \subset P_i)$, it has an inverse limit which is just $(\varprojlim X_i \subset \varprojlim P_i)$. But this again should be clear. □

The *principle of diagonal embedding* will prove to be crucial in many situations:

Proposition 1.1.4 *If $(X \subset P)$ and $(X' \subset P')$ are two formal embeddings, any morphism $f : X' \to X$ may be inserted in a diagram of formal embeddings*

$$\begin{array}{ccc} X' & \hookrightarrow & P' \\ \| & & \uparrow w \\ X' & \hookrightarrow & P'' \\ \downarrow f & & \downarrow v \\ X & \hookrightarrow & P \end{array}$$

Proof: We may identify X' with the graph of f inside $X' \times X$ and embed this product into $P'' = P' \times P$. Then, we take $v = p_2$ and $w = p_1$. □

Note that the morphism $P'' \to P'$ will inherit any universal property of P.

We extend definition 1.1.1 to formal embeddings as follows:

Definition 1.1.5 *A morphism of formal embeddings $(f, v) : (Y, Q) \to (X, P)$ is said to be* flat *(resp.* smooth*, resp.* étale*, resp.* separated*, resp.* proper*, resp.* finite*) if v is flat (resp. smooth, resp. étale, resp. separated, resp. proper, resp. finite) at all $y \in Y$. We might also say that at v is* flat *(resp.* smooth*, resp.* étale*, resp.* separated*, resp.* proper*, resp.* finite*) at Y.*

Note that v is separated at Y if and only if $\overline{Y}$ is separated over P_k. Moreover, v is proper (resp. finite) at Y if and only if the restriction of v to any irreducible component of $\overline{Y}$ is proper (resp. finite) over P_k. When Y is quasi-compact, this just means that $\overline{Y}$ itself is proper (resp. finite) over P_k.

Finally, note that if $K \hookrightarrow K'$ is an isometric embedding of complete ultrametric fields, $\mathcal{V}'$ denotes the valuation ring of K' and k' its residue field, there exists an extension functor

$$\begin{array}{ccc} \mathrm{Fmb}(\mathcal{V}) & \longrightarrow & \mathrm{Fmb}(\mathcal{V}') \\ (X \subset P) & \longmapsto & (X_{k'} \subset P_{\mathcal{V}'}). \end{array}$$

Tubes

As already mentioned, we will consider K-analytic spaces in the sense of Berkovich (see for example section 1 of [3] or appendix 4.2). We assume that they are strictly analytic and call them *analytic varieties* over K. We will denote by $\mathrm{An}(K)$ the category of analytic varieties over K, and more generally, if V is an analytic variety over K, we will denote by $\mathrm{An}(V) := \mathrm{An}(K)_{/V}$ the category of analytic varieties over V.

As shown in [4], section 1, there is a *generic fiber* functor

$$\begin{array}{ccc}\mathrm{FSch}(\mathcal{V}) & \longrightarrow & \mathrm{An}(K)\\ P & \longmapsto & P_K\end{array}$$

which is easily seen to be left exact; when $P = \mathrm{Spf}(A)$, we simply have $P_K = \mathcal{M}(A_K)$. If P is a formal $\mathcal{V}$-scheme, there is a natural *specialization map*

$$\begin{array}{ccc}P_K & \xrightarrow{\mathrm{sp}} & P\\ x & \longmapsto & \widetilde{x};\end{array}$$

when $P = \mathrm{Spf}(A)$, any $x \in P_K$ induces a continuous morphism $A_K \to \mathcal{K}(x)$ which reduces to a morphism $A_k \to k(x)$ whose kernel is $\widetilde{x} \in P_k$ (whose underlying space is identified with that of P). Note that specialization is anticontinuous when $\mathrm{FSch}(\mathcal{V})$ is endowed with its Zariski topology and $\mathrm{An}(K)$ is endowed with its analytic topology. More precisely, the inverse image of an open subset is a closed analytic domain and the inverse image of a closed subset is an open subset (as the next local description shows). In fact, this will not be a problem for us because we will implicitly endow $\mathrm{FSch}(\mathcal{V})$ with the coarse topology (all presheaves are sheaves).

If $X \subset P$ is a formal embedding, we will consider the *tube* $]X[_P := \mathrm{sp}^{-1}(X)$ of X in P. This definition will be generalized later on. When $P = \mathrm{Spf}(A)$ and

$$X := \{x \in P, \left\{\begin{array}{ll}\forall i = 1,\ldots,r, & \overline{f}_i(x) = 0\\ \exists j = 1,\ldots,s, & \overline{g}_j(x) \neq 0\end{array}\right.\},$$

we have

$$]X[_P := \{x \in P_K, \left\{\begin{array}{ll}\forall i = 1,\ldots,r, & |f_i(x)| < 1\\ \exists j = 1,\ldots s, & |g_j(x)| = 1\end{array}\right.\}.$$

Note that if P' denotes the maximal admissible formal subscheme of P and $X' = X \cap P'$, then $]X'[_{P'} =]X[_P$. This is why, in practice, we may generally work with admissible formal schemes.

Recall also that if X is an algebraic variety over K, we may consider its analytification X^{an} which is an analytic variety over K. Now, if Y is a locally finitely presented scheme over $\mathcal{V}$, we may consider its completion $\widehat{Y}$ and there is a canonical map $(\widehat{Y})_K \to (Y_K)^{\mathrm{an}}$. It is not difficult to check that this is an open immersion when Y is separated and even an isomorphism when Y is proper.

Example: In Monsky-Washnitzer's situation (example following definition 1.1.2), we could derive from a presentation of a smooth affine $\mathcal{V}$-algebra A, a formal embedding

$$X := \mathrm{Spec}(A_k) \subset P := \widehat{\mathbf{P}^N_{\mathcal{V}}}.$$

We have $P_k = \mathbf{P}^N_k$ and $P_K = \mathbf{P}^{N,\mathrm{an}}_K$ and specialization map

$$\begin{array}{ccc}\mathbf{P}^{N,\mathrm{an}}_K & \xrightarrow{\mathrm{sp}} & \widehat{\mathbf{P}^N_{\mathcal{V}}}\\ (x_0,\ldots,x_N) & \longmapsto & (\widetilde{x}_0,\ldots,\widetilde{x}_N)\end{array}$$

whenever $\max|x_i| = 1$. If we set $V := \mathrm{Spec} A^{\mathrm{an}}_K$, one easily checks that

$$]X[_P = \mathcal{M}(\widehat{A}_K) = V \cap \mathbf{B}^N(0, 1^+).$$

Proposition 1.1.6 *The functor*

$$\begin{array}{ccc} \mathrm{Fmb}(\mathcal{V}) & \longrightarrow & \mathrm{An}(K) \\ (X \subset P) & \longmapsto &]X[_P \end{array}$$

is left exact.

Proof: Clearly, this functor sends the final object $(\mathrm{Spec} k \subset \mathrm{Spf}\mathcal{V})$ of $\mathrm{Fmb}(\mathcal{V})$ to the final object $\mathcal{M}(K)$ of $\mathrm{An}(K)$. Assume now that we are given two morphisms $(X \subset P) \to (Z \subset R)$ and $(Y \subset Q) \to (Z \subset R)$. We want to show that

$$]X \times_Z Y[_{P \times_R Q} =]X[_P \times_{]Z[_R}]Y[_Q.$$

We have

$$X \times_Z Y = (P \times_R Y) \cap (X \times_R Q)$$

and

$$]X[_P \times_{]Z[_R}]Y[_Q = (P_K \times_{R_K}]Y[_Q) \cap (]X[_P \times_{R_K} Q_K).$$

Since the tube is just an inverse image, it commutes with intersection and we are therefore reduced to showing that

$$]X \times_{R_k} Q_k[_{P \times_R Q} =]X[_P \times_{R_K} Q_K$$

which follows for example from the local description of the tube. $\square$

The last proposition describes the analytic counterpart of the notion of separateness and properness introduced in definition 1.1.5. We simply recall that *locally separated* means that the diagonal embedding is an immersion and send the reader to appendix 4.2 for the notion of *interior*.

Proposition 1.1.7 *Let $v : P' \to P$ be a morphism of formal schemes and X' a subvariety of P'.*

1. *If v is separated at X', then $]X'[_{P'}$ has a locally separated neighborhood V' in P'_K relative to P_K.*
2. *If v is proper at X', then $]X'[_{P'}$ is contained in the interior of P'_K relative to P_K.*

And the converse is also true when the formal schemes are admissible.

Proof: Both results directly follow from Temkin's work: when the formal schemes are admissible, they follow from proposition 2.5 and theorem 4.1 of [27] (see also remark 5.8 of [28]). In general, we may replace P and P' by their maximal admissible formal subschemes. $\square$

1.2 Overconvergent varieties

The category of formal embeddings in not rich enough for our purpose. We need some analytic structure. This leads to the notion of overconvergent variety that we introduce here. Note that the definition of morphisms that we will consider now will have to be weakened in the future. We call them formal morphisms.

Definition 1.2.1 *An* overconvergent variety *is a pair made of a formal embedding $X \subset P$ over $\mathcal{V}$ and a morphism of analytic varieties $\lambda : V \to P_K$ over K. The* tube *of X in V is $]X[_V := \lambda^{-1}(]X[_P)$.*

An overconvergent variety can be represented by the diagram

$$X \hookrightarrow P \stackrel{\mathrm{sp}}{\leftarrow} P_K \stackrel{\lambda}{\leftarrow} V$$

and we will denote by

$$i_{X,V} :]X[_V \hookrightarrow V$$

the inclusion map.

We will usually write $(X \subset P \leftarrow V)$. We might also forget λ or sp in the notations and just write $\mathrm{sp} : V \to P$ or $\lambda : V \to P$, in which case we will write $]X[_V = \mathrm{sp}^{-1}(X)$ or $]X[_V = \lambda^{-1}(X)$. Also, if $x \in]X[_V$, we will write $\widetilde{x} = \mathrm{sp}(\lambda(x))$. Finally, when no confusion should arise, we will simply call i_X, i_V or even i the inclusion map.

Example: In the Monsky-Washnitzer situation (example following definition 1.1.2), we saw that there is a formal embedding

$$X := \mathrm{Spec}(A_k) \subset P := \widehat{\mathbf{P}^N_{\mathcal{V}}}.$$

We may also consider the inclusion morphism

$$V := \mathrm{Spec}(A_K)^{\mathrm{an}} \hookrightarrow (\mathbf{P}^N_K)^{\mathrm{an}} = (\widehat{\mathbf{P}^N_{\mathcal{V}}})_K = P_K.$$

in order to get an overconvergent variety

$$X \subset P \stackrel{\mathrm{sp}}{\leftarrow} P_K \hookleftarrow V.$$

Actually, for each $\lambda > 1$, we may set

$$V_\lambda := \mathbf{B}^N(0, \lambda) \cap V$$

and we get another overconvergent variety

$$X \subset P \stackrel{sp}{\leftarrow} P_K \hookleftarrow V_\lambda.$$

Note that V_λ is affinoid so that we can write $V_\lambda = \mathcal{M}(A_\lambda)$. Actually, we have

$$\varinjlim A_\lambda = A_K^\dagger \subset \widehat{A}_K$$

where $A^\dagger$ denotes of the weak completion of A (see the original article [24] for the notion of weak completion).

We now come to the definition of morphisms. As already mentioned, we will consider, later on, another category with the same objects but more maps.

Definition 1.2.2 *A* formal morphism *of overconvergent analytic varieties is a commutative diagram*

$$\begin{array}{ccccccc} X' & \hookrightarrow & P' & \stackrel{\mathrm{sp}}{\longleftarrow} & P'_K & \stackrel{\lambda'}{\longleftarrow} & V' \\ \downarrow f & & \downarrow v & & \downarrow v_K & & \downarrow u \\ X & \hookrightarrow & P & \stackrel{\mathrm{sp}}{\longleftarrow} & P_K & \stackrel{\lambda}{\longleftarrow} & V \end{array}$$

where f is a morphism of algebraic varieties, v is a morphism of formal schemes and u is a morphism of analytic varieties.

We will usually write

$$(f \subset v, u) : (X' \subset P' \stackrel{\lambda'}{\leftarrow} V') \to (X \subset P \stackrel{\lambda}{\leftarrow} V).$$

The following result is straightforward but very useful in practice because it generally allows us to split some questions into two simpler ones: working on the formal embeddings side or on the analytic varieties side.

Proposition 1.2.3 *Any formal morphism of overconvergent varieties is the composition of a morphism of the form*

$$(f \subset v, \mathrm{Id}_{V'}) : (X' \subset P' \leftarrow V') \to (X \subset P \leftarrow V').$$

and a morphism of the form

$$(\mathrm{Id}_X \subset \mathrm{Id}_P, u) : (X \subset P \leftarrow V') \to (X \subset P \leftarrow V).$$

Proof: Any morphism as above splits as

$$\begin{array}{ccccccc} X' & \hookrightarrow & P' & \xleftarrow{\mathrm{sp}} & P'_K & \xleftarrow{\lambda'} & V' \\ \downarrow f & & \downarrow v & & \downarrow & & \| \\ X & \hookrightarrow & P & \xleftarrow{\mathrm{sp}} & P_K & \xleftarrow{\lambda \circ u} & V' \\ \| & & \| & & \| & & \downarrow u \\ X & \hookrightarrow & P & \xleftarrow{\mathrm{sp}} & P_K & \xleftarrow{\lambda} & V. \end{array} \qquad \square$$

Proposition 1.2.4 *We have the following results:*

1. *With obvious composition, overconvergent varieties and formal morphisms form a category* $\mathrm{An}(\mathcal{V})$ *with finite inverse limits. Moreover, if* $K \hookrightarrow K'$ *is an isometric embedding, there is a natural base extension functor* $\mathrm{An}(\mathcal{V}) \to \mathrm{An}(\mathcal{V}')$.

2. *The forgetful functor*

$$\begin{array}{ccc} \mathrm{An}(\mathcal{V}) & \longrightarrow & \mathrm{An}(K) \\ (X \subset P \leftarrow V) & \longmapsto & V \end{array}$$

is exact with fully faithful right adjoint

$$\begin{array}{ccc} \mathrm{An}(K) & \longrightarrow & \mathrm{An}(\mathcal{V}) \\ V & \longmapsto & (\mathrm{Spec}(k) \subset \mathrm{Spf}(\mathcal{V}) \leftarrow V). \end{array}$$

3. *The forgetful functor*

$$\begin{array}{ccc} \mathrm{An}(\mathcal{V}) & \longrightarrow & \mathrm{Fmb}(\mathcal{V}) \\ (X \subset P \leftarrow V) & \longmapsto & X \subset P \end{array}$$

is exact with left adjoint

$$\begin{array}{ccc} \mathrm{Fmb}(\mathcal{V}) & \longrightarrow & \mathrm{An}(\mathcal{V}) \\ X \subset P & \longmapsto & (X \subset P \leftarrow \emptyset). \end{array}$$

and right adjoint

$$\begin{array}{ccc} \mathrm{Fmb}(\mathcal{V}) & \longrightarrow & \mathrm{An}(\mathcal{V}) \\ X \subset P & \longmapsto & (X \subset P \leftarrow P_K). \end{array}$$

Moreover, both adjoints are fully faithful.

Proof: All these assertions are easily checked. More precisely, it is clear that we do have a category and the existence of the base extension should also be clear. The existence of the finite inverse limits as well as the left exactness of the functors mean that the inverse limit of a diagram

$$X_i \hookrightarrow P_i \stackrel{\mathrm{sp}}{\leftarrow} P_{iK} \stackrel{\lambda_i}{\leftarrow} V_i$$

indexed by some finite set I is simply

$$\varprojlim X_i \hookrightarrow \varprojlim P_i \stackrel{\mathrm{sp}}{\leftarrow} \varprojlim P_{iK} \stackrel{\lambda}{\leftarrow} \varprojlim V_i.$$

And this is quite obvious. It remains to verify the adjointness and full faithfulness properties which is an easy exercise. □

A formal morphism of overconvergent varieties

$$(f \subset v, u) : (X' \subset P' \leftarrow V') \to (X \subset P \leftarrow V).$$

will induce a morphism

$$]f[_u :]X'[_{V'} \to]X[_V$$

between the tubes: we have a commutative diagram

$$\begin{array}{ccccccccc} X' & \hookrightarrow & P' & \stackrel{\mathrm{sp}}{\longleftarrow} & P'_K & \stackrel{\lambda'}{\longleftarrow} & V' & \hookleftarrow &]X'[_{P'} \\ \downarrow f & & \downarrow v & & \downarrow v_K & & \downarrow u & & \downarrow]f[_u \\ X & \hookrightarrow & P & \stackrel{\mathrm{sp}}{\longleftarrow} & P_K & \stackrel{\lambda}{\longleftarrow} & V & \hookleftarrow &]X[_P. \end{array}$$

Since $]f[_u$ is just the morphism induced by u, we will sometimes write $u :]X'[_{V'} \to]X[_V$. Also, when $u = \mathrm{Id}_V$, we will write $]f[_V :]X'[_V \to]X[_V$.

Proposition 1.2.5 *The functor*

$$\begin{array}{ccc} \mathrm{An}(\mathcal{V}) & \longrightarrow & \mathrm{An}(K) \\ (X \subset P \leftarrow V) & \longmapsto &]X[_V \end{array}$$

is left exact.

Proof: Since pull back is left exact, our assertion therefore follows from proposition 1.1.6. □

Let us fix some vocabulary and notations for the rest of this article (see also appendix 4.1). A contravariant functor on a category C will be called a *presheaf* on C. Unless otherwise specified, the target category will always be the category Sets of sets. Also, a natural transformation between contravariant functors will be called a morphisms of presheaves. We will denote by $\widehat{C}$ the category of presheaves of sets on C. It is possible that C is endowed with a topology (often generated by a pretopology) in which case we call it a *site*. And then, a presheaf that satisfies a suitable glueing property will be called a *sheaf* (see appendix 4.1 for a more precise formulation). We will denote by $\widetilde{C}$ the category of sheaves of sets on C. This a full subcategory of $\widehat{C}$. This is the *topos* associated to C.

For example, we will endow $\mathrm{An}(K)$ with the analytic topology. This is the (big) analytic site of K whose topology is generated by the pretopology made of open coverings. The corresponding topos will be denoted K_{An}. An object $\mathcal{F}$ of K_{An} is given by a usual sheaf (of sets) $\mathcal{F}_V$ on each analytic variety V and a family of compatible maps $u^{-1}\mathcal{F}_V \to \mathcal{F}_W$ for each morphism $u : W \to V$ such that $(\mathcal{F}_V)_{|W} = \mathcal{F}_W$ whenever W is an open subset of V.

In general, if we are given a functor $g : C' \to C$, the induced functor

$$g^{-1}: \qquad \begin{array}{ccc} \widehat{C} & \longrightarrow & \widehat{C'} \\ T & \longmapsto & T \circ g \end{array}$$

has a right adjoint g_* (and also a left adjoint). If both C and C' are actually sites and g_* preserves sheaves, then we say that g is *cocontinuous*. Quite often, this condition can be checked by considerations on coverings (see appendix 4.1 again for some details).

Definition 1.2.6 *The* analytic topology *on* $\mathrm{An}(\mathcal{V})$ *is the coarsest topology making cocontinuous the forgetful functor*

$$\begin{array}{ccc} \mathrm{An}(\mathcal{V}) & \longrightarrow & \mathrm{An}(K) \\ (X \subset P \leftarrow V) & \longmapsto & V. \end{array}$$

We might sometimes denote by $\mathcal{V}_{\mathrm{An}}$ the corresponding topos. There is a very simple description of this topology:

Proposition 1.2.7 *The analytic topology on* $\mathrm{An}(\mathcal{V})$ *is generated by the following pretopology: families*

$$\{(X \subset P \leftarrow V_i) \to (X \subset P \leftarrow V)\}_{i \in I}$$

where $V = \cup_i V_i$ *is an open covering.*

Proof: Note first that such families do define a pretopology. We may therefore use the following criterion: the forgetful functor is cocontinuous if whenever $\{V_i \to V\}_{i\in I}$ is a covering, the family of all

$$(X' \subset P' \leftarrow V') \to (X \subset P \leftarrow V)$$

such that $V' \to V$ factors through some V_i, is a covering. This condition is equivalent to

$$(X' \subset P' \leftarrow V') \to (X \subset P \leftarrow V)$$

factoring through some $(X \subset P \leftarrow V_i)$. Thus, we see that the forgetful functor is cocontinuous if and only if whenever $\{V_i \to V\}_{i\in I}$ is a covering, so is

$$\{(X \subset P \leftarrow V_i) \to (X \subset P \leftarrow V)\}_{i \in I}. \quad \square$$

Recall that a site is *standard* if it has finite inverse limits and the topology is coarser than the canonical topology. This last condition simply means that for any objet $X \in C$, the presheaf $Y \mapsto \mathrm{Hom}(Y, X)$ is a sheaf. This is a very pleasant situation because it allows us to consider C as a full subcategory of $\widetilde{C}$.

Corollary 1.2.8 *The site* $\mathrm{An}(\mathcal{V})$ *is a standard site.*

Proof: We have to show that the analytic topology is coarser than the canonical topology on $\mathrm{An}(\mathcal{V})$. We take any $(X \subset P \leftarrow V) \in \mathrm{An}(\mathcal{V})$ and we have to prove that the presheaf

$$(X' \subset P' \leftarrow V') \mapsto \mathrm{Hom}((X' \subset P' \leftarrow V'), (X \subset P \leftarrow V))$$

is a sheaf. Thus, we are given a covering

$$V' = \cup_i V'_i$$

and a compatible family of morphisms

$$\{(f \subset v, u_i) : (X' \subset P' \leftarrow V'_i) \to (X \subset P \leftarrow V)\}_{i \in I}.$$

As Berkovich showed (see proposition 1.3.2 of [3] for example), the analytic topology is coarser than the canonical topology on $\mathrm{An}(K)$, and it follows that the $u'_i s$ glue to a morphism $u : V' \to V$. It is clear that (f, u) is a morphism. □

Unless otherwise specified, categories of schemes and formal schemes are always endowed with the *coarse* topology - and *not* the Zariski topology.

Recall that a cocontinuous functor $g : C' \to C$ gives rise to a morphism of toposes $\widetilde{g} : \widetilde{C}' \to \widetilde{C}$ (a functor $\widetilde{g}_* : \widetilde{C}' \to \widetilde{C}$ with exact left adjoint $\widetilde{g}^{-1} : \widetilde{C} \to \widetilde{C}'$). A functor $g : C' \to C$ may also be *continuous*: it means that g^{-1} preserves sheaves. The induced functor that we may denote by $f_* : \widetilde{C} \to \widetilde{C}'$ (so that f_* denotes the restriction of g^{-1}) has a left adjoint f^{-1}. If this adjoint is exact, we obtain a morphism of topos $f : \widetilde{C} \to \widetilde{C}'$. We will also say that we have a *morphism of sites* $f : C \to C'$. Note that f^{-1} extends g; this is why we usually do not use the letter g and write from the beginning $f^{-1} : C' \to C$.

Proposition 1.2.9 *We have the following results:*

1. *The forgetful functor*

$$\begin{array}{ccc} \mathrm{An}(\mathcal{V}) & \longrightarrow & \mathrm{An}(K) \\ (X \subset P \leftarrow V) & \longmapsto & V \end{array}$$

   is left exact, continuous and cocontinuous, giving rise to a morphism of sites

$$\mathrm{An}(K) \to \mathrm{An}(\mathcal{V}).$$

   and a morphism of toposes

$$\widetilde{\mathrm{An}(\mathcal{V})} \to \widetilde{\mathrm{An}(K)}.$$

2. *The forgetful functor*

$$\begin{array}{ccc} \mathrm{An}(\mathcal{V}) & \longrightarrow & \mathrm{Fmb}(\mathcal{V}) \\ (X \subset P \leftarrow V) & \longmapsto & X \subset P \end{array}$$

   is exact, continuous and cocontinuous, giving rise to a morphism of sites

$$\mathrm{An}(\mathcal{V}) \to \mathrm{Fmb}(\mathcal{V}).$$

   and another one in the other direction $\mathrm{Fmb}(\mathcal{V}) \to \mathrm{An}(\mathcal{V})$ *which is a section of the former.*

3. *The tube functor*

$$\begin{aligned} \mathrm{An}(\mathcal{V}) &\longrightarrow \mathrm{An}(K) \\ (X,V) &\longmapsto]X[_V \end{aligned}$$

is left exact and continuous, giving rise to a morphism of sites $\mathrm{An}(K) \to \mathrm{An}(\mathcal{V})$.

Proof: All exactness properties follow from propositions 1.2.4 and 1.2.5. When a left exact functor preserves covering families (for given pretopologies) it is automatically continuous. All continuity assertion therefore easily follow from proposition 1.2.7. Since $\mathrm{Fmb}(\mathcal{V})$ is endowed with the coarse topology the functor in the first assertion is automatically cocontinuous. Finally, the functor of the second assertion is cocontinuous by definition. □

1.3 Strict neighborhoods

We just defined overconvergent varieties $(X \subset P \leftarrow V)$ but the notion of formal morphism is too rigid. We are actually only interested in what happens in a neighborhood of $]X[_V$ in V. Moreover, we want that, in the Monsky-Washnitzer situation (see example after definition 1.2.1), any morphism on the weak completions of the rings induces a morphism on the corresponding overconvergent varieties. In order to do that, we need more flexibility. Thus, we will introduce the notion of strict neighborhood and make them invertible in order to obtain the category of overconvergent varieties.

Let $(X \subset P \leftarrow V)$ be an overconvergent variety and W an analytic domain in V. If W is a neighborhood of $]X[_V$ in V, we will say that W is a *neighborhood* of X in V. The relation with Berthelot's definition of strict neighborhood is highlighted by the next result.

Recall (see appendix 4.2 for details) that an analytic variety V over K has a Grothendieck topology which is finer than the usual one (consisting of analytic domains). We will denote by V_G the corresponding space. We will call a covering of V admissible if it is admissible for this Grothendieck topology. For example, a covering $V = V_1 \cup V_2$ by two analytic domains will be admissible if and only if V_1 is a neighborhood of the points which are not in V_2 and conversely.

Proposition 1.3.1 *Let* $(X \subset P \leftarrow V)$ *be an overconvergent variety and* W *an analytic domain in* V. *Recall that* $\overline{X}$ *denotes the Zariski closure of* X *in* P. *Then,* W *is a neighborhood of* X *in* V *if and only if the covering*

$$]\overline{X}[_V =]\overline{X}[_W \cup]\overline{X}\setminus X[_V$$

is admissible. Actually, it will have an open refinement.

Proof: Assume first that W is a neighborhood of X in V. Replacing W by some open neighborhood, we may assume that W is open in V. In this case, the above covering is simply an open covering and we are done. Conversely, if the covering is admissible and $x \in]X[_V$, then $x \in]\overline{X}[_W$ but $x \notin]\overline{X}\setminus X[_V$. It follows that $]\overline{X}[_W$ is a neighborhood of x in $]\overline{X}[_V$, and *a fortiori* that W is a neighborhood of x in V. □

Recall that if V is a Hausdorff analytic variety over K, the set V_0 of rigid points of V inherits the structure of a rigid analytic variety. The inclusion $V_0 \hookrightarrow V$ induces an isomorphism of toposes $\widetilde{V_0} \simeq \widetilde{V_G}$ (see appendix 4.2 again).

Recall also (see definition 3.1.1 of [23]) that, if $X \subset P$ is a formal embedding, a *strict neighborhood* of $]X[_{P0}$ in $]\overline{X}[_{P0}$ is an admissible open subset V_0 of $]\overline{X}[_{P0}$ such that the covering

$$]\overline{X}[_{P0} = V_0 \cup]\overline{X} \backslash X[_{P0}$$

is admissible. We will also say that V_0 is a *strict neighborhood* of X in P.

Corollary 1.3.2 *If X is quasi-compact and $X \subset P$ is a formal embedding, there exists a cofinal family of paracompact neighborhoods V of X in P such that the family of all V_0 is a cofinal family of strict neighborhoods of X in P.*

Proof: The proposition tells us that $V \subset]\overline{X}[_P$ is a neighborhood of X in P if and only if the covering

$$]\overline{X}[_P = V \cup]\overline{X} \backslash X[_P$$

is admissible. This characterization is quite similar to the definition of strict neighborhoods above. Recall now that Berthelot defined what he calls *standard* strict neighborhoods (see [23], definition 3.4.3). One easily checks that their definition makes sense in Berkovich theory as well and that they are paracompact. In rigid geometry, it is shown in [23], proposition 3.4.1, that they define a cofinal system of strict neighborhoods. The analog result holds with the same proof in Berkovich theory. □

Definition 1.3.3 *A formal morphism*

$$\begin{array}{ccccccc}
X & \hookrightarrow & P' & \xleftarrow{\text{sp}} & P'_K & \xleftarrow{\lambda'} & V' \\
\| & & \downarrow v & & \downarrow v_K & & \downarrow u \\
X & \hookrightarrow & P & \xleftarrow{\text{sp}} & P_K & \xleftarrow{\lambda} & V
\end{array}$$

of overconvergent varieties is a strict neighborhood *if u is the inclusion of a neighborhood of X in V and $]X[_{V'} =]X[_V$.*

Note that there is no explicit condition on the morphism v. As the next proposition shows, strict neighborhoods act at two different levels: they will allow the replacement of P by some other formal scheme and the replacement of V by a neighborhood of X in V.

Proposition 1.3.4 *Any strict neighborhood of overconvergent varieties is the composition of a formal morphism*

$$(\mathrm{Id}_X \subset v, Id_{V'}) : (X \subset P' \leftarrow V') \to (X \subset P \leftarrow V')$$

that induces an equality on the tubes, and a formal morphism of the form

$$(\mathrm{Id}_X \subset \mathrm{Id}_P, u) : (X \subset P \leftarrow V') \to (X \subset P \leftarrow V).$$

where u is the inclusion of a neighborhood V' of X in V.

Proof: Follows from proposition 1.2.3. □

More precisely, a strict neighborhood splits as

$$\begin{array}{ccccccccc}
X & \hookrightarrow & P' & \xleftarrow{\mathrm{sp}} & P'_K & \longleftarrow & V' & \hookleftarrow &]X[_{V'} \\
\| & & \downarrow & & \downarrow & & \| & & \| \\
X & \hookrightarrow & P & \xleftarrow{\mathrm{sp}} & P_K & \longleftarrow & V' & \hookleftarrow &]X[_{V'} \\
\| & & \| & & \| & & \downarrow & & \| \\
X & \hookrightarrow & P & \xleftarrow{\mathrm{sp}} & P_K & \longleftarrow & V & \hookleftarrow &]X[_{V}.
\end{array}$$

Lemma 1.3.5 *We have the following results:*

1. *Any identity map is a strict neighborhood in* $\mathrm{An}(\mathcal{V})$.
2. *Any composition of strict neighborhoods is a strict neighborhood in* $\mathrm{An}(\mathcal{V})$.
3. *Any diagram*

$$\begin{array}{ccc}
 & & (X \subset P' \leftarrow V') \\
 & & \downarrow v \\
(Y \subset Q \leftarrow W) & \longrightarrow & (X \subset P \leftarrow V)
\end{array}$$

   in $\mathrm{An}(\mathcal{V})$*, where* v *is a strict neighborhood, can be completed into a commutative square*

$$\begin{array}{ccc}
(Y \subset Q' \leftarrow W') & \longrightarrow & (X \subset P' \leftarrow V') \\
\downarrow w & & \downarrow v \\
(Y \subset Q \leftarrow W) & \longrightarrow & (X \subset P \leftarrow V)
\end{array}$$

   where w *is a strict neighborhood.*

4. *Any diagram*

$$(Y \subset Q \leftarrow W) \overset{u_1}{\underset{v_2}{\rightrightarrows}} (X \subset P' \leftarrow V') \xrightarrow{w} (X \subset P \leftarrow V)$$

   where v *is a strict neighborhood and* $v \circ u_1 = v \circ u_2$ *can be extended on the left as*

$$(Y \subset Q' \leftarrow W') \xrightarrow{u} (Y \subset Q \leftarrow W) \overset{u_1}{\underset{v_2}{\rightrightarrows}} (X \subset P' \leftarrow V')$$

   where u *is a strict neighborhood such that* $v_1 \circ u = v_2 \circ u$.

Proof: First and second assertion are trivial. For the third one, we may simply choose the fibered product and for the last one, the equalizer. It is easily checked that these constructions do give strict neighborhoods as expected. □

Proposition 1.3.6 *The category* $\mathrm{An}(\mathcal{V})$ *admits calculus of right fractions with respect to strict neighborhoods.*

Proof: The lemma shows that all the conditions of ([19], I, 2.2.2) are satisfied. □

Definition 1.3.7 *The category of fractions* $\mathrm{An}^\dagger(\mathcal{V})$ *of* $\mathrm{An}(\mathcal{V})$ *with respect to strict neighborhoods is the* category of overconvergent varieties *over* $\mathcal{V}$.

Thus, a morphism from an overconvergent variety $(X' \subset P' \leftarrow V')$ to an overconvergent variety $(X \subset P \leftarrow V)$ is a commutative diagram

$$\begin{array}{ccccc}
X' & \hookrightarrow & P' & \leftarrow & V' \\
\| & & \uparrow & & \uparrow \\
X' & \hookrightarrow & P'' & \leftarrow & W' \\
\downarrow f & & \downarrow & & \downarrow u \\
X & \hookrightarrow & P & \leftarrow & V
\end{array}$$

where the upper map is a strict neighborhood.

In particular, we see that any morphism defines a morphism of algebraic varieties $f : X' \to X$ and a morphism of analytic varieties $u : W' \to V$ defined on some neighborhood of X' in V'. We will show that this pair satisfies a compatibility condition that is due to A. Besser (definition 4.4 of [12]):

Definition 1.3.8 *Let* $(X \subset P \overset{\lambda}{\leftarrow} V)$ *and* $(X' \subset P' \overset{\lambda'}{\leftarrow} V')$ *be two overconvergent varieties. Let* $f : X' \to X$ *be a morphism of algebraic varieties,* W' *a neighborhood of* X' *in* V' *and* $u : W' \to V$ *a morphism of analytic varieties. The pair* (f, u) *is* pointwise compatible *if*

$$\forall x' \in]X'[_{V'}, \quad \lambda(\widetilde{u(x')}) = f(\widetilde{\lambda'(x')}).$$

It is said to be geometrically pointwise compatible *if this property still holds after any isometric extension of* K.

Now we can state the next result that gives a more down-to-earth description of the category $\mathrm{An}^\dagger(\mathcal{V})$:

Proposition 1.3.9 *Let* $(X \subset P \overset{\lambda}{\leftarrow} V)$ *and* $(X' \subset P' \overset{\lambda'}{\leftarrow} V')$ *be two overconvergent varieties. Then any morphism between them induces a pair of geometrically pointwise compatible morphisms* (f, u). *Conversely, let* $f : X' \to X$ *be a morphism of algebraic varieties,* W' *a neighborhood of* X' *in* V' *and* $u : W' \to V$ *a morphism of analytic varieties. If* (f, u) *is geometrically pointwise compatible, it comes from a unique morphism of overconvergent varieties.*

Proof: As we saw above, a morphism is a commutative diagram

$$\begin{array}{ccccc}
X' & \hookrightarrow & P' & \overset{\lambda'}{\leftarrow} & V' \\
\| & & \uparrow & & \uparrow \\
X' & \hookrightarrow & P' & \overset{\lambda''}{\leftarrow} & W' \\
\downarrow f & & \downarrow & & \downarrow u \\
X & \hookrightarrow & P & \overset{\lambda}{\leftarrow} & V
\end{array}$$

where the upper map is a strict neighborhood. In particular, we have $]X'[_{W'} =]X'[_{V'}$. If $x' \in]X'[_{V'}$, then commutativity downstairs implies that $f(\widetilde{\lambda''(x')} = \lambda(\widetilde{u(x')}))$ and commutativity upstairs gives $\widetilde{\lambda'(x')} = \widetilde{\lambda''(x')}$. Thus we obtain pointwise compatibility. If we are given any isometric extension $K \hookrightarrow K'$, we can extend the whole situation and geometric pointwise compatibility follows.

Conversely, assume that we are given a geometrically pointwise compatible pair of morphisms (f, u) with $f : X' \to X$ and $u : W' \to V$. If we consider the diagonal embedding of X' into $P' \times P$ and similarly embed W' diagonally into $P'_K \times P_K$, we obtain a commutative diagram

$$\begin{array}{ccccc} X' & \hookrightarrow & P' & \xleftarrow{\lambda'} & V' \\ \| & & \uparrow{\scriptstyle p_1} & & \uparrow \\ X' & \hookrightarrow & P' \times P & \xleftarrow{\lambda''} & W' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle u} \\ X & \hookrightarrow & P & \xleftarrow{\lambda} & V. \end{array}$$

We want to show that the upper morphism is a strict neighborhood. In other words, we have to check that $]X'[_{W'} =]X'[_{V'}$. Since the first one sits as an analytic domain inside the second, it is sufficient to show that the induced map $]X'[_{W'} \to]X'[_{V'}$ is surjective. Thus, we are given $x' \in]X'[_{V'}$ and we want to show that it comes from some (unique) $y' \in]X'[_{W'}$.

First of all, since W' is a neighborhood of X' in V', we know that x' comes from a unique $y' \in W'$ and we want to show that $y' \in]X'[_{W'}$ or in other words that $\widetilde{\lambda''(y')} \in X'$. We may extend the basis to the residue field $\mathcal{K}(y')$ and assume that y' is rational so that $\widetilde{\lambda''(y')}$ is determined by its projections onto P' and P. Then, it follows from pointwise compatibility that

$$\widetilde{\lambda''(y')} = (\widetilde{\lambda'(x')}, f(\widetilde{\lambda'(x')}) \in X'. \quad \square$$

Note that it is possible that a non trivial embedding of an analytic domain induces a bijection on all rigid points: simply remove one non rigid point. This is why it does not seem reasonable to expect the above proposition to hold without the "geometric" compatibility.

This proposition shows that the formal schemes play a secondary role and we will therefore generally simply write (X, V) for an overconvergent variety and

$$(f, u) : (X', V') \to (X, V)$$

for a morphism. We will also generally assume that u is actually defined on V' since we may always replace it by a smaller neighborhood of X'. In general, there will not be any problem either in assuming that we actually have a formal morphism.

Example: Back again to our Monsky-Washnitzer situation (see the example following definition 1.2.1). We assume now that we are given two finitely presented algebras A and A' over $\mathcal{V}$ and we use the same notations as above. A morphism $f : X' \to X$ corresponds to an algebra homomorphism $A_k \to A'_k$. If A is smooth, this homomorphism lifts to a homomorphism $A^\dagger \to A'^\dagger$ which induces $A_\lambda \to A'_\mu$ for λ and μ close enough to 1 and therefore gives a morphism $u : V'_\mu \to V_\lambda \subset V$. Geometric pointwise compatibility is easily checked and we do get a morphism

$$(f, u) : (X', V') \to (X, V)$$

which in general is *not* a formal morphism.

Proposition 1.3.10 *Finite inverse limits exist in* $\mathrm{An}^\dagger(\mathcal{V})$ *and the canonical functor*

$$\mathrm{An}(\mathcal{V}) \to \mathrm{An}^\dagger(\mathcal{V})$$

is left exact.

Proof: Apply proposition 3.1 of [19] and its first corollary. □

In practice, the inverse limit of a diagram

$$X_i \hookrightarrow P_i \stackrel{\mathrm{sp}}{\leftarrow} P_{iK} \stackrel{\lambda_i}{\leftarrow} V_i$$

indexed by some finite set I is simply

$$\varprojlim X_i \hookrightarrow \prod P_i \stackrel{\mathrm{sp}}{\leftarrow} \prod P_{iK} \stackrel{\lambda}{\leftarrow} \varprojlim V_i.$$

Of course, if the maps in the diagram are formal morphisms, we may use inverse limits instead of products in the middle.

Proposition 1.3.11 *1. The functor*

$$\begin{aligned} \mathrm{Fmb}(\mathcal{V}) &\longrightarrow \mathrm{An}^\dagger(\mathcal{V}) \\ (X \subset P) &\longmapsto (X, P_K) \end{aligned}$$

is left exact.

2. The forgetful functor

$$\begin{aligned} \mathrm{An}^\dagger(\mathcal{V}) &\longrightarrow \mathrm{Sch}(k) \\ (X, V) &\longmapsto X \end{aligned}$$

(is left exact and) has a fully faithful left adjoint

$$\begin{aligned} \mathrm{Sch}(k) &\longrightarrow \mathrm{An}^\dagger(\mathcal{V}) \\ X &\longmapsto (X, \emptyset) \end{aligned}$$

3. The tube functor

$$\begin{aligned} \mathrm{An}^\dagger(\mathcal{V}) &\longrightarrow \mathrm{An}(K) \\ (X, V) &\longmapsto]X[_V \end{aligned}$$

is left exact.

Proof: The first assertion is obtained by composition. The second one follows from the fact that $\emptyset$ is an initial object. Finally, the last assertion follows from the above description of finite inverse limits. □

It is also important to remark that the assignments

$$(X \subset P \leftarrow V) \rightsquigarrow (X \subset P)$$

as well as

$$(X \subset P \leftarrow V) \rightsquigarrow V$$

are not functorial anymore.

Proposition 1.3.12 *A morphism $(f,u):(X',V')\to(X,V)$ is an isomorphism in* $\mathrm{An}^\dagger(\mathcal{V})$ *if and only if f is an isomorphism, u induces an isomorphism between neighborhoods of X' in V' and X in V, respectively, and the induced map $]f[_u :]X'[_{V'} \to]X[_V$ is surjective. If this is the case, then $]f[_u$ is actually an isomorphism.*

Proof: If (f,u) is an isomorphism, there exists an inverse (g,v) for (f,u) and it follows from proposition 1.3.9 that f is an isomorphism and that u induces an isomorphism between neighborhoods. Conversely, if f is an isomorphism and u induces an isomorphism between neighborhoods, they induce an embedding of analytic domains

$$]f[_u :]X'[_{V'} \hookrightarrow]X[_V.$$

We may assume that u itself is bijective. Then it is clear that the inverse g of f and the inverse v of u define a morphism if and only if $]f[_u$ is an isomorphism, which happens exactly when it is surjective. □

Corollary 1.3.13 *If*

$$X \hookrightarrow P \overset{\mathrm{sp}}{\leftarrow} P_K \overset{\lambda}{\leftarrow} V$$

is an overconvergent variety and λ factors through P'_K where P' is a formal subscheme of P containing X, we get an isomorphism

$$(X \subset P' \leftarrow V) \simeq (X \subset P \leftarrow V)$$

in $\mathrm{An}^\dagger(\mathcal{V})$. □

Example: In the Monsky-Washnitzer situation (example after definition 1.2.1), we can replace $P := \widehat{\mathbf{P}^N_{\mathcal{V}}}$ with the completion P' of the Zariski closure of $\mathrm{Spec}(A)$ in $\mathbf{P}^N_{\mathcal{V}}$ and get an isomorphic overconvergent variety. We see that X is open in P'_k and V is open in P'_K. This is a more pleasant situation to work with.

Recall that a formal blowing up induces an isomorphism on the generic fibers.

Corollary 1.3.14 *If $(X \subset P \leftarrow V)$ is an overconvergent variety and $P' \to P$ is a formal blowing up centered outside X, then the induced morphism*

$$(X \subset P' \leftarrow V) \to (X \subset P \leftarrow V)$$

is an isomorphism in $\mathrm{An}^\dagger(\mathcal{V})$. □

Corollary 1.3.15 *Any overconvergent variety is isomorphic in* $\mathrm{An}^\dagger(\mathcal{V})$ *to some overconvergent variety $(X \subset P \leftarrow V)$ where, if we denote as usual by $\overline{X}$ the Zariski closure of X in P, $\overline{X}\backslash X$ is a divisor in $\overline{X}$ and $]\overline{X}[_V = V$.* □

1.4 The overconvergent site

We introduced in the previous section the category of overconvergent varieties by making strict neighborhoods invertible. This category inherits a topology that we study now.

Recall that if C' is a site and $g: C' \to C$ is any functor, the *image topology* on C is the coarsest topology that makes g continuous.

Definition 1.4.1 *The* analytic topology *on* $\mathrm{An}^\dagger(\mathcal{V})$ *is the image of the analytic topology of* $\mathrm{An}(\mathcal{V})$*. The site is called the* overconvergent site*, the corresponding topos* $\mathcal{V}_{\mathrm{An}^\dagger}$ *is the* overconvergent topos *and its objects are called* overconvergent sheaves*.*

Proposition 1.4.2 *The analytic topology on* $\mathrm{An}^\dagger(\mathcal{V})$ *is defined by the following pretopology: families of formal morphisms*

$$\{(X \subset P_i \leftarrow V_i) \to (X \subset P \leftarrow V)\}_{i \in I}$$

where $\{V_i\}_{i\in I}$ *is a an open covering of a neighborhood of* X *in* V *and* $]X[_V = \cup]X[_{V_i}$*.*

Moreover, up to isomorphism, any such covering is a covering for the pretopology of $\mathrm{An}(\mathcal{V})$*.*

Proof: The last assertion follows from the fact that such a family is isomorphic in $\mathrm{An}^\dagger(\mathcal{V})$ to the family

$$\{(X \subset P \leftarrow V_i) \to (X \subset P \leftarrow V')\}_{i \in I}$$

with $V' = \cup V_i$.

Let us verify now that our families do define a pretopology. It is clear that the identity has this form. Moreover, transitivity is satisfied as one easily checks. We need to verify that base change is satisfied and this is done in two steps. First of all, base change by a formal morphism clearly gives a family of the same type: this follows from left exactness of our functors. Also, composition on the right with a strict neighborhood also has the same type.

Finally, since up to isomorphism in $\mathrm{An}^\dagger(\mathcal{V})$, any such family is a covering for the pretopology of $\mathrm{An}(\mathcal{V})$, it follows that our pretopology generates the coarsest topology making the canonical functor $\mathrm{An}(\mathcal{V}) \to \mathrm{An}^\dagger(\mathcal{V})$ continuous. □

Proposition 1.4.3 *The canonical functor*

$$\mathrm{An}(\mathcal{V}) \to \mathrm{An}^\dagger(\mathcal{V})$$

is the inverse image for an embedding of sites

$$\mathrm{An}^\dagger(\mathcal{V}) \hookrightarrow \mathrm{An}(\mathcal{V}).$$

This embedding induces an equivalence between $\mathcal{V}_{\mathrm{An}^\dagger}$ *and the full subcategory of sheaves* $\mathcal{F}$ *on* $\mathrm{An}(\mathcal{V})$ *such that*

$$\mathcal{F}(X \subset P, V) = \mathcal{F}(X \subset P', V')$$

whenever $(X \subset P', V') \to (X \subset P, V)$ *is a strict neighborhood.*

Proof: Since the functor is left exact and continuous, it provides a morphism of sites $\mathrm{An}^\dagger(\mathcal{V}) \to \mathrm{An}(\mathcal{V})$. Moreover, the universal property of the category of fractions tells us that this embedding induces an equivalence between $\widehat{\mathrm{An}^\dagger(\mathcal{V})}$ and the full subcategory of *presheaves* $\mathcal{F}$ such that $\mathcal{F}(X \subset P, V) = \mathcal{F}(X' \subset P', V')$ whenever

$$(X' \subset P', V') \to (X \subset P, V)$$

is a strict neighborhood. Thus, it only remains to show that a presheaf $\mathcal{F}$ on $\mathrm{An}^\dagger(\mathcal{V})$ is a sheaf if and only if its image in $\mathrm{An}(\mathcal{V})$ is a sheaf. This follows from the fact that, up to isomorphism, any covering for the pretopology of $\mathrm{An}^\dagger(\mathcal{V})$ is a covering in $\mathrm{An}(\mathcal{V})$. □

Proposition 1.4.4 *The site* $\mathrm{An}^\dagger(\mathcal{V})$ *is a standard site.*

Proof: It follows from proposition 1.3.10 that our site has fibered products and it remains to show that the topology is coarser than the canonical topology. We use the description of proposition 1.3.9. We let $(X, V) \in \mathrm{An}^\dagger(\mathcal{V})$ and we show that the presheaf

$$(X', V') \mapsto \mathrm{Hom}((X', V'), (X, V))$$

is a sheaf on $\mathrm{An}^\dagger(\mathcal{V})$. So we assume that we are given a covering

$$\{(X', V'_i) \to (X', V')\}_{i \in I}$$

for our pretopology and a compatible family of morphisms

$$\{(f, u_i) : (X, V'_i) \to (X, V)\}_{i \in I}.$$

Up to isomorphism, we may assume that $V' = \cup V'_i$ and glue the u_i's. □

We recall again that categories of schemes and formal schemes are endowed with the coarse topology (which makes specialization continuous).

Proposition 1.4.5 1. *The functor*

$$\begin{array}{ccc} \mathrm{Fmb}(\mathcal{V}) & \longrightarrow & \mathrm{An}^\dagger(\mathcal{V}) \\ (X \subset P) & \longmapsto & (X \subset P \leftarrow P_K) \end{array}$$

is left exact and continuous, giving rise to a morphism of sites

$$\mathrm{sp} : \mathrm{An}^\dagger(\mathcal{V}) \to \mathrm{Fmb}(\mathcal{V}).$$

2. *The forgetful functor*

$$\begin{array}{ccc} \mathrm{An}^\dagger(\mathcal{V}) & \longrightarrow & \mathrm{Sch}(k) \\ (X, V) & \longmapsto & X \end{array}$$

is left exact and continuous, giving rise to a morphism of sites

$$I : \mathrm{Sch}(k) \to \mathrm{An}^\dagger(\mathcal{V}).$$

3. *The tube functor*

$$\begin{array}{ccc} \mathrm{An}^\dagger(\mathcal{V}) & \longrightarrow & \mathrm{An}(K) \\ (X, V) & \longmapsto &]X[_V \end{array}$$

is left exact and continuous, giving rise to a morphism of sites $\mathrm{An}(K) \to \mathrm{An}^\dagger(\mathcal{V})$.

Proof: Exactness properties were shown in proposition 1.3.11. Using proposition 1.4.3, the continuity assertions follow from proposition 1.2.9. □

Proposition 1.4.6 *Let* $K \hookrightarrow K'$ *be an isometric embedding and* $\mathcal{V}'$, k' *denote the valuation ring and residue field of* K' *respectively. Then there is an extension functor*

$$\begin{array}{ccc} \mathrm{An}^\dagger(\mathcal{V}) & \longrightarrow & \mathrm{An}^\dagger(\mathcal{V}') \\ (X \subset P \leftarrow V) & \longmapsto & (X_{k'} \subset P_{\mathcal{V}'} \leftarrow V_{K'}) \end{array}$$

which is left exact and continuous, giving rise to a morphism of sites

$$\mathrm{An}^\dagger(\mathcal{V}') \to \mathrm{An}^\dagger(\mathcal{V}).$$

Proof: Follows directly from our definitions. □

Note that the analogous statement for $\mathrm{An}(\mathcal{V})$ is also true.

When T is a presheaf on a category C, we may consider the *restricted category* $C_{/T}$ whose objects are pairs (X, u) where X is an object of C and $u \in T(X)$. A morphism $(X', u') \to (X, u)$ is a morphism $f : X' \to X$ such that $T(f)(u) = u'$. Of course, when T is a representable presheaf identified with the object that it represents, this definition is compatible with the previous one.

Definition 1.4.7 *An* overconvergent presheaf *is a presheaf* T *on* $\mathrm{An}^\dagger(\mathcal{V})$*. An* overconvergent variety over T *is an object of* $\mathrm{An}^\dagger(T) := \mathrm{An}^\dagger(\mathcal{V})_{/T}$.

We will give some examples below.

Recall that if C is a site and $g : C' \to C$ any functor, then the *induced topology* on C' is the finest topology that makes g continuous. Note that when g is left exact and C' has fibered products, a family in C' is a covering family for the induced topology if and only if its image in C is a covering family. This applies in particular to the case of a restriction functor $C_{/T} \to C$ where C is a site and T a presheaf on C.

Definition 1.4.8 *If* T *is an overconvergent presheaf, then* $\mathrm{An}^\dagger(T)$*, endowed with the induced topology, is the* overconvergent site over T*. The corresponding topos* $T_{\mathrm{An}^\dagger}$ *is the* overconvergent topos over T *and its objects are called* overconvergent sheaves on T.

It is a general fact that any morphism of overconvergent presheaves $f : T' \to T$ will induce a morphism of toposes

$$f_{\mathrm{An}^\dagger} : T'_{\mathrm{An}^\dagger} \to T_{\mathrm{An}^\dagger}.$$

Note that if $\widetilde{T}$ denotes the sheaf associated with T, we obtain a canonical isomorphism $T_{\mathrm{An}^\dagger} \simeq \widetilde{T}_{\mathrm{An}^\dagger}$. In practice however, it is more convenient to work with presheaves whose descriptions are generally simpler.

As a first example of overconvergent presheaf, we can consider the case of an overconvergent variety (C, O) identified with the sheaf that it represents. We obtain the category of overconvergent varieties over (C, O). An object is simply a morphism $(X, V) \to (C, O)$ and a morphism in $\mathrm{An}^\dagger(C, O)$ is simply a usual morphism that commutes with the given ones. Of course, any morphism of overconvergent varieties $(f, u) : (C', O') \to (C, O)$ will induce a morphism of toposes

$$u_{\mathrm{An}^\dagger} : (C', O')_{\mathrm{An}^\dagger} \to (C, O)_{\mathrm{An}^\dagger}$$

with

$$u_{\mathrm{An}^\dagger}^{-1}(X, V) = (C' \times_C X, O' \times_O V).$$

As a particular case, if S is a formal $\mathcal{V}$-scheme, we will call

$$\mathrm{An}^\dagger(S) := \mathrm{An}^\dagger(S_k, S_K)$$

the *category of overconvergent varieties over* S. By functoriality, any morphism of formal $\mathcal{V}$-schemes $v : S' \to S$ provides us with a morphism of toposes

$$v_{\mathrm{An}^\dagger} : S'_{\mathrm{An}^\dagger} \to S_{\mathrm{An}^\dagger}.$$

The next proposition shows that we could replace everywhere $\mathrm{Spf}\mathcal{V}$ by some formal scheme S and get a relative theory. But we will not do that because it sounds simpler to see the relative situation as a particular instance of the restriction process.

Proposition 1.4.9 *Let S be a formal $\mathcal{V}$-scheme. Then, up to isomorphism, an overconvergent variety over S is a pair made of a formal embedding $X \subset P$ over S and a morphism of analytic varieties $\lambda : V \to P_K$ over S_K. And a morphism is just a formal morphism $(f \subset v, u) : (X' \subset P', V') \to (X \subset P, V)$ of overconvergent varieties which is defined over S.*

Proof: We know that, up to isomorphism, any morphism of overconvergent varieties is a formal morphism. For example, if $(X \hookrightarrow P \leftarrow V)$ is an overconvergent variety over S, we can embed X in $P \times S$ and V in $(P \times S)_K$ in order to get another overconvergent variety $(X \hookrightarrow P \times S \leftarrow V)$ over S which is isomorphic to the original one. Everything else follows. $\square$

If (C, O) is an overconvergent variety, the functor $I : \mathrm{Sch}(k) \to \mathrm{An}^\dagger(\mathcal{V})$ that appeared in proposition 1.4.5 extends formally to a morphism of sites

$$I_{C,O} : \mathrm{Sch}(C) \to \mathrm{An}^\dagger(C, O)$$

with $I_{C,O}^{-1}(X, V) = X$.

Proposition 1.4.10 *If (C, O) is an overconvergent variety, the functor*

$$I_{C,O*} : \widehat{\mathrm{Sch}(C)} \to (C, O)_{\mathrm{An}^\dagger}$$

is continuous and left exact, giving rise to a morphism of topos

$$U_{C,O} : (C, O)_{\mathrm{An}^\dagger} \to \widehat{\mathrm{Sch}(C)}.$$

And we have a sequence of adjoint functors

$$I_{C,O}^{-1} \quad , \quad I_{C,O*} = U_{C,O}^{-1} \quad , \quad U_{C,O*}.$$

with $U_{C,O} \circ I_{C,O} = \mathrm{Id}$. In particular, $I_{C,O}$ is an embedding of topos.

Proof: The functor $I_{C,O*}$ being a direct image of morphism of sites is automatically left exact. Moreover, it is also automatically continuous since we have the coarse topology on the left hand side. And one checks directly that $I_{C,O*}$ is fully faithful. $\square$

We will define now for a given overconvergent variety (C, O) and an algebraic variety X over C the overconvergent sheaf X/O as well as the corresponding site and topos. We will use the restriction morphism

$$j_{C,O} : (C, O)_{\mathrm{An}^\dagger} \to \mathcal{V}_{\mathrm{An}^\dagger}.$$

Definition 1.4.11 *If (C, O) is an overconvergent variety and X an algebraic variety over C, then*

$$X/O := j_{C,O\,!} I_{C,O\,*} X$$

is the overconvergent sheaf of overconvergent varieties over X above (C, O).

Thus, we may consider the category $\mathrm{An}^\dagger(X/O)$ of overconvergent varieties over X/O and the topos $(X/O)_{\mathrm{An}^\dagger}$. We decided not to mention C in the notations because it plays a non significant role but we must not forget that it is also a part of the data. If S is a formal $\mathcal{V}$-scheme and X is an S_k-scheme, we will also write $X/S := X/S_K$. Thus, we obtain the category $\mathrm{An}^\dagger(X/S)$ of overconvergent varieties over X/S and the corresponding topos $(X/S)_{\mathrm{An}^\dagger}$. Actually, if $S = \mathrm{Spec} R$, we will write X/R or X/R_K.

It should be remarked that any C-morphism $f : X' \to X$ induces, by functoriality, a morphism of overconvergent sheaves $X'/O \to X/O$ and therefore also a morphism of toposes

$$f_{\mathrm{An}^\dagger} : (X'/O)_{\mathrm{An}^\dagger} \to (X/O)_{\mathrm{An}^\dagger}.$$

In the particular case of the structural morphism, since $I_{C,O*}C = (C,O)$, we get a morphism of presheaves $X/O \to (C,O)$ giving rise to a canonical map

$$j_{X/O} : (X/O)_{\mathrm{An}^\dagger} \to (C,O)_{\mathrm{An}^\dagger}.$$

For further use, note that the functor $X \mapsto X/O$ is left exact as composite of two left exact functors. It means that if X_1 and X_2 are two algebraic varieties over X, we have

$$(X_1 \times_X X_2)/O = X_1/O \times_{X/O} X_2/O.$$

Proposition 1.4.12 *Let (C,O) be an overconvergent variety and X an algebraic variety over C. We have an equivalence of categories*

$$\mathrm{An}^\dagger(X/O) \simeq \mathrm{Sch}(X) \times_{\mathrm{Sch}(C)} \mathrm{An}^\dagger(C,O)$$

Proof: We denote by $\widehat{j}_{C,O}$ the morphism induced by $j_{C,O}$ on presheaves and consider the presheaf

$$\widehat{X/O} := \widehat{j}_{C,O!}I_{C,O*}X.$$

We will prove that

$$\mathrm{An}^\dagger(\widehat{X/O}) \simeq \mathrm{Sch}(X) \times_{\mathrm{Sch}(C)} \mathrm{An}^\dagger(C,O).$$

As a consequence, one easily sees that $\widehat{X/O}$ is actually a sheaf so that $X/O = \widehat{X/O}$ and the proof is finished. More precisely, this follows from the fact that the presheaf associated to the object (C,O) is a sheaf (the topology is coarser than the canonical topology) and the description of the pretopology in proposition 1.4.2.

By definition, an overconvergent variety over $\widehat{X/O}$ is given by an object (U,V) of $\mathrm{An}^\dagger(\mathcal{V})$ and a section of the presheaf $\widehat{j}_{C,O!}I_{C,O*}X$ on this object. Using the explicit description of the functor $\widehat{j}_{C,O!}$ (see appendix 4.1 for example), we see that such a section is given by some structural morphism

$$(U,V) \to (C,O)$$

and a section of $I_{C,O*}X$ on (U,V). By definition, this section corresponds to a morphism from U to X over C. Summarizing, we get an overconvergent variety (U,V) over (C,O) and a morphism $U \to X$ which is a factorization of the structural morphism $U \to C$ through X. $\square$

Said differently, we see that, up to isomorphism, an overconvergent variety over X/O is a pair of morphisms

$$(U \to X, \quad V \to O)$$

such that the composite

$$(U \to X \to C, \quad V \to O)$$

is a morphism of overconvergent varieties. Then, a morphism $(U', V') \to (U, V)$ of overconvergent varieties over X/O is a pair made of a morphism $f : U' \to U$ of algebraic varieties over X and a morphism $u : V' \to V$ of analytic varieties over O such that (f, u) is a morphism of overconvergent varieties over (C, O).

It also follows from the proposition that, given any overconvergent variety (X, V) over (C, O), there is a canonical morphism of presheaves $(X, V) \to X/O$ giving rise to a morphism of sites

$$(X, V)_{\mathrm{An}^\dagger} \to (X/O)_{\mathrm{An}^\dagger}.$$

Proposition 1.4.13 *Let $(C', O') \to (C, O)$ be a morphism of overconvergent varieties. Then,*

1. *If $X' \to C'$ is any morphism of algebraic varieties, we have*

$$X'/O' \simeq X'/O \times_{C'/O} (C', O').$$

2. *If $X \to C$ is any morphism and $X' := X \times_C C'$, we have*

$$X'/O' \simeq X/O \times_{(C,O)} (C', O').$$

Proof: The first assertion easily follows from the proposition. For the second one, we write

$$X'/O' = X'/O \times_{C'/O} (C', O') = (X \times_C C')/O \times_{C'/O} (C', O')$$
$$= X/O \times_{(C,O)} C'/O \times_{C'/O} (C', O') = X/O \times_{(C,O)} (C', O'). \quad \square$$

Proposition 1.4.14 *Let (C, O) be an overconvergent variety, $f : Y \to X$ be a morphism of algebraic varieties over C and (X', V') an overconvergent variety over X/O. Then,*

$$f_{\mathrm{An}^\dagger}^{-1}(X', V') = Y/O \times_{X/O} (X', V') = (Y \times_X X')/V'.$$

Proof: We have

$$Y/O \times_{X/O} (X', V') = Y/O \times_{X/O} X'/O \times_{X'/O} (X', V')$$
$$= (Y \times_X X')/O \times_{X'/O} (X', V') = (Y \times_X X')/V'. \quad \square$$

As an example, we can give a description of inverse image with respect to an immersion of algebraic varieties.

Corollary 1.4.15 *Let (C, O) be an overconvergent variety and $\alpha : Y \hookrightarrow X$ an immersion of algebraic varieties over C. Then, if $(X', V') \in \mathrm{An}^\dagger(X/O)$, we have*

$$\alpha_{\mathrm{An}^\dagger}^{-1}(X', V') = (Y', V')$$

where Y' is the inverse image of Y inside X'.

Proof: Since Y' is a subvariety of X', we have $Y'/V' = (Y', V')$. $\square$

We will also need a modified version of our sheaf X/O.

If $(X, V) \to (C, O)$ is a morphism of overconvergent varieties, we will denote by X_V/O the image *presheaf* of the canonical morphism $(X, V) \to X/O$ of overconvergent (pre-) sheaves. When $V = P_K$, we will write X_P instead of X_V. Thus, by definition, the category $\mathrm{An}^\dagger(X_V/O)$ is essentially the full subcategory of $\mathrm{An}^\dagger(X/O)$ consisting of all (X', V') such that the canonical map $X' \to X$ lifts to *some* morphism $(X', V') \to (X, V)$. In other words, it is the category of overconvergent varieties over (C, O) that factors through a specific $X' \to X$ and through a possible $V' \to V$. Another way to see it is to think of objects living in $\mathrm{An}^\dagger(X, V)$ with morphisms being in $\mathrm{An}^\dagger(X/O)$.

Of course, any morphism $(f, u) : (X', V') \to (X, V)$ of overconvergent varieties over (C, O) induces a morphism of analytic presheaves $f_u : X'_{V'}/O \to X_V/O$ which in turn gives a morphism of toposes

$$f_{u\mathrm{An}^\dagger} : (X'_{V'}/O)_{\mathrm{An}^\dagger} \to (X_V/O)_{\mathrm{An}^\dagger}.$$

We will need the following:

Proposition 1.4.16 *If*

$$\begin{array}{ccc} (X', V') & \longrightarrow & (C', O') \\ \downarrow & & \downarrow \\ (X, V) & \longrightarrow & (C, O) \end{array}$$

is a cartesian diagram, there is a canonical isomorphism

$$X'_{V'}/O' \simeq X_V/O \times_{(C,O)} (C', O').$$

Proof: By definition, X_V/O is the presheaf image of the canonical morphism $X/O \to (C, O)$. Thus if we extend the basis, we obtain, thanks the second part of proposition 1.4.13, that $X_V/O \times_{(C,O)} (C', O')$ is the presheaf image of the canonical morphism

$$X'/O' \simeq X/O \times_{(C,O)} (C', O') \to (C', O')$$

which is exactly $X'_{V'}/O'$. $\square$

1.5 The local section theorem

We show in this section that, under geometric conditions (properness and smoothness), a formal morphism is locally left invertible in $\mathrm{An}^\dagger(\mathcal{V})$. In other words, it is a covering morphism in the overconvergent site. We derive some important consequences. Unfortunately, we will need a result that is only valid for good analytic varieties. Thus, we make the following definition:

Definition 1.5.1 *An overconvergent variety* $(X \subset P \leftarrow V)$ *is said to be* good *if any point of* $]X[_V$ *has an affinoid neighborhood in* V*. We will say that a formal embedding* $X \subset P$ *is* good *if the overconvergent variety* (X, P_K) *is.*

We send the reader to the last part of appendix 4.2 for a review of standard properties of morphisms in Berkovich theory.

Recall that an analytic variety is said to be *good* if it is locally affinoid. The condition in the proposition therefore says that V is good in a neighborhood of X.

Example: In the Monsky-Washnitzer situation (example following definition 1.2.1), we do get a good overconvergent variety because $V := (\mathrm{Spec} A_K)^{\mathrm{an}}$ is algebraic and an algebraic variety is always good.

Recall that we call a morphism *boundaryless* if it is closed in the sense of Berkovich. This notion is local for the Grothendieck topology.

Proposition 1.5.2 *Let* $(X \subset P \leftarrow V)$ *be an overconvergent variety,*

$$(f, v) : (X' \subset P') \to (X \subset P)$$

a proper morphism of formal embeddings and V' *a neighborhood of* X *in* $P'_K \times_{P_K} V$*. Then* v *induces a boundaryless morphism between neighborhoods of* X' *and* X *in* V' *and* V *respectively. Moreover, if* $(X \subset P \leftarrow V)$ *is good, so is* $(X' \subset P' \leftarrow V')$*.*

Proof: Shrinking V' if necessary, we may assume that it is open in $P'_K \times_{P_K} V$. It follows from proposition 1.1.7 that v induces a boundaryless morphism from a neighborhood of X' in P' to a neighborhood of X in P. Pulling back to V and intersecting with V', we obtain a boundaryless morphism between neighborhoods of X' and X in V' and V respectively.

If $(X \subset P \leftarrow V)$ is good, we may assume that V itself is good and that the morphism $V' \to V$ is boundaryless. It follows that V' is good and therefore, that the overconvergent variety $(X' \subset P' \leftarrow V')$ is good. □

Recall that a morphism $W \to V$ between two analytic varieties is *universally flat* if it is, locally for the Grothendieck topology of V and W, of the form $\mathcal{M}(B) \to \mathcal{M}(A)$ with $A \to B$ flat (note that this is equivalent to the usual definition of being flat and staying flat after any base extension).

Proposition 1.5.3 *Let* $(X \subset P \overset{\lambda}{\leftarrow} V)$ *be an overconvergent variety,*

$$(f, v) : (X' \subset P') \to (X \subset P)$$

a flat morphism of formal embeddings and V' *a neighborhood of* X *in* $P'_K \times_{P_K} V$*. Then* v *induces a universally flat morphism between neighborhoods of* X' *and* X *in* V' *and* V*, respectively.*

Proof: Our condition means that v is flat in a neighborhood of X'. Thus, given any $x' \in]X'[_{V'}$, we can find affine neighborhoods Q' of $\tilde{x}'$ in P' and Q of $f(\tilde{x}')$ in P such that v induces a flat morphism $Q' \to Q$:

$$\begin{array}{ccccc} \tilde{x}' & \hookrightarrow & Q' & \hookrightarrow & P' \\ \downarrow & & \downarrow & & \downarrow v \\ f(\tilde{x}') & \hookrightarrow & Q & \hookrightarrow & P. \end{array}$$

The corresponding morphism $Q'_K \to Q_K$ is automatically universally flat. If we pull back along $\lambda : V \to P_K$ and intersect with V' upstairs, we obtain a universally flat morphism $W' \to W$ between analytic domains inside V' and V:

$$\begin{array}{ccccc} x' & \hookrightarrow & W' & \hookrightarrow & V' \\ \downarrow & & \downarrow & & \downarrow u \\ v(x') & \hookrightarrow & W & \hookrightarrow & V. \end{array}$$

It follows that $u : V' \to V$ is universally flat at x'. Since this is true for all points of $]X'[_{P'}$ and that universal flatness is a local notion (even Zariski local, unpublished result of Ducros), we may shrink again V and V' in order to get a universally flat morphism $V' \to V$. $\square$

Recall that a morphism $W \to V$ of analytic varieties $W \to V$ is *formally smooth* (resp. *formally étale*) if, locally for the Grothendieck topology, it satisfies the usual lifting condition (this is equivalent to quasi-smooth or rig-smooth).

Lemma 1.5.4 *Let $X \subset P$ be a smooth (resp. étale) a formal embedding. Then, there exists a formally smooth (resp. formally étale) neighborhood V of X in P_K.*

Proof: Let x be a point of $]X[_P$ and Q a smooth affine neighborhood of $\widetilde{x}$ in P. Since Ω^1_Q is locally free (resp. 0), so is $\Omega^1_{Q_K}$ and it follows from proposition 6.23 of [17] that Q_K is formally smooth (resp. formally étale). Then, $V \cap Q_K$ is also formally smooth (resp. formally étale) and we see that V is formally smooth (resp. formally étale) at x. This is true for any point x in $]X[_P$ and it follows that V is formally smooth (resp. formally étale) at each $x \in]X[_P$. Since formal smoothness (resp. formal étaleness) is a local notion, we may shrink V a little bit in order to get a formally smooth (resp. formally étale) neighborhood of $]X[_P$ in P_K. $\square$

Proposition 1.5.5 *Let $(X \subset P \leftarrow V)$ be an overconvergent variety,*

$$(f, v) : (X' \subset P') \to (X \subset P)$$

a smooth (resp. étale) morphism of formal embeddings and V' a neighborhood of X' in $P'_K \times_{P_K} V$. Then v induces a formally smooth (resp. formally étale) morphism between neighborhoods of X' and X in V' and V, respectively.

Proof: Thanks to proposition 1.5.3, we may assume that v induces a universally flat morphism $V' \to V$ between neighborhoods of X' and X in V' and P_K, respectively. Using proposition 6.27 of [17], we may therefore assume that $(X \subset P \leftarrow V)$ is reduced to the point $(\mathrm{Spec} k \subset \mathrm{Spf} \mathcal{V} \leftarrow \mathcal{M}(K))$, in which case, this is just the assertion of Lemma 1.5.4. $\square$

Recall that a morphism of analytic varieties is *smooth* (resp. *étale*) if it is formally smooth (resp. formally étale) and boundaryless. Again, this notion is local for the Grothendieck topology.

Corollary 1.5.6 *Let $(X \subset P \leftarrow V)$ be an overconvergent variety,*

$$(f, v) : (X' \subset P') \to (X \subset P)$$

a proper smooth (resp. finite étale) morphism of formal embeddings and V' a neighborhood of X' in $P'_K \times_{P_K} V$. Then v induces a smooth (resp. étale) morphism between neighborhoods of X' and X in V' and V, respectively.

Proof: Since a smooth (resp. étale) morphism is simply a formally smooth (resp. formally étale) boundaryless morphism, this assertion follows from propositions 1.5.2 and 1.5.5. □

We can now state the weak fibration theorem:

Proposition 1.5.7 (Weak Fibration Theorem) *Let $(X \subset P \leftarrow V)$ be an overconvergent variety, $v : P' \to P$ be a morphism of formal embeddings of X and V' a neighborhood of X' in $P'_K \times_{P_K} V$. If v is smooth (resp. étale) at X, the fibers of the induced morphism $]X[_{V'} \to]X[_V$ are strict open polydiscs (resp. $]X[_{V'} \simeq]X[_V$).*

Proof: Let $x \in]X[_V$ and $\mathcal{V}(x)$ be the ring of integers of $\mathcal{K}(x)$, the completed residue field at x. We may extend scalars by $\mathrm{Spf}(\mathcal{V}(x)) \to P$ and therefore assume that $P = \mathrm{Spf}(\mathcal{V})$, $V = \mathcal{M}(K)$ and $X = \mathrm{Spec}(k)$. We have to show that $]\widetilde{x}[_{P'}$ is an open polydisc. We may replace P' by any open neighborhood of $\widetilde{x}$ and therefore assume that there is an étale morphism $P' \to \widehat{\mathbf{A}}^d_{\mathcal{V}}$ sending $\widetilde{x}$ to 0. It follows from Lemma 4.4 of [6] that this morphism induces an isomorphism

$$]\widetilde{x}[_{P'} \simeq]0[_{\widehat{\mathbf{A}}^d_{\mathcal{V}}} = \mathbf{B}^d(0, 1^-). \quad \square$$

Lemma 1.5.8 *Let $(X \subset P \leftarrow V)$ be an overconvergent variety and $v : P' \to P$ be a morphism of formal embeddings of X that is separated at X. Then, there exists a neighborhood V' of X' in $P'_K \times_{P_K} V$ such that, if $u : V' \to V$ denotes the induced map, then for all $x \in]X[_V$, we have*

$$u^{-1}(]\widetilde{x}[_V) \cap V' =]\widetilde{x}[_{V'}.$$

In particular, we have

$$u^{-1}(]X[_V) \cap V' =]X[_{V'}.$$

Proof: Recall that our assumption means that the induced map $\overline{X}^{P'} \to \overline{X}^P$ is separated. We take $V' :=]\overline{X}^{P'}[_{P'_K \times_{P_K} V}$. It is therefore sufficient to show that

$$v^{-1}(\widetilde{x}) \cap \overline{X}^{P'} = \{\widetilde{x}\}$$

and then pull back. Thus, we are led to check that the dense open immersion

$$X \hookrightarrow v^{-1}(X) \cap \overline{X}^{P'}$$

is bijective. But the projection

$$v^{-1}(X) \cap \overline{X}^{P'} \to X$$

is separated as a pull-back of a separated map. A dense open immersion that admits a separated section is necessarily an isomorphism. □

We will need the following lemma which is inspired from proposition 3.7.5 of [3] (where the varieties are implicitly assumed to be good).

Lemma 1.5.9 *Let*

$$u : (V', x') \to (V, x)$$

be a smooth morphism of germs of good *analytic varieties over K. Any isomorphism of germs*

$$\varphi_x : (u^{-1}(x), x') \simeq (\mathbf{A}^d_{\mathcal{K}(x)}, 0)$$

over $\mathcal{K}(x)$ extends to an isomorphism of germs

$$(V', x') \simeq (\mathbf{A}^d_V, (0, x))$$

over (V, x).

Proof: We may assume that u comes from a morphism of affinoid algebras $A \to A'$ and that our isomorphism comes from a morphism

$$\varphi_x^* : \mathcal{K}(x)\{T_1/R, \ldots, T_d/R\} \to \mathcal{K}(x)\widehat{\otimes}_A A'.$$

for R big enough. Since the image of $\mathcal{O}_{V,x} \otimes_A A'$ is dense in $\mathcal{K}(x)\widehat{\otimes}_A A'$, there exist

$$\varphi_1, \ldots, \varphi_d \in \mathcal{O}_{V,x} \otimes_A A'$$

such that $\|\varphi_x^*(T_i) - \overline{\varphi}_i\| < R$.

After an automorphism of $\mathcal{K}(x)\{T_1/R, \ldots, T_d/R\}$, we may assume that $\varphi_x^*(T_i) = \overline{\varphi}_i$. Moreover, shrinking V (and consequently V') if necessary, we may also assume that $\varphi_1, \ldots, \varphi_d \in A'$ and we can consider the induced morphism $\varphi : V' \to \mathbf{A}^d_V$. By construction, it induces the inclusion φ_x between the fibers at x. It follows from Lemma 3.7.7 of [3] that this is an isomorphism in a neighborhood of x'. $\square$

It will be convenient to make the following definition:

Definition 1.5.10 *Let $(X \subset P \leftarrow V)$ be an overconvergent variety and $f : X' \to X$ a morphism of algebraic varieties. A* geometric realization *of f over V is a formal morphism of overconvergent varieties*

$$\begin{array}{ccccc} X' & \hookrightarrow & P' & \leftarrow & V' \\ \downarrow f & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \leftarrow & V \end{array}$$

where v is proper smooth at X' and V' is a neighborhood of X' in $P'_K \times_{P_K} V$. We will then say that f is (geometrically) realizable*.*

Note that, up to isomorphism, and unless we need special properties of V', we may always assume that $V' = P'_K \times_{P_K} V$. In particular, a geometric realization is completely determined by the morphism of formal schemes $v : P' \to P$. Finally, note that corollary 1.5.6 above says that a geometric realization induces a smooth morphism between a neighborhood of X' in V' and a neighborhood of X in V.

Theorem 1.5.11 *If (X, V) is a good overconvergent variety, any geometric realization $(X, V') \to (X, V)$ of Id_X has locally a section in $\mathrm{An}^\dagger(\mathcal{V})$.*

Proof: Shrinking V and V' if necessary we may assume that they are good and, thanks to corollary 1.5.6, that v induces a smooth morphism $u : V' \to V$. Shrinking them a little more, we may assume thanks to lemma 1.5.8 that $u^{-1}(]\widetilde{y}[_V) =]\widetilde{y}[_{V'}$ for all $y \in]X[_V$. In particular, any local section of u will define a local section in $\mathrm{An}^\dagger(\mathcal{V})$.

Since $u^{-1}(]X[_V) =]X[_{V'}$, it follows from proposition 1.5.7 that there exists for each $x \in]X[_V$, an isomorphism

$$\varphi_x : u^{-1}(x) \simeq \mathbf{B}^d_{\mathcal{K}(x)}(0, 1^-).$$

We set $x' := \varphi_x^{-1}(0)$. It follows from lemma 1.5.9 that there exists a neighborhood W' of x' in V' such that φ_x extends to some open immersion $\varphi : W' \hookrightarrow \mathbf{A}^d_V$. We may then chose for s the zero section of $\mathbf{A}^d_V$. $\square$

If (C, O) is an overconvergent variety and X is an algebraic variety over C, we introduced in definition 1.4.11 the sheaf X/O of overconvergent varieties on X over (C, O). If $(X, V) \to (C, O)$ is a morphism of overconvergent varieties, there is a canonical morphism of overconvergent sheaves $(X, V) \to X/O$ and we defined X_V/O as the image presheaf of the canonical morphism $(X, V) \to X/O$ of overconvergent (pre-) sheaves.

Corollary 1.5.12 *With the assumptions and notations of the theorem, given a morphism $(X, V) \to (C, O)$ of overconvergent varieties, we have an isomorphism*

$$(X_{V'}/O)_{\mathrm{An}^\dagger} \simeq (X_V/O)_{\mathrm{An}^\dagger}.$$

Proof: It follows from the theorem that the morphism $(X, V') \to (X, V)$ is an epimorphism of overconvergent sheaves. It follows that the morphism $\widetilde{X_{V'}/O} \hookrightarrow \widetilde{X_V/O}$ induced on their sheaf images inside X/O, which is a monomorphism, is actually an isomorphism of sheaves. Therefore, we get an equivalence on toposes (recall that if T is an overconvergent presheaf, we always have $T_{\mathrm{An}^\dagger} = \widetilde{T}_{\mathrm{An}^\dagger}$). $\square$

Definition 1.5.13 *If T is any overconvergent presheaf, the* good overconvergent site $\mathrm{An}^\dagger_g(T)$ *over T is the full subcategory of $\mathrm{An}^\dagger(T)$ consisting of good overconvergent varieties, endowed with the induced topology.*

We will denote by $T_{\mathrm{An}^\dagger_g}$ the corresponding *good overconvergent topos*. Note that the inclusion functor

$$\mathrm{An}^\dagger_g(T) \hookrightarrow \mathrm{An}^\dagger(T)$$

is left exact, continuous and cocontinuous. If T is an overconvergent presheaf, we will denote by T_g its restriction to $\mathrm{An}^\dagger_g(\mathcal{V})$. Then, there is an isomorphism of sites

$$\mathrm{An}^\dagger_g(\mathcal{V})_{/T_g} \simeq \mathrm{An}^\dagger_g(T).$$

Proposition 1.5.14 *Let (C, O) be a good overconvergent variety and X an algebraic variety over C. If $(X, V) \to (C, O)$ is a geometric realization of X over O, then (X, V) is a covering of X/O in $\mathrm{An}^\dagger_g(\mathcal{V})$.*

Proof: We have to show that, given any good overconvergent variety $(Y \subset Q \leftarrow W)$ over X/O, there exists locally, a morphism to $(X \subset P \leftarrow V)$:

$$\begin{array}{ccccc} Y & \hookrightarrow & Q & \leftarrow & W \\ \downarrow & & & & \downarrow \\ X & \hookrightarrow & P & \leftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ C & \hookrightarrow & S & \leftarrow & O \end{array}$$

We may assume that the map $(Y, W) \to (C, O)$ is formal and introduce the graph

$$(Y \subset Q' := Q \times_S P \leftarrow W' := W \times_{Q_K} P_K)$$

and the corresponding projections:

$$\begin{array}{ccccc} Y & \hookrightarrow & Q & \leftarrow & W \\ \| & & \uparrow & & \uparrow \\ Y & \hookrightarrow & Q' & \leftarrow & W' \\ \downarrow & & \downarrow & & \downarrow \\ X & \hookrightarrow & P & \leftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ C & \hookrightarrow & S & \leftarrow & O \end{array}$$

By construction, the upper morphism satisfies the hypothesis of theorem 1.5.11. Therefore, it has locally a section and we are done. $\square$

Corollary 1.5.15 *With the assumptions and notations of the proposition, there is an equivalence of toposes*

$$(X_V/O)_{\mathrm{An}_g^\dagger} \simeq (X/O)_{\mathrm{An}_g^\dagger}.$$

Proof: It follows from proposition 1.5.14 that $(X_V/O)_g$ is a covering of $(X/O)_g$ and therefore, the sheaf associated to $(X_V/O)_g$ is exactly $(X/O)_g$. $\square$

Chapter 2

Coefficients

In section 1, we define the realizations of an overconvergent sheaf and explain how one can recover an overconvergent sheaf from its realizations. These realizations live on the tubes and not on some neighborhood.

In section 2, we study the notion of coherence in a neighborhood of a subspace of a locally compact topological space and apply the results to overconvergent varieties.

In section 3, we introduce the sheaf of overconvergent functions and the notion of overconvergent crystal. We prove some standard properties of these crystals.

Section 4 is devoted to a comparison theorem between overconvergent stratifications (that will correspond to overconvergent crystals) and usual stratifications (that will correspond to modules with integrable connections). We show that the forgetful functor is fully faithful when we stick to coherent objects.

In section 5, we put together the results of chapter 1 and those of section 3 in order to connect overconvergent crystals to modules with connection. We will show that finitely presented overconvergent modules correspond to modules with overconvergent integrable connections. As a corollary, we get the equivalence between overconvergent modules of finite presentation and overconvergent isocrystals in the sense of Berthelot.

2.1 Realization of sheaves

It is always convenient to see a crystal as a family of usual sheaves with transition maps. We show here that the overconvergent sheaves admit such a description.

Let V be a topological space and T a subset of V. An inclusion $V'' \subset V'$ of open subsets of V is called a *T-isomorphism* if $V' \cap T = V'' \cap T$. A family V_i' of open subsets of V is called an open *T-covering* of an open subset V' if for each i, we have $V_i' \cap T \subset V'$ and also $V' \cap T \subset \cup_i V_i'$.

Lemma 2.1.1 *Let V be a topological space and T a subset of V. Then*

1. *The category* Open(V) *of open subsets of V admits calculus of right fractions with respect to T-isomorphisms.*

2. *The image topology on the category of fractions* $\mathrm{Open}(V)_T$ *is generated by the pretopology of open* T*-coverings.*

3. *The inclusion* $T \subset V$ *induces an equivalence of sites*

$$\mathrm{Open}(V)_T \simeq \mathrm{Open}(T)$$

when T *is equipped with the induced topology.*

Proof: It is clear that T-isomorphisms form a subcategory which is stable by intersection with an open subset. The first assertion formally follows. Now, the inclusion $T \subset V$ clearly induces an equivalence of categories

$$\mathrm{Open}(V)_T \simeq \mathrm{Open}(T).$$

And open T-coverings in $\mathrm{Open}(V)$ correspond to open coverings in T. $\square$

Any analytic variety V over K will be endowed with its analytic topology and we will denote by V_{an} the small topos of sheaves on V.

Proposition 2.1.2 *Let* (X, V) *be an overconvergent variety. The obvious functor* $V' \mapsto (X, V')$ *on* $\mathrm{Open}(V)$ *induces a functor*

$$\mathrm{Open}(V)_{]X[_V} \to \mathrm{An}^\dagger(X, V).$$

which is continuous, cocontinuous and left exact.

Proof: It is clear that if $V' \subset V''$ is an $]X[_V$-isomorphism, then $(X, V') \hookrightarrow (X, V'')$ is a strict neighborhood. Moreover, it follows directly from our definitions that the induced functor is continuous and cocontinuous: any open $]X[_V$-covering of some open subset $V' \subset V$ gives rise to an analytic open covering of (X, V') and conversely. It is also clearly left exact. $\square$

Corollary 2.1.3 *If* (X, V) *is an overconvergent variety, the functor* $V' \mapsto (X, V')$ *defines a morphism of sites*

$$\varphi_{X,V} : \mathrm{An}^\dagger(X, V) \to \mathrm{Open}(V)_{]X[_V} \simeq \mathrm{Open}(]X[_V).$$

and a morphism of toposes

$$\psi_{X,V} :]X[_{V\mathrm{an}} \to (X, V)_{\mathrm{An}^\dagger}$$

giving rise to a sequence of adjoint functors at the topos level:

$$\varphi_{X,V}^{-1} \quad , \quad \varphi_{X,V*} = \psi_{X,V}^{-1} \quad , \quad \psi_{X,V*}. \quad \square$$

Proposition 2.1.4 *If* $(f, u) : (X', V') \to (X, V)$ *is a morphism of overconvergent varieties, the diagram*

$$\begin{array}{ccc} (X', V')_{\mathrm{An}^\dagger} & \xrightarrow{\varphi_{X',V'}} &]X'[_{V'\mathrm{an}} \\ \downarrow & & \downarrow]f[_u \\ (X, V)_{\mathrm{An}^\dagger} & \xrightarrow{\varphi_{X,V}} &]X[_{V\mathrm{an}} \end{array}$$

is commutative.

Proof: This follows from the fact that, if W is an open subset of V, then $(f,u)^{-1}(X,W) = (X', u^{-1}(W))$. $\square$

Corollary 2.1.5 *If $(f,u) : (X', V') \to (X,V)$ is a morphism of overconvergent varieties and $\mathcal{F}$ any sheaf on $]X[_V$, one has*

$$\Gamma((X',V'), \varphi_{X,V}^{-1}\mathcal{F}) = \Gamma(]X'[_{V'},]f[_u^{-1}\mathcal{F}). \quad \square$$

Proposition 2.1.6 *If (X,V) is an overconvergent variety, we have*

$$\varphi_{X,V} \circ \psi_{X,V} = Id.$$

In particular, $\psi_{X,V}$ is an embedding of toposes.

Proof: We have

$$(\varphi_{X,V} \circ \psi_{X,V})^{-1}(\mathcal{F}) = \psi_{X,V}^{-1}(\varphi_{X,V}^{-1}\mathcal{F}) = \varphi_{X,V*}(\varphi_{X,V}^{-1}\mathcal{F}).$$

Thus, if V' is an open subset of V, we see that

$$\Gamma(]X[_{V'}, (\varphi_{X,V} \circ \psi_{X,V})^{-1}(\mathcal{F})) = \Gamma(]X[_{V'}, \varphi_{X,V*}(\varphi_{X,V}^{-1}\mathcal{F}))$$

$$= \Gamma((X,V'), \varphi_{X,V}^{-1}\mathcal{F}) = \Gamma(]X[_{V'}, \mathcal{F}). \quad \square$$

For further use, we introduce the following terminology:

Definition 2.1.7 *Let T be an overconvergent presheaf over an overconvergent variety (C,O), then the* projection *of $T_{\mathrm{An}^\dagger}$ onto $]C[_O$ is the canonical morphism of toposes*

$$p_T : T_{\mathrm{An}^\dagger} \to (C,O)_{\mathrm{An}^\dagger} \to]C[_{O\mathrm{an}}.$$

Deriving this functor will give the *absolute overconvergent cohomology.*

We now turn to the definition of the realizations.

Definition 2.1.8 *Let (X,V) be an overconvergent variety and $\mathcal{F}$ an overconvergent sheaf on (X,V). Then the the* realization *of $\mathcal{F}$ on $]X[_V$ is the sheaf*

$$\mathcal{F}_{X,V} := \varphi_{X,V*}\mathcal{F}.$$

If T is an overconvergent presheaf, (X,V) an overconvergent variety over T and $\mathcal{F}$ an overconvergent sheaf on T, the realization *$\mathcal{F}_{X,V}$ of $\mathcal{F}$ on (X,V) is defined as follows: we first take the inverse image of $\mathcal{F}$ by the restriction morphism*

$$(X,V)_{\mathrm{An}^\dagger} \to T_{\mathrm{An}^\dagger}$$

and then, its realization on (X,V).

We will often write $\mathcal{F}_V$ instead of $\mathcal{F}_{X,V}$. In the case $V = P_K$, we will write $\mathcal{F}_{X \subset P}$ or $\mathcal{F}_P$ and call it the *realization* of $\mathcal{F}$ on $X \subset P$ or P.

There is a very simple description of the realization of an overconvergent sheaf $\mathcal{F}$ on (X, V): note first that, since the topology of $]X[_V$ is induced by the topology of V, any open subset of $]X[_V$ is of the form $]X[_{V'}$ with V' open in V. Then, we simply have

$$\Gamma(]X[_{V'}, \mathcal{F}_{X,V}) = \mathcal{F}(X, V').$$

Also, using realizations, it is not difficult to describe the morphism $\varphi_{X,V}$. Of course, by definition, for any overconvergent sheaf $\mathcal{F}$ on (X, V), we have $\varphi_{X,V*}\mathcal{F} = \mathcal{F}_{X,V}$. But also, if $\mathcal{F}$ is a sheaf on $]X[_V$ and $(f, u) : (X', V') \to (X, V)$ is a morphism of overconvergent varieties, then

$$(\varphi_{X,V}^{-1}\mathcal{F})_{X',V'} =]f[_u^{-1}\mathcal{F}_{X,V}.$$

Now, if T is an overconvergent presheaf, $(f, u) : (X', V') \to (X, V)$ a morphism of overconvergent varieties over T and $\mathcal{F} \in T_{\mathrm{An}^\dagger}$, functoriality gives us a morphism

$$\phi_{fu} :]f[_u^{-1}\mathcal{F}_{X,V} \to \mathcal{F}_{Y,W}$$

on $]X'[_{V'}$. Of course the morphisms ϕ_{fu} satisfy the usual compatibility condition. These data uniquely determine $\mathcal{F}$ as the following proposition states.

Proposition 2.1.9 *If T is an overconvergent presheaf, the category $T_{\mathrm{An}^\dagger}$ is equivalent to the following category:*

1. *An object is a collection of sheaves $\mathcal{F}_{X,V}$ on $]X[_V$ for each $(X, V) \in \mathrm{An}^\dagger(T)$ and morphisms $\phi_{fu\mathcal{F}} :]f[_u^{-1}\mathcal{F}_{X,V} \to \mathcal{F}_{X',V'}$ for each $(f, u) : (X', V') \to (X, V)$, satisfying the usual cocycle condition and $(\mathcal{F}_{X,V})_{|V'} = \mathcal{F}_{X,V'}$ when V' is an open subset of V.*
2. *A morphism is a collection of morphisms $\mathcal{F}_{X,V} \to \mathcal{G}_{X,V}$ compatible with the morphisms ϕ_{fu}.*

Proof: This is completely standard. □

Proposition 2.1.10 *If T is an overconvergent presheaf, the topos $T_{\mathrm{An}^\dagger}$ has enough points. More precisely, if $(X, V) \in \mathrm{An}^\dagger(T)$ and $x \in]X[_V$, then the functor $\mathcal{F} \mapsto \mathcal{F}_{V,x}$ is a fiber functor and they form a conservative family.*

Proof: The point $x \in]X[_V$ defines a point of the topos $]X[_{V\mathrm{an}}$ and composition with $\psi_{X,V}$ followed by the structural morphism $(X, V)_{\mathrm{An}^\dagger} \to T_{\mathrm{An}^\dagger}$ gives a point of our topos. We have to show that this family of points is conservative. Since the points of a topological space form a conservative family, it is sufficient to note that the "family" of realization functors $\{\mathcal{F} \mapsto \mathcal{F}_{X,V}\}_{(X,V)\in\mathrm{An}^\dagger(T)}$ is faithful. □

Finally, we consider the case of an immersion of algebraic varieties.

Proposition 2.1.11 *Let (C, O) be an overconvergent variety, $\alpha : Y \hookrightarrow X$ an immersion over C, and $(X', V') \in \mathrm{An}^\dagger(X/O)$. Let Y' be the inverse image of Y in X' and $\alpha' : Y' \hookrightarrow X'$ the inclusion map. If $\mathcal{F}$ is a sheaf on $\mathrm{An}^\dagger(X/O)$, then*

$$(\alpha_{\mathrm{An}^\dagger *}\mathcal{F})_{V'} =]\alpha'[_*\mathcal{F}_{V'}.$$

Proof: We saw in corollary 1.4.15 that $\alpha_{\mathrm{An}^\dagger}^{-1}(X', V') = (Y', V')$, and it follows that $(\alpha_{\mathrm{An}^\dagger *}\mathcal{F})(U, V) = \mathcal{F}(Y', V')$. And the same is true for any open subset of V'. Thus we see that

$$i_{X'*}(\alpha_{\mathrm{An}^\dagger *}\mathcal{F})_{V'} = i_{Y'*}\mathcal{F}_{V'} = i_{X'*}]\alpha'[_*\mathcal{F}_{V'}.$$

Pulling back by $i_{X'}$ gives the result. □

Surprisingly enough, we obtain the following corollary (heuristically, we expect exactness for closed immersions):

Corollary 2.1.12 *With the assumptions and notations of the proposition, if $\alpha : Y \hookrightarrow X$ is an open immersion, then $\alpha_{\mathrm{An}^\dagger *}$ is exact.*

Proof: In this case $]\alpha'[_*$ is the inclusion of a closed subset. □

2.2 Coherent sheaves

We prove here some general results about coherent sheaves on locally compact spaces and we apply them to Berkovich analytic varieties.

We first need to make clear some terminology. A topological space is said to be *compact* if it is quasi-compact and Hausdorff. A topological space is said to be *locally compact* if any point has a compact neighborhood. A locally compact space need not be Hausdorff but nevertheless, any point has a basis of compact neighborhoods. Recall that a topological space is *paracompact* (resp. *countable at infinity*) if it is Hausdorff and any open covering has a locally finite open refinement (resp. if it is a countable union of compact subsets). Locally compact spaces that are countable at infinity are always paracompact.

Proposition 2.2.1 *Let $(V, \mathcal{O}_V)$ be a ringed space and $i : T \hookrightarrow V$ the inclusion of a subspace. Assume either that V is paracompact or that V is Hausdorff and T is compact. If $\mathcal{F}$ is an $\mathcal{O}_V$-module of finite presentation and $\mathcal{G}$ any $\mathcal{O}_V$-module, the canonical map*

$$\varinjlim \mathrm{Hom}_{\mathcal{O}_{V'}}(\mathcal{F}_{|V'}, \mathcal{G}_{|V'}) \to \mathrm{Hom}_{i^{-1}\mathcal{O}_V}(i^{-1}\mathcal{F}, i^{-1}\mathcal{G}),$$

where V' runs through all open neighborhoods of T, is an isomorphism.

Proof: Since $\mathcal{F}$ is finitely presented, we have

$$i^{-1}\mathcal{H}\mathrm{om}_{\mathcal{O}_V}(\mathcal{F}, \mathcal{G}) = \mathcal{H}\mathrm{om}_{i^{-1}\mathcal{O}_V}(i^{-1}\mathcal{F}, i^{-1}\mathcal{G})$$

and therefore,

$$\mathrm{Hom}_{i^{-1}\mathcal{O}_V}(i^{-1}\mathcal{F}, i^{-1}\mathcal{G}) = \Gamma(T, i^{-1}\mathcal{H}\mathrm{om}_{\mathcal{O}_V}(\mathcal{F}, \mathcal{G})).$$

With our hypothesis, it follows from [22], proposition 2.5, that

$$\Gamma(T, i^{-1}\mathcal{H}\mathrm{om}_{\mathcal{O}_V}(\mathcal{F}, \mathcal{G})) = \varinjlim \Gamma(V', \mathcal{H}\mathrm{om}_{\mathcal{O}_V}(\mathcal{F}, \mathcal{G})_{|V'})$$

and we know that local hom commute with localization. □

Corollary 2.2.2 *With the assumptions and notations of the proposition, if $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_V$-modules of finite presentation such that $i^{-1}\mathcal{F} = i^{-1}\mathcal{G}$, there exists an open neighborhood V' of T in V such that $\mathcal{F}_{|V'} = \mathcal{G}_{|V'}$.* □

Proposition 2.2.3 *Let $(V, \mathcal{O}_V)$ be a locally compact ringed space and $i : T \hookrightarrow V$ the inclusion of a locally closed subspace. If $\mathcal{O}_V$ is a coherent ring, then $i^{-1}\mathcal{O}_V$ is also a coherent ring.*

Proof : After replacing V by an open subset, we may assume that T is closed in V. We have to show that if W is an open subset of V and $m : i^{-1}\mathcal{O}_W^N \to i^{-1}\mathcal{O}_W$ a morphism, then $\ker m$ is of finite type. Since V is locally compact, given any $x \in T$, there exists an open subset $\mathcal{U}$ of V and a compact subset K of V such that $x \in \mathcal{U} \subset K \subset W$. Since $K \cap T$ is compact, it follows from proposition 2.2.1 that there exists an open neighborhood $\mathcal{U}'$ of $K \cap T$ in W and a morphism $n : \mathcal{O}_{\mathcal{U}'}^N \to \mathcal{O}_{\mathcal{U}'}$ whose restriction to $K \cap T$ coincides with the restriction of m. The same thing is obviously true over $\mathcal{U} \cap T$. Since $\mathcal{O}_V$ is coherent, the kernel of n is of finite type. Since pull back is exact, the kernel of the restriction of m to $\mathcal{U} \cap T$ is also of finite type. And we are done. $\square$

Proposition 2.2.4 *Let $(V, \mathcal{O}_V)$ be a locally compact ringed space and $i : T \hookrightarrow V$ the inclusion of a closed subspace. Assume that V is countable at infinity or that T is compact. If $\mathcal{F}$ is an $i^{-1}\mathcal{O}_V$-module of finite presentation, there exists an open neighborhood V' of T in V and an $\mathcal{O}_{V'}$-module of finite presentation $\mathcal{G}$ such that $\mathcal{F} = i_{V'}^{-1}\mathcal{G}$.*

Proof: Let $\mathcal{F}$ be an $i^{-1}\mathcal{O}_V$-module of finite presentation. By local compactness, for each $x \in T$, there exists a sequence $x \in \mathcal{U} \subset K \subset W$ with $\mathcal{U}, W$ open in V and K compact and a morphism $m : i^{-1}\mathcal{O}_W^r \to i^{-1}\mathcal{O}_W^s$ with $\mathcal{F}_{|W \cap T} = \mathrm{coker}(m)$. Of course, we may assume that W is Hausdorff. Since $K \cap T$ is compact, it follows from proposition 2.2.1 that there exists an open neighborhood $\mathcal{U}'$ of $K \cap T$ in V and a morphism $n : \mathcal{O}_{\mathcal{U}'}^r \to \mathcal{O}_{\mathcal{U}'}^s$ whose restriction to $K \cap T$ coincides with the restriction of m. Let $\mathcal{G}' := \mathrm{coker} n$. We may replace $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{U}'$ in order to have $\mathcal{U} \subset \mathcal{U}'$. If we set $\mathcal{G} := \mathcal{G}'_{|\mathcal{U}}$, we have $i^{-1}\mathcal{G} = \mathcal{F}_{|\mathcal{U} \cap T}$.

Thus, if we assume that T is compact, we see that we can find a finite open covering $\{\mathcal{U}_k\}_{k=1}^n$ of T in V, and for each k an $\mathcal{O}_{\mathcal{U}_k}$ module of finite presentation $\mathcal{G}_k$ such that $i^{-1}\mathcal{G}_k = \mathcal{F}_{|\mathcal{U}_k \cap T}$. More precisely, there exists for each k an open subset $\mathcal{U}'_k$ of V with $\mathcal{U}_k \subset \mathcal{U}'_k$, a compact subset K_k of V with $K_k \cap T \subset \mathcal{U}'_k$ and a finitely presented module $\mathcal{G}'_k$ on $\mathcal{U}'_k$ whose restriction to $K_k \cap T$ coincides with the restriction of $\mathcal{F}$. In particular, the restrictions of $\mathcal{G}'_k$ and $\mathcal{G}'_l$ to $K_k \cap K_l \cap T$ coincide. It follows from Corollary 2.2.2 that there exists an open neighborhood $\mathcal{U}'_{kl}$ of $K_k \cap K_l \cap T$ in V such that $(\mathcal{G}'_k)_{|\mathcal{U}'_{kl}} = (\mathcal{G}'_l)_{|\mathcal{U}'_{kl}}$. Now, since there are only a finite number of them, we may shrink each $\mathcal{U}'_k$ in order to have for each pair (k, l), $\mathcal{U}'_k \cap \mathcal{U}'_l \subset \mathcal{U}'_{kl}$. We can then, for each k, replace $\mathcal{U}_k$ by $\mathcal{U}_k \cap \mathcal{U}'_k$ and still get a covering of T. We have for each pair (k, l), $(\mathcal{G}_k)_{|\mathcal{U}_k \cap \mathcal{U}_l} = (\mathcal{G}_l)_{|\mathcal{U}_k \cap \mathcal{U}_l}$. It follows that the $\mathcal{G}_k$ glue together in order to give a finitely presented $\mathcal{O}_{V'}$-module $\mathcal{G}$ such that $\mathcal{F} = i_{V'}^{-1}\mathcal{G}$.

We now consider the second case, namely we assume that V is countable at infinity. In other words, we assume that V is an increasing union of compact subsets $K_n, n \in \mathbf{N}$. Since, for each $n \in \mathbf{N}$, $T \cap K_n$ is compact in K_n, it follows from the first case that the restriction of our finitely presented $i^{-1}\mathcal{O}_V$-module $\mathcal{F}$ to $K_n \cap T$ extends to some finitely presented module $\mathcal{G}_n$ on some open neighborhood V_n of $K_n \subset T$ in V. By induction, we may assume that $V_n \subset V_{n+1}$ and, using Corollary 2.2.2 again, that $\mathcal{G}_{n+1|V_n} = \mathcal{G}_n$ so that they glue in order to give a finitely presented module $\mathcal{G}$ on $V' = \cup V_n$. $\square$

If $\mathcal{O}_V$ is a sheaf of rings on some topological space V, we write $\mathrm{Coh}(\mathcal{O}_V)$ for the category of coherent $\mathcal{O}_V$-modules on V.

Corollary 2.2.5 *Let $(V, \mathcal{O}_V)$ be a locally compact ringed space with $\mathcal{O}_V$ coherent. Let $i : T \hookrightarrow V$ be the inclusion of a closed subspace. Assume that V is countable at infinity or that V is Hausdorff and T compact. Then, the restriction functors* $\mathrm{Coh}(\mathcal{O}_{V'}) \to \mathrm{Coh}(i^{-1}\mathcal{O}_V)$ *where V' runs trough all inclusions of open neighborhoods of T in V induce an equivalence of categories*

$$\varinjlim \mathrm{Coh}(\mathcal{O}_{V'}) \simeq \mathrm{Coh}(i^{-1}\mathcal{O}_V).$$

Proof: Using the previous proposition, this is an immediate consequence of proposition 2.2.1. $\square$

We will now apply these results to our overconvergent varieties. Such an object is made of a locally closed immersion $X \hookrightarrow P$ of an algebraic variety into a formal scheme and a morphism $\lambda : V \to P_K$ of analytic varieties. We are mainly interested in the inclusion of the tube

$$i_X :=]X[_V := \lambda^{-1}\mathrm{sp}^{-1}(X) \hookrightarrow V.$$

We have the first fundamental result:

Proposition 2.2.6 *If (X, V) is an overconvergent variety, then $i_X^{-1}\mathcal{O}_V$ is a coherent ring.*

Proof: Since an analytic variety is locally compact, this is a direct application of proposition 2.2.3. $\square$

In order to study coherent modules, we need to introduce now some topological properties of overconvergent varieties.

Definition 2.2.7 *An overconvergent variety (X, V) is said to be*

1. paracompact *(resp.* countable at infinity*), if there exists a neighborhood V' of X in V which is paracompact (resp. countable at infinity) with $]X[_V$ closed in V'.*
2. Hausdorff *(resp.* compact*) if there exists a neighborhood V' of X in V which is Hausdorff (resp. compact).*

We will need the notion of *closed tube* of radius $\eta < 1$ with $\eta \in \sqrt{|K^\times|}$ for a given overconvergent variety $X \hookrightarrow P \overset{\lambda}{\leftarrow} V$. When $P = \mathrm{Spf}(A)$ and

$$X := \{x \in P, \left\{ \begin{array}{ll} \forall i = 1, \ldots, r, & \overline{f}_i(x) = 0 \\ \exists j = 1, \ldots, s, & \overline{g}_j(x) \neq 0 \end{array} \right. \},$$

we have

$$[X]_{V\eta} := \{x \in V, \left\{ \begin{array}{ll} \forall i = 1, \ldots, r, & |f_i(\lambda(x))| \leq \eta \\ \exists j = 1, \ldots s, & |g_j(\lambda(x))| = 1 \end{array} \right. \}.$$

This is *not* independent of the description of X but these closed tubes glue for η close to 1 when (X, V) is compact (see proposition 2.3.2 of [23] for the rigid analog).

Proposition 2.2.8 *Any compact overconvergent variety is countable at infinity.*

Proof: We may assume that V itself is compact. We have $]X[_V$ closed in $]\overline{X}[_V$ and $]\overline{X}[_V$ open in V. It is therefore sufficient to remark that the tube $]\overline{X}[_V$ is the increasing union of the closed tubes $[\overline{X}]_{V\eta}$, that these closed tubes are closed subsets of V, which is compact, and are therefore compact themselves. Of course, there exists a countable cofinal subfamily. $\square$

Corollary 2.2.9 *1. If $X \subset P$ is a formal embedding with X quasi-compact, the associated overconvergent variety (X, P_K) is countable at infinity.*

2. Locally, any overconvergent variety (X, V) is countable at infinity.

Proof: For the first assertion, we may replace P with a quasi-compact neighborhood of $\overline{X}$ and assume that P itself is quasi-compact in which case, P_K is compact. For the second one, it is sufficient to use the fact that V is locally compact. $\square$

The next step consists in applying the results of corollary 2.2.5.

Proposition 2.2.10 *If (X, V) is an overconvergent variety which is countable at infinity, we have an equivalence of categories*

$$\varinjlim \mathrm{Coh}(\mathcal{O}_{V'}) \simeq \mathrm{Coh}(i_X^{-1}\mathcal{O}_V)$$

when V' runs trough all inclusions of open neighborhoods of X in V.

Proof: This is a particular case of corollary 2.2.5. $\square$

With the use of Berkovich theory, we do not really need Berthelot's $j^\dagger$ construction because strict neighborhoods are replaced with standard neighborhoods. We make this more precise now.

Lemma 2.2.11 *Let $i : T \hookrightarrow V$ be the inclusion of a closed subset into a topological space whose points are all closed. If $\mathcal{F}$ is any sheaf on V, we have*

$$\varinjlim j'_* j'^{-1}\mathcal{F} = i_{X*} i_X^{-1}\mathcal{F},$$

where j' runs through all immersions of neighborhoods of T in V.

Proof: In a topological space whose points are closed, any subset is an intersection of open subsets. Then, the assertion easily follows by looking at the stalks. $\square$

Proposition 2.2.12 *Let (X, V) be an overconvergent variety and $\mathcal{F}$ any sheaf on V. If $]X[_V$ is closed in V, we have*

$$\varinjlim j'_* j'^{-1}\mathcal{F} = i_{X*} i_X^{-1}\mathcal{F}.$$

where j' runs through all immersions of neighborhoods of X in V.

Proof: Use lemma 2.2.11. $\square$

Note that the condition in the proposition is really mild because we may always replace V with $]\overline{X}[_V$ and that $]X[_V$ is closed in $]\overline{X}[_V$.

We consider now the functor $j^\dagger$ of Berthelot (see proposition 5.1.2 of [23]) that we will denote by $j_0^\dagger$: if $X \subset P$ is a formal embedding, V_0 a strict neighborhood of $]X[_{P0}$ in $]\overline{X}[_{P0}$ and $\mathcal{F}$ is any sheaf on V, we set

$$j_{X0}^\dagger \mathcal{F} := \varinjlim j'_{0*} j_0'^{-1} \mathcal{F}$$

where j'_0 runs through all immersions of strict neighborhoods of $]X[_{P0}$ in V_0.

Corollary 2.2.13 *If $X \subset P$ be a* good *formal embedding with X quasi-compact, there is a canonical equivalence of categories*

$$\mathrm{Coh}(i_X^{-1}\mathcal{O}_{P_K}) \simeq \mathrm{Coh}(j_{X0}^\dagger \mathcal{O}_{]X[_{P0}})$$

Proof: We showed in corollary 1.3.2 that neighborhoods of $]X[_P$ contained in $]\overline{X}[_P$ correspond essentially to strict neighborhoods. Using theorem 5.4.4 of [23] and proposition 2.2.10, we are reduced to showing that if V is a neighborhood of X in P, then

$$\mathrm{Coh}(\mathcal{O}_V) \simeq \mathrm{Coh}(\mathcal{O}_{V_0}).$$

Since we may replace V_0 by V_G on the right and assume that V is good, this follows from proposition 1.3.4 of [3]. $\square$

The reader who knows about rigid cohomology should be convinced now that $i_X^{-1}\mathcal{O}_V$ is the correct sheaf of ring that we should study now.

2.3 Crystals

We introduce now the ring of overconvergent function as well as the notion of overconvergent crystal.

Proposition 2.3.1 *Any morphism $(f,u) : (X', V') \to (X, V)$ of overconvergent varieties induces a morphism of ringed spaces*

$$(]f[_u^\dagger,]f[_{u*}) : (]X'[_{V'}, i_{X'}^{-1}\mathcal{O}_{V'}) \to (]X[_V, i_X^{-1}\mathcal{O}_V).$$

and we have for all $i_X^{-1}\mathcal{O}_V$-modules $\mathcal{F}$,

$$]f[_u^\dagger \mathcal{F} = i_{X'}^{-1} u^* i_{X*}\mathcal{F}.$$

Proof: Pulling back by $i_{X'}$ the canonical map $u^{-1}\mathcal{O}_V \to \mathcal{O}_{V'}$ gives

$$]f[_u^{-1} i_X^{-1}\mathcal{O}_V = i_{X'}^{-1} u^{-1}\mathcal{O}_V \to i_{X'}^{-1}\mathcal{O}_{V'}$$

and we get a morphism of ringed spaces as asserted. If $\mathcal{F}$ is a sheaf on $]X[_V$, we have

$$]f[_u^{-1}\mathcal{F} =]f[_u^{-1} i_X^{-1} i_{X*}\mathcal{F} = i_{X'}^{-1} u^{-1} i_{X*}\mathcal{F}.$$

Thus, if $\mathcal{F}$ is a $i_X^{-1}\mathcal{O}_V$-module, we see that

$$]f[_u^\dagger \mathcal{F} = i_{X'}^{-1}\mathcal{O}_{V'} \otimes_{i_{X'}^{-1}u^{-1}\mathcal{O}_V} i_{X'}^{-1} u^{-1} i_{X*}\mathcal{F}$$

$$= i_{X'}^{-1}(\mathcal{O}_{V'} \otimes_{u^{-1}\mathcal{O}_V} u^{-1} i_{X*}\mathcal{F}) = i_{X'}^{-1} u^* i_{X*}\mathcal{F}. \quad \square$$

Actually, in practice, we will often write $u^\dagger$ and u_* instead of $]f[_u^\dagger$ and $]f[_{u*}$. On the other hand, when $u = \mathrm{Id}_V$, we will write $]f[_V^\dagger$ and $]f[_{V*}$.

For further use, note the following:

Corollary 2.3.2 *Let (X, V) be an overconvergent variety, $\mathcal{F}$ a $i_X^{-1}\mathcal{O}_V$-module and $\alpha : Y \hookrightarrow X$ the inclusion of a subvariety, then $]\alpha[_V^\dagger \mathcal{F} =]\alpha[_V^{-1}\mathcal{F}$.*

Proof: We have

$$]\alpha[_V^\dagger \mathcal{F} = i_{X'}^{-1} i_{X*}\mathcal{F} =]\alpha[_V^{-1} i_X^{-1} i_{X*}\mathcal{F} =]\alpha[_V^{-1}\mathcal{F}. \quad \square$$

The next result also follows from the proposition:

Corollary 2.3.3 *The presheaf of rings*

$$\mathcal{O}_{\mathcal{V}}^\dagger : (X, V) \mapsto \Gamma(]X[_V, i_X^{-1}\mathcal{O}_V)$$

is a sheaf on $\mathrm{An}^\dagger(\mathcal{V})$. $\square$

Definition 2.3.4 *The sheaf $\mathcal{O}_{\mathcal{V}}^\dagger$ is the* sheaf of overconvergent functions. *If T is an overconvergent presheaf, the restriction $\mathcal{O}_T^\dagger$ of $\mathcal{O}_{\mathcal{V}}^\dagger$ to $\mathrm{An}^\dagger(T)$ is the* sheaf of overconvergent functions on T. *An $\mathcal{O}_T^\dagger$-module will be called an* overconvergent module *on T.*

By definition, the realization of $\mathcal{O}_{\mathcal{V}}^\dagger$ on some overconvergent variety (X, V) is just $i_X^{-1}\mathcal{O}_V$. Note also that if $f : T' \to T$ is a morphism of analytic presheaves, we have

$$f_{\mathrm{An}^\dagger}^{-1}\mathcal{O}_T^\dagger = \mathcal{O}_{T'}^\dagger.$$

If E is an overconvergent module on (X, V), then E_V is a $i_X^{-1}\mathcal{O}_V$-module and for all morphisms

$$(f, u) : (X', V') \to (X, V),$$

the morphism $\phi_{uE} :]f[_u^{-1} E_V \to E_{V'}$ on $]X'[_{V'}$ extends to a $i_{X'}^{-1}\mathcal{O}_{V'}$-linear map

$$\phi_{uE}^\dagger :]f[_u^\dagger E_V \to E_{V'}.$$

Proposition 2.3.5 *Let (X, V) be an overconvergent variety. Then,*

1. *There is a canonical morphism of ringed spaces*

$$(\varphi_{X,V}^*, \varphi_{X,V*}) : (\mathrm{An}^\dagger(X, V), \mathcal{O}_{(X,V)}^\dagger) \to (]X[_V, i_X^{-1}\mathcal{O}_V).$$

2. *If $\mathcal{F}$ is any $i_X^{-1}\mathcal{O}_V$-module and $(f, u) : (X', V') \to (X, V)$ is any morphism, then the realization of $\varphi_{X,V}^*\mathcal{F}$ on (X', V') is $]f[_u^\dagger \mathcal{F}$.*

3. *If E is an overconvergent module on (X,V), then the realization of the adjunction map*
$$\varphi^*_{X,V}\varphi_{X,V*}E \to E$$
along some $(f,u) : (X',V') \to (X,V)$ is the transition map $\phi^\dagger_{uE} :]f[^\dagger_u E_V \to E_{V'}$.

Proof: We know that
$$i_X^{-1}\mathcal{O}_V = \varphi_{X,V*}\mathcal{O}^\dagger_{(X,V)}.$$
By adjunction, we obtain a map
$$\varphi^{-1}_{X,V} i_X^{-1}\mathcal{O}_V \to \mathcal{O}^\dagger_{(X,V)}$$
and the first assertion formally follows. By definition, we will have
$$\varphi^*_{X,V}\mathcal{F} = \mathcal{O}^\dagger_{(X,V)} \otimes_{\varphi^{-1}_{X,V} i_X^{-1}\mathcal{O}_V} \varphi^{-1}_{X,V}\mathcal{F}.$$
Since realization on (X',V') is the the pull-back by $\psi_{X',V'}$, it commutes with tensor products and we see that
$$(\varphi^*_{X,V}\mathcal{F})_{X',V'} = i_{X'}^{-1}\mathcal{O}_{V'} \otimes_{]f[_u^{-1} i_X^{-1}\mathcal{O}_V}]f[_u^{-1}\mathcal{F} =]f[^\dagger_u\mathcal{F}.$$
The case $\mathcal{F} = E_V$ gives the last result. □

Proposition 2.3.6 *If T is an overconvergent presheaf, the category of overconvergent modules on T is equivalent to the following category:*

1. *An object is a collection of $i_X^{-1}\mathcal{O}_V$-modules $E_{X,V}$ on $]X[_V$ for each $(X,V) \in \mathrm{An}^\dagger(T)$ and $i_{X'}^{-1}\mathcal{O}_{V'}$-linear maps $\phi^\dagger_{fuE} :]f[^\dagger_u E_{X,V} \to E_{X',V'}$ for each morphism*
$$(f,u) : (X',V') \to (X,V)$$
of overconvergent varieties over T, satisfying the usual cocycle conditions and such that $(E_{X,V})_{|V'} = E_{X,V'}$ whenever V' is an open subset of V.

2. *A morphism is a collection of $i_X^{-1}\mathcal{O}_V$-linear maps $E_{X,V} \to E'_{X,V}$ compatible with the morphisms $\phi^\dagger_{fu}$.*

Proof: As usual. □

Definition 2.3.7 *Let T be an overconvergent presheaf. An overconvergent module E on T is an* overconvergent crystal *if all the transition maps $\phi^\dagger_{fuE}$ are isomorphisms.*

We denote this full subcategory by $\mathrm{Crys}^\dagger(T)$. Note that an overconvergent module E on T is a crystal if and only if for all $(X,V) \in \mathrm{An}^\dagger(T)$, $E_{/(X,V)}$ is an overconvergent crystal on (X,V). It is actually sufficient to check it for some covering $\{(X_i,V_i)\}_{i\in I}$ of T.

We should also remark that any morphism of analytic presheaves $f : T' \to T$ provides a functor
$$f^{-1}_{\mathrm{An}^\dagger} : \mathrm{Crys}^\dagger(T') \to \mathrm{Crys}^\dagger(T).$$

Proposition 2.3.8 *If (X, V) is an overconvergent variety, the functors $\varphi^*_{X,V}$ and $\varphi_{X,V*}$ induce an equivalence of categories between* $\mathrm{Crys}^\dagger(X, V)$ *and the category of $i_X^{-1}\mathcal{O}_V$-modules on $]X[_V$. In particular,* $\mathrm{Crys}^\dagger(X, V)$ *is an abelian category with tensor product, internal hom and enough injectives.*

Proof: It follows from the second assertion of proposition 2.3.5 that if $\mathcal{F}$ is any $i_X^{-1}\mathcal{O}_V$-module, then $\varphi^*_{X,V}\mathcal{F}$ is a crystal and the adjunction map $\varphi_{X,V*}\varphi^*_{X,V}\mathcal{F} \to \mathcal{F}$ is bijective. Now, if E is an overconvergent crystal on (X, V), we know from the last assertion of proposition 2.3.5 that the realization of the adjunction map $\varphi^*_{X,V}\varphi_{X,V*}E \to E$ along any $(f, u) : (X', V') \to (X, V)$ is the transition map $\phi^\dagger_{uE} :]f[^\dagger_u E_V \to E_{V'}$, which is by hypothesis, an isomorphism. It follows that this adjunction map is also an isomorphism. □

Corollary 2.3.9 *If T is an overconvergent presheaf, the category* $\mathrm{Crys}^\dagger(T)$ *is an additive subcategory of the category of overconvergent modules which is stable under cokernel, extensions and tensor product.*

Proof: We may clearly assume that $T = (X, V)$ in which case everything follows from the right exactness of $\varphi^*_{X,V}$ on the category of overconvergent modules. □

Proposition 2.3.10 *Let T be an overconvergent presheaf. If E is an overconvergent crystal on T and E' is any overconvergent module, then for each overconvergent variety (X, V) over T, we have*

$$\mathcal{H}\mathrm{om}_{\mathcal{O}^\dagger_T}(E, E')_{X,V} = \mathcal{H}\mathrm{om}_{i_X^{-1}\mathcal{O}_V}(E_{X,V}, E'_{X,V}).$$

Proof: It is again sufficient to consider the case $T = (X, V)$. Then, our assertion formally follows from proposition 2.3.8. Namely, we have

$$\mathcal{H}\mathrm{om}_{\mathcal{O}^\dagger_{(X,V)}}(E, E')_{X,V} = \varphi_{X,V*}\mathcal{H}\mathrm{om}_{\mathcal{O}^\dagger_{(X,V)}}(\varphi^*_{X,V}\varphi_{X,V*}E, E')$$

$$= \mathcal{H}\mathrm{om}_{\varphi_{X,V*}\mathcal{O}^\dagger_{(X,V)}}(\varphi_{X,V*}E, \varphi_{X,V*}E') = \mathcal{H}\mathrm{om}_{i_X^{-1}\mathcal{O}_V}(E_{X,V}, E'_{X,V}). \quad \square$$

At this point, we need to introduce some finiteness conditions.

Proposition 2.3.11 *Let T be an overconvergent presheaf. An overconvergent module E on T is finitely presented if and only if it is a crystal and for all overconvergent varieties (X, V) over T, $E_{X,V}$ is a coherent $i_X^{-1}\mathcal{O}_V$-module. Moreover, E is locally free of rank r if and only if for all (X, V) over T, $E_{X,V}$ is a locally free $i_X^{-1}\mathcal{O}_V$-module of rank r.*

Proof: The question being local on $\mathrm{An}^\dagger(T)$, we may assume that $T = (X, V)$. Then, since both $\varphi_{X,V*}$ and $\varphi^*_{X,V}$ are additive and right exact, our assertions follow directly from the fact that $\mathcal{O}^\dagger_{X,V}$ is a crystal. □

If T is an overconvergent presheaf, we will denote by $\mathrm{Mod}^\dagger_{\mathrm{fp}}(T)$ the category of overconvergent modules of finite presentation on T. It is a full subcategory of $\mathrm{Crys}^\dagger(T)$.

Proposition 2.3.12 *Let T be an overconvergent presheaf. If $E, E' \in \mathrm{Mod}^{\dagger}_{\mathrm{fp}}(T)$, then*

$$\mathcal{H}\mathrm{om}_{\mathcal{O}^{\dagger}_{T}}(E, E') \in \mathrm{Mod}^{\dagger}_{\mathrm{fp}}(T).$$

In particular, it is a crystal.

Proof: If $(f, u) : (X', V') \to (X, V)$ is a morphism over T, we have

$$]f[^{\dagger}_{u}\mathcal{H}\mathrm{om}_{\mathcal{O}^{\dagger}_{T}}(E, E')_{X,V} =]f[^{\dagger}_{u}\mathcal{H}\mathrm{om}_{i_X^{-1}\mathcal{O}_V}(E_{X,V}, E'_{X,V}).$$

Since $E_{X,V}$ is coherent, we have

$$]f[^{\dagger}_{u}\mathcal{H}\mathrm{om}_{i_X^{-1}\mathcal{O}_V}(E_{X,V}, E'_{X,V}) = \mathcal{H}\mathrm{om}_{i_{X'}^{-1}\mathcal{O}_V}(]f[^{\dagger}_{u}E_{X,V},]f[^{\dagger}_{u}E'_{X,V}).$$

Since we are dealing with crystals, we have

$$\mathcal{H}\mathrm{om}_{i_{X'}^{-1}\mathcal{O}_V}(]f[^{\dagger}_{u}E_{X,V},]f[^{\dagger}_{u}E'_{X,V}) = \mathcal{H}\mathrm{om}_{i_{X'}^{-1}\mathcal{O}_V}(E_{X',V'}, E'_{X',V'}).$$

And we get

$$]f[^{\dagger}_{u}\mathcal{H}\mathrm{om}_{\mathcal{O}^{\dagger}_{T/S}}(E, E')_{X,V} = \mathcal{H}\mathrm{om}_{\mathcal{O}^{\dagger}_{T}}(E, E')_{X',V'}. \quad \square$$

We will finish this section with the study of immersions of algebraic varieties. We need some preliminary results and we start with the following elementary lemma.

Lemma 2.3.13 *Let $i : T \hookrightarrow V$ be the inclusion of a closed subspace in a topological space, $\mathcal{A}$ a sheaf of rings on V, and $\mathcal{F}$ and $\mathcal{G}$ two $\mathcal{A}$-modules. Then*

$$i_* i^{-1}(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}) = i_* i^{-1}\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}.$$

Proof: Stalks are identical on both sides. $\square$

The following lemma is analogous to proposition 2.1.4 of [10] (see also proposition 5.3.8 of [23]).

Lemma 2.3.14 *Let $(f, u) : (X', V') \to (X, V)$ be a morphism of overconvergent varieties. Assume that $]X[_V$ is closed in V and that $u^{-1}(]X[_V) =]X'[_{V'}$. Let us write $i :]X[_V \hookrightarrow V$ and $i' :]X'[_{V'} \hookrightarrow V'$ for the inclusion maps. Then, if $\mathcal{F}$ is any $\mathcal{O}_V$-module, we have*

$$u^* i_* i^{-1}\mathcal{F} \simeq i'_* i'^{-1} u^* \mathcal{F}.$$

In particular, if $\mathcal{F}$ is a $i_X^{-1}\mathcal{O}_V$-module, we have

$$u^* i_* \mathcal{F} \simeq i'_*]f[^{\dagger}_{u}\mathcal{F}.$$

Proof: Using the previous lemma, our statement is a formal consequence of the fact that, with our hypothesis (a cartesian square with a closed embedding), we have for any sheaf, $u^{-1} i_* \mathcal{F} = i'_* u^{-1}\mathcal{F}$. $\square$

Proposition 2.3.15 *Let (C, O) be an overconvergent variety and $\alpha : Y \hookrightarrow X$ an open immersion over C. If E is an overconvergent crystal on Y/O then $\alpha_{\mathrm{An}*}E$ is an overconvergent crystal on X/O. In other words, $\alpha_{\mathrm{An}*}$ induces a functor*

$$\alpha_{\mathrm{An}*} : \mathrm{Crys}^{\dagger}(Y/O) \to \mathrm{Crys}^{\dagger}(X/O).$$

Proof: We are given an overconvergent crystal E on Y/O, and a morphism $(f, u) = (X', V') \to (U, V)$ in $\mathrm{An}^\dagger(X/O)$. We want to show that

$$]f[_u^\dagger(\alpha_{\mathrm{An}^*}E)_V = (\alpha_{\mathrm{An}*}E)_{V'}.$$

We call Y' the pull-back of Y in X', and denote by $\alpha' : Y' \hookrightarrow X'$ the inclusion map. Using proposition 2.1.11, our formula can be rewritten

$$]f[_u^\dagger]\alpha[_{V*}E_{Y\cap U,V} =]\alpha'[_{V'*}E_{Y',V'}.$$

It is clearly sufficient to do the case $U = X$. If $g : Y' \to Y$ denotes the map induced by f, we have $E_{Y',V'} =]g[_u^\dagger E_{Y,V}$. We are thus reduced to show that

$$]f[_u^\dagger \circ]\alpha[_{V*} =]\alpha'[_{V'*} \circ]g[_u^\dagger.$$

In other words, we want a base change isomorphism for the commutative diagram with cartesian left square:

$$\begin{array}{ccccc}]Y'[_{V'} & \xrightarrow{]\alpha'[} &]X'[_{V'} & \xrightarrow{i_{X'}} & V' \\ \downarrow{\scriptstyle]g[} & & \downarrow{\scriptstyle]f[} & & \downarrow{\scriptstyle u} \\ ]Y[_V & \xrightarrow{]\alpha[} &]X[_V & \xrightarrow{i_X} & V. \end{array}$$

We may assume that our morphism is formal and split the verification in two parts using proposition 1.3.4.

So we assume first that $u = \mathrm{Id}_V$. We simply have here $]f[^\dagger =]f[^{-1}$ and $]g[^\dagger =]g[^{-1}$. Thus, we are reduced to check that

$$]f[^{-1} \circ]\alpha'[_* =]\alpha[_* \circ]g[^{-1}$$

which follows from the fact that $]Y]_V$ is closed in $]X[_V$ (and the diagram is cartesian).

We assume now that $P' = P$ and $f = \mathrm{Id}_X$, in which case, also $g = \mathrm{Id}_Y$. Shrinking V and V' if necessary, we may also assume that $]X[_V$ and $]X[_{V'}$ are closed in V and V'. Moreover, since i_{X*} is fully faithful, it is sufficient to prove that

$$i_{X*} \circ]\mathrm{Id}_X[_u^\dagger \circ]\alpha[_{V*} = i_{X*} \circ]\alpha'[_{V'*} \circ]\mathrm{Id}_Y[_u^\dagger.$$

We are in the situation of applying lemma 2.3.14 both to X and Y. On the left hand side, we get

$$i_{X*} \circ]\mathrm{Id}_X[_u^\dagger \circ]\alpha[_{V*} = u^* \circ i_{X*} \circ]\alpha[_{V*} = u^* \circ i_{Y*}$$

and on the right hand side, we get

$$i_{X*} \circ]\alpha'[_{V'*} \circ]\mathrm{Id}_Y[_u^\dagger = i_{Y'*} \circ]\mathrm{Id}_Y[_u^\dagger = u^* \circ i_{Y*}. \quad \square$$

2.4 Stratifications

We fix an overconvergent variety $(C \subset S \leftarrow O)$ over $\mathcal{V}$. We will use the notion of stratification as a bridge between crystals and modules with integrable connection. The first point is to relate overconvergent stratifications to usual stratification (analogous to HPD stratification versus usual stratification in crystalline cohomology). We do this here.

Note that if $(X \subset P \leftarrow V)$ is an overconvergent variety over $(C \subset S \leftarrow O)$, then $(X, V) \times_{X/O} (X, V)$ is representable by $(X \subset P \times P \leftarrow V \times_O V)$. When the structural morphism is a formal morphism of overconvergent varieties, we may (and will) replace $P \times P$ with $P \times_S P$.

Definition 2.4.1 *Let (X, V) an overconvergent variety over (C, O).*

We write $V^2 := V \times_O V$ and denote by

$$p_1, p_2 : (X, V^2) \to (X, V)$$

the projections.

An overconvergent stratification *on a $i_X^{-1}\mathcal{O}_V$-module $\mathcal{F}$ is an isomorphism*

$$\epsilon : p_2^\dagger \mathcal{F} \simeq p_1^\dagger \mathcal{F}$$

on $]X[_{V^2}$, called the Taylor isomorphism *of E, satisfying the usual cocycle condition on triple products. A* morphism of overconvergent stratified modules *is a morphism of $i_X^{-1}\mathcal{O}_V$-modules compatible with the data.*

We denote this category by $\mathrm{Strat}^\dagger(X, V/O)$. We will also set:

$$\mathcal{H}^\dagger(\mathcal{F}) := \ker \left[\epsilon \circ p_2^{-1} - p_1^{-1} : p_{]X[_V *}\mathcal{F} \to p_{]X[_{V^2}*} p_1^\dagger \mathcal{F}\right]$$

where $p_{]X[_V} :]X[_V \to]C[_O$ and $p_{]X[_{V^2}} :]X[_{V^2} \to]C[_O$ denote the projections.

Lemma 2.4.2 *If $(f, u) : (X', V') \to (X, V)$ is a morphism of overconvergent varieties with u universally flat in a neighborhood of X' in V', then $]f[_u^\dagger$ is exact.*

Proof: Our assumption implies that, shrinking V and V' if necessary, the morphism $u^{-1}\mathcal{O}_V \to \mathcal{O}_{V'}$ is flat. Pulling back by $i_{X'}^{-1}$, we see that

$$]f[_u^{-1} i_X^{-1} \mathcal{O}_V = i_{X'}^{-1} u^{-1} \mathcal{O}_V \to i_{X'}^{-1} \mathcal{O}_{V'}$$

is also flat. And this is what we want. $\square$

Proposition 2.4.3 *Let (X, V) be an overconvergent variety over (C, O). Then, the category $\mathrm{Strat}^\dagger(X, V/O)$ is an additive category with cokernels and the forgetful functor from $\mathrm{Strat}^\dagger(X, V/O)$ to the category of $i_X^{-1}\mathcal{O}_V$-modules is right exact and faithful. Moreover, if V is universally flat in a neighborhood of X over O, then $\mathrm{Strat}^\dagger(X, V/O)$ is even an abelian category and the forgetful functor is exact.*

Proof: It is clear that we have an additive category and that the forgetful functor is faithful. We will show, when V is universally flat over O, the existence of a stratification on the kernel of a morphism of overconvergent stratified modules $m : \mathcal{F} \to \mathcal{G}$, and also that this new structure turns $\ker m$ into a kernel in the category of overconvergent stratified modules. The analogous result for cokernels is shown exactly in the same way (without the flatness assumption).

If V is universally flat over O, the projections p_1 and p_2 are universally flat. It follows from the lemma that $p_1^\dagger$ and $p_2^\dagger$ are exact. Thus, we see that the Taylor isomorphisms induce an isomorphism $p_2^\dagger(\ker m) \simeq p_1^\dagger(\ker m)$ which is obviously a stratification on $\ker m$. Clearly, if the image of a morphism of overconvergent stratified modules $\mathcal{G} \to \mathcal{F}$ is contained in $\ker m$, the induced map $\mathcal{G} \to \ker m$ is compatible with the stratifications. $\square$

We will need an infinitesimal version of this notion of overconvergent stratification. We want first to recall some notions concerning analytic varieties over K. If W is a fixed analytic variety over K and V an analytic variety over W, we will write $V^{(n)}$ for the n-th infinitesimal neighborhood of V in $V \times_W V$ and $p_1^{(n)}, p_2^{(n)} : V^{(n)} \to V$ for the projections. By definition, there is an exact sequence

$$0 \to \Omega^1_{V/W} \to \mathcal{O}_{V^{(1)}} \to \mathcal{O}_V \to 0.$$

We can define stratifications, modules with (integrable) connection (and the sheaf of differential operators $\mathcal{D}_{V/W}$) as usual. Everything behaves as expected.

If $(X \subset P \leftarrow V) \in \mathrm{An}^\dagger(C, O)$, we may consider

$$(X \subset P \times P \leftarrow V^{(n)}) \in \mathrm{An}^\dagger(C, O).$$

Definition 2.4.4 *Let* $(X, V) \in \mathrm{An}^\dagger(C, O)$. *A* stratification *on a* $i_X^{-1}\mathcal{O}_V$*-module* $\mathcal{F}$ *is a compatible sequence of* Taylor isomorphisms

$$\{\epsilon^{(n)} : p_2^{(n)\dagger}\mathcal{F} \simeq p_1^{(n)\dagger}\mathcal{F}\}_{n\in\mathbf{N}}$$

that satisfy the cocycle condition on triple products (and $\epsilon^{(0)} = Id_{\mathcal{F}}$*). A* morphism of stratified modules *is a morphism of* $i_X^{-1}\mathcal{O}_V$*-modules that is compatible with the data.*

For each $n \in \mathbf{N}$, we will write

$$\mathcal{H}^{(n)}(\mathcal{F}) := \ker\left[(\epsilon^{(n)} \circ (p_2^{(n)})^{-1}) - (p_1^{(n)})^{-1} : p_{]X[_V*}\mathcal{F} \to p_{]X[_{V^{(n)}}*}p_1^{(n)\dagger}\mathcal{F}\right].$$

Definition 2.4.5 *Let* $(X, V) \in \mathrm{An}^\dagger(C, O)$. *A* connection *on a* $i_X^{-1}\mathcal{O}_V$*-module* $\mathcal{F}$ *is an* $\mathcal{O}_O$*-linear map*

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_{V/O}$$

satisfying the Leibniz rule. A horizontal *map is a* $i_X^{-1}\mathcal{O}_V$*-linear map compatible with the connections.* Integrability *is defined in the usual way.*

Stratified $i_X^{-1}\mathcal{O}_V$-modules form a category $\mathrm{Strat}(X, V/O)$ and $i_X^{-1}\mathcal{O}_V$-modules with integrable connection make a category $\mathrm{MIC}(X, V/O)$. As usual, there is an obvious forgetful functor $\mathrm{Strat}(X, V/O) \to \mathrm{MIC}(X, V/O)$ sending $\{\epsilon^{(n)}\}_{n\in\mathbf{N}}$ to the morphism

$$\nabla = (\epsilon^{(1)} \circ (p_2^{(1)})^{-1}) - (p_1^{(1)})^{-1} : \mathcal{F} \to \mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_{V/O} \subset p_1^{(2)\dagger}\mathcal{F}.$$

When $\mathrm{Char}K = 0$ and V is smooth in the neighborhood of $]X[_V$, it is not hard to see that we get an equivalence

$$\mathrm{Strat}(X, V/O) \simeq \mathrm{MIC}(X, V/O)$$

and that $\mathcal{H}^{(n)}(\mathcal{F})$ is independent of n, for $n > 0$; and canonically isomorphic to $\mathcal{F}^{\nabla=0}$.

We can do better with some finiteness conditions. In general, if $\mathcal{F}$ and $\mathcal{G}$ are two $i_X^{-1}\mathcal{O}_V$-modules with integrable connections with $\mathcal{F}$ coherent, then $\mathcal{H}\mathrm{om}_{i_X^{-1}\mathcal{O}_V}(\mathcal{F}, \mathcal{G})$ has a natural integrable connection given by $\nabla(m)(x) = \nabla(m(x)) - (m \otimes \mathrm{Id})(x)$. Moreover, we have

$$\Gamma(]X[_V, \mathcal{H}\mathrm{om}_{i_X^{-1}\mathcal{O}_V}(\mathcal{F}, \mathcal{G})^{\nabla=0}) = \mathrm{Hom}_{\mathrm{MIC}(X,V/O)}(\mathcal{F}, \mathcal{G}).$$

In the same way, if $\mathcal{F}$ and $\mathcal{G}$ are stratified $i_X^{-1}\mathcal{O}_V$-modules with $\mathcal{F}$ coherent, then $\mathcal{H}om_{i_X^{-1}\mathcal{O}_V}(\mathcal{F},\mathcal{G})$ has a natural stratification and we have

$$\varprojlim \Gamma\left(]X[_V, \mathcal{H}^{(n)}\left(\mathcal{H}om_{i_X^{-1}\mathcal{O}_V}(\mathcal{F},\mathcal{G})\right)\right) = \mathrm{Hom}_{\mathrm{Strat}(X,V/O)}(\mathcal{F},\mathcal{G})$$

Moreover, the canonical functor

$$\mathrm{Strat}(X,V/O) \to \mathrm{MIC}(X,V/O)$$

is compatible with these constructions.

We now come back to overconvergent stratifications: by restriction, there is an obvious functor

$$\mathrm{Strat}^\dagger(X,V/O) \to \mathrm{Strat}(X,V/O)$$

and a compatible family of canonical maps $\mathcal{H}^\dagger(\mathcal{F}) \to \mathcal{H}^{(n)}(\mathcal{F})$. Assume that $\mathcal{F}$ and $\mathcal{G}$ are two overconvergent stratified $i_X^{-1}\mathcal{O}_V$-modules with $\mathcal{F}$ coherent. Then $\mathcal{H}om_{i_X^{-1}\mathcal{O}_V}(\mathcal{F},\mathcal{G})$ has a natural overconvergent stratification and

$$\Gamma\left(]X[_V, \mathcal{H}^\dagger\left(\mathcal{H}om_{i_X^{-1}\mathcal{O}_V}(\mathcal{F},\mathcal{G})\right)\right) = \mathrm{Hom}_{\mathrm{Strat}^\dagger(X,V/S)}(\mathcal{F},\mathcal{G})$$

Again, the canonical functor

$$\mathrm{Strat}^\dagger(X,V/O) \to \mathrm{Strat}(X,V/O)$$

is compatible with these constructions. Finally, by composition, we have

$$\mathrm{Strat}^\dagger(X,V/O) \to \mathrm{Strat}(X,V/O) \to \mathrm{MIC}(X,V/O)$$

and we will denote by $\mathrm{MIC}^\dagger(X,V/O)$ the full subcategory generated by the image of this functor.

Definition 2.4.6 *Let (X,V) be an overconvergent variety over (C,O). If*

$$(\mathcal{F},\nabla) \in \mathrm{MIC}^\dagger(X,V/O),$$

we say that the connection is overconvergent.

In the next proposition, we will need to localize with respect to the Grothendieck topology of an analytic variety V. Recall that there exists a canonical morphism $\pi_V : V_G \to V$ of ringed spaces. By functoriality, if $i : T \hookrightarrow V$ is the inclusion of an analytic domain and $\mathcal{F}$ a sheaf on V, then

$$\Gamma(T, i^{-1}\mathcal{F}) = \Gamma(T, \pi_V^{-1}\mathcal{F}).$$

If $\mathcal{F}$ is an $\mathcal{O}_V$-module, it is common to write $\mathcal{F}_G = \pi_V^*\mathcal{F}$. Recall also that, when V is good, we get an equivalence on coherent sheaves: $\mathrm{Coh}(\mathcal{O}_V) \simeq \mathrm{Coh}(\mathcal{O}_{V_G})$.

Proposition 2.4.7 *If V is a* good *analytic variety and $\mathcal{F}$ is a coherent sheaf on V, then the canonical map $\pi_V^{-1}\mathcal{F} \to \mathcal{F}_G$ is injective.*

Proof: It is sufficient to show that if W is an affinoid domain inside V, the canonical map

$$\Gamma(W, \pi_V^{-1}\mathcal{F}) \to \Gamma(W, \mathcal{F}_G)$$

is injective. By definition, we have

$$\Gamma(W, \pi_V^{-1}\mathcal{F}) = \varinjlim_{W \subset U} \Gamma(U, \mathcal{F})$$

where U runs through analytic open subsets of V that contain W. It is therefore sufficient to prove that if U is an open neighborhood of W in V and $s \in \Gamma(U, \mathcal{F})$ is sent to $0 \in \Gamma(W, \mathcal{F}_G)$, then there exists an open neighborhood U' of W in V such that $s_{|U'} = 0$. Of course, this will follow if we show that for each $x \in W$, the stalk of s at x is zero. But if we denote by $\mathcal{F}_W$ the coherent sheaf on W induced by $\mathcal{F}$ (i.e. $\mathcal{F}_W := \pi_{W*}(\mathcal{F}_G)_{|W}$), the image of s in the stalk of $\mathcal{F}_W$ at x is zero. It is therefore sufficient to check that the map

$$\mathcal{F}_x \to \mathcal{F}_{Wx}$$

is injective. Since V is good, we may assume that V and W are both affinoid with $A := \Gamma(V, \mathcal{O}_V)$ and $M := \Gamma(V, \mathcal{F})$. And we want to show that the map

$$\mathcal{O}_{V,x} \otimes_A M \to \mathcal{O}_{W,x} \otimes_A M$$

is injective. This is an immediate consequence of the flatness of the inclusion map $W \hookrightarrow V$.
$\square$

Proposition 2.4.8 *Let*

$$\begin{array}{ccccc} X & \hookrightarrow & P & \leftarrow & V \\ \downarrow f & & \downarrow v & & \downarrow \\ C & \hookrightarrow & S & \leftarrow & O \end{array}$$

be a formal morphism of overconvergent varieties with f quasi-compact, v smooth at X, O locally separated and V a good *neighborhood of X in $P_K \times_{S_K} O$. If $\mathcal{F}$ is a coherent overconvergent stratified module on $(X, V)/O$, then*

$$\mathcal{H}^\dagger(\mathcal{F}) = \varprojlim \mathcal{H}^{(n)}(\mathcal{F}).$$

Proof: We start with a topological reduction. The question being local on O which is locally compact, we may assume that the image of the canonical map $O \to S_K$ is relatively compact. It is then contained in some S'_K where S' is a quasi-compact formal scheme. Since the morphism $f : X \to C$ is assumed to be quasi-compact we see that $X' := X \times_S S'$ is quasi-compact and so is its closure $\overline{X'}$ in P. In particular, we can find a quasi-compact open neighborhood P' of $\overline{X'}$ in P. By construction, P'_K is a compact neighborhood of X in $P_K \times_{S_K} O$. It follows from proposition 2.2.8 that the overconvergent variety (X, V) is countable at infinity.

We denote by $p_1 :]X[_{V^2} \to]X[_V$ and $p_1^{(n)} :]X[_{V^{(n)}} \to]X[_V$ the first projections. In order to lighten the notations, we will identify the underlying spaces of $]X[_{V^{(n)}}$ and $X[_V$. In particular, $p_{1*}^{(n)}$ becomes the identity. By definition, it is sufficient to show that the natural map

$$p_{]X[_{V^2}/O*} p_1^\dagger \mathcal{F} \to \varprojlim p_{]X[_{V^{(n)}}/O*} p_1^{(n)\dagger} \mathcal{F}$$

is injective. It is even sufficient to check that the canonical map

$$p_{1*}p_1^\dagger \mathcal{F} \to \varprojlim p_1^{(n)\dagger}\mathcal{F}$$

is injective. Since (X, V) is countable at infinity, we can use proposition 2.2.10. Thus, shrinking V if necessary, it is therefore sufficient to consider sheaves of the form $i^{-1}\mathcal{F}$ where $\mathcal{F}$ is a coherent module on V of and $i :]X[_V \hookrightarrow V$ denotes the inclusion map. In this situation, we want to show that the map

$$p_{1*}i^{-1}p_1^*\mathcal{F} \to \varprojlim i^{-1}p_1^{(n)*}\mathcal{F}$$

is injective where the map i on the left hand side denotes the inclusion $]X[_{V^2} \hookrightarrow V^2$ and the maps p_1 and $p_1^{(n)}$ now denote the first projections $V^2 \to V$ and $V^{(n)} \to V$.

We want to use the Grothendieck topology but we need to be very careful because $]X[_V$ might not be good. First of all, since V is good, $\mathcal{F}$ extends uniquely to a coherent sheaf $\mathcal{F}_G = \pi_V^*\mathcal{F}$ for the Grothendieck topology of V. Moreover, since O is locally separated and V is good, then V^2 also is good. If W is an analytic open subset of $]X[_V$, since V^2 is good and $p_1^*\mathcal{F}_G$ coherent, it follows from proposition 2.4.7 that there is a natural injective map

$$\begin{aligned} \Gamma(W, p_{1*}i^{-1}p_1^*\mathcal{F}) &= \Gamma(p_1^{-1}(W), i^{-1}p_1^*\mathcal{F}) \\ = \Gamma(p_1^{-1}(W), \pi_{V^2}^{-1}p_1^*\mathcal{F}) &\hookrightarrow \Gamma(p_1^{-1}(W), \pi_{V'^2}^*p_1^*\mathcal{F}) \\ = \Gamma(W, p_{1*}p_1^*\pi_V^*\mathcal{F}) &= \Gamma(W, p_{1*}p_1^*\mathcal{F}_G). \end{aligned}$$

For the same reason, we also have an injection

$$\Gamma(W, \varprojlim i^{-1}p_1^{(n)*}\mathcal{F}) \hookrightarrow \Gamma(W, \varprojlim p_1^{(n)*}\mathcal{F}_G).$$

It is therefore sufficient to show that the morphism

$$p_{1*}p_1^*\mathcal{F}_G \to \varprojlim p_1^{(n)*}\mathcal{F}_G$$

is injective. Now the question is local for the Grothendieck topology on $]X[_V$. It is therefore also local on P and we may assume that P is affine, that X is closed in P, and that we have local coordinates $t_1, \ldots, t_n$. They induce local coordinates $\tau_1, \ldots, \tau_n$ on $P \times_S P$ with respect to the first projection p_1. Using Lemma 4.4 of [6], we get an isomorphism

$$]X[_{V^2} \simeq]X[_V \times_K \mathbf{B}^n(0, 1^-).$$

and we may use lemma 2.4.9 below. □

Lemma 2.4.9 *Let V be a good analytic variety and $\mathbf{B}$ any non trivial polydisc (open or closed) with coordinates $t_1, \ldots, t_n$. Denote by*

$$p : V \times_K \mathbf{B} \to V$$

and

$$p^{(n)} : V \times_K \mathcal{M}(K[\underline{t}]/(\underline{t})^n) \to V,$$

the projections. If $\mathcal{F}$ is a coherent $\mathcal{O}_V$-module, the canonical map

$$p_*p^*\mathcal{F} \to \varprojlim_n p^{(n)*}\mathcal{F}$$

is injective.

Proof: Since we work with coherent sheaves and good analytic varieties, we may freely use the Grothendieck topology. Taking inverse limits, it is sufficient to consider the case of a closed polydisc. We may even assume that it has radius 1. Moreover, it is sufficient to check that, when V is affinoid, the map

$$\Gamma(V, p_*p^*\mathcal{F}) \to \Gamma(V, \varprojlim_n p^{(n)*}\mathcal{F})$$

is injective. If we let $A := \Gamma(V, \mathcal{O}_V)$ and $M := \Gamma(V, \mathcal{F})$, this map is the canonical map

$$M \otimes_A A\{\underline{t}\} \to M \otimes_A A[[\underline{t}]].$$

By induction on the number of generators of M, we may assume that M is a quotient of A. It is therefore an affinoid algebra and we are reduced to the case $M = A$ which is part of the definition of the ring of convergent power series. □

Corollary 2.4.10 *With the assumptions and notations of the proposition, the canonical functor*

$$\mathrm{Strat}^\dagger(X, V/O) \to \mathrm{Strat}(X, V/O)$$

is fully faithful on coherent modules.

Proof: Follows from the above description of morphisms in both categories. □

2.5 Crystals and connections

We fix an overconvergent variety $(C \subset S \leftarrow O)$. Then, the results of the first chapter and the previous section, will allow us to prove that overconvergent modules of finite presentation do correspond to modules with an overconvergent integrable connection on a geometric realization.

Recall that if (X, V) is an overconvergent variety over (C, O), we defined the overconvergent presheaf X_V/O as the image of (X, V) in X/O. We may therefore consider the category $\mathrm{An}^\dagger(X_V/O)$ of overconvergent varieties over (C, O) that factors through a specific $X' \to X$ and through a possible $V' \to V$.

Lemma 2.5.1 *Let $f : X' \to X$ be a morphism of varieties over C that extends in two ways to morphisms*

$$(f, u_1) : (X', V') \to (X, V)$$

and

$$(f, u_2) : (X', V') \to (X, V)$$

of overconvergent varieties over (C, O). If E is an overconvergent crystal on X_V/O, there is a canonical isomorphism

$$\epsilon_V : u_2^\dagger E_V \simeq u_1^\dagger E_V.$$

Proof: Just take

$$\epsilon_V := \phi_{u_1}^\dagger \circ (\phi_{u_2}^\dagger)^{-1} : u_2^\dagger E_V \simeq E_{V'} \simeq u_1^\dagger E_V. \quad \square$$

Recall that we introduced in definition 2.1.7, for an overconvergent presheaf T over an overconvergent variety (C, O), the projection $p_T : T_{\mathrm{An}^\dagger} \to]C[_O$.

Lemma 2.5.2 *Let (X, V) be an overconvergent variety over (C, O). If O' is an open subset of O and V' its inverse image inside V, then*

$$\Gamma(]C[_{O'}, p_{X_V/O*}E) = \Gamma(X_{V'}/O', E).$$

Proof: By definition, $p_{X_V/O}$ is the composition of the restriction map induced by the morphism of overconvergent presheaves $X_V/O \to (C, O)$ and the canonical morphism $\mathrm{An}^\dagger(C, O) \to]C[_O$. Inverse image of $]C[_{O'}$ through the later is simply (C, O') and inverse image of (C, O') through the former is $X_{V'}/O'$ as we saw in proposition 1.4.16. $\square$

Proposition 2.5.3 *Let (X, V) be an overconvergent variety over (C, O). Then, the functor*

$$\begin{array}{ccc} \mathrm{Crys}^\dagger(X_V/O) & \longrightarrow & \mathrm{Strat}^\dagger(X, V/O) \\ E & \longmapsto & (\mathcal{F}, \epsilon) \end{array}$$

with $\mathcal{F} := E_V$ and

$$\epsilon : p_2^\dagger E_V \simeq p_1^\dagger E_V$$

is an equivalence of categories. Moreover, we have

$$p_{X_V/O*}E \simeq \mathcal{H}^\dagger(\mathcal{F}).$$

Proof: Let $(\mathcal{F}, \epsilon)$ be an overconvergent stratified module. If $(f, u) : (X', V') \to (X, V)$ is a morphism in $\mathrm{An}^\dagger(C, O)$, we set $E_{V'} := u^\dagger\mathcal{F}$. This seems to depend on u but if we are given two such morphisms $(f, u_1) : (X', V') \to (X, V)$ and $(f, u_2) : (X', V') \to (X, V)$, we can consider the diagonal morphism $(f, u) : (X', V') \to (X, V^2)$. Pulling back the Taylor isomorphism gives a canonical isomorphism $u_2^\dagger\mathcal{F} \simeq u_1^\dagger\mathcal{F}$. This shows that our definition is essentially independent of the choices. One easily checks that this gives a quasi-inverse to our functor.

The second assertion can be deduced from the first one but we can also prove it directly: using lemma 2.5.2, it is sufficient to show that $\Gamma(X_V/O, E)$ is the kernel of

$$\epsilon \circ p_2^{-1} - p_1^{-1} : \Gamma(]X[_V, E_V) \to \Gamma(]X[_{V^2}, p_1^\dagger E_V).$$

But we have

$$\Gamma(X_V/O, E) = \varprojlim \Gamma((X', V'), E)$$

$$= \varprojlim \Gamma(]X'[_{V'}, E_{X',V'}) = \varprojlim \Gamma(]X'[_{V'},]f[_u^\dagger E_{X,V})$$

Since any morphism factors through its graph, we see that

$$\Gamma(X_V/O, E) = \ker \left[\Gamma(]X[_V, E_V) \begin{array}{l} \to \Gamma(]X[_{V^2}, p_1^\dagger E_V) \\ \to \Gamma(]X[_{V^2}, p_2^\dagger E_V) \end{array} \right]$$

which is what we want. $\square$

Corollary 2.5.4 *Let (X, V) be an overconvergent variety over (C, O). The realization functor to from $\mathrm{Crys}^\dagger(X_V/O)$ to the category of $i_X^{-1}\mathcal{O}_V$-modules is right exact and faithful. Moreover, if V is universally flat in a neighborhood of X over O, then $\mathrm{Crys}^\dagger(X_V/O)$ is even an abelian category and the realization functor is exact.*

Proof: Follows from proposition 2.4.3. □

Note that there is a sequence of functors

$$\mathrm{Crys}^\dagger(X_V/O) \simeq \mathrm{Strat}^\dagger(X, V/O)$$

$$\to \mathrm{Strat}(X, V/O) \to \mathrm{MIC}(X, V/O)$$

whose essential image is the category $\mathrm{MIC}^\dagger(X, V/O)$ of $i_X^{-1}\mathcal{O}_V$-modules with an overconvergent integrable connections on $]X[_V/O$. And there is also a sequence of morphisms

$$p_{X_V/O*}E \simeq \mathcal{H}^\dagger(E_V) \to \varprojlim \mathcal{H}^{(n)}(E_V) \to E_V^{\nabla=0}.$$

In both cases, the last arrow is bijective when $\mathrm{Char}K = 0$.

Proposition 2.5.5 *If (X, V) is an overconvergent variety over (C, O), then an overconvergent module E on X_V/O is finitely presented (resp. locally free of rank r) if and only if it is a crystal and E_V is coherent (resp. locally free of rank r). Moreover, if V is universally flat in the neighborhood of $]X[_V$ over O, then $\mathrm{Mod}^\dagger_{\mathrm{fp}}(X_V/O)$ is an abelian subcategory of $\mathrm{Crys}^\dagger(X_V/O)$.*

Proof: The first assertion follows from proposition 2.3.11 because here, if $(f, u) : (X', V') \to (X, V)$ is any morphism of overconvergent varieties and E_V is coherent, then $E_{V'} =]f[_u^\dagger E_V$ will also be coherent. The second assertion formally follows since $\mathrm{Coh}(i_X^{-1}\mathcal{O}_V)$ is an abelian subcategory of the category of all $i_X^{-1}\mathcal{O}_V$-modules and that, under the flatness condition, the realization functor is exact and faithful. □

Note also that if E, E' are two overconvergent modules of finite presentation on X_V/O, then the overconvergent stratified module associated to $\mathcal{H}\mathrm{om}_{\mathcal{O}^\dagger_{X_V/O}}(E, E')$ is

$$\mathcal{H}\mathrm{om}_{i_X^{-1}\mathcal{O}_V}(E_{X,V}, E'_{X,V}).$$

Proposition 2.5.6 *Assume* $\mathrm{Char}K = 0$. *Let*

$$\begin{array}{ccccc} X & \hookrightarrow & P & \leftarrow & V \\ \downarrow f & & \downarrow v & & \downarrow \\ C & \hookrightarrow & S & \leftarrow & O \end{array}$$

be a formal morphism of overconvergent varieties with f quasi-compact, v smooth at X, O locally separated and V a good *neighborhood of X in $P_K \times_{S_K} O$. Then,*

1. *There is an equivalence of categories*

$$\mathrm{Mod}^\dagger_{\mathrm{fp}}(X_V/O) \simeq \mathrm{MIC}^\dagger_{\mathrm{coh}}(X, V/O)$$

between finitely presented modules on the overconvergent site of X_V over (C, O) and coherent $i_X^{-1}\mathcal{O}_V$-modules with an overconvergent integrable connection on $]X[_V/O$.

2. *If E is an overconvergent module of finite presentation on X_V/O, we have*

$$p_{X_V/O*}E = E_V^{\nabla=0}.$$

Proof: It follows from proposition 2.5.3 that $\mathrm{Mod}^\dagger_{\mathrm{fp}}(X_V/O)$ is equivalent to the category of coherent overconvergent stratified modules and we proved in corollary 2.4.10 that, under our hypothesis, the forgetful functor to stratified modules is fully faithful. Finally, as already mentioned after definition 2.4.5, stratified modules are equivalent to modules with integrable connections since we assumed $\mathrm{Char}K = 0$ and v smooth in the neighborhood of X.

The second assertion may be seen as a consequence of fulfaithfulness:

$$p_{X_V/O*}E = \mathcal{H}om(\mathcal{O}^\dagger_{X_V/O}, E) \simeq \mathcal{H}om_\nabla(i_X^{-1}\mathcal{O}, E_V) = E_V^{\nabla=0}. \quad \square$$

Recall (see definition 2.2.5 of [10] or definition 7.2.10 and proposition 7.2.13 of [23]) that if $X \subset P$ is a formal embedding, an *overconvergent isocrystal* on $(X \subset \overline{X} \subset P/S)$ is a coherent $j^\dagger_{X0}\mathcal{O}_{]\overline{X}[_{P0}}$-module with an integrable connection (relative to S_{K0}) whose Taylor series converges on a strict neighborhood of the diagonal. They form a category $\mathrm{Isoc}^\dagger(X \subset \overline{X} \subset P/S)$. This is shown (corollary 7.3.12 of [23]) to be independent of P when P is smooth at X over S in which case the category is denoted $\mathrm{Isoc}^\dagger(X \subset \overline{X}/S)$.

Corollary 2.5.7 *Assume* $\mathrm{Char}K = 0$ *and S is separated. Let X be a quasi-compact separated algebraic variety over S_k. Let $X \hookrightarrow P$ be a good formal embedding into a formal S-scheme which is smooth at X. Then, we have a canonical equivalence of categories*

$$\mathrm{Mod}^\dagger_{\mathrm{fp}}(X_P/S) \simeq \mathrm{Isoc}^\dagger(X \subset \overline{X}/S).$$

Proof: It follows from corollary 2.2.13 that $\mathrm{Isoc}^\dagger(X \subset \overline{X}/S)$ is equivalent to the category of coherent $i^{-1}\mathcal{O}_{P_K}$-modules with an integrable connection whose Taylor series converges on a neighborhood of the diagonal. We may then apply the proposition. $\square$

We will now apply the results of section 1.5 to crystals. If T is an overconvergent presheaf, we considered the good overconvergent site $\mathrm{An}^\dagger_g(T)$ made of good overconvergent varieties over T. By restriction, it becomes a ringed site and we can define crystals in the usual way. We will denote by $\mathrm{Crys}^\dagger_g(T)$ the category of overconvergent crystals in $\mathrm{An}^\dagger_g(T)$ and $\mathrm{Mod}^\dagger_{\mathrm{g,fp}}(T)$ the category of finitely presented modules on this ringed site. Note that when T has a covering by good overconvergent varieties, we obtain an equivalence on crystals but this is not the case in general. However, this applies to the case when T is represented by a good overconvergent variety (X, V) or when $T = X_V/O$ if moreover, we are given a morphism $(X, V) \to (C, O)$. When this is the case, we will remove the index "g" from the notations.

Recall that a geometric realization of a morphism $f : X' \to X$ is a formal morphism of overconvergent varieties

$$\begin{array}{ccccc} X' & \hookrightarrow & P' & \leftarrow & V' \\ \downarrow & & \downarrow u & & \downarrow \\ X & \hookrightarrow & P & \leftarrow & V \end{array}$$

where u is proper smooth at X' and V' is a neighborhood of X' in $P'_K \times_{P_K} V$.

Proposition 2.5.8 1. *Let (X, V) be a good overconvergent variety over (C, O) and $(X, V') \to (X, V)$ a geometric realization of the identity. Then, there is an equivalence of categories*

$$\mathrm{Crys}^\dagger(X_V/O) \simeq \mathrm{Crys}^\dagger(X_{V'}/O).$$

2. *Let (C, O) be a good overconvergent variety and $(X, V) \to (C, O)$ a geometric realization of a morphism $X \to C$. Then, there is an equivalence of categories*

$$\mathrm{Crys}^\dagger_g(X/O) \simeq \mathrm{Crys}^\dagger(X_V/O)$$

Proof: The first assertion follows from Corollary 1.5.12 (since the assumptions of the theorem are satisfied) and the second one from Corollary 1.5.15 (since the assumptions of the proposition are satisfied). □

Theorem 2.5.9 *Assume that* $\mathrm{Char}K = 0$. *Let (C, O) be a good overconvergent variety and X an algebraic variety over C. If $(X, V) \to (C, O)$ is a geometric realization of X over C, then*

1. *There is an equivalence of categories*

$$\mathrm{Mod}^\dagger_{\mathrm{g,fp}}(X/O) \simeq \mathrm{MIC}^\dagger_{\mathrm{coh}}(X, V/O)$$

between finitely presented modules on the good overconvergent site of X over (C, O) and coherent $i_X^{-1}\mathcal{O}_V$-modules with an overconvergent integrable connection on $]X[_V/O$.

2. *If E is an module of finite presentation on $(X/O)_g$, we have*

$$p_{X/O*}E = E_V^{\nabla=0}.$$

Proof: This follows from propositions 2.5.6 and 2.5.8. □

More precisely, we have the following sequence of functors

$$\mathrm{Crys}^\dagger_g(X/O) \simeq \mathrm{Crys}^\dagger(X_V/O)$$

$$\simeq \mathrm{Strat}^\dagger(X, V/O) \hookrightarrow \mathrm{Strat}(X, V/O) \simeq \mathrm{MIC}(X, V/O)$$

and a sequence of isomorphisms

$$p_{X/O*}E \simeq p_{X_V/O*}E \simeq \mathcal{H}^\dagger(E_V) \simeq \varprojlim \mathcal{H}^{(n)}(E_V) \simeq E_V^{\nabla=0}$$

(in both cases, the last arrow also is bijective because $\mathrm{Char}K = 0$).

Proposition 2.5.10 *Assume that* $\mathrm{Char}K = 0$ *and that (C, O) is good. If X is a realizable algebraic variety over (C, O), then* $\mathrm{Mod}^\dagger_{\mathrm{g,fp}}(X/O)$ *is an abelian category. Moreover, if $f : X' \to X$ is morphism of realizable varieties over (C, O), then*

$$f^{-1}_{\mathrm{An}^\dagger} : \mathrm{Mod}^\dagger_{\mathrm{g,fp}}(X/O) \to \mathrm{Mod}^\dagger_{\mathrm{g,fp}}(X'/O)$$

is exact.

Proof: The first assertion results from proposition 2.5.5. For the second one, we may choose a realization both for X and for X'. We may then use the diagonal embedding and assume that f extends to a morphism of formal schemes which is smooth in the neighborhood of X. In particular, it induces a universally flat morphism on neighborhoods, and therefore an exact functor. □

Recall (corollary 8.1.9 of [23]) that if $X \subset P$ is a formal embedding over S that is proper and smooth at X, the category $\mathrm{Isoc}^\dagger(X \subset \overline{X}/S)$ does not depend on $\overline{X}$ and is denoted by $\mathrm{Isoc}^\dagger(X/S)$.

Proposition 2.5.11 *Assume* $\mathrm{Char} K = 0$. *Let* S *be a good formal scheme. If* X *is a realizable algebraic variety over* S_k, *we have a canonical equivalence of categories*

$$\mathrm{Mod}^{\dagger}_{\mathrm{g,fp}}(X/S) \simeq \mathrm{Isoc}^{\dagger}(X/S)$$

Proof: Follows from proposition 2.5.8 and corollary 2.5.7. □

Remark: As a corollary, we recover the fact that the category $\mathrm{Isoc}^{\dagger}(X/S)$ is essentially independent of the choices, which is one of the key points in rigid cohomology.

Last remark: We can easily remove the realizable assumption. This is almost a triviality if we consider definition 8.1.3 of [23] and not really more complicated if we use the former one of Berthelot (definition 2.3.2 (iii) of [10]) and proposition 3.6.6 below.

Chapter 3

Cohomology

In section 3.1, we state and prove Berthelot's strong fibration theorem using Berkovich theory. The main ingredient (invariance of neighborhoods by a finite étale map) is deduced from general properties of Berkovich analytic varieties. The rest of the proof follows the original lines. These results are only used in section 3.5.

The main point in section 3.2 is a technical base change for finite quasi-immersions that is necessary in order to pull differential operators to the crystal level in the following section.

In section 3.3, we study differential operators on strict neighborhoods, pull them back to differential operators at the crystal level and push them to the analytic site. It is necessary at this point to use derived functors. Anyway, we show that the cohomology of the derived linearization of a complex with differential operators is captured by the cohomology of the original complex.

Section 3.4 is totally independent of the previous ones and is devoted to the comparison of the behavior of overconvergence for the analytic and for the Grothendieck topology. This is necessary because the strong fibration theorem that we proved in section 1 holds only locally for the Grothendieck topology. And we need this result in the proof of the main theorem in the following section.

Section 3.5 contains the main results. We prove that the cohomology of an overconvergent module of finite presentation can be computed via de Rham cohomology and derive some consequences such as the crucial fact that this cohomology coincides with rigid cohomology.

In section 3.6, we show that, even if the usual topology of an analytic variety is not rich enough to recover the Zariski topology of its reduction, the cohomology of crystals on the overconvergent site is local for the Zariski topology of X. As a consequence, we obtain the main comparison theorem with rigid cohomology.

3.1 The strong fibration theorem

We reprove here in the context of Berkovich theory the strong fibration theorem of Berthelot. The strategy of the proof is exactly the same as Berthelot's.

The following result is the key point. The classical proof is quite involved (Theorem 1.3.5 of [10] or Theorem 3.4.12 of [23]). We see it here as an add-on to the first part of section 1.5.

Proposition 3.1.1 *Let $(X \subset P \leftarrow V)$ be an overconvergent variety, $v : P' \to P$ be a morphism of embeddings of X that is finite étale at X and V' a neighborhood of X in $P'_K \times_{P_K} V$. Then, the induced morphism of overconvergent varieties is an isomorphism*

$$(X, V') \simeq (X, V).$$

Proof: Shrinking V and V' if necessary, we may assume thanks to corollary 1.5.6 that v induces an étale morphism $V' \to V$. Moreover, it follows from proposition 1.5.7 that v induces an isomorphism $]X[_{V'} \simeq]X[_V$. Now, the result follows from lemma 3.1.2 below. □

Lemma 3.1.2 *An étale morphism of analytic varieties $V' \to V$ that induces an isomorphism on analytic domains $T' \simeq T$ induces an isomorphisms between open neighborhoods of T' and T in V' and V respectively.*

Proof: We consider the morphism of germs of analytic varieties $(V', T') \to (V, T)$. It is a surjective quasi-immersion in the sense of Berkovich ([3], definition 4.3.3) and it follows from proposition 4.3.4 of [3] that it induces an isomorphism on the small étale topos

$$(V', T')_{\mathrm{et}} \simeq (V, T)_{\mathrm{et}}.$$

Since the morphism of germs is étale, it formally follows that it is an isomorphism of germs. □

If $(X \subset P \hookrightarrow V)$ is an overconvergent variety, we may embed X in $\widehat{\mathbf{A}}^n_P$ using the zero section in order to get another formal embedding and we obtain a geometric realization of Id_X:

$$\begin{array}{ccccc} X & \hookrightarrow & \widehat{\mathbf{A}}^n_P & \longleftarrow & \mathbf{B}^n_V \\ \| & & \downarrow & & \downarrow \\ X & \hookrightarrow & P & \longleftarrow & V. \end{array}$$

The strong fibration theorem says that, locally, any geometric realization of the identity looks like this one. Unfortunately, "locally" means with respect to the Grothendieck (and the Zariski) topology.

More precisely, we have:

Theorem 3.1.3 (Strong Fibration Theorem) *If X is an algebraic variety over k, any geometric realization $(X, V') \to (X, V)$ of Id_X is locally for the Zariski topology on X and for the Grothendieck topology on V, isomorphic to $(X, \mathbf{B}^n_V)$.*

Almost all the ideas in the following proof are already in Berthelot's original preprint [10].

Proof: We start with a geometric realization

$$\begin{array}{ccccc} X & \hookrightarrow & P' & \longleftarrow & V' \\ \| & & \downarrow & & \downarrow \\ X & \hookrightarrow & P & \longleftarrow & V. \end{array}$$

It means that P' is proper and smooth over P at X and that V' is a neighborhood of X in $P'_K \times_{P_K} V$.

Since the question is local for the Grothendieck topology, we may assume that V is affinoid. Using corollary 1.3.15, we may assume that the locus at infinity of X, both in P and in P', is the support of a divisor. Also, being local for the Grothendieck topology of V, the question is local for the Zariski topology of P. In particular, we may assume that P is affine and that the locus at infinity of X in P is actually a hypersurface. Finally, removing extra components, we may also assume that P' is quasi-compact.

Now, let $Y = \overline{X}^{P'}$ be the Zariski closure of X in P'_k. By hypothesis, the map $Y \to P_k$ induced by u, is proper. Using corollary 1.3.14, we may therefore assume thanks to Chow's lemma (corollary I.5.7.14 of [26]) that the map $Y \to P_k$ is projective. Thus, there exists a commutative diagram

$$\begin{array}{ccccc} & & Y & \hookrightarrow & \mathbf{P}^N_P \\ & \nearrow & \downarrow & & \downarrow p \\ X & & & & \\ & \searrow & \downarrow & & \\ & & P_k & \hookrightarrow & P. \end{array}$$

The closed immersion $X \hookrightarrow p^{-1}(X) = \mathbf{P}^N_X$ is a section of the canonical projection which is smooth. Since the question is local on X, we may assume that there exists an open subset U of $\mathbf{P}^N_X$ such that X is defined in U by a regular sequence. If $\overline{X} := \overline{X}^P$ denotes the Zariski closure of X in P, we may assume that $U = D^+(\bar{s}) \cap \mathbf{P}^N_{\overline{X}}$ with $s \in \Gamma(\mathbf{P}^N_P, \mathcal{O}(m))$ for some m and that the regular sequence is induced by $t_1, \ldots, t_d \in \Gamma(\mathbf{P}^N_P, \mathcal{O}(n))$ for some n. Then, by construction, the induced morphism

$$v : P'' := V^+(t_1, \ldots, t_d) \to P$$

is finite and étale at X and lifts the map $Y \to \overline{X}$. We embed Y in the product $P''' := P'' \times_P P'$ and consider the following cartesian diagram of embeddings of X:

$$\begin{array}{ccc} P' & \xleftarrow{p_1} & P''' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p_2} \\ P & \xleftarrow[v]{} & P''. \end{array}$$

Proposition 3.1.1 tells us that both horizontal arrows induce an isomorphism on the corresponding overconvergent varieties. We may therefore replace u by p_2 and assume that u induces an isomorphism on $Y \simeq \overline{X}$. The question is henceforth local on Y too. This was the hard part.

Since the question is local on X, we may assume that the conormal sheaf $\check{\mathcal{N}}$ of X in $u^{-1}(X)$ is free. Since the question is local on Y, we can replace P' by an affine neighborhood of Y. We may then lift a basis of $\check{\mathcal{N}}$ to a sequence $s_1, \ldots, s_n$ of elements of the ideal of Y in P'. By construction, they induce a morphism $P' \to \widehat{\mathbf{A}}^n_P$ of embeddings of X which is finite étale at X. And we get as expected an isomorphism of overconvergent varieties over (X, V) using again proposition 3.1.1. □

In order to apply this theorem, we will have to localize with respect to the Zariski topology of X when (X, V) is an overconvergent variety. This is possible thanks to the next result.

Proposition 3.1.4 *Let (X,V) be an overconvergent variety and $\{X_k\}_{k\in I}$ a locally finite open covering of X with inclusion maps*

$$\alpha_k : X_k \hookrightarrow X, \quad \alpha_{kl} : X_k \cap X_l \hookrightarrow X, \quad \ldots$$

If $\mathcal{F}$ is an abelian sheaf on $]X[_V$, there is a long exact sequence

$$0 \to \mathcal{F} \to \prod_k]\alpha_k[_{V*}]\alpha_k[_V^{-1}\mathcal{F} \to \prod_{k,l}]\alpha_{kl}[_{V*}]\alpha_{kl}[_V^{-1}\mathcal{F} \to \cdots$$

Proof: This assertion is local on V. Since the specialization map is anticontinuous, the tubes of the irreducible components of X form an open covering of $]X[_V$. We may thus assume that X is irreducible and, in particular, quasi-compact, in which case our covering is finite. Again, since the specialization map is anticontinuous, we obtain a finite closed covering and our sequence is simply the Mayer-Vietoris sequence for this closed covering. □

Corollary 3.1.5 *Let $(f,u) : (X', V') \to (X, V)$ be a morphism of overconvergent varieties, $\mathcal{F}'$ a complex of abelian sheaves on $]X'[_{V'}$ and $\{X'_k\}_{k\in I}$ a locally finite open covering of X'. Then, there is a spectral sequence*

$$E_1^{r,s} = \bigoplus \mathrm{R}^s u_{|]X'_{\underline{k}}[_{V'}*}\mathcal{F}'_{|]X'_{\underline{k}}[_{V'}*} \Rightarrow \mathrm{R}^{r+s}u_*\mathcal{F}'.$$

Proof: Of course, we use the fact that all maps $]\alpha'_{\underline{k}}[:]X'_{\underline{k}}[_{V'} \to]X'[_{V'}$ are inclusions of closed subsets and therefore that all $]\alpha'_{\underline{k}}[_*$ are exact. □

3.2 Functoriality

In this section, we prove some functoriality results that are used in the next one.

Recall that if (X,V) is an overconvergent variety, there exists a morphism of ringed toposes

$$(\varphi_{X,V}^*, \varphi_{X,V*}) : ((X,V)_{\mathrm{An}^\dagger}, \mathcal{O}^\dagger_{(X,V)}) \to (]X[_{V^{\mathrm{an}}}, i_X^{-1}\mathcal{O}_V)$$

where $\varphi_{X,V*}$ denotes the realization functor which is exact and preserves injectives.

Recall also that when $(f,u) : (X',V') \to (X,V)$ is a morphism of overconvergent varieties, there is a restriction map that we will denote by

$$u_{\mathrm{An}^\dagger} : ((X',V')_{\mathrm{An}^\dagger}, \mathcal{O}^\dagger_{(X',V')}) \to ((X,V)_{\mathrm{An}^\dagger}, \mathcal{O}^\dagger_{(X,V)})$$

which is naturally a morphism of ringed toposes. Moreover, $u_{\mathrm{An}^\dagger}^{-1}$, which has a left adjoint, is also exact and preserves injectives.

Realization on (X',V') of an overconvergent sheaf $\mathcal{F}$ defined on (X,V) is by definition

$$\mathcal{F}_{X',V'} := \varphi_{X',V'*}u_{\mathrm{An}^\dagger}^{-1}\mathcal{F}.$$

By composition, this is an exact functor that preserves injectives.

Finally, note that the morphism $(f,u) : (X',V') \to (X,V)$ will also induce a morphism of ringed toposes

$$(u^\dagger, u_*) : (]X'[_{V'^{\mathrm{an}}}, i_{X'}^{-1}\mathcal{O}_{V'}) \to (]X[_{V^{\mathrm{an}}}, i_X^{-1}\mathcal{O}_V).$$

and that we always have

$$(\varphi^*_{X,V}\mathcal{F})_{X',V'} = u^\dagger\mathcal{F}$$

(we previously used to write $]f[^\dagger_u$ and $]f[_{u*}$ but we need to lighten the notations a little bit). There is a first elementary result:

Proposition 3.2.1 *If $(f,u) : (X',V') \to (X,V)$ is a morphism of overconvergent varieties and $\mathcal{F}$ is an $i^{-1}_{X,V}\mathcal{O}_V$-module, there is a canonical isomorphism*

$$u^{-1}_{\mathrm{An}^\dagger}\varphi^*_{X,V}\mathcal{F} = \varphi^*_{X',V'}u^\dagger\mathcal{F}.$$

Proof: Simply follows from functoriality: there is a commutative diagram of ringed toposes

$$\begin{array}{ccc} (]X'[_{V'})_{\mathrm{an}} & \xleftarrow{\varphi_{X',V'}} & (X',V')_{\mathrm{An}^\dagger} \\ \downarrow u & & \downarrow u_{\mathrm{An}^\dagger} \\ (]X[_V)_{\mathrm{an}} & \xleftarrow[\varphi_{X,V}]{} & (X,V)_{\mathrm{An}^\dagger}. \end{array}$$ □

There is also a first easy base change theorem:

Proposition 3.2.2 *Let*

$$\begin{array}{ccc} (Y',W') & \xrightarrow{(f',u')} & (Y,W) \\ \downarrow (g',v') & & \downarrow (g,v) \\ (X',V') & \xrightarrow{(f,u)} & (X,V) \end{array}$$

be a cartesian diagram of overconvergent varieties. If $\mathcal{F}'$ is an analytic sheaf on (X',V'), we have

$$(u_{\mathrm{An}^\dagger *}\mathcal{F}')_{Y,W} = u'_*\mathcal{F}_{Y',W'}.$$

Actually, if $\mathcal{F}'$ is a complex of abelian sheaves, we even get

$$(\mathrm{R}u_{\mathrm{An}^\dagger *}\mathcal{F}')_{Y,W} = \mathrm{R}u'_*\mathcal{F}'_{Y',W'}.$$

Proof: The first assertion follows follows from the formal identity in any topos

$$v^{-1}_{\mathrm{An}^\dagger}u_{\mathrm{An}^\dagger *} = u'_{\mathrm{An}^\dagger *}v'^{-1}_{\mathrm{An}^\dagger}$$

after applying $\varphi_{Y,W*}$. For the second one, it is sufficient to recall that the realization functors are exact and preserve injectives. □

Recall from [3], Definition 4.3.3, that a *finite quasi-immersion* of analytic varieties is a finite morphism which is a homeomorphism onto its image with purely inseparable residue field extensions. This property is stable under composition and base change. Projections of infinitesimal neighborhoods are non trivial examples of quasi-immersions.

Definition 3.2.3 *A morphism $(f,u) : (X',V') \to (X,V)$ of overconvergent varieties is* finite *(resp. a* finite quasi-immersion*) if, after replacing V and V' by strict neighborhoods of X and X' respectively, we have*

1. *u is finite (resp. a finite quasi-immersion)*
2. *$u^{-1}(]X[_V) =]X'[_{V'}$*

Proposition 3.2.4 *Let $(f,u) : (X',V') \to (X,V)$ be a finite quasi-immersion of overconvergent varieties and $\mathcal{F}'$ an $i_{X'}^{-1}\mathcal{O}_{V'}$-module. Then the adjunction map*

$$\varphi_{X,V}^* u_* \mathcal{F}' \to u_{\mathrm{An}^\dagger *} \varphi_{X',V'}^* \mathcal{F}'$$

is an isomorphism. Moreover, we have

$$\mathrm{R}^q u_* \mathcal{F}' = 0 \quad \text{and} \quad \mathrm{R}^q u_{\mathrm{An}^\dagger *} \varphi_{X',V'}^* \mathcal{F}' = 0 \quad \text{for} \quad q > 0.$$

There should be an analogous result for finite morphisms and coherent sheaves but we will not need it.

Proof: Of course, we choose V and V' such that (f,u) satisfies the conditions of definition 3.2.3. Let $(g,v) : (Y,W) \to (X,V)$ be any morphism of overconvergent varieties, $(g',v') : (Y',W') \to (X',V')$ the pull-back along (f,u), and $(f',u') : (Y',W') \to (Y,W)$ the corresponding map. We are in the situation of proposition 3.2.2 and it follow that

$$(\mathrm{R}u_{\mathrm{An}^\dagger *} \varphi_{X',V'}^* \mathcal{F}')_{Y,W} = \mathrm{R}u'_*(\varphi_{X',V'}^* \mathcal{F}')_{Y',W'}.$$

In particular, in order to prove the last assertion, it is actually sufficient to show that

$$\mathrm{R}^q u_* \mathcal{F}' = 0 \quad \text{for} \quad q > 0$$

and then apply it to u'_* and $(\varphi_{X',V'}^* \mathcal{F}')_{Y',W'}$ (finite quasi-immersions are preserved under base change).

We also have to show that we always have

$$\Gamma((Y,W), \varphi_{X,V}^* u_* \mathcal{F}') \simeq \Gamma((Y,W), u'_*(\varphi_{X',V'}^* \mathcal{F}')_{(Y',W')}).$$

And this means that

$$\Gamma(]Y[_W, v^\dagger u_* \mathcal{F}') \simeq \Gamma(]Y[_W, u'_* v'^\dagger \mathcal{F}')$$

with the cartesian diagram

$$\begin{array}{ccc}]Y'[_{W'} & \xrightarrow{u'} &]Y[_W \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ ]X'[_{V'} & \xrightarrow{u} &]X[_V. \end{array}$$

We are therefore reduced to checking that

$$v^\dagger u_* \mathcal{F}' \simeq u'_* v'^\dagger \mathcal{F}'$$

on $]Y[_W$.

Of course, it is sufficient to consider sheaves of the form $i_{X'}^{-1}\mathcal{F}'$. Thus, changing notations, we want to show that, if $\mathcal{F}'$ is an $\mathcal{O}_{V'}$-module, we have

$$v^\dagger u_* i_{X'}^{-1} \mathcal{F}' \simeq u'_* v'^\dagger i_{X'}^{-1} \mathcal{F}'$$

and

$$\mathrm{R}^q u_* i_{X'}^{-1}\mathcal{F}' = 0 \quad \text{for} \quad q > 0$$

with the cartesian diagram

$$\begin{array}{ccc} W' & \xrightarrow{u'} & W \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v} \\ V' & \xrightarrow{u} & V. \end{array}$$

Since the map u is just the inclusion of a closed subset (it is a finite homeomorphism), it follows from the second condition of definition 3.2.3 that the cohomology vanishes and that $u_* i_{X'}^{-1}\mathcal{F}' = i_X^{-1} u_*\mathcal{F}'$. By definition, we obtain

$$v^\dagger u_* i_{X'}^{-1}\mathcal{F}' = v^\dagger i_X^{-1} u_*\mathcal{F}' = i_Y^{-1} v^* u_*\mathcal{F}'.$$

For the same reasons, since we always have $v'^\dagger i_{X'}^{-1}\mathcal{F}' = i_{Y'}^{-1} v'^*\mathcal{F}'$, we get

$$u'_* v'^\dagger i_{X'}^{-1}\mathcal{F}' = u'_* i_{Y'}^{-1} v'^*\mathcal{F}' = i_Y^{-1} u'_* v'^*\mathcal{F}'.$$

It is therefore sufficient to show that

$$v^* u_*\mathcal{F}' = u'_* v'^*\mathcal{F}'$$

on W. This can be checked on each fiber. If $x \in W$ is not in the image of W', then $v(x)$ is not in the image of V' and both sides vanish. If $x \in W'$, then

$$(v^* u_*\mathcal{F}')_{u'(x)} = \mathcal{O}_{W,u'(x)} \otimes_{\mathcal{O}_{V,vu'(x)}} (u_*\mathcal{F}')_{vu'(x)} =$$

$$= \mathcal{O}_{W,u'(x)} \otimes_{\mathcal{O}_{V,uv'(x)}} (u_*\mathcal{F}')_{uv'(x)} = \mathcal{O}_{W,u'(x)} \otimes_{\mathcal{O}_{V,uv'(x)}} \mathcal{F}'_{v'(x)}$$

At last, we use the fact that u is finite to obtain

$$u'_* v'^*\mathcal{F}'_{u'(x)} = (v'^*\mathcal{F}')_x = \mathcal{O}_{W',x} \otimes_{\mathcal{O}_{V',v'(x)}} \mathcal{F}'_x$$

$$= (\mathcal{O}_{W,u'(x)} \otimes_{\mathcal{O}_{V,vu'(x)}} \mathcal{O}_{V',v'(x)}) \otimes_{\mathcal{O}_{V',v'(x)}} \mathcal{F}'_x = \mathcal{O}_{W,u'(x)} \otimes_{\mathcal{O}_{V,vu'(x)}} \mathcal{F}'_{v'(x)}. \quad \square$$

3.3 Differential operators

We introduce in this section the notions of differential operator and derived linearization of a complex of differential operators in our context. Then, we apply the theory to the de Rham complex.

We fix a morphism $(X, V) \to (C, O)$ of overconvergent varieties.

As we already did, we will consider the infinitesimal neighborhood of order n of V which is the analytic subvariety $V^{(n)}$ defined by $\mathcal{I}^{n+1}$ if V is defined by $\mathcal{I}$ in $V \times_O V$. The diagonal embedding $\delta : V \hookrightarrow V \times_O V$ and the projections $p_i : V \times_O V \to V$ induce morphisms $\delta^{(n)} : V \hookrightarrow V^{(n)}$ and $p_i^{(n)} : V^{(n)} \to V$. From $p_i \circ \delta = \mathrm{Id}_V$, we deduce that $p_1^{(n)}$ and $p_2^{(n)}$ are identical as continuous maps (both are inverse to the homeomorphism $\delta^{(n)}$). Actually, if we use $\delta^{(n)}$ to identify the underlying topological spaces of $V^{(n)}$ and V, then $p_i^{(n)}$ becomes the identity as continuous maps both for $i = 1$ and $i = 2$ so that $p_{2*}^{(n)} = p_{1*}^{(n)}$. Of course, the projections act differently on sections so that $p_2^{(n)*} \neq p_1^{(n)*}$.

We will embed (X,V) into $(X,V^{(n)})$ using $\delta^{(n)}$ and consider also the projection maps $p_i^{(n)} : (X,V^{(n)}) \to (X,V)$. All these maps do nothing on the underlying topological spaces and only play a role on sections.

Definition 3.3.1 *An $i_C^{-1}\mathcal{O}_O$-linear morphism $d : \mathcal{F} \to \mathcal{G}$ between two $i_X^{-1}\mathcal{O}_V$-modules is a* differential operator (of order at most n) *if it factors as*

$$d: \qquad \mathcal{F} \xrightarrow{p_2^{(n)*}} p_{2*}^{(n)} p_2^{(n)\dagger}\mathcal{F} = p_{1*}^{(n)} p_2^{(n)\dagger}\mathcal{F} \xrightarrow{\overline{d}} \mathcal{G}.$$

where $\overline{d}$ is $i_X^{-1}\mathcal{O}_V$-linear

Note that the equality sign is *not* linear: the abelian sheaves are the same but they carry different $i_X^{-1}\mathcal{O}_V$-module structures. Note also that $\overline{d}$ is unique: locally, we have

$$\overline{d}(\overline{f \otimes g} \otimes m) = fd(gm).$$

The composition of two differential operators is a differential operator. More precisely, if $d : \mathcal{F} \to \mathcal{G}$ and $d' : \mathcal{G} \to \mathcal{H}$ are two differential operators of order at most n and m respectively, then $d' \circ d$ is a differential operator of order at most $m+n$ and

$$\overline{d' \circ d} = \overline{d'} \circ p_2^{(m)\dagger}(\overline{d}).$$

We will mainly be concerned with differential operators of order at most 1: it just means that, locally, we have

$$d(fgm) + fgd(m) = fd(gm) + gd(fm).$$

The standard example of differential operators of order at most 1 is given by the differentials in a de Rham complex

$$\mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^{\bullet}_{V/O}$$

when $\mathcal{F}$ is an $i_X^{-1}\mathcal{O}_V$-module with an integrable connection.

We can extend this notion of differential operator to overconvergent modules as follows:

Definition 3.3.2 *An $\mathcal{O}^{\dagger}_{(C,O)}$-linear morphism $d : E \to E'$ between two $\mathcal{O}^{\dagger}_{(X,V)}$-modules is a* differential operator (of order at most n) *if it factors as the composite of two $\mathcal{O}^{\dagger}_{(X,V)}$-linear morphisms*

$$E \xrightarrow{p_{2\mathrm{An}^\dagger}^{(n)*}} p^{(n)}_{2\mathrm{An}^\dagger *} p^{(n)-1}_{2\mathrm{An}^\dagger} E = p^{(n)}_{1\mathrm{An}^\dagger *} p^{(n)-1}_{2\mathrm{An}^\dagger} E \xrightarrow{\overline{d}} E'.$$

Of course, here again, the equality sign in this definition is *not* linear.

Proposition 3.3.3 *The functors $\varphi_{X,V*}$ and $\varphi^*_{X,V}$ induce an equivalence between the category of overconvergent crystals on (X,V) with differential operators and the category of $i_X^{-1}\mathcal{O}_V$-modules and differential operators.*

Proof: If $d : \mathcal{F} \to \mathcal{G}$ is a differential operator of order at most n, we can apply the functor $\varphi^*_{X,V}$ to $\overline{d}$ and get a morphism of crystals

$$\varphi^*_{X,V}(\overline{d}) : \varphi^*_{X,V}(p^{(n)}_{1*}p^{(n)\dagger}_2\mathcal{F}) \to \varphi^*_{X,V}(\mathcal{G}).$$

We have thanks to proposition 3.2.1 and 3.2.4,

$$p^{(n)}_{1\mathrm{An}^\dagger *}p^{(n)^{-1}}_{2\mathrm{An}^\dagger}\varphi^*_{X,V}\mathcal{F} = p^{(n)}_{1\mathrm{An}^\dagger *}\varphi^*_{X,V^{(n)}}(p^{(n)\dagger}_2\mathcal{F}) \simeq \varphi^*_{X,V}(p^{(n)}_{1*}p^{(n)\dagger}_2\mathcal{F}).$$

And we compose on the left with $p^{(n)^*}_{2\mathrm{An}^\dagger}$ in order to get a differential operator at the crystal level. Since we already know from proposition 2.3.8 that our functor induce an equivalence between crystals on (X, V) and of $i_X^{-1}\mathcal{O}_V$-modules when we consider only linear maps, the conclusion is immediate. $\square$

The following lemma shows that the category of overconvergent modules and differential operators has enough injectives.

Lemma 3.3.4 *Any complex $(E^\bullet, d)$ of overconvergent modules and differential operators has a resolution $(I^\bullet, d)$ made of injective overconvergent modules and differential operators.*

Proof: It is sufficient to show that if $E \hookrightarrow I$ and $E' \hookrightarrow I'$ are two inclusion maps into injective overconvergent modules, then any differential operator $d : E \to E'$ of order at most n, extends to a differential operator $I \to I'$ of the same order. By functoriality, it is clearly sufficient to extend

$$\overline{d} : p^{(n)}_{1\mathrm{An}^\dagger *}p^{(n)^{-1}}_{2\mathrm{An}^\dagger}E \to E'.$$

Since I' is an injective module, this will follow from the fact that both $p^{(n)}_{1\mathrm{An}^\dagger *}$ and $p^{(n)^{-1}}_{2\mathrm{An}^\dagger}$ are left exact. $\square$

Now that we can pull a complex of differential operators back to $(X, V)_{\mathrm{An}^\dagger}$, we want to push it to $(X/O)_{\mathrm{An}^\dagger}$. We first need an explicit description of the restriction morphism

$$j_{X,V} : (X, V)_{\mathrm{An}^\dagger} \to (X/O)_{\mathrm{An}^\dagger}.$$

Lemma 3.3.5 *If $\mathcal{F}$ is an analytic sheaf on (X, V) and (X', V') is an overconvergent variety over X/O, we have*

$$(j_{X,V*}\mathcal{F})_{X',V'} = p_{1*}\mathcal{F}_{X',V'\times_O V}$$

where $p_1 : V' \times_O V \to V'$ denotes the first projection. If $\mathcal{F}$ is a complex of abelian sheaves, we even get

$$(\mathrm{R}j_{X,V*}\mathcal{F})_{X',V'} = \mathrm{R}p_{1*}\mathcal{F}_{X',V'\times_O V}.$$

Proof: The first assertion follows from the fact that $j^{-1}_{X,V}(X', V') = (X', V' \times_O V)$ and therefore

$$\Gamma((X', V'), j_{X,V*}\mathcal{F}) = \Gamma(j^{-1}_{X,V}(X', V'), \mathcal{F}) = \Gamma((X', V' \times_O X), \mathcal{F})$$

$$= \Gamma(]X'[_{V'\times_O V}, \mathcal{F}_{X',V'\times V}) = \Gamma(]X'[_{V'}, p_{1*}\mathcal{F}_{X',V'\times_O V}).$$

More precisely, this equality should be applied when V' is replaced with some open subset. For the second assertion, we recall that the realization functors are exact and preserve injectives. $\square$

In order to justify the next definition, we also prove the following.

Lemma 3.3.6 *If $d : E \to E'$ is a differential operator between overconvergent modules on (X, V), the morphism*

$$j_{X,V*}(d) : j_{X,V*}E \to j_{X,V*}E'$$

is $\mathcal{O}^{\dagger}_{X/O}$-linear.

Proof: This follows from the definition of a differential operator and the fact that the diagram

$$(X, V^{(n)}) \underset{p_2^{(n)}}{\overset{p_1^{(n)}}{\rightrightarrows}} (X, V) \xrightarrow{j_{X,V}} X/O$$

is commutative. □

Definition 3.3.7 *If $\mathcal{F}$ is a sheaf of $i_X^{-1}\mathcal{O}_V$-modules, the* linearization *of $\mathcal{F}$ is*

$$L(\mathcal{F}) = j_{X,V*}\varphi^*_{X,V}\mathcal{F}.$$

If $d : \mathcal{F} \to \mathcal{G}$ is a differential operator of finite order, its linearization *is*

$$L(d) : j_{X,V*}\varphi^*_{X,V}(d) : L(\mathcal{F}) \to L(\mathcal{G}).$$

If $(\mathcal{F}^\bullet, d)$ is a complex of $i_X^{-1}\mathcal{O}_V$-modules and differential operators, the derived linearization *of $\mathcal{F}$ is*

$$\mathrm{R}L(\mathcal{F}^\bullet) = \mathrm{R}j_{X,V*}\varphi^*_{X,V}\mathcal{F}^\bullet.$$

Note that RL is not the right derived functor of L (and anyway that L is not left exact in general). Thus, we have to be careful when one moves from L to RL. It also follows from lemma 3.3.6 that the linearization of a differential operator *is* linear. We now give a more explicit description of this linearization.

Proposition 3.3.8 *If $\mathcal{F}$ is a sheaf of $i_X^{-1}\mathcal{O}_V$-modules and (X', V') is an overconvergent variety over X/O, we have*

$$L(\mathcal{F})_{X',V'} = p_{1*}p_2^{\dagger}\mathcal{F}$$

where $p_1 : V' \times_O V \to V'$ and $p_2 : V' \times_O V \to V$ denote the projections. Actually, if $(\mathcal{F}^\bullet, d)$ is a complex of $i_X^{-1}\mathcal{O}_V$-modules and differential operators, we have

$$\mathrm{R}L(\mathcal{F}^\bullet)_{X',V'} = \mathrm{R}p_{1*}p_2^{\dagger}\mathcal{F}^\bullet.$$

Proof: Results from lemma 3.3.5. □

If $d : \mathcal{F} \to \mathcal{G}$ is a differential operator, we can also give an explicit description of the corresponding linear map

$$p_{1*}p_2^{\dagger}\mathcal{F} \to p_{1*}p_2^{\dagger}\mathcal{G}$$

on $]X'[_{V'\times_O V}$. Of course, the main point is to describe the pull back morphism

$$\delta : p_{1*}p_2^{\dagger}\mathcal{F} \to p_{1*}p_2^{\dagger}p_2^{(n)\dagger}\mathcal{F}$$

that will be composed on the right with $\overline{d}$. Thus, we consider the following diagram (where all products are taken relative to O)

$$\begin{array}{ccccccc}
V' \times V^{(n)} & \xrightarrow{\mathrm{Id}\times i_n} & V' \times V \times V & \underset{p_{12}}{\overset{p_{13}}{\rightrightarrows}} & V' \times V & \xrightarrow{p_1} & V' \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} & \searrow{\scriptstyle p_3} & \downarrow{\scriptstyle p_2} & & \\
V^{(n)} & \xrightarrow{i_n} & V \times V & \underset{p_1}{\overset{p_2}{\rightrightarrows}} & V & &
\end{array}.$$

and call $p_k^{(n)} = p_k \circ i_n$ and $p_{1k}^{(n)} = p_{1k} \circ (\mathrm{Id} \times i_n)$. We consider the map

$$p_{13}^{(n)*} : p_2^\dagger \mathcal{F} \to p_{13*}^{(n)} p_{13}^{(n)\dagger} p_2^\dagger \mathcal{F} = p_{13*}^{(n)} p_2^\dagger p_2^{(n)\dagger} \mathcal{F}$$

and apply p_{1*} in order to get

$$\delta = p_{13}^{(n)*} : p_{1*} p_2^\dagger \mathcal{F} \to p_{1*} p_2^\dagger p_2^{(n)\dagger} \mathcal{F}.$$

Recall that $p_{X/O}$ is defined as the composition of the restriction from map from $(X/O)_{\mathrm{An}^\dagger}$ to $(C, O)_{\mathrm{An}^\dagger}$ the realization at (C, O):

$$p_{X/O} : \qquad \begin{array}{ccc} (X/O)_{\mathrm{An}^\dagger} & \longrightarrow & (]C[_O)_{\mathrm{an}} \\ (X/O') & \longleftarrow\!\!\!\dashv &]C[_{O'}. \end{array}$$

We will also denote by $p_{]X[_V} :]X[_V \to]C[_O$ the map induced on the tubes.

Proposition 3.3.9 *If $\mathcal{F}$ is a sheaf of $i_X^{-1}\mathcal{O}_V$-modules, there is a canonical isomorphism*

$$p_{X/O*} L(\mathcal{F}) \simeq p_{]X[_V *} \mathcal{F}$$

on $]C[_O$. Actually, if $(\mathcal{F}^\bullet, d)$ is a complex of $i_X^{-1}\mathcal{O}_V$-modules and differential operators, we also have

$$\mathrm{R}p_{X/O*} \mathrm{R}L(\mathcal{F}^\bullet) \simeq \mathrm{R}p_{]X[_V *} \mathcal{F}^\bullet.$$

Proof: We consider the commutative diagram

$$\begin{array}{ccc}
(X, V)_{\mathrm{An}^\dagger} & \xrightarrow{\varphi_{X,V}} & (]X[_V)_{\mathrm{an}} \\
\downarrow{\scriptstyle j_{X,V}} & & \downarrow{\scriptstyle p_{]X[_V}} \\
(X/O)_{\mathrm{An}^\dagger} & \xrightarrow{p_{X/O}} & (]C[_O)_{\mathrm{an}}
\end{array}$$

We have

$$p_{X/O*} L(\mathcal{F}) = p_{X/O*} j_{X,V*} \varphi_{X,V}^* \mathcal{F} = p_{]X[_V *} \varphi_{X,V*} \varphi_{X,V}^* \mathcal{F} = p_{]X[_V *} \mathcal{F}$$

because $\varphi_{X,V*}\varphi_{X,V}^* \mathcal{F} = \mathcal{F}$.

Similarly, when $(\mathcal{F}^\bullet, d)$ is a complex of $i_X^{-1}\mathcal{O}_V$-modules and differential operators, we have

$$\mathrm{R}p_{X/O*} \mathrm{R}L(\mathcal{F}^\bullet) = \mathrm{R}p_{X/C,O*} \mathrm{R}j_{X,V*} \varphi_{X,V}^* \mathcal{F}^\bullet$$

$$= \mathrm{R}(p_{X/O} \circ j_{X,V})_* \varphi_{X,V}^* \mathcal{F}^\bullet = \mathrm{R}(p_{]X[_V} \circ \varphi_{X,V})_* \varphi_{X,V}^* \mathcal{F}^\bullet$$

$$= \mathrm{R}p_{]X[_V*}\varphi_{X,V*}\mathrm{R}\varphi^*_{X,V}\mathcal{F}^\bullet = \mathrm{R}p_{]X[_V*}\varphi_{X,V*}\varphi^*_{X,V}\mathcal{F}^\bullet = \mathrm{R}p_{]X[_V*}\mathcal{F}^\bullet. \quad \square$$

We end this section with an application of these methods to the de Rham complex. As already mentioned, if $\mathcal{F}$ is an $i_X^{-1}\mathcal{O}_V$-module with an integrable connection, its de Rham complex

$$\mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}$$

is a complex of differential operators of order at most 1. We may therefore consider its derived linearization

$$\mathrm{R}L(\mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}).$$

We can give an explicit description of this object (we stick to the case of crystals):

Proposition 3.3.10 *If E an overconvergent crystal on X/O, we have the following:*

1. *If (X', V') is an overconvergent variety over X/O, then*
$$\mathrm{R}L(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O})_{X',V'} = \mathrm{R}p_{1*}(p_1^\dagger E_{X',V'} \otimes_{i_{X'}^{-1}\mathcal{O}_{V'\times V}} i_{X'}^{-1}\Omega^\bullet_{V'\times V/V'}).$$
2. *There is a canonical augmentation*
$$E \to \mathrm{R}L(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}).$$

Actually, both results hold before deriving.

Proof: Since E is a crystal, we have

$$p_2^\dagger(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}) = p_2^\dagger E_{X,V} \otimes_{i_{X'}^{-1}\mathcal{O}_{V'\times V}} i_{X'}^{-1}\Omega^\bullet_{V'\times V/V'}$$

$$= E_{X',V'\times V} \otimes_{i_{X'}^{-1}\mathcal{O}_{V'\times V}} i_{X'}^{-1}\Omega^\bullet_{V'\times V/V'} = p_1^\dagger E_{X',V'} \otimes_{i_{X'}^{-1}\mathcal{O}_{V'\times V}} i_{X'}^{-1}\Omega^\bullet_{V'\times V/V'}$$

and we may apply $\mathrm{R}p_{1*}$ in order to get the first assertion thanks to proposition 3.3.8.

Note that it is sufficient to prove the second assertion before deriving and then compose with the canonical morphism $L \to \mathrm{R}L$. Since E is a crystal, we have

$$j_{X,V}^{-1}E = \varphi^*_{X,V}\varphi_{X,V*}j_{X,V}^{-1}E = \varphi^*_{X,V}E_{X,V}$$

and by adjunction, we obtain a map

$$E \to j_{X,V*}\varphi^*_{X,V}E_{X,V} = L(E_{X,V}).$$

We have to show that the composite map

$$E \to L(E_{X,V}) \to L(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_{V/O})$$

is zero. This can be done locally and, thanks to the first part, we are reduced to check that

$$E_{X',V'} \to p_{1*}p_1^\dagger E_{X',V'} \to p_{1*}(p_1^\dagger E_{X',V'} \otimes_{i_{X'}^{-1}\mathcal{O}_{V'\times V}} i_{X'}^{-1}\Omega^\bullet_{V'\times V/V'})$$

is zero. This is linear in $E_{X',V'}$ and we are reduced to show that the composite map

$$i_{X'}^{-1}\mathcal{O}_{V'} \to p_{1*}i_{X'}^{-1}\mathcal{O}_{V'\times V} \to p_{1*}i_{X'}^{-1}\Omega^\bullet_{V'\times V/V'}$$

is zero. Here again, we need to be careful since the base change map $i_{X'}^{-1}p_{1*} \to p_{1*}i_{X'}^{-1}$ is not bijective in general. But we do not care because it is *sufficient* that the composition

$$i_{X'}^{-1}\mathcal{O}_{V'} \to i_{X'}^{-1}p_{1*}\mathcal{O}_{V'\times V} \to i_{X'}^{-1}p_{1*}\Omega^\bullet_{V'\times V/V'}$$

is zero. Finally, we may drop the $i_{X'}^{-1}$ and are reduced to the augmentation from the constant sheaf to the usual de Rham complex. $\square$

3.4 Grothendieck topology and overconvergence

In proposition 3.5.5 below, we will need to localize with respect to the Grothendieck topology. Unfortunately, the dictionary of section 1.3 of [3] is not sufficient for us and we need to expand it a little bit. This is what we do here.

We start with a geometrical result.

Lemma 3.4.1 *Let $(X \subset P \leftarrow V)$ be an overconvergent variety. We assume that P is affine and that $\infty_X := \overline{X} \setminus X$ is a hypersurface $\overline{g} = 0$ in $\overline{X}$ with g a function on P. We also assume that V is affinoid and that $|\overline{X}|_V = V$. Then, the affinoid domains*

$$V_\epsilon := \{x \in V, \quad |g(\lambda(x))| \geq \epsilon\},$$

for $\epsilon \xrightarrow{<} 1$ and $\epsilon \in \sqrt{|K^\times|}$, form a cofinal family of neighborhoods of X in V.

Proof: Since

$$]X[_V := \{x \in V, \overline{g}(\lambda(x)) \neq 0\} = \{x \in V, |g(\lambda(x))| \geq 1\},$$

it is clear that each V_ϵ is an affinoid neighborhood of X in V. Let V' be an open neighborhood of X in V. Denote by T, T' and T_ϵ the complement of $]X[_V$ in V, V' and V_ϵ respectively. Clearly, $T = \cup T_\epsilon$ is an open covering and T' is compact as a closed subset of V. It follows that there must exist ϵ such that $T' \subset T_\epsilon$ which means that $V_\epsilon \subset V'$. □

If $u : V' \to V$ is a morphism of analytic varieties over K, then $u_G : V'_G \to V_G$ will denote the corresponding morphism of ringed sites (see appendix 4.2 for details). Recall also that there is an obvious natural morphism of ringed sites $\pi_V : V_G \to V$ and that we sometimes write $\mathcal{F}_G := \pi_V^* \mathcal{F}$.

If (X, V) is an overconvergent variety and $\mathcal{F}$ is a sheaf on V, we proved in proposition 2.2.12 that, when $]X[_V$ is closed in V, we have

$$i_{X*} i_X^{-1} \mathcal{F} = \varinjlim j'_* j'^{-1} \mathcal{F}$$

when j' runs through all inclusions of neighborhoods of X in V. In a similar way, if $\mathcal{G}$ is a sheaf on V_G, one sets

$$j_X^\dagger \mathcal{G} = \varinjlim j'_{G*} j'^{-1}_G \mathcal{G}.$$

We use below the notion of family of subsets *of finite type*: it just means that any subset meets only a finite number of the others. Note that a locally finite covering by quasi-compact subsets is automatically of finite type.

Lemma 3.4.2 *Let $(X \subset P \leftarrow V)$ be an overconvergent variety. Assume that V is separated and countable at infinity and that $]\overline{X}[_V = V$. If $\mathcal{G}$ is a coherent sheaf on V_G, then*

$$\forall q \geq 0, \quad \varinjlim \mathrm{H}^q(V'_G, \mathcal{G}) \simeq \mathrm{H}^q(V_G, j_X^\dagger \mathcal{G})$$

where V' runs through all the neighborhoods of X in V.

The result actually holds if V is locally separated and paracompact (this follows from proposition 3.4.3 below).

Proof: Note first that this equality is always true when V is compact because then, V_G is quasi-compact and quasi-separated. In particular, this holds in the affinoid case. If, moreover, X has a cofinal family of affinoid neighborhoods in V, we see that the left hand side is zero for $q > 0$ and it follows that the right-hand side is zero too.

We now come back to the general case. After blowing up ∞_X in P, we may find a locally finite affine covering of $P = \cup_\alpha P_\alpha$ such that ∞_{X_α} is a hypersurface in $\overline{X}_\alpha := \overline{X} \cap P_\alpha$. Since V is countable at infinity, we can find a countable admissible affinoid covering of finite type $V = \cup_{i\in\mathbf{N}} W_i$ such that the image of W_i in P_K is contained in some $P_{\alpha K}$. Then, it follows from lemma 3.4.1 that there exists for each $i \in \mathbf{N}$ a decreasing family $\{W_{i,\epsilon}\}_{\epsilon \xrightarrow{<} 1}$ of affinoid neighborhoods of X in W that are cofinal. If $\underline{\epsilon} := \{\epsilon_i\}_{i\in\mathbf{N}}$ is any sequence inside $[0, 1[$, we let $V_{\underline{\epsilon}} := \cup W_{i,\epsilon_i}$. This is a covering of finite type by affinoid domains and therefore an admissible covering of an analytic domain. Note that $V_{\underline{\epsilon}}$ is a neighborhood of X in V because this question is local for the Grothendieck topology of V and that $V_{\underline{\epsilon}} \cap W_i \supset W_{i,\epsilon_i}$ Moreover, if V' is an open neighborhood of X in V, then, for each $i \in \mathbf{N}$, $V' \cap W_i$ is an open neighborhood of X in W_i and there exists ϵ_i such that $W_{i,\epsilon_i} \subset V'$. It follows that there exists $\underline{\epsilon}$ such that $V_{\underline{\epsilon}} \subset V'$.

For fixed $i_1, \ldots, i_p \in \mathbf{N}$, the analytic domains $W_{\underline{i,\epsilon}} := W_{i_1,\epsilon_1} \cap \cdots \cap W_{i_p,\epsilon_p}$ with $\epsilon_1, \ldots, \epsilon_p < 1$, form a cofinal system of neighborhoods of X in $W_{\underline{i}} := W_{i_1} \cap \cdots \cap W_{i_p}$. Since our variety is separated, all the analytic domains $W_{\underline{i,\epsilon}}$ are affinoid, and we have for all $q > 0$,

$$\mathrm{H}^q(W_{\underline{i,\epsilon},G}, \mathcal{G}) = 0 \quad \text{and} \quad \mathrm{H}^q(W_{\underline{i},G}, j_X^\dagger \mathcal{G}) = 0$$

(use the remark at the beginning of this proof). It follows that for any sequence $\underline{\epsilon}$, we have

$$\mathrm{H}^q(V_{\underline{\epsilon},G}, \mathcal{G}) = \check{\mathrm{H}}^q(\{W_{i,\epsilon_i,G}\}, \mathcal{G}) \quad \text{and} \quad \mathrm{H}^q(V, j_X^\dagger \mathcal{G}) = \check{\mathrm{H}}^q(\{W_{i,G}\}, j_X^\dagger \mathcal{G}))$$

Since filtered direct limits are exact, it is therefore sufficient to show that

$$\forall q \geq 0, \quad \varinjlim_{\underline{\epsilon}} \check{C}^q(\{W_{i,\epsilon_{i},G}\}, \mathcal{G}) \simeq \check{C}^q(\{W_{i,G}\}, j_X^\dagger \mathcal{G}).$$

We are therefore reduced to prove that for all $i_1, \ldots, i_p \in \mathbf{N}$, we have

$$\varinjlim_{\underline{\epsilon}} \prod_{\underline{i}} \Gamma(W_{\underline{i,\epsilon},G}, \mathcal{G}) \simeq \prod_{i} \varinjlim_{\epsilon} \Gamma(W_{\underline{i},\epsilon,G}, \mathcal{G})$$

with $W_{\underline{i},\epsilon} := W_{i_1,\epsilon} \cap \cdots \cap W_{i_p,\epsilon}$. This is an easy exercise on commutation of products and filtered direct limits. □

Proposition 3.4.3 *Let (X, V) is an overconvergent variety with $]X[_V$ closed in V. Then,*

1. *If $\mathcal{F}$ is any sheaf on V, we have*

$$\pi_{V*} j_X^\dagger \pi_V^{-1} \mathcal{F} = i_{X*} i_X^{-1} \mathcal{F}.$$

2. *If $\mathcal{F}$ is an $\mathcal{O}_V$-module and V is good, we have*

$$\pi_{V*} j_X^\dagger \pi_V^* \mathcal{F} = i_{X*} i_X^{-1} \mathcal{F}.$$

3. If moreover, $]\overline{X}[_V = V$ and $\mathcal{F}$ is coherent, then

$$\mathrm{R}\pi_{V*}j_X^\dagger \pi_V^* \mathcal{F} = i_{X*}i_X^{-1}\mathcal{F}.$$

Proof: The first assertion is checked on the stalks and follows from the facts that an analytic variety is locally compact and that global sections on compact sets commute with direct limits.

For the second assertion, we use the same method and the fact that with our additional hypothesis, any point has a basis of affinoid neighborhoods. Then it is sufficient to notice that, if W is a good compact analytic domain in V, and in particular an affinoid domain, we have

$$\Gamma(W_G, j_X^\dagger \mathcal{F}_G) = \Gamma(]X[_W, i_X^{-1}\mathcal{F}).$$

In order to prove the last assertion, since any good analytic variety has a basis of open subsets that are separated and countable at infinity, it is sufficient to show that, if V is separated and countable at infinity, then

$$\forall q \geq 0, \quad \mathrm{H}^q(V_G, j_X^\dagger \mathcal{F}_G) = \mathrm{H}^q(]X[_V, i_X^{-1}\mathcal{F}).$$

We proved in lemma 3.4.2 that

$$\forall q \geq 0, \quad \varinjlim \mathrm{H}^q(V'_G, \mathcal{F}_G) \simeq \mathrm{H}^q(V_G, j_X^\dagger \mathcal{F}_G)$$

where where V' runs through all the open neighborhoods of X in V. On the other hand, since V is paracompact and $]X[_V$ is closed in V, we also have (see for example Remark 2.6.9 of [22])

$$\forall q \geq 0, \quad \varinjlim \mathrm{H}^q(V', \mathcal{F}) \simeq \mathrm{H}^q(]X[_V, i_X^{-1}\mathcal{F}).$$

Our assertion therefore follows from proposition 1.3.6.ii of [3]. □

In fact, we will need a relative version of this result:

Corollary 3.4.4 *Let $(X', V') \to (X, V)$ be a morphism of good overconvergent varieties with $]\overline{X}[_V = V$ (resp. $]\overline{X}'[_{V'} = V'$). Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be a complex with coherent terms on V (resp. V') and differentials defined on V_G (resp. V'_G). Assume that there is an isomorphism $j_X^\dagger \mathcal{F}_G \simeq \mathrm{R}u_{G*}j_{X'}^\dagger \mathcal{F}_{G'}$. Then, we also have $i_{X*}i_X^{-1}\mathcal{F} \simeq \mathrm{R}u_* i_{X'*}i_{X'}^{-1}\mathcal{F}'$.*

Proof: We apply $\mathrm{R}\pi_{V*}$ on both sides of the first equality in order to get, thanks to proposition 3.4.3,

$$i_{X*}i_X^{-1}\mathcal{F} \simeq \mathrm{R}\pi_{V*}j_X^\dagger \pi_V^* \mathcal{F} \simeq \mathrm{R}\pi_{V*}\mathrm{R}u_{G*}j_{X'}^\dagger \pi_{V'}^* \mathcal{F}'$$

$$\simeq \mathrm{R}u_* \mathrm{R}\pi_{V'*}j_{X'}^\dagger \pi_{V'}^* \mathcal{F}' = \mathrm{R}u_* i_{X'*}i_{X'}^{-1}\mathcal{F}'. \quad \square$$

3.5 Cohomology

We will show now that overconvergent cohomology can be computed as de Rham cohomology. Due to the restriction on corollary 3.4.4, we will have to work with the good overconvergent sites.

We assume throughout this section that Char$K = 0$ and we let $(C \subset S \leftarrow O)$ be a *good* overconvergent variety.

Recall that we defined the good analytic site $\mathrm{An}^\dagger_g(T)$ on an overconvergent presheaf T as the subsite of good overconvergent varieties over T. We will freely use for $\mathrm{An}^\dagger_g(T)$ any result proved for $\mathrm{An}^\dagger(T)$. The proofs are identical (and usually easier).

Definition 3.5.1 *1. Let X be an algebraic variety over C with structural morphism $p : X \to C$. Consider the projection*

$$p_{X/O} : \quad \begin{array}{ccc} (X/O)_{\mathrm{An}^\dagger_g} & \longrightarrow & (]C[_O)_{\mathrm{an}} \\ (X/O')_g & \longleftarrow\!\!\!\shortmid &]C[_{O'} \end{array}$$

composed of the restriction map and the morphism $\varphi_{C,O}$. Then, the absolute cohomology *of a complex of abelian sheaves $\mathcal{F}$ on $(X/O)_g$ is $\mathrm{R}p_{X/O*}\mathcal{F}$.*

2. Let $f : X' \to X$ be a morphism of algebraic varieties over C. Consider the induced morphism of overconvergent presheaves $f : X'/O \to X/O$ and the corresponding morphism of toposes

$$f_{\mathrm{An}^\dagger} : (X'/O)_{\mathrm{An}^\dagger_g} \to (X/O)_{\mathrm{An}^\dagger_g}.$$

Then, the relative cohomology *of a complex of abelian sheaves $\mathcal{F}'$ on $(X'/O)_g$ is $\mathrm{R}f_{\mathrm{An}^\dagger *}\mathcal{F}'$.*

Relative cohomology can be recovered from absolute cohomology; and, conversely, absolute cohomology can also be derived from relative cohomology:

Proposition 3.5.2 *1. If $f : X' \to X$ be a morphism of algebraic varieties over C, $\mathcal{F}'$ is a complex of abelian sheaves on $(X'/O)_g$ and (U, V) a good overconvergent variety over X/O, then*

$$(\mathrm{R}f_{\mathrm{An}^\dagger *}\mathcal{F}')_{U,V} = \mathrm{R}p_{X'\times_X U/V*}\mathcal{F}'_{|X'\times_X U/V}.$$

2. If $p : X \to C$ is a morphism of algebraic varieties and $\mathcal{F}$ is a complex of abelian sheaves on $(X/O)_g$, then

$$\mathrm{R}p_{X/O*}\mathcal{F} = (\mathrm{R}p_{\mathrm{An}^\dagger *}\mathcal{F})_{C,O}.$$

Proof: This is just base change for restriction maps. □

Recall that we assume that Char$K = 0$ and that (C, O) is a good overconvergent variety. The notion of geometric realization was introduced in definition 1.5.10.

Theorem 3.5.3 *Let X be an algebraic variety over C and E be an overconvergent module of finite presentation on $(X/O)_g$. If $(X, V) \to (C, O)$ a geometric realization of X, there is a canonical isomorphism*

$$\mathrm{R}p_{X/O*}E \simeq \mathrm{R}p_{]X[_V*}(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O})$$

where $p_{]X[_V} :]X[_V \to]C[_O$ denotes the morphism induced on the tubes.

Proof: It is sufficient, thanks to proposition 3.3.9, to prove proposition 3.5.4 below. □

Proposition 3.5.4 *In the situation of the theorem, the augmentation map is an isomorphism*

$$E \simeq \mathrm{R}L(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}).$$

on $\mathrm{An}^\dagger_g(X/O)$.

Proof: We have to show that for any good overconvergent variety (X', V') over X/O, we have

$$E_{X',V'} \simeq \mathrm{R}L(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O})_{X',V'}.$$

If we write $\mathcal{F}' := E_{X',V'}$, proposition 3.3.10 allows us to rewrite this isomorphism as

$$\mathcal{F}' \simeq \mathrm{R}p_{1*}(p_1^\dagger \mathcal{F}' \otimes_{i_{X'}^{-1}\mathcal{O}_{V'\times V}} i_{X'}^{-1}\Omega^\bullet_{V'\times V/V'}).$$

Note that $p_1 : V' \times_O V \to V'$ is a geometric realization of the identity of X'. Our assertion therefore follows from the Poincaré lemma below. □

Proposition 3.5.5 (Poincaré Lemma) *Let* (X, V) *be a good overconvergent variety and* $(f, u) : (X, V') \to (X, V)$ *be a geometric realization of* Id_X. *If* $\mathcal{F}$ *is a coherent* $i_X^{-1}\mathcal{O}_V$*-module, then*

$$\mathcal{F} \simeq \mathrm{R}u_*(u^\dagger \mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_{V'}} i_X^{-1}\Omega^\bullet_{V'/V}).$$

Proof: It follows form corollary 3.1.5 that the question is local on X. We want to show that it is local for the Grothendieck topology of V also. This is not clear. First of all, we may replace V with $]\overline{X}[_V$. Also, since the question is local for the analytic topology, we may assume that V is countable at infinity and replace, as we usually do, $\mathcal{F}$ by $i_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is a coherent $\mathcal{O}_V$-module. After pushing by i_{X*}, we are reduced, thanks to proposition 2.2.12, to showing that

$$i_{X*}i_X^{-1}\mathcal{F} \simeq \mathrm{R}u_*(i_{X*}i_X^{-1}u^*\mathcal{F} \otimes_{\mathcal{O}_{V'}} \Omega^\bullet_{V'/V}).$$

Since $\mathcal{F}$ is coherent and V and V' are good (use proposition 1.5.3 for V'), it is sufficient to prove thanks to corollary 3.4.4 that

$$j_X^\dagger \mathcal{F}_G \simeq \mathrm{R}u_{G*}(j_X^\dagger u_G^* \mathcal{F}_G \otimes_{\mathcal{O}_{V'_G}} \Omega^\bullet_{V'/V,G}).$$

Now, the question is local for the Grothendieck topology of V. Thanks to the strong fibration theorem 3.1.3, we may therefore assume that P is affine, that the locus at infinity of X is a hypersurface, that V is affinoid with $V =]\overline{X}[_V$, and that $P' = \widehat{\mathbf{A}}^n_P$.

One may then proceed by induction on n. The case $n = 1$ results from lemma 3.5.6 below. For the induction process, we use the Gauss-Manin connection as in lemma 6.5.5 of [23]. □

Lemma 3.5.6 *Let* $(X \subset P \leftarrow V)$ *be an overconvergent variety. Assume that* P *is affine, that the locus at infinity of* X *is a hypersurface, that* V *is affinoid and that* $]\overline{X}[_V = V$. *If* $\mathcal{F}$ *is a coherent* $i_X^{-1}\mathcal{O}_V$*-module, there is canonical isomorphism*

$$\Gamma(]X[_V, \mathcal{F}) \simeq \mathrm{R}\Gamma(]X[_V \times \mathbf{D}(0, 1^-), p_1^\dagger \mathcal{F} \xrightarrow{\partial/\partial t} p_1^\dagger \mathcal{F}).$$

Proof: The proof goes exactly as its rigid counterpart. Since this is quite long and technical, we send the reader to proposition 6.5.7 of [23]. □

We can now derive several corollaries.

Corollary 3.5.7 *Let $f : X' \to X$ be a morphism of algebraic varieties over C, E' an overconvergent module of finite presentation on $(X'/O)_g$ and (U, V) a good overconvergent variety over X/O.*

Let $U' := U \times_X X'$ and $(p_1, u) : (U', V') \to (U, V)$ be a geometric realization of the first projection. Then, there is a canonical isomorphism

$$(\mathrm{R}f_{\mathrm{An}^\dagger *}E')_{(U,V)} \simeq \mathrm{R}u_*(E'_{U',V'} \otimes_{i_{U'}^{-1}\mathcal{O}_{V'}} i_{U'}^{-1}\Omega^\bullet_{V'/V}).$$

Proof: We know from proposition 3.5.2 that

$$(\mathrm{R}f_{\mathrm{An}^\dagger *}E)_{(U,V)} = \mathrm{R}p_{U'/V*}E_{|U'/V}$$

and we apply theorem 3.5.3. □

Recall that if V is a Hausdorff analytic variety over K, we may consider the morphism of toposes

$$\pi_0 : \widetilde{V_0} \simeq \widetilde{V_G} \xrightarrow{\pi_V} \widetilde{V}.$$

If $\mathcal{F}$ is a complex of abelian groups on V_0, we will write $\mathcal{F}^{\mathrm{an}} := \mathrm{R}\pi_{0*}\mathcal{F}$.

We now recall the definition of absolute rigid cohomology (definitions 8.2.5, 7.4.4 and 6.2.1 of [23]). We are given a morphism $p : X \to S_k$ of algebraic varieties and an overconvergent isocrystal E on X/S. We assume that p extends to a morphism $v : P \to S$ that is proper and smooth at X and we denote by $v_K :]\overline{X}[_P \to S_K$ the induced map. The overconvergent isocrystal E has a realization E_P on P which is a coherent $j^\dagger_{X0}\mathcal{O}_{]\overline{X}[_{P0}}$-module with an integrable connection and

$$\mathrm{R}p_{\mathrm{rig}}E := \mathrm{R}v_{K0*}(E_P \otimes_{\mathcal{O}_{]\overline{X}[_{P0}}} \Omega^\bullet_{]\overline{X}[_{P0}/]\overline{S}_K}).$$

In general, it is necessary to rely on descent methods such that in [16].

Proposition 3.5.8 *Let S be a good formal scheme and $p : X \to S_k$ a realizable morphism of algebraic varieties. Let E be an overconvergent isocrystal on X/S. Then, we have*

$$(\mathrm{R}p_{\mathrm{rig}}E)^{\mathrm{an}} = \mathrm{R}p_{X/S*}E$$

Of course, we use proposition 2.5.11 to identify the category of overconvergent modules of finite presentation on $(X/S)_g$ and the category of overconvergent isocrystals on X/S.

Proof: Realizable means that p extends to a morphism $v : P \to S$ that is proper and smooth at X. Now, we see E as an overconvergent module on $(X/S)_g$ and consider its realization E_P on P. The question being local on S_K, we may assume that S is quasi-compact and therefore also that P is quasi-compact. Then, there exists a good open neighborhood V of $]X[_P$ in $]\overline{X}[_P$ and a coherent module with an integrable connection $\mathcal{F}$ on V such that $E_P = i_X^{-1}\mathcal{F}$. Since V is good, $\mathcal{F}$ extends uniquely to a coherent module with an integrable connection $\mathcal{F}_0$ on V_0. Recall from corollary 1.3.2 that strict neighborhoods in rigid geometry correspond essentially to neighborhoods in Berkovich theory. In particular, if we still denote by $v_K : V \to S_K$ the induced morphism, we have thanks to proposition 6.2.2 of [23],

$$\mathrm{R}p_{\mathrm{rig}}E := \mathrm{R}v_{K0*}(j^\dagger_{X0}\mathcal{F}_0 \otimes_{\mathcal{O}_{V_0}} \Omega^\bullet_{V_0/S_{K0}}).$$

Since V is good, it follows from proposition 3.4.3 (and the equivalence $\widetilde{V_0} \simeq \widetilde{V_G}$) that

$$(j^\dagger_{X0}\mathcal{F}_0 \otimes_{\mathcal{O}_{V_0}} \Omega^\bullet_{V_0/S_{K0}})^{\mathrm{an}} = i_{X*}i_X^{-1}\mathcal{F} \otimes_{\mathcal{O}_V} \Omega^\bullet_{V/S_K}$$

and therefore,

$$(\mathrm{R}p_{\mathrm{rig}}E)^{\mathrm{an}} = \mathrm{R}v_{K*}(i_{X*}i_X^{-1}\mathcal{F} \otimes_{\mathcal{O}_V} \Omega^\bullet_{V/S_K}) = \mathrm{R}v_{K*}i_{X*}(i_X^{-1}\mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/S_K})$$

$$= \mathrm{R}v_{K*}(E_P \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/S_K}) = \mathrm{R}p_{X/S*}E. \quad \square$$

Remark: As a corollary of this proposition, we recover the fact that the rigid cohomology of an overconvergent isocrystal on X/S is independent of the choice of the formal embedding $X \hookrightarrow P$ into a formal scheme that is proper and smooth at X over S.

Last remark: One can remove the condition that X/S is realizable by using some glueing which is possible as shown in section 3.6 below. More precisely, Berthelot defines rigid cohomology as Čech cohomology relative to a Zariski open covering by realizable varieties ([8]). Alternatively, we can use definition given at the end of section 10.4 of [16]. It is then necessary to translate the results of section 3.6 in terms of cohomological descent as D. Brown does in [14].

3.6 Zariski localization

In this section, we show that the cohomology of an overconvergent crystal is local for the Zariski topology (on the algebraic side). As a consequence, we get that rigid cohomology coincides with our cohomology even when there is no geometric realization. Everything below also holds when we work with the good overconvergent site.

We let (C, O) be an overconvergent variety as usual but we first give a general definition:

Definition 3.6.1 *Let T be an overconvergent presheaf. An overconvergent sheaf $\mathcal{F}$ on T is said to be* of Zariski type *if for any overconvergent variety (X, V) over T and any open immersion $\alpha : U \hookrightarrow X$, we have $]\alpha[_V^{-1}\mathcal{F}_{X,V} = \mathcal{F}_{U,V}$.*

Clearly, this property will then be satisfied for any locally closed immersion.

Recall that if X is an algebraic variety over C and $\alpha : U \hookrightarrow X$ is a locally closed immersion, then

$$\alpha_{\mathrm{An}^\dagger} : (U/O)_{\mathrm{An}^\dagger} \hookrightarrow (X/O)_{\mathrm{An}^\dagger}$$

denotes the corresponding morphism of toposes (this is a restriction map).

Lemma 3.6.2 *Let X be an algebraic variety over C. Let $\mathcal{F}$ be a sheaf of Zariski type on X/O and $\alpha : U \hookrightarrow X$ be a locally closed immersion over C. Let (X', V') be an overconvergent variety over X/O, $f : X' \to X$ the structural map, $U' = f^{-1}(U)$ and $\alpha' : U' \hookrightarrow X'$ the inclusion map. Then, we have*

$$(\alpha_{\mathrm{An}^\dagger *}\alpha^{-1}_{\mathrm{An}^\dagger}\mathcal{F})_{X',V'} =]\alpha'[_*]\alpha'[^{-1}\mathcal{F}_{X',V'}.$$

Proof: It follows from proposition 2.1.11 that

$$(\alpha_{\mathrm{An}^\dagger *}\alpha_{\mathrm{An}^\dagger}^{-1}\mathcal{F})_{X',V'} =]\alpha'[_*(\alpha_{\mathrm{An}^\dagger}^{-1}\mathcal{F})_{U',V'}$$
$$=]\alpha'[_*\mathcal{F}_{U',V'} =]\alpha'[_*]\alpha'[^{-1}\mathcal{F}_{X',V'}. \quad \square$$

Proposition 3.6.3 *Let X be an algebraic variety over C and $\mathcal{F}$ an abelian sheaf of Zariski type on X/O. If $\{X_k\}_{k\in I}$ is a locally finite open covering of X, there is a long exact sequence*

$$0 \to \mathcal{F} \to \prod_k \alpha_{k\mathrm{An}^\dagger *}\alpha_{k\mathrm{An}^\dagger}^{-1}\mathcal{F} \to \prod_{k,l} \alpha_{kl\mathrm{An}^\dagger *}\alpha_{kl\mathrm{An}^\dagger}^{-1}\mathcal{F} \to \cdots$$

where $\alpha_k : X_k \hookrightarrow X, \alpha_{kl} : X_k \cap X_l \hookrightarrow X, \ldots$ denote the inclusion maps.

Proof: The assertion can be checked on some (X', V') with $f : X' \to X$. We may clearly assume that $X' = X$ and we are reduced, thanks to the previous lemma, to the exact sequence of proposition 3.1.4. $\square$

Thus, we get a spectral sequence on absolute cohomology:

Corollary 3.6.4 *With the assumptions of the proposition, there is a spectral sequence*

$$E_1^{r,s} = \bigoplus_{|\underline{k}|=r+1} \mathrm{R}^s p_{X_{\underline{k}}/O*}\mathcal{F}_{|X_{\underline{k}}} \Rightarrow \mathrm{R}^{r+s} p_{X/O*}\mathcal{F}$$

where

$$X_{\underline{k}} := X_{k_1} \cap \cdots \cap X_{k_{r+1}}. \quad \square$$

We also mention the spectral sequence of relative cohomology:

Corollary 3.6.5 *Let $f : X' \to X$ be a morphism of algebraic varieties over C, $\mathcal{F}'$ an abelian sheaf of Zariski type on X'/O, $\{X'_k\}_{k\in I}$ a locally finite open covering of X' and $f : X' \to X$ any C-morphism. Then, there is a spectral sequence*

$$E_1^{r,s} = \bigoplus_{|\underline{k}|=r+1} \mathrm{R}^s f_{|X'_{\underline{k}}\mathrm{An}^\dagger *}\mathcal{F}'_{|X'_{\underline{k}}} \Rightarrow \mathrm{R}^{r+s} f_{\mathrm{An}^\dagger *}\mathcal{F}'. \quad \square$$

What is important is that the above results apply to crystals as the following proposition shows. Moreover, we also see that, being a crystal is of local nature for the Zariski topology.

Proposition 3.6.6 *Let X be an algebraic variety over C and E an overconvergent module on X/O. Let $\{X_k\}_{k\in I}$ be a locally finite open covering of X. Then E is a crystal if and only if it is of Zariski type and for each k, $E_{|X_k}$ is a crystal.*

Proof: It follows from corollary 2.3.2 that an overconvergent crystal is of Zariski type. And we also know that the inverse image of a crystal is a crystal.

Conversely, assume that E is an overconvergent module of Zariski type and that for each k, $E_{|X_k}$ is a crystal. Let $(f, u) : (U', V') \to (U, V)$ be any morphism of overconvergent varieties over X and $g : U \to X$ the canonical map. For each $k = (k_1, \ldots, k_r)$, let

$$X_{\underline{k}} = X_{k_1} \cap \cdots \cap X_{k_r}, \quad U_{\underline{k}} = g^{-1}(X_{\underline{k}}), \quad U'_{\underline{k}} = f^{-1}(U_{\underline{k}}),$$

$f_{\underline{k}} : U'_{\underline{k}} \to U_{\underline{k}}$ the restriction of f and finally, $\alpha_k : U_{\underline{k}} \hookrightarrow U, \alpha'_{\underline{k}} : U'_{\underline{k}} \hookrightarrow U'$ the inclusion maps. Since $E_{|X_{\underline{k}}}$ is a crystal, we have $]f_{\underline{k}}[_u^\dagger E_{U_{\underline{k}},V} = E_{U'_{\underline{k}},V'}$. Since E is Zariski type, it follows that

$$]\alpha'_{\underline{k}}[^{-1}]f[_u^\dagger E_{U,V} =]f_{\underline{k}}[_u^\dagger]\alpha_{\underline{k}}[^{-1} E_{U,V} =]\alpha'_{\underline{k}}[^{-1} E_{U',V'}$$

and we can apply proposition 3.1.4. Note that the first equality follows from the fact that $]\alpha[^{-1} =]\alpha[^\dagger$ when α is an immersion as was shown in corollary 2.3.2. □

We can now state the main comparison theorem:

Theorem 3.6.7 *Assume* Char$K = 0$. *Let S be a good formal scheme, X is an algebraic variety over k and $p : X \to S_k$ a morphism of algebraic varieties. Then, we have a canonical equivalence of categories*

$$\mathrm{Mod}^\dagger_{\mathrm{g,fp}}(X/S) \simeq \mathrm{Isoc}^\dagger(X/S).$$

If E is an overconvergent isocrystal on X/S, we have

$$(\mathrm{R}p_{\mathrm{rig}}E)^{\mathrm{an}} = \mathrm{R}p_{X/S*}E.$$

Proof: Using the results of this section, this follows immediately from propositions and 2.5.11 and 3.5.8. □

Finally, there is a particular case of the theorem that is worth stating:

Corollary 3.6.8 *If X is an algebraic variety over k and E an overconvergent isocrystal on X/K, we have for all $i \in \mathbf{N}$,*

$$\mathrm{H}^i_{\mathrm{rig}}(X/K, E) = \mathrm{H}^i((X/\mathcal{V})_{\mathrm{An}^\dagger_g}, E). \quad \square$$

And in particular, we obtain

$$\mathrm{H}^i_{\mathrm{rig}}(X/K) = \mathrm{H}^i((X/\mathcal{V})_{\mathrm{An}^\dagger_g}, \mathcal{O}^\dagger_{X/\mathcal{V}}).$$

Chapter 4

Appendix

4.1 Sites and toposes

For the convenience of the reader, we give here a brief review of the basics of topos theory that is used in this article. Of course, almost everything can be found in [1] (see also chapter 0 of [20]). Note that we do not discuss set-theoretical questions.

Topology

A **presheaf** on a category C is a contravariant functor from C to Sets. A **morphism of presheaves** is a natural transformation between them. We denote by $\hat{C}$ the category of presheaves on C. Thanks to the Yoneda lemma, we can (and will) embed C into $\hat{C}$ by $X \mapsto \mathrm{Hom}(-, X)$. A **sieve** of $X \in C$ is a subobject R of X in $\hat{C}$. A **topology** τ on C is a family of sieves, called **τ-covering sieves** which

1. is stable by pull-back: if $R \hookrightarrow X$ is a covering sieve, then for all $Y \to X$, $R \times_X Y \hookrightarrow Y$ is a covering sieve.
2. is of local nature: if $R \hookrightarrow X$ is a sieve and if there exists a covering sieve $S \hookrightarrow X$ such that, for all $Y \to S$, then $R \times_X Y \hookrightarrow Y$ is a covering sieve, then $R \hookrightarrow X$ itself is a covering sieve.
3. contains C.

A category C endowed with a topology τ is a **site**. A topology σ is **coarser** than a topology τ if $\sigma \subset \tau$. Alternatively, we will say that τ is **finer** than σ. The **coarse topology** on a category C is the topology for which the sieves are just the objects of C. Any intersection of topologies is a topology. Given any family of sieves, there exists a finest and a coarsest topology for which they become coverings.

A **sheaf** on a site C is a presheaf F such that for all covering sieves R of X, the canonical map

$$F(X) = \mathrm{Hom}(X, F) \to \mathrm{Hom}(R, F)$$

is bijective. It is actually sufficient to check this property for a generating family of sieves which is stable by pull back. The full subcategory of $\hat{C}$ made of sheaves on C is the **topos** $\tilde{C}$.

The **canonical topology** on a category C is the finest topology for which the presheaves $\mathrm{Hom}(-, X)$ are sheaves. The topology of a site C is coarser than the canonical topology if and only if $C \subset \tilde{C}$. A site is said to be **standard** if it has fibered products and the topology is coarser than the canonical topology. If C is a site, then $\tilde{C}$ is a standard site for the canonical topology: actually, we have $\tilde{\tilde{C}} = \tilde{C}$. Finally, note that if C is endowed with the coarse topology, the corresponding topos is $\hat{C}$.

The **sieve of X generated by a family** $\{f_k : X_k \to X\}$ of morphisms of C, is the union of the images of the morphisms f_k in $\hat{C}$. If C is a site, a **covering family** is a family that generates a covering sieve. A **pretopology** on a category C is a set of families of morphisms $\{X_k \to X\}$, which

1. is stable by pull-back: if $\{X_k \to X\}$ is a covering family, then for all $Y \to X$, $\{X_k \times_X Y \to Y\}$ (exists and) is a covering family.
2. is stable by composition: if $\{X_k \to X\}$ is a covering family and for each k, $\{X_{kl} \to X_k\}$ is a covering family, then $\{X_{kl} \to X\}$ is covering family.
3. contains the identities.

The **topology generated by the pretopology** is the coarsest topology for which the families of the pretopology are covering families.

If C is a site with fibered products, then the set of all covering families is a pretopology that generates the topology of C. In general, if C is a site defined by a pretopology, the sieves generated by the covering families are cofinal among all covering sieves. Moreover, a sheaf is simply a presheaf T such that for any family $\{X_k \to X\}_k$ in the pretopology, the sequence

$$T(X) \to \prod_k T(X_k) \substack{\to \\ \to} \prod_{k,l} T(X_k \times_X X_l)$$

is exact. Actually, it is sufficient to check this property for a generating set of families.

Note that if C is a site, we may also endow $\widehat{C}$ with the coarser topology that is finer than the canonical topology and such that any covering family in C is still a covering family in $\widehat{C}$. Then, a family $\{T_k \to T\}$ will be a **covering family** in $\widehat{C}$ if and only if the family $\{\widetilde{T}_k \to \widetilde{T}\}$ is a covering in $\widetilde{C}$ (notations below).

Morphisms

If $g : C \to C'$ is a functor, the composition functor

$$\begin{aligned} \hat{g}^{-1} : \hat{C}' &\longrightarrow \hat{C} \\ T' &\longmapsto T' \circ g \end{aligned}$$

has a left adjoint $\hat{g}_!$ and a right adjoint $\hat{g}_*$. Note that $\hat{g}_!$ extends g. As we will see below, it is sometimes convenient to write f^{-1} instead of g for the original functor. In this case, we will write

$$\hat{f}^{-1} := \hat{g}_!, \quad \hat{f}_* := \hat{g}^{-1}, \quad \hat{f}^! := \hat{g}_*$$

A functor $f^{-1} : C \to C'$ between two sites is **continuous** if $\hat{f}_*$ preserves sheaves. If this is the case, then $\hat{f}_*$ induces a functor $\widetilde{f}_* : \tilde{C}' \to \tilde{C}$ which has a left adjoint $\widetilde{f}^{-1} : \tilde{C} \to \tilde{C}'$. Note also that f^{-1} is always continuous when C is endowed the coarse topology. In general, if f^{-1} is continuous *and* $\widetilde{f}^{-1} : \tilde{C} \to \tilde{C}'$ is exact, then the pair $(\widetilde{f}^{-1}, \widetilde{f}_*)$ is called a **morphism of sites** $f :=: C' \to C$. A morphism of sites f is an **embedding of sites** if f_* is fully faithful. A **morphism of toposes** is a morphism of sites between two toposes (with their canonical topology).

If $f^{-1} : C' \to C$ is continuous, it preserves covering families and the converse is also true when C' has fibered products and f^{-1} is left exact. Moreover, in this case, we do get a morphism of sites $f : C \to C'$. This applies in particular to toposes and we see that a morphism of sites $f : C' \to C$ always induces a morphism of toposes $\widetilde{f} : \widetilde{C'} \to \widetilde{C}$. Recall also that the inclusion $\tilde{C} \hookrightarrow \hat{C}$ is the direct image of a morphism of toposes whose left adjoint will be written $T \mapsto \tilde{T}$. If T is a presheaf on C, then $\tilde{T}$ is the **associated sheaf**. Actually, if $f^{-1} : C' \to C$ is continuous and $T \in \tilde{C}'$, then

$$\widetilde{f}^{-1}(T) = \widetilde{\hat{f}^{-1}(T)}.$$

A functor $g : C' \to C$ between two sites is **cocontinuous** if $\hat{g}_*$ preserves sheaves. If $g : C' \to C$ is a cocontinuous functor, then $\hat{g}_*$ induces a functor $\widetilde{g}_* : \tilde{C}' \to \tilde{C}$ which extends to a morphism of toposes $\widetilde{g} : \tilde{C}' \to \tilde{C}$.

When both sites are defined by a pretopology, we have the following criterion to tell wether g is cocontinuous or not: given an object X' in C' and a covering family $\{X_i \to g(X')\}_{i \in I}$, we have to show that the family of all $u : Y' \to X'$ such that $g(u)$ factors through some X_i is a covering family of X''. This is the case for example if any covering family of $g(X')$ is the image of a covering family of X'.

If C is a site and $f^{-1} : C' \to C$ is any functor, the **induced topology** on C' is the finest topology that makes f^{-1} continuous. If the functor $f^{-1} : C' \to C$ is left exact and C' has fibered products, a family in C' is a covering family for the induced topology if and only if its image in C is a covering family. Note that, in general, the topology of a site C is the topology induced by the canonical functor

$$\begin{array}{ccc} C & \longrightarrow & \tilde{C} \\ X & \longmapsto & \tilde{X}. \end{array}$$

If C' is a site and $f^{-1} : C' \to C$ is any functor, there exists a coarsest topology on C that makes f^{-1} continuous. It is called the **image topology** on C. Note also that if C' is a site and $g : C' \to C$ any functor, there exists a finest topology on C that makes g cocontinuous.

Restriction

If C is a category and T an object in C, the **restricted category** $C_{/T}$ is the category of arrows $X \to T$ and morphisms compatible with these arrows. More generally, if T is a presheaf on C, then $C_{/T}$ is the full subcategory of $\hat{C}_{/T}$ made of arrows $X \to T$ with $X \in C$. In other words, an object of $C_{/T}$ is a pair (X, u) where $X \in C$ and $u \in T(X)$ and a morphism $(Y, v) \to (X, u)$ is a morphism $f : Y \to X$ such that $T(f)(u) = v$.

It C is a site, T a presheaf on C and $C_{/T}$ is endowed with the induced topology, the forgetful functor $j_T : C_{/T} \to C$ is continuous and cocontinuous. In particular, it induces a morphism of toposes

$$j_T : \widetilde{C_{/T}} \to \tilde{C}$$

and j_T^{-1} has a left adjoint $j_{T!}$. If we denote by $\widehat{j}_T$ the corresponding morphism on presheaves, we have

$$(\widehat{j}_{T!}F)(X) = \coprod_{u \in T(X)} F(X,u).$$

and $j_{T!}F = \widetilde{\widehat{j}_{T!}F}$. It is important to remark that, on abelian sheaves, there exists also a left adjoint $j_{T!}^{ab}$ which is exact so that j_T^{-1} preserves injectives.

We have an equivalence of toposes $\widetilde{C_{/T}} \simeq \tilde{C}_{/\tilde{T}}$ and j_T^{-1} corresponds to product with $\tilde{T}$. Any morphism of presheaves $u : T' \to T$ induces a functor $j_u : C_{/T'} \to C_{/T}$ which gives a morphism of toposes when C is a site. Actually, this is a particular case of a restriction map as before because we may see the morphism u as a presheaf on $C_{/T}$. More generally, if $f : C' \to C$ is a morphism of sites, T a presheaf on C, T' a presheaf on C' and $T' \to \hat{f}^{-1}(T)$ a morphism, there is a canonical morphism of toposes $\widetilde{C'_{/T'}} \to \widetilde{C_{/T}}$.

Finally, as for open embeddings in topological spaces, if we are given a cartesian diagram of presheaves

$$\begin{array}{ccc} T_1 \times_T T_2 & \xrightarrow{p_1} & T_1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle u_1} \\ T_2 & \xrightarrow{u_2} & T, \end{array}$$

then we have a general base change theorem that reads

$$j_{u_1}^{-1} j_{u_2*}\mathcal{F} \simeq j_{p_1*} j_{p_2}^{-1}\mathcal{F}.$$

For complexes of abelian sheaves, we also have

$$j_{u_1}^{-1}\mathrm{R}j_{u_2*}\mathcal{F} \simeq \mathrm{R}j_{p_1*} j_{p_2}^{-1}\mathcal{F}$$

since inverse images preserve injectives.

4.2 Analytic varieties

We give here a brief review of Berkovich theory (see [3], section 1). A very good introduction is also given in [5]. We will consider only fields with *non-trivial* valuations and *strictly analytic spaces* defined *directly* over the base field. We call them analytic varieties.

Let K be a complete ultrametric field with *non-trivial* absolute value.

Affinoid varieties

An **affinoid algebra over** K is a quotient A of a **Tate algebra**

$$K\{T_1, \ldots, T_n\} := \{\sum_{\underline{i}=\underline{0}}^{\infty} a_{\underline{i}} \underline{T}^{\underline{i}}, \quad a_{\underline{i}} \to 0 \quad \text{when} \quad \underline{i} \to \infty\}.$$

over K. The topology induced by the gauss norm

$$\|\sum_{\underline{i}=\underline{0}}^{\infty} a_{\underline{i}} \underline{T}^{\underline{i}}\| := \max |a_{\underline{i}}|$$

only depends on A (and not on the presentation as a quotient of a Tate algebra). Moreover, any morphism between two affinoid algebras is continuous.

The **Gelfand spectrum** of a an affinoid algebra A is the set $\mathcal{M}(A)$ of continuous semi-absolute values (or multiplicative semi-norms) on A/K, that is,

$$\mathcal{M}(A) = \{x \in \mathcal{C}^0(A, \mathbf{R}_{\geq 0}), x(a) = |a| \text{ if } a \in K,$$
$$x(f+g) \leq \max\{x(f), x(g)\} \text{ and } x(fg) = x(f)x(g) \text{ when } f, g \in A\}.$$

Any $x \in V$ has a prime kernel $\mathfrak{p}$ and we will denote by $\mathcal{K}(x)$ the completion of the fraction field of $A/\mathfrak{p}$. Its valuation ring will be written $\mathcal{V}(x)$ and its residue field $k(x)$. We will denote by $f(x) \in \mathcal{K}(x)$ the image of $f \in A$ so that $x(f) = |f(x)|$. The Gelfand spectrum $V = \mathcal{M}(A)$ is endowed with the topology of simple convergence (the coarsest topology making continuous all maps $x \mapsto |f(x)|$ with $f \in A$). It becomes a non-empty compact (Hausdorff) topological space. We clearly get a functor $A \mapsto \mathcal{M}(A)$ from affinoid algebras to topological spaces.

An **affinoid variety** V is a triple made of a topological space, an affinoid algebra A and a homeomorphism $V \simeq \mathcal{M}(A)$. We will then write $\mathcal{O}(V) := A$. A morphism of affinoid varieties $u : V' \to V$ is a pair made of a (continuous) map and a homomorphism $\varphi : A := \mathcal{O}(V) \to A' := \mathcal{O}(V')$ making commutative

$$\begin{array}{ccc} V' & \xrightarrow{u} & V \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{M}(A') & \xrightarrow{\mathcal{M}(\varphi)} & \mathcal{M}(A). \end{array}$$

Clearly, the functors $A \mapsto \mathcal{M}(A)$ and $V \mapsto \mathcal{O}(V)$ establish an equivalence between the category of affinoid algebras and the category of affinoid varieties. If V is an affinoid variety, we say that a subset $W \subset V$ is an **affinoid domain** if the functor

$$C \mapsto \{u : \mathcal{M}(C) \to V, \quad u \text{ is affinoid and } \mathrm{Im}(u) \subset W\}$$

is representable by an affinoid algebra B. If this is the case, we actually get an homeomorphism $\mathcal{M}(B) \simeq W$ and W becomes an affinoid variety. Affinoid domains form a basis of compact subsets for the topology of V.

If A is an affinoid algebra, we may consider the relative Tate algebra

$$A\{T_1, \ldots, T_N\} = \{\sum_{\underline{i}=\underline{0}}^{\infty} f_{\underline{i}} \underline{T}^{\underline{i}}, \quad f_{\underline{i}} \to 0, \underline{i} \to \infty\}.$$

The **closed polydisc** on $V \simeq \mathcal{M}(A)$ is

$$\mathbf{B}_V^N(0, 1^+) := \mathcal{M}(A\{T_1, \ldots, T_N\})$$

and we will write $x_i := T_i(x)$. The **open polydisc** on V is

$$\mathbf{B}_V^N(0, 1^-) := \{x' \in \mathbf{B}_V^N(0, 1^+), \quad \forall i = 1, \ldots, N, \quad |x_i| < 1\}$$

We will say that a morphism of affinoid varieties $V' \to V$ is **finite** (resp. a **closed immersion**) if the map $\mathcal{O}(V) \to \mathcal{O}(V')$ is finite (resp. surjective). We say that $x' \in V'$ is **inner** with respect to an affinoid morphism $u : V' \to V$ if there exists a closed immersion $V' \hookrightarrow \mathbf{B}_V^N(0, 1^+)$ such that $x' \in \mathbf{B}_V^N(0, 1^-)$. For example, the Gauss norm on $K\{T_1, \ldots, T_n\}$ defines a point of $\mathbf{B}^N(0, 1^+)$ which is *not* inner (with respect to $\mathcal{M}(K)$).

Quasi-nets

A **quasi-net** on a topological space V is a set τ of subsets of V such that any point x of V has a neighborhood which is a finite union of elements of τ containing x. For example, $\{W_1, W_2\}$ is a quasi-net on V if and only if W_1 is a neighborhood of any point not in W_2 and conversely. For example, in $\mathbf{R}$, any finite covering of a closed interval by closed intervals is a quasi-net.

If $W \subset V$, we write $\tau_W = \{W' \in \tau, W' \subset W\}$. We say that W is a τ**-admissible subset** if τ_W is a quasi-net on W (for the induced topology). We say that τ is a **net** if any finite intersection W of elements of τ is τ-admissible. In this case, a covering of a τ-admissible subset W by τ-admissible subsets is a τ**-admissible covering** if it defines a quasi-net on W.

I include a proof of the following result because I could not find any reference:

Proposition *If τ is a net on V, then τ-admissible subsets and τ-admissible coverings define a Grothendieck topology on V for which τ is a basis. Moreover, it satisfies axioms G_0, G_1, G_2 of [13].*

Proof : It is clear that V and $\emptyset$ are admissible. This is G_0.

We now prove that G_1 holds. It means that admissibility of a subset is of local nature for the Grothendieck topology. Thus, we are given an admissible subset W of V, an admissible covering $W = \cup W_i$ of W and a subset $W' \subset W$ such that for all i, $W' \cap W_i$ is admissible. We have to show that W' is admissible. Let $x \in W'$. Since the set of W_i's is a quasi-net on W and $x \in W$, there exists an open subset $U \subset V$ and a subfamily $W_{i_1}, \ldots, W_{i_n}$ with $x \in W_{i_j}$ such that

$$x \in U \cap W \subset \cup W_{i_j}.$$

For each $j = 1, \ldots, n$, we have $x \in W' \cap W_{i_j}$ which is admissible. Thus, there exists finitely many $W'_{j,k} \in \tau$ such that $x \in W'_{j,k} \subset W' \cap W_{i_j}$ and an open subset $U_j \subset V$ such that

$$x \in U_j \cap W_{i_j} \cap W' \subset \cup_k W'_{j,k}.$$

Now, we let $U' = U \cap (\cap U_j)$ and we have

$$x \in U' \cap W' = U \cap (\cap U_j) \cap W' = (U \cap W) \cap (\cap U_j) \cap W'$$

$$\subset (\cup W_{i_j}) \cap (\cap U_j) \cap W' \subset \cup (W_{i_j} \cap U_j \cap W') \subset \cup W'_{j,k}.$$

The next step is G_2: we are given a covering $W = \cup W_i$ of an admissible subset by admissible subsets and a refinement $W = \cup W'_i$ which is admissible. We have to check that the first covering is admissible. This is easy. If $x \in W$, there exists an open subset $U \subset V$ and a subfamily $W'_{i_1}, \ldots, W'_{i_n}$ with $x \in W'_{i_j}$ such that $x \in U \cap W \subset \cup W'_{i_j}$. Now, for each i_j, there exists a k_j such that $W'_{i_j} \subset W_{k_j}$. It follows that $x \in W_{k_j}$ and that $x \in U \cap W \subset \cup W_{k_j}$.

It still remains to show that we do have a Grothendieck topology. It is clear that any admissible subset is a covering of itself. Also, admissibility of subsets is stable under finite intersection: the question is local for the Grothendieck topology by G_1. We are therefore reduced to the case of two elements of τ. And we are done since we assumed that τ is a net. Transitivity of admissibility of coverings is proved exactly as G_1 above. Finally, the trace of an admissible covering on a subset is admissible since the trace of a quasi-net on any subset is a quasi-net (and admissibility of subsets is stable under finite intersection). □

Analytic varieties

Let V be a locally Hausdorff topological space. An **affinoid atlas** on V is a net τ of affinoid varieties and inclusion of affinoid domains. There is an obvious ordering on affinoid atlases on V. It can be shown that any affinoid atlas extends to a unique maximal affinoid atlas. Note also that the restriction of an affinoid atlas to any subnet is an affinoid atlas.

An **analytic variety** over k is a locally Hausdorff topological space V endowed with a maximal affinoid atlas τ. The elements of τ are called **affinoid domains** of V, the τ-admissible subsets are called **analytic domains** and τ-admissible coverings are simply called **admissible covering**. When endowed with its Grothendieck topology, we will denote our space by V_G. Of course, an affinoid variety comes naturally with a structure of analytic variety.

Any open subset of an analytic variety V is an analytic domain: this question being G-local reduces to the affinoid case. It follows that the identity map $\pi_V : V_G \to V$ is continuous. Actually, it is a morphism of sites. The pull back map $\mathcal{F} \mapsto \pi_V^{-1}\mathcal{F}$ is (exact and) fully faithful.

One can show that the presheaf $W \mapsto \mathcal{O}(W)$ on affinoid domains extends uniquely to a sheaf of rings $\mathcal{O}_{V_G}$ on V_G called the structural sheaf. This sheaf induces a sheaf of rings $\mathcal{O}_V := \pi_{V*}\mathcal{O}_{V_G}$ on V turning π_V into a morphism of ringed spaces

$$(\pi_V*, \pi_{V*}) : (V_G, \mathcal{O}_{V_G}) \longrightarrow (V, \mathcal{O}_V).$$

On usually writes $\mathcal{F}_G := \pi_V^*\mathcal{F}$. Note that the functor $\mathcal{F} \mapsto \mathcal{F}_G$ is not fully faithful in general (even on coherent sheaves).

A **valued ringed space over** K is a set V endowed with a Grothendieck topology, a sheaf of K-algebras $\mathcal{O}_V$ and for each point $x \in V$, a semi-absolute value on $\mathcal{O}_{V,x}$. A **morphism of valued ringed spaces over** K is a morphism of Grothendieck ringed spaces compatible with the semi-absolute values. Any analytic variety V has a natural structure of valued ringed space, namely

$$\mathcal{O}_{V_G,x} = \varinjlim_{x\in W} \mathcal{O}(W),$$

with $x \in W$ affinoid, comes with a semi-absolute value $f \mapsto |f(x)|$. A **morphism of analytic varieties** $u : V' \to V$ is a morphism of valued ringed spaces $u_G : V'_G \to V_G$.

One can show that the functor $A \to \mathcal{M}(A)$ from affinoid algebras to analytic varieties is fully faithful. We may therefore identify the category of affinoid varieties with a full subcategory of the category of analytic varieties. This implies that any morphism of analytic varieties is continuous: since being open is a local property for the grothendieck topology, one reduces to the affine case.

Actually, to give a morphism of analytic varieties $V' \to V$ amounts to the following data (this is Berkovich's original definition):

1. affinoid atlases τ and τ' on V and V', respectively

2. for each $W' \in \tau'$, a morphism of affinoid varieties $W' \to W$ with $W \in \tau$.

Moreover, we require these morphisms to be compatible whenever this is meaningful.

Finally, note that we defined a morphism of analytic variety as a morphism of valued ringed spaces $u_G : V'_G \to V_G$; note that it actually induces a morphism of ringed spaces $u : (V', \mathcal{O}_{V'}) \to (V, \mathcal{O}_V)$.

Note: a point x of an analytic variety V is called a **rigid point** if $\mathcal{K}(x)$ is a finite extension of K. The set of rigid points is denoted by V_0. When V a Hausdorff, there exists a unique structure of rigid analytic variety on V_0 such that gives a bijection between affinoid domains (resp. admissible affinoid coverings) of V and affinoid open subsets (resp. admissible affinoid coverings) of V_0. This is functorial and the inclusion map $V_0 \hookrightarrow V$ induces an equivalence of toposes $\widetilde{V}_G \simeq \widetilde{V}_0$. Note however that this inclusion map $V_0 \hookrightarrow V$ is not continuous: if ξ is not a rigid point of V, then $W = V \setminus \xi$ is an open subset of V but the induced map $W_0 \to V_0$ is not an open immersion. For example, if we remove the "generic" point from the unit disc, we get the disjoint union of the residue classes mapping bijectively onto the rigid disc which is not an open immersion of rigid analytic spaces.

Properties

The category of analytic varieties has fibered products and we have

$$\mathcal{M}(B) \times_{\mathcal{M}(A)} \mathcal{M}(C) = \mathcal{M}(B \widehat{\otimes}_A C).$$

Also, if we are given an isometry $K \hookrightarrow K'$, there is an extension functor $V \mapsto V_{K'}$ such that

$$\mathcal{M}(A)_{K'} = \mathcal{M}(K' \widehat{\otimes}_K A).$$

If $u : V' \to V$ is any morphism and $W \subset V$ is an analytic domain, then $u^{-1}(W) \simeq V' \times_V W$. Similarly, if $x \in V$, then $u^{-1}(x) \simeq V'_{\mathcal{K}(x)}$.

A morphism of analytic varieties $u : V' \to V$ is **finite** (resp. **a closed immersion**) if the induced map $u^{-1}(W) \to W$ is a finite morphism (resp. a closed immersion) of affinoid varieties whenever W is an affinoid domain inside V.

A morphism $u : V' \to V$ of analytic varieties is said to be **(locally) separated** if the diagonal map is a (locally) closed immersion. An analytic variety is said to be **good** if any point has an affinoid neighborhood (or equivalently a basis of affinoid neighborhoods). Note that fibered product of good over locally separated is always good. Good analytic varieties play an important role because the functor $\mathcal{F} \mapsto \mathcal{F}_G$ is fully faithful when V is good. It is even an equivalence if we stick to coherent sheaves.

We say that a point $x' \in V'$ is **inner** with respect to a morphism $u : V' \to V$ if there exists an affinoid domain W in V and an affinoid neighborhood W' of x' in $u^{-1}(W)$ such that x' is inner with respect to $W' \to W$. The **interior** of V' with respect to V is the set of inner points and its **boundary** is the complement of the interior. The morphism is said to be **boundaryless** (*closed* in the terminology of Berkovich) if the boundary is empty. The map u is said to be **proper** if it is topologically proper and boundaryless. Both notions are local for the Grothendieck topology. A finite morphism is always proper and a proper morphism is separated.

A morphism $u : V' \to V$ is **universally flat** (resp. **formally smooth**, **formally étale**) if given any point $x' \in V'$, there exists an affinoid domain W' in V' containing x' and an affinoid domain W of V with $u(W') \subset W$ such that the induced morphism $\mathcal{O}(W) \to \mathcal{O}(W')$ is flat (resp. formally smooth, formally étale). Then, if u induces a morphism $W' \to W$ between analytic domains in V' an V respectively, the induced map will also be universally flat (resp. formally smooth, formally étale). Finally, a morphism $u : V' \to V$ is **smooth** (resp. **étale**) if it is formally smooth (resp. formally étale) and boundaryless.

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