Constructible $\nabla$-modules on curves

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Abstract

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristic with perfect residue field. Let $X$ be a geometrically connected smooth proper curve over $\mathcal{V}$. We introduce the notion of constructible convergent $\nabla$-module on the analytification $X^\text{an}_K$ of the generic fiber of $X$. A constructible module is an $\mathcal{O}_{X^\text{an}_K}$-module which is not necessarily coherent, but becomes coherent on a stratification by locally closed subsets of the special fiber $X_k$ of $X$. The notions of connection, of (over-) convergence and of Frobenius structure carry over to this situation. We describe a specialization functor from the category of constructible convergent $\nabla$-modules to the category of $\mathcal{D}^\dagger_{\hat{X}^\text{an}_K}$-modules. We show that specialization induces an equivalence between constructible $F$-$\nabla$-modules and perverse holonomic $F$-$\mathcal{D}^\dagger_{\hat{X}^\text{an}_K}$-modules.

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Introduction

When we look for a category of coefficients for a cohomological theory, we usually get one which is too small inside another one which is too big and we aim at the perfect one that will sit in between.

For example, if we are interested in the singular cohomology of an algebraic variety over $\mathbb{C}$, we have on one side the category of local systems of finite dimensional vector spaces and on the other side the category of all sheaves of vector spaces. The perfect category will be the category of constructible sheaves. Similarly, for de Rham cohomology, we have on one side the category of coherent modules endowed with an integrable connection and on the other side the category of all $\mathcal{D}$-modules. The perfect one will be the category of regular holonomic $\mathcal{D}$-modules. Last, we may consider the category of finitely presented crystals and the category of all modules on the infinitesimal site. In between, we have the category of constructible pro-crystals (unpublished manuscript of Deligne [16]). We have equivalences of categories between local systems, modules with an integrable connection and finitely presented crystals (we need to add some regularity conditions when moving from analytic to algebraic side). At the derived category level, we also have an equivalence between constructible sheaves, regular holonomic $\mathcal{D}$-modules and constructible pro-crystals ([18] and [16]).

If we are interested in the $p$-adic cohomology of a variety of characteristic $p > 0$, there is no analog to the first theory. The closest would be the étale $p$-adic cohomology which is not satisfying. There is a very good analog to the second theory, which is the theory of arithmetic $\mathcal{D}$-modules (and overconvergent isocrystals) developed by Berthelot, Virrion, Huyghe, Trihan, Caro and others (see [8] for an overview). More recently, I started to develop a crystalline theory in [23]. I showed that the category of overconvergent isocrystals is equivalent to the category of finitely presented crystals on the overconvergent site. When I was visiting the university of Tokyo in October 2009, Shiho suggested that we should look for the perfect category in this new theory, and prove that it is equivalent to the one developed by Berthelot (an overconvergent Deligne-Kashiwara correspondence). I had a very naive idea of what this category of constructible overconvergent crystals should be and gave a lecture on this topic in March 2010 at Oxford University. The aim of this article is to show that the strategy works for curves.

More precisely, if $X$ is a smooth proper curve over a complete discrete valuation ring $\mathcal{V}$ of mixed characteristic $p$ with fraction field $K$ and perfect residue field $k$, we introduce the notion of constructible convergent $\nabla$-module $E$ on $X_{\mathcal{K}}$ and show that its specialization $\text{Rsp}_e E$ is a perverse (complex of) $\mathcal{D}^{\dagger}_{X_{\mathcal{Q}}}$-modules. We show in theorem 9.11 that specialization induces an equivalence between constructible $F$-$\nabla$-modules on $X_{\mathcal{K}}$ and perverse holonomic $F$-$\mathcal{D}^{\dagger}_{X_{\mathcal{Q}}}$-modules. It should be remarked that the overconvergent isocrystals on an open subset $U$ of $X_k$ form a full subcategory of the category of all constructible convergent $\nabla$-modules on $X_{\mathcal{K}}$ and that our theorems extend Berthelot’s equivalence theorems for specialization of (over-) convergent isocrystals.
Our definitions are very general and apply to any formal $\mathcal{V}$-scheme. A constructible over-convergent $\nabla$-module is an (not necessarily coherent) $\mathcal{O}_{\hat{X}_{\mathbb{Q}}}$-module $E$ with a convergent (automatically integrable here) connection and we require that there exists a finite covering of the special fibre $X_k$ by locally closed subsets $Y$ such that, if $i_Y : ]Y[ \rightarrow X_{\mathbb{R}}^p$ denotes the inclusion map, then $i_Y^{-1}E$ is a coherent $i_Y^{-1}\mathcal{O}_{\hat{X}_{\mathbb{R}}}$-module. In this definition, we view $X_{\mathbb{R}}^p$ as a Berkovich analytic space (so that restriction corresponds to restriction to a strict neighborhood) but we will define $\text{Rsp}_E := \text{Rsp}_E E_0$ where $E_0$ denotes the corresponding module for the rigid topology.

We can give an explicit description of constructible convergent $\nabla$-modules and their specialization. If $D$ is a smooth relative divisor on $X$ with affine open complement Spec $A$, we can consider $A_K^\dagger$, which is the generic fiber of the weak completion of $A$, and for each $a \in D_K$, the Robba ring $\mathcal{R}_a$ on the residue field $K(a)$. Then, a constructible convergent $\nabla$-module is equivalent to the following data for some sufficiently big divisor $D$: an over-convergent $\nabla$-module $M$ (of finite type) over $A_K^\dagger$ and for each $a \in D_K$, a finite dimensional $K(a)$-vector space $H_a$ and a horizontal $A_K^\dagger$-linear map

$$M \rightarrow \mathcal{R}_a \otimes_{K(a)} H_a. \quad (0.1)$$

Now, we can describe $\text{Rsp}_E$ as follows. Let $\mathcal{U}' = \text{Spf} A'$ be an affine open subset of $\hat{X}$. First of all, if $D' = \mathcal{U}' \cap \hat{D}$ is defined in $\mathcal{U}'$ by an equation $f$, we may consider the weak completion $\mathcal{A}'[1/f]$. Also, we denote by $\delta_a$ the quotient of the Robba ring $\mathcal{R}_a$ by the ring $\mathcal{O}_a^\text{an}$ of convergent functions on the open unit disc. We extend scalars on the left, project on the right and add all the maps in order to get the complex

$$\mathcal{A}'[1/f] \otimes_{A_K^\dagger} M \rightarrow \bigoplus_{a \in D_K^\dagger} \delta_a \otimes_{K(a)} H_a. \quad (0.2)$$

This is exactly the value of $\text{Rsp}_E$ on $\mathcal{U}'$. It is naturally a complex of $\mathcal{D}_{\hat{X}_{\mathbb{Q}}}$-modules on $\mathcal{U}'$. It is perverse in the sense that it has $\mathcal{O}_{\hat{X}_{\mathbb{Q}}}$-flat cohomology in degree 0, finite support in degree 1 and no other cohomology.

As already mentioned, we use Berkovich theory of ultrametric geometry (see [1] and [2] or [17]) and will try to recall the basic constructions when we meet them. We will generally avoid reference to standard results proved in the context of rigid analytic geometry and rather reprove them here in this new language when necessary. Anyway, the reader should keep in mind that a significant part of what follows is merely a reorganization of some of Berthelot’s material using Berkovich theory instead of Tate’s.

Note that I will use the terminology finite module for finitely presented module and finite torsion module if we ever have to consider a module which is finite as set. I will call $\nabla$-module a module endowed with a (integrable) connection. We denote with an upper $*$ the inverse image for a morphism of ringed spaces when the upper $-1$ is used for inverse image for a morphism of topological spaces. Also, we will only consider real numbers $\eta$ inside $\sqrt{|K|}$ where $K$ is our fixed ultrametric field. Finally, I will systematically identify a sheaf on a topological space reduced to one point with its global sections.

Many thanks to Daniel Caro and Pierre Berthelot who helped me a lot when I was struggling with arithmetic $\mathcal{D}$-modules. I am indebted too to Florian Ivorra who helped me clarify some questions related to adjunction of derived functors. I also recall that the problem which is solved here was pointed out by Atsushi Shiho. Finally, I want to thank the referee for his very careful reading of the manuscript and all his comments.
1 Constructible modules

We fix a complete ultrametric field $K$ with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{m}$ and residue field $k$. We will assume that $K$ is non trivial and fix some $\pi \in K$ with $0 < |\pi| < 1$. We will also assume that $K$ has characteristic zero, that $k$ has positive characteristic $p$, that the valuation is discrete and that $k$ is perfect. When Frobenius enters the game, we also fix an isometry $\sigma$ on $K$ that lifts the absolute Frobenius of $k$.

We let $X$ be a geometrically connected proper smooth $\mathcal{V}$-scheme of relative dimension one.

We may consider its special fiber $X_k$ as well as its generic fiber $X_{\hat{K}}$ which are proper smooth geometrically connected curves and we will be particularly interested in the Berkovich analytification $X_{\mathbb{A}}$ of $X_k$. For the moment, we will simply use the fact that, since $X$ is proper, we can identify $X_{\mathbb{A}}$ with the generic fiber $\hat{X}$ of the completion of $X$. In particular, $X_{\mathbb{A}}$ is covered (for the Grothendieck topology) by the affinoid domains $\mathcal{U}_K = \mathcal{M}(\mathcal{A}_K)$ (set of continuous multiplicative semi-norms) if $X$ is covered by some affine subsets $\mathcal{U} = \operatorname{Spf} \mathcal{A}$ (set of open primes).

We will consider the specialization map

$$\operatorname{sp} : X_{K_{\mathbb{A}}} = \hat{X}_K \rightarrow \hat{X} \simeq X_k$$ (1.1)

(where the last morphism is a homeomorphism). Locally, the specialization of a point $x \in \mathcal{U}_K$ is the open prime

$$p = \{ f \in \mathcal{A}, \; |f(x)| < 1 \} \in \operatorname{Spf} \mathcal{A} = \mathcal{U}$$ (1.2)

(see section 1 of [B] for the general construction). Note that if $x$ is a semi-norm on a ring and $f$ a function in this ring, we write $|f(x)|$ instead of $x(f)$.

When $Y$ is a subset of $X_k$, we will consider its tube

$$\lbrack Y \rbrack := \operatorname{sp}^{-1}(Y) \subset X_{K_{\mathbb{A}}}.$$ (1.3)

This is mostly used when $Y$ is locally closed, in which case $\lbrack Y \rbrack$ is an analytic domain in $X_{\mathbb{A}}$. We will give an explicit description below, but it should first be remarked that, since $X_k$ is a connected curve, any locally closed subset $Y$ of $X_k$ is necessarily open or closed. Moreover, any open subset $U \subsetneq X_k$ is affine. Also, any closed subset $Z \subsetneq X_k$ is a finite set of closed points.

When $U$ is open in $X_k$, then $\lbrack U \rbrack$ is a closed subset of $X_{\mathbb{A}}$ (for the Berkovich topology): more precisely, if $\mathcal{U}$ is the formal lifting of $U$, then $\lbrack U \rbrack = \mathcal{U}_K$ (which is affinoid and therefore compact). Conversely, if $Z$ is a closed subset of $X_k$, then $\lbrack Z \rbrack$ is an open subset of $X_{\mathbb{A}}$ (for the Berkovich topology): more precisely, if $Z$ is defined inside some formal affine open subset $\mathcal{U}$ by an equation $f = 0 \mod \mathfrak{m}$, then $\lbrack Z \rbrack$ is defined in $\mathcal{U}_K$ by $|f(x)| < 1$ (being open is local for Grothendieck topology). It will also be convenient to consider the tubes $\lbrack Z \rbrack_\eta$ and $\lbrack Z \rbrack_\eta$ of radius $\eta < 1$ defined by $|f(x)| \leq \eta$ and $|f(x)| < \eta$ respectively (they are independent of the choice of $f$ when $\eta$ is close to 1).

The specialization map $\operatorname{sp} : X_{\mathbb{A}} \rightarrow X_k$ is surjective and may be used to give a description of $X_{\mathbb{A}}$. First of all, there is only one point above the generic point $\xi$ of $X_k$ (and none if we would use rigid analytic spaces) that we shall therefore denote by $\lbrack \xi \rbrack$ and call the generic point of $X_{\mathbb{A}}$ even though it depends on the model. Actually, if $U \subset X_k$ is a non empty affine subset, then $\lbrack \xi \rbrack$ is the Shilov boundary of $\lbrack U \rbrack$ (the point where any function
reaches its maximum). It is also the usual boundary of \( |U| \) as subset of the topological space \( X_K^{an} \) and the absolute boundary of \( |U| \) in Berkovich sense. For example, in the case of \( X = \mathbb{P}_k^1 \), then \( |\xi| \) is the Gauss norm.

By definition, \( X_K^{an} \setminus |\xi| \) is the disjoint union of all open subsets \( |x| \) for \( x \) a closed point in \( X_K \). These open subsets \( |x| \) are called the residual classes. In other words, the generic point is used to connect the residual classes together (in particular, it follows from proposition \[1.1\] below, that \( X_K^{an} \) is simply connected). We can also mention that \( \cap |U| = |\xi| \) when \( U \) runs through all the non empty (affine) open subsets of \( X_K \).

Before giving an explicit description of the residual classes, recall that there exists also a canonical map \( X_K^{an} \to X_K \). If \( \text{Spec} \, A \subset X \) is an open subset, the analytification of \( \text{Spec} \, A_K \) is the set \( \mathcal{M}^{alg}(A_K) \) of all multiplicative semi-norms \( x \) on \( A \) and \( x \) is sent to its kernel

\[
p = \{ f \in A, \quad |f(x)| = 0 \} \in \text{Spec} \, A_K.
\]

The map \( X_K^{an} \to X_K \) induces a bijection between the rigid points of \( X_K^{an} \) (whose residue fields are finite over \( K \)) and the closed points of \( X_K \) (note that all the non rigid points are then sent to the generic point of \( X_K \)). We will implicitly identify the closed points of \( X_K \) with the rigid points of \( X_K^{an} \).

We call a closed point \( a \in X_K \) a lifting of a closed point \( x \in X_k \) if \( \text{sp}(a) = x \). Note that \( K(a) \) is a finite extension of \( K \) and inherits a structure of complete ultrametric field whose ring of integers will be denoted by \( \mathcal{V}(a) \). The point \( a \) is said to be unramified if \( K(a)/K \) is an unramified extension (i.e \( \mathcal{V}(a) / \mathcal{V} \) is étale over \( \mathcal{V} \)).

Here is a standard result on residual classes:

**Proposition 1.1** If \( x \in X_k \) is a closed point and \( a \) is an unramified lifting of \( x \), then \( |x| \simeq D_{K(a)}(0, 1^+) \). Actually, we have \( |x|_\eta \simeq D_{K(a)}(0, \eta^+) \) for \( \eta < 1 \) and close to 1.

**Proof:** This follows from Berthelot’s weak fibration theorem (see lemma 4.4 of [5]) as we can now recall. When \( x \) is a rational point, we can write \( x \) as the unique zero of an étale map

\[
t : \mathcal{U} = \text{Spf} \, A \to \mathbb{A}_K^1.
\]

The corresponding morphism \( \mathcal{U}_K \to D_K(0, 1^+) \) will induce an isomorphism \( |x|_\eta \simeq D_1(0, \eta^+) \).

In general, we first have to extend the basis and consider the projection \( X_{\mathcal{V}(a)} \to X \). We pick up some point \( \gamma \) over \( x \). Then, the corresponding morphism \( X_{K(a)}^{an} \to X_K^{an} \) will induce an isomorphism \( |y|_\eta \simeq |x|_\eta \). \( \square \)

We are finished with the study of finite closed subsets and we consider now the case of a non empty affine open subset \( U \subset X_K \). We may always lift non canonically \( U \) to an affine open subset \( \text{Spec} \, A \subset X \). Since \( k \) is perfect, we may even assume that \( \text{Spec} \, A \) is the open complement of (the support of) a smooth (relative) divisor \( D \subset X \). In other words, the complement of \( U \) in \( X_K \) is \( D_k := \{ x_1, \ldots, x_n \} \) and the complement of \( \text{Spec} \, A_K \) in \( X_K \) is \( D_K = \{ a_1, \ldots, a_n \} \) where each \( a_i \) is an unramified lifting of \( x_i \). In this case, choosing \( A \) amounts to fixing the “center” \( a_i \) of each disc \( |x_i| \).

Actually, we will only use the weak completion \( A_k^+ \) of \( A \) and more precisely its generic fiber \( A_k^+ \). In fact, if \( Z \) denotes the closed complement of \( U \), the subsets \( V_\lambda := X_K^{an} \setminus |Z|_\lambda \) form a cofinal family of affinoid neighborhoods of \( |U| \) and \( A_k^+ = \varprojlim \lambda A_\lambda \) if we write \( V_\lambda =: M(A_\lambda) \). In particular, the algebra \( A_k^+ \) does not depend on the algebraic lifting but only
on $U$. However, the geometric construction can be useful and we will stick to the original
definition of Monsky and Washnitzer.

It is also easy to see that the subsets $V_\lambda$ for various $\lambda$ and $U$ form a cofinal family of
affinoid neighborhoods of the generic point $|\xi|$. Actually, since $X_K^\text{an}$ is compact, any open
neighborhood $V$ of $|\xi|$ has a compact complement $C \subset \cup|x| = \cup|x_\lambda$ and therefore, $C \subset \cup_{\text{finite}}|x_\lambda| = T[\lambda]$. It follows that $V_\lambda := X_K^\text{an}| \setminus T[\lambda] \subset V$. In particular, we have

$$\mathcal{O}_{X_K^\text{an}}|_\xi = \lim_\rightarrow A_K^\dagger$$

when Spec $A$ runs through the non empty affine open subsets of $X$. Note that $\mathcal{O}_{X_K^\text{an}}|_\xi$ is
a henselian field and that $|f(|\xi|)| = \|f\|_{|U|}$ if $f \in A_K^\dagger$ and Spec $A$ is a lifting of $U$.

If $Y$ is a subset of $X_k$, we will denote by

$$i_Y : |Y| \hookrightarrow X_K^\text{an}$$

the corresponding inclusion. Recall from above that, since we use Berkovich topology, $i_U$ is a closed embedding when $U$ is an open subset in $X_k$ and that $i_Z$ is an open embedding when $Z$ is a closed subset of $X_k$.

**Definition 1.2** An $\mathcal{O}_{X_K^\text{an}}$-module $E$ is constructible (resp. constructible-free) if there exists a finite covering of $X_k$ by locally closed subsets $Y$ such that $i_Y^{-1}E$ is a coherent (resp. free) $i_Y^{-1}\mathcal{O}_{X_K^\text{an}}$-module.

This definition makes sense in a very general situation but we really want to stick to curves here in order to give an explicit description of these constructible modules. Nevertheless, the following is purely formal:

**Proposition 1.3** Constructible (resp. constructible-free) modules form an abelian (resp. additive) subcategory stable under extensions and tensor product.

**Proof:** Since the restriction maps $E \mapsto i_Y^{-1}E$ are exact and commute with tensor product, all these assertions follow from the analogous results for coherent (resp. free) modules. □

Note that if $Y' \subset Y$ and $i_Y^{-1}E$ is coherent, free or locally free, so is $i_{Y'}^{-1}E$ by restriction. It should also be remarked that if $Z \subset X_k$ is a closed subset, then $i_Z^{-1}E = E|_Z$ is the usual restriction to an open subset. However, when $U \subset X_k$ is open, then $i_U$ is a closed embedding and $i_U^{-1}E$ is the sheaf of sections defined in some neighborhood of the compact subset $|U|$. More on this soon.

At some point, we will need to use some theorems that were formulated in the language of rigid analytic geometry. It will therefore be necessary to have a dictionary at our disposal. We want to explain this right now. We may consider the Grothendieck topology on $X_K^\text{an}$: admissible open subsets are analytic domains and coverings are Tate coverings (see for example [2], page 25-26). If $V$ is an analytic domain of $X_K^\text{an}$, we will denote by $V_G$ the associated Grothendieck space. There is an obvious continuous map of Grothendieck spaces $\pi_V : V_G \to V$ which is simply the identity on underlying sets. Actually, $\pi_V^1$ is exact (because the embedding of an analytic domain is then universally flat) and fully faithful (see Proposition 1.3.4 of [2]) when $V$ is locally affinoid. If this is the case, it even induces
an equivalence on coherent sheaves. We will write \( E_G := \pi_\ast V E \). Finally, the set \( V^{\mathrm{rig}} \) of rigid points of \( V \) (points whose residue field is finite over \( K \)) has a natural structure of rigid analytic variety and the topos of \( V^{\mathrm{rig}} \) is equivalent to the topos of \( V_G \) (affinoid domains and affinoid coverings coincide). We will denote by \( E_0 \) the rigid sheaf corresponding to \( E_G \) (if \( W \) is an affinoid domain inside \( V \), we have \( E_0(W^{\mathrm{rig}}) = E_G(W) \)).

The next comparison theorem was proved in corollary 2.2.13 of [23] but it will not be long to recall how it works.

**Proposition 1.4** If \( U \) is an open subset of \( X_k \), the functor \( E \mapsto \left(i_U \ast E\right)_0 \) induces an equivalence between the category of coherent \( i_U^{-1} \mathcal{O}_{X_k^{\mathrm{an}}} \)-modules on \( |U| \) and the category of coherent \( j^0 \mathcal{O}_{X_k^{\mathrm{rig}}} \)-modules.

**Proof:** If \( Z \) is the closed complement of \( U \) and \( \lambda < 1 \), we consider the affinoid domain \( V_\lambda = X_k^{\mathrm{an}} \setminus Z[\lambda] \). Since \( X_k^{\mathrm{an}} \) is locally compact, the category of coherent \( i_U^{-1} \mathcal{O}_{X_k^{\mathrm{an}}} \)-modules is equivalent to the direct limit of the category of coherent \( \mathcal{O}_{V_\lambda} \)-modules (see corollary 2.2.5 of [23] for example). And we also know that the category of coherent \( j^0 \mathcal{O}_{X_k^{\mathrm{rig}}} \)-modules is equivalent to the direct limit of the category of coherent \( \mathcal{O}_{V_\lambda^{\mathrm{rig}}} \)-modules (theorem 5.4.4 of [22]). □

In general, we will not use this comparison theorem because most results are a lot simpler to prove in the Berkovich topology and it sounds unnatural to use the rigid topology in order to derive them. For example, we have the following (so called theorem A and B) which is essentially proposition 5.4.8 of [22].

**Proposition 1.5** If \( U \) is an affine open subset of \( X_k \), then \( i_U^{-1} \mathcal{O}_{X_k^{\mathrm{an}}} \) is a coherent sheaf of rings. Moreover, if \( \text{Spec} \, A \subset X \) is an algebraic lifting of \( U \), then

\[
\Gamma(|U|, i_U^{-1} \mathcal{O}_{X_k^{\mathrm{an}}}) = A_K^1.
\]

Finally, the functor \( \Gamma(|U|, -) \) induces an equivalence between coherent (resp. locally free, resp. free) \( i_U^{-1} \mathcal{O}_{X_k^{\mathrm{an}}} \)-modules and finite (resp. finite projective, resp. finite free) \( A_K^1 \)-modules.

**Proof:** Since \( |U| \) is a compact subset of the locally compact space \( X_k^{\mathrm{an}} \) and \( \mathcal{O}_{X_k^{\mathrm{an}}} \) is coherent, the first assertion is purely formal (see for example proposition 2.2.3 of [23]).

For the same reason, any coherent \( i_U^{-1} \mathcal{O}_{X_k^{\mathrm{an}}} \)-module \( E \) is the restriction of a coherent module on some neighborhood \( V \) of \( |U| \) in \( X_k^{\mathrm{an}} \) and we get an equivalence of categories if we allow the restriction of \( V \) (see corollary 2.2.5 of [23]).

As mentioned above, there exists a cofinal family of affinoid neighborhoods \( V_\lambda = \mathcal{M}(A_\lambda) \) with \( A_K^1 = \lim A_\lambda \). The rest of the proposition follows then immediately from the analogous results on affinoid varieties and the properties of filtered direct limits. □

**Lemma 1.6** If \( E \) is a constructible (resp. constructible free) module and \( Z \subset X_k \) is a finite closed subset, then \( i_Z^{-1} E \) is coherent (resp. locally free).

**Proof:** We may assume that \( Z \) is reduced to one point \( x \). Then, there exists a locally closed subset \( Y \subset X_k \) with \( x \in Y \) and \( i_Y^{-1} E \) coherent (resp. free) on \( |Y| \). We may then restrict to \( |x| \). □
Proposition 1.7 If $E$ is a constructible module, there exists a non-empty open subset $U \subset X$ such that $i^{-1}_U E$ is a free $i^{-1}_U \mathcal{O}_{X^m_K}$-module.

Proof: By definition, if $\xi$ is the generic point of $X_k$, there exists a locally closed subset $U \subset X_k$, which is necessarily open, such that $\xi \in U$ and $i^{-1}_U E$ is a coherent $i^{-1}_U \mathcal{O}_{X^m_K}$-module. It follows that the stalk $E_{\xi}$ of $E$ at $\xi$ is a finite dimensional vector space (recall that $\mathcal{O}_{X^m_K,\xi}$ is a field). At this point, it is convenient to write $i^{-1}_U E = i^{-1}_U F$ where $i: U \rightarrow V$ is the inclusion of $U$ in some neighborhood and $F$ is a coherent $\mathcal{O}_V$-module. Since $\mathcal{F}|_{\xi} = E_{\xi}$ is a finite dimensional vector space and $\mathcal{F}$ is coherent, we know that there exists a neighborhood of $\xi$ in $V$ on which $\mathcal{F}$ becomes free. We saw above that the subsets $V_\lambda = X^m_K \setminus J[\lambda]$ with $T$ finite closed and $\lambda < 1$ form a cofinal family of neighborhoods of $\xi$. Therefore, there exists such a $T$ and $\lambda$ with $\mathcal{F}|_{V_\lambda}$ free. We may then remove a finite number of closed points in $U$ and assume that $U \cap T = \emptyset$ in order to get $i^{-1}_U E$ free. □

Corollary 1.8 Let $E$ be a constructible module. Then, the following are equivalent:

1. $E$ is constructible-free
2. $i^{-1}_Z E$ is locally free whenever $Z \subset X_k$ is a finite closed subset.
3. $i^{-1}_x E$ is free whenever $x \in X_k$ is a closed point

Proof: We know from proposition [1.1] that if $x \in X_k$ is any closed point, then $|x|$ is a disc. Moreover, since the valuation is discrete, a coherent locally free module on a disk is automatically free (see for example proposition 16.1.4 of [21]). It follows that assertion 2 and 3 are equivalent. And we showed in lemma [1.6] that they are automatically fulfilled when $E$ is constructible-free. Conversely, since $E$ is constructible, we know from the proposition that there exists a non-empty open subset $U \subset X_k$ such that $i^{-1}_U E$ is free. If we assume that assertion 3 holds, then, in particular, for all $x \notin U$, $i^{-1}_x E$ is free. Now, $U$ and the points $x \notin U$ form a finite covering of $X_k$ by locally closed subsets. □

Corollary 1.9 An $\mathcal{O}_{X^m_K}$-module $E$ is constructible-free if and only if there exists a finite covering of $X_k$ by locally closed subsets $Y$ such that $i^{-1}_Y E$ is a locally free $i^{-1}_Y \mathcal{O}_{X^m_K}$-module.

Proof: The condition is clearly necessary. Conversely, any such module is constructible and we may apply the previous corollary. □

Corollary 1.10 An $\mathcal{O}_{X^m_K}$-module $E$ is constructible (resp. constructible-free) if and only if there exists a non-empty affine open subset $U \subset X_k$ with closed complement $Z$ such that $i^{-1}_U E$ is a coherent (resp. locally free) $i^{-1}_U \mathcal{O}_{X^m_K}$-module and $i^{-1}_Z Z E$ is coherent (resp. locally free). Moreover, we may assume that $i^{-1}_U E$ is free.

Proof: It follows from the previous corollary that the condition is necessary. Conversely, if $E$ is constructible, we can find such a $U$ thanks to the proposition and then apply lemma [1.6] to the complement $Z$ of $U$ in $X_k$. □

Corollary 1.11 Let $E$ be a constructible-free module and $Y \subset X_k$ a locally closed subset. If $i^{-1}_Y E$ is coherent, it is necessarily locally free.
Proof: If \( a \in Y \) specializes to \( x \in X_k \), then \( a \in [x] \) and we know that \( i^{-1}_x E \) is free. It follows that the stalk \( E_a \) of \( E \) at \( a \) is free. A coherent module whose stalks are free is necessarily locally free. \( \square \)

If \( Z \subset X_k \) is a closed subset, then \( i_Z : [Z] \hookrightarrow X_K^{an} \) is an open immersion and we may consider the extension by zero \( i_Z! \) outside \( Z \) which is a left adjoint functor to \( i^{-1}_Z \).

**Proposition 1.12** An \( \mathcal{O}_{X_K^{an}} \)-module \( E \) is constructible (resp. constructible free) if and only if there exists an exact sequence

\[
0 \to i_Z! E_Z \to E \to i_{U*} E_U \to 0
\]

where \( U \) is a non-empty affine open subset of \( X_k \) with closed complement \( Z \), \( E_U \) is a coherent (resp. locally free) \( i^{-1}_U \mathcal{O}_{X_k^{an}} \)-module and \( E_Z \) is coherent (resp. locally free). Moreover, we may assume that \( i^{-1}_U E \) is free.

Proof: If we are given such and exact sequence, we may pull back along \( i_U \) and \( i_Z \) in order to obtain coherent modules \( E_U \simeq i^{-1}_U E \) and \( E_Z \simeq i^{-1}_Z E \). And conversely, using corollary [1.10] we can find a non-empty affine open subset \( U \subset X_k \) with closed complement \( Z \) such that \( i^{-1}_U E \) is a coherent (resp. locally free) \( i^{-1}_U \mathcal{O}_{X_K^{an}} \)-module and \( i^{-1}_Z E \) is coherent (resp. locally free). There is an exact sequence

\[
0 \to i_Z! i^{-1}_Z E \to E \to i_{U*} i^{-1}_U E \to 0 \quad (1.10)
\]

and we can set \( E_U := i^{-1}_U E \) and \( E_Z := i^{-1}_Z E \). \( \square \)

2 Classification

Recall that if \( Y \) is a locally closed subset of \( X_k \), we denote by \( i_Y : [Y] \hookrightarrow X_K^{an} \) the inclusion map. Recall also that \( \xi \) denotes the generic point of \( X_k \).

**Definition 2.1** Let \( T \subset X_k \) be a non empty finite closed subset. If \( \mathcal{F} \) is a coherent \( \mathcal{O}_T \)-module, we will write \( \mathcal{O}^{an}(\mathcal{F}) := \Gamma([T], \mathcal{F}) \). The Robba module of \( \mathcal{F} \) is

\[
\mathcal{R}(\mathcal{F}) := (i_T*, \mathcal{F})_{[\xi]} \quad (2.1)
\]

and the Dirac space of \( \mathcal{F} \) is defined by the short exact sequence

\[
0 \to \mathcal{O}^{an}(\mathcal{F}) \to \mathcal{R}(\mathcal{F}) \to \delta(\mathcal{F}) \to 0 \quad (2.2)
\]

We will write \( \mathcal{O}_T^{an} := \Gamma([T], \mathcal{O}_T) \) and call

\[
\mathcal{R}_T := (i_T*, \mathcal{O}_T)_{[\xi]} \quad (2.3)
\]

the Robba ring and the Dirac space, respectively, of \( T \).

**Proposition 2.2** If \( x \) is a closed point of \( X_k \) with unramified lifting \( a \), then \( \mathcal{R}_x \) is (isomorphic to) the usual Robba ring \( \mathcal{R}_a \) over \( K(a) \). In general, we have

\[
\mathcal{R}_T(\mathcal{F}) = \lim_{\to} \Gamma([T]\lambda, \mathcal{F}) \quad (2.4)
\]
Proof: As already mentioned, the subsets $X_K^{an}|Z[\lambda]$ form a cofinal family of affinoid neighborhoods of $|\xi|$ when $Z$ runs through the finite subsets of $X_k$ and $\lambda < 1$. It follows that

$$R_T(K) = (i_{T*}\mathcal{F}|T)[|\xi|] = \lim_{\longrightarrow} \Gamma(X_K^{an}|Z[\lambda], i_{T*}\mathcal{F}|T[\lambda]) = \lim_{\longrightarrow} \Gamma([T]\|T[\lambda], \mathcal{F}).$$

(2.5)

In particular, we obtain

$$R_x = \lim_{\longrightarrow} \Gamma([x]\|x[\lambda], \mathcal{F}|x)).$$

(2.6)

Since $|x| \simeq \mathbf{D}_{K(a)}(0, 1^{-})$ and $|x| \mathbf{D}_{K(a)}(0, \lambda)$, this is the (usual) Robba ring $R_a$ over $K(a)$. □

If $T$ is a non empty finite closed subset of $X_k$, the adjunction map

$$\mathcal{O}_{X_K^{an}} \to i_{T*}i_T^{-1}\mathcal{O}_{X_K^{an}}$$

will induce on the stalks a canonical morphism $\mathcal{O}_{X_K^{an}|\xi|} \to R_T$. In particular, if $U$ is a non empty affine open subset of $X_k$ with algebraic lifting $\text{Spec } A$, there is a canonical morphism

$$A_K^\dag \to \mathcal{O}_{X_K^{an}|\xi|} \to R_T.$$

(2.8)

Definition 2.3 Let $T$ be a non empty finite closed subset of $X_k$. If $E$ is a constructible module on $X_K^{an}$, then the Robba fiber of $E$ at $T$ is

$$R_T(E) := R_T \otimes \mathcal{O}_{X_K^{an}|\xi|} E[|\xi|].$$

(2.9)

Let $U$ be a non empty affine open subset of $X_k$ with algebraic lifting $\text{Spec } A$ and $M$ a finite $A_K^\dag$-module. Then, the Robba fiber of $M$ at $T$ is

$$R_T(M) := R_T \otimes_{A_K^\dag} M.$$

(2.10)

Note that the Robba fiber of a constructible module $E$ is a free module (of finite rank) because $\mathcal{O}_{X_K^{an}|\xi|}$ is a field. Note also that if $\text{Spec } B$ is some affine open subset of $\text{Spec } A$, and $M$ is a finite $A_K^\dag$-module, then

$$R_T(B_K^\dag \otimes_{A_K^\dag} M) = R_T(M).$$

(2.11)

Finally, if $i_U^{-1}E$ is coherent and $M = \Gamma([U], i_U^{-1}E)$, then $R_T(M) = R_T(E)$. Be careful however that $R_T(E) \neq R(E|[T])$ in general (unless $E$ is coherent) but we have the following:

Lemma 2.4 Let $U$ be a non empty affine open subset of $X_k$ with algebraic lifting $\text{Spec } A$ and $T$ be a non empty finite closed subset of $U$. Let $M$ be a finite $A_K^\dag$-module and $M_T$ the restriction to $[T]$ of the corresponding $i_U^{-1}\mathcal{O}_{X_K^{an}}$-module. Then, the Robba fiber of $M$ at $T$ is identical to the Robba module of $M_T$:

$$R_T(M) = R(M_T).$$

(2.12)

Proof: Since we are working with right exact functors, we may assume that $M = A_K^\dag$ in which case $M_T = \mathcal{O}_{|T|}$ and we fall back onto the definition. □
Definition 2.5 Let $U$ be a non empty affine open subset of $X_k$ with closed complement $Z$ and Spec $A$ an algebraic lifting of $U$.

A kit on $U$ is a triple made of a finite $A_K^1$-module $M$, a coherent module $F$ on $|Z|$ and an $R_Z$-linear map $R_Z(M) \to R(F)$. It is said to be free if $M$ is projective and $F$ is locally free.

A morphism of kits on $U$ is a pair made of a linear map $M' \to M$ and a linear map $F' \to F$ making commutative the obvious diagram.

In practice, we may just say that $R_Z(M) \to R(F)$ ‘is’ a kit, $M$ and $F$ being understood as being part of the data.

Definition 2.6 1. Let $U$ be a non empty affine open subset of $X_k$ with closed complement $Z$ and Spec $A$ an algebraic lifting of $U$. Let $R_Z(M) \to R(F)$ be a kit on $U$. Let $T \subset U$ be any finite closed subset and $W = \text{Spec } B$ some affine open subset of $V$ which is a lifting of $U \setminus T$. Then, if $N := B_K^1 \otimes A_K^1 M$, the kit

$$R_{Z \cup T}(N) = R_{Z \cup T}(M) = R_Z(M) \oplus R_T(M) \to R(F) \oplus R(M_T). \quad (2.13)$$

is called the restriction of $R_Z(M) \to R(F)$ to $U \setminus T$.

2. Two kits $R_Z(M) \to R(F)$ on $U$ and $R_Z(M') \to R(F')$ on $U'$ are said to be equivalent if their restrictions to $U \cap U'$ are isomorphic.

Proposition 2.7 The category of constructible (resp. constructible-free) modules on $X^n_K$ is equivalent to the category of kits (resp. free kits) modulo equivalence.

Proof: If we are given a closed subset $Z$ of $X_k$ with closed complement $U$, we know that $|U|$ is a closed subset with open complement $|Z|$. It is then a general fact that the category of $O_{X^n_K}$-modules $E$ on $X^n_K$ is equivalent to the category of triples made of an $i_Z^{-1}O_{X^n_K}$-module $E_U$, an $i_Z^{-1}O_{X^n_K}$-module $E_Z$ and an $i_U^{-1}O_{X^n_K}$-linear map $E_U \to i_U^{-1}i_Z^1 E$. This equivalence is easily seen on the following morphism of short exact sequences:

$$0 \longrightarrow i_Z^1 E_Z \longrightarrow i_U^{-1}i_Z^1 E_Z \longrightarrow i_U^{-1}i_U^{-1}i_Z^1 E_Z \longrightarrow 0 \quad (2.14)$$

More precisely, $E$ is obtained by pushing the given map $E_U \to i_U^{-1}i_Z^1 E_Z$ to $X$ and taking fiber product with the adjunction map $i_Z^1 E_Z \to i_U^{-1}i_U^{-1}i_Z^1 E_Z$; and conversely, given $E$, we can consider the restriction to $U$ of the adjunction map $E \to i_Z^1 i_U^{-1} E$.

We know from corollary 1.10 that $E$ is constructible if and only if we can find such $U$ and $Z$ with both $E_U$ and $E_Z$ coherent. Then, we simply set $F = E_Z$ and $M = \Gamma(|U|, E_U)$. Now, if $i_{\xi}\colon [\xi] \to [U]$ denotes the inclusion map, we have

$$i_U^{-1}i_Z^1 E_Z = i_{\xi}^{-1}i_{\xi}^{-1}i_U^{-1}i_Z^1 E_Z = i_{\xi}^{-1}R(F) \quad (2.15)$$

and it follows that, canonically,

$$\text{Hom}_{i_U^{-1}O_{X^n_K}}(E_U, i_U^{-1}i_Z^1 E_Z) = \text{Hom}_{i_U^{-1}O_{X^n_K}}(E_U, i_{\xi}^{-1}R(F)) \quad (2.16)$$
≃ \text{Hom}_{\mathcal{O}_{X_{K}^{\text{an}}}[\xi]}(E_{U}[\xi], \mathcal{R}(\mathcal{F})) \simeq \text{Hom}_{\mathcal{O}_{Z}}(\mathcal{R}_{Z}(M), \mathcal{R}(\mathcal{F})) \quad (2.17)

because \( E_{U}[\xi] = \mathcal{O}_{X_{K}^{\text{an}}}[\xi] \otimes_{\mathcal{A}_{K}} M \). Thus, we see that morphisms \( E_{U} \rightarrow i_{U}^{-1}i_{Z_{*}}E_{Z} \) correspond bijectively to morphisms \( \mathcal{R}_{Z}(M) \rightarrow \mathcal{R}(\mathcal{F}) \).

Of course, one easily checks that if we shrink \( U \), the corresponding kit will be the restriction of \( \mathcal{R}_{Z}(M) \rightarrow \mathcal{R}(\mathcal{F}) \).

□

In practice, a morphism \( \mathcal{R}_{Z}(M) \rightarrow \mathcal{R}(\mathcal{F}) \) corresponds to a morphism \( i_{U*}E_{U} \rightarrow i_{\xi*}\mathcal{R}(\mathcal{F}) \), and we can pull back the exact sequence

\[ 0 \rightarrow i_{Z!}\mathcal{F} \rightarrow i_{Z*}\mathcal{F} \rightarrow i_{\xi*}\mathcal{R}(\mathcal{F}) \rightarrow 0 \quad (2.18) \]

in order to get

\[ 0 \rightarrow i_{Z!}E_{Z} \rightarrow E \rightarrow i_{U*}E_{U} \rightarrow 0. \quad (2.19) \]

3 Specialization

The specialization map \( sp : X_{K}^{\text{an}} \rightarrow X_{k} \) is not continuous for the Berkovich topology of \( X_{K}^{\text{an}} \) and the Zariski topology of \( X_{k} \) (actually, it is anticontinuous: inverse image of closed is open and conversely). It is therefore necessary to use the Grothendieck topology of \( X_{K}^{\text{an}} \) in order to define direct image.

**Definition 3.1** If \( E \) is any \( \mathcal{O}_{X_{K}^{\text{an}}} \)-module, the specialization of \( E \) is

\[ \tilde{sp}_{*}E := sp_{*}E_{G} \quad (3.1) \]

where \( E_{G} \) denotes the module associated to \( E \) for the Grothendieck topology.

**Proposition 3.2** The specialization functor \( \tilde{sp}_{*} \) is left exact. Moreover, if \( E \) is any module on \( X_{K}^{\text{an}} \), then

\[ R\tilde{sp}_{*}E = Rsp_{*}E_{G} = Rsp_{*}E_{0} \quad (3.2) \]

where \( E_{G} \) (resp. \( E_{0} \)) denotes the module associated to \( E \) for the Grothendieck topology (resp. on the corresponding rigid space).

**Proof:** The functor is left exact as composition of two left exact functors. Moreover, since \( E \mapsto E_{G} \) is even exact, we have \( R\tilde{sp}_{*}E := Rsp_{*}E_{G} \). Finally, we have \( \Gamma([U], E_{G}) = \Gamma([U]^{rig}, E_{0}) \) if \( U \) is any affine open subset of \( X_{k} \).

Recall now from [6], 4.2.4, that if \( T \) is a finite closed subset of \( X_{k} \), then the sheaf \( \mathcal{O}_{X}((T)) \) may be defined as follows. First of all, if \( j : X_{k} \setminus Z \hookrightarrow X_{k} \) denotes the inclusion map, then \( \mathcal{O}_{X}((T)) \) is a subsheaf of \( j_{*}j^{-1}\mathcal{O}_{X} \). Moreover, if \( T \) is defined in \( U = \text{Spf} \mathcal{A} \) by \( f = 0 \mod m \), then

\[ \Gamma(U, \mathcal{O}_{X}((T))) := \lim_{\to} A\{\pi/f^{r}\} \quad (3.3) \]

so that

\[ \Gamma(U, \mathcal{O}_{X}((T))_{Q}) := A[1/f]^{1}_{Q}. \quad (3.4) \]

The next result is an analog for the Berkovich theory of a theorem of Berthelot (proposition 4.3.2 of [6], using proposition 1.4).
Proposition 3.3 If $U$ is a non empty open subset of $X_k$ with closed complement $Z$, then
\[
R\hat{\mathcal{p}}_*i_{U*}i_U^{-1}\mathcal{O}_{X_K^{an}} = \mathcal{O}_{\hat{X}}((1)Z)\mathbb{Q}.
\] (3.5)
Moreover, the functor $R\hat{\mathcal{p}}_*i_{U*}$ induces an equivalence between coherent $i_U^{-1}\mathcal{O}_{X_K^{an}}$-modules and coherent $\mathcal{O}_{\hat{X}}((1)Z)\mathbb{Q}$-modules.

Proof: If $E$ is a coherent $i_U^{-1}\mathcal{O}_{X_K^{an}}$-module, we may always consider it as the restriction to $|U|$ of some coherent sheaf $\mathcal{F}$ defined on some neighborhood $V_\lambda = X_K^{an}\setminus|Z[\lambda]|$ of $|U|$. By definition, $R^q\hat{\mathcal{p}}_*i_{U*}E$ is the sheaf associated to
\[
U' \mapsto H^q(|U'|, i_U^{-1}i_{U*}E)
\] (3.6)
If we set $V'_\lambda = |U'|\cap V_\lambda$ and denote by $i_\lambda : V'_\lambda \hookrightarrow V_\lambda$ the inclusion map, it is a general topological result (proposition 2.5 of [20] for example) that
\[
H^q(|U'|, i_U^{-1}i_{U*}E) = \lim_{\lambda \to 0} H^q(V'_\lambda, i_\lambda^*\mathcal{F}).
\] (3.7)
Since $V'_\lambda$ is affinoid, we have $H^q(|U'|, i_U^{-1}i_{U*}E) = 0$ for $q > 0$ and therefore, $R^q\hat{\mathcal{p}}_*i_{U*}E_U = 0$ for $q > 0$. Moreover, we have $V'_\lambda = |U'|\cap Z[\lambda]$ if $Z' = Z\cap U'$. If we call $U' = \text{Spf} \mathcal{A}'$ the formal lifting of $U'$ and if $Z'$ is defined by $f = 0 \mod \mathfrak{m}$ in $U'$, we have $V'_\lambda = \mathcal{M}(\mathcal{A}_K'\{\pi/f^r\})$ for $\lambda = |\pi|^{1/r}$ and therefore,
\[
\Gamma(|U'|, i_U^{-1}\mathcal{O}_{X_K^{an}}) = \lim_{\lambda \to 0} \mathcal{A}_K'\{\pi/f^r\} = \Gamma(U, \mathcal{O}_{\hat{X}}((1)Z)\mathbb{Q}).
\] (3.8)
The last assertion then follows from theorem A and B for $\mathcal{O}_{\hat{X}}((1)Z)\mathbb{Q}$ (see proposition 4.3.2 of [6]).

In practice, if Spec $A$ is an algebraic lifting of $U$, a coherent $i_U^{-1}\mathcal{O}_{X_K^{an}}$-module $E$ corresponds to a finite $\mathcal{A}_K'$-module $M$. Now, let $U'$ be any affine open subset of $X$ with formal lifting $U' = \text{Spf} \mathcal{A}'$ and $f$ an equation for $Z$ in $U'$. Then, there is a canonical map $\mathcal{A}^! \to \mathcal{A}'[1/f]^!$ and
\[
\Gamma(U', \hat{\mathcal{p}}_*i_{U*}E) = \mathcal{A}'[1/f]^!\mathbb{Q} \otimes_{\mathcal{A}_K'} M.
\] (3.9)

We now study the case of a finite closed subset $Z \subset X_k$. We will write
\[
H^*_Z := \mathcal{O}_{\hat{X}}((1)Z)/\mathcal{O}_{\hat{X}}.
\] (3.10)
We do not need it at this point, but we may notice that, in the theory of arithmetic $\mathcal{D}$-modules, we have
\[
R\Gamma^!_{X}\mathcal{O}_{\hat{X}} \simeq H_Z^!\mathbb{Q}[-1].
\] (3.11)
This cohomology with support is closely related to the notion of Dirac space introduced in definition [27] as we shall see right now.

Lemma 3.4 If $Z$ is a finite closed subset of $X_k$, we have
\[
H^*_Z\mathbb{Q} = \bigoplus_{x \in Z} i_x \delta_x
\] (3.12)
where $i_x : \{x\} \hookrightarrow \hat{X}$ denotes the inclusion map.
Proof: Since $\mathcal{O}_{\tilde{X}}(\tilde{1}Z)$ and $\mathcal{O}_{\tilde{X}}$ coincide outside $Z$, it is sufficient to show that there is an isomorphism on the stalks $H^1_{\tilde{Z}Q,x} \simeq \delta_x$ when $x \in Z$. We will now use the fact that if $U$ is an affine neighborhood of $x$, then $|U| \setminus |x|$ and $|x|$ form an open covering of $|U|$ with intersection $|x| \setminus |x|$ and that this covering is acyclic for $\mathcal{O}_{|U|}$. We call $U = \text{Spf } \mathcal{A}$ the formal lifting of $U$ and assume that $x$ is defined in $U$ by an equation $f = 0 \mod m$. We will also denote by $\mathcal{R}_{x,\lambda} := \Gamma(|x|, \mathcal{O}_{|x|})$ and $\delta_{x,\lambda} = \mathcal{R}_{x,\lambda}/\mathcal{O}_{x,\lambda}$ with $\mathcal{O}_{x,\lambda} = \Gamma(|x|, \mathcal{O}_{|x|})$ as before. If $\lambda = |\pi|^{1/r}$, then since we have an open covering which is acyclic for the structural sheaf, in the following morphism of exact sequences, the first square is bicartesian and it follows that the last map is an isomorphism

$$
\begin{array}{c}
0 \\ \downarrow \\
A_\mathcal{Q} \\ \downarrow \\
\mathcal{O}^\text{an}_x \\ \downarrow \\
\mathcal{R}_{x,\lambda} \\ \downarrow \\
\delta_{x,\lambda} \\
\downarrow \\
0.
\end{array}
$$

(3.13)

Taking limit on $\lambda < 1$ (and all $U \ni x$), we obtain the expected isomorphism $H^1_{\tilde{Z}Q,x} = \delta_x$. □

Lemma 3.5 If $Z$ is a non empty finite closed subset of $X_k$, then

$$
\text{RSp}_x^*i_Z!\mathcal{O}_Z \simeq H^1_{\tilde{Z}Q}[-1] = \text{RL}^1_{\tilde{Z}\mathcal{O}_\tilde{X}Q}. 
$$

(3.14)

Proof: If $U$ denotes the complement of $Z$ in $X_k$, there is an exact sequence

$$
0 \to i_Z!i_Z^{-1}\mathcal{O}_{X_k^p} \to \mathcal{O}_{X_k^p} \to i_U^*i_U^{-1}\mathcal{O}_{X_k^p} \to 0
$$

(3.15)

from which we derive a triangle

$$
\begin{array}{c}
\text{RSp}_x^*i_Z!\mathcal{O}_Z \\
\to \\
\text{RSp}_x^*i_Z^{-1}\mathcal{O}_{X_k^p} \\
\to \\
\text{RSp}_x^*i_U^*i_U^{-1}\mathcal{O}_{X_k^p}.
\end{array}
$$

(3.16)

and then, using proposition 3.2 and the definition of $H^1_{\tilde{Z}}$ in (3.10), we obtain

$$
\text{RSp}_x^*i_Z!\mathcal{O}_Z \simeq [\mathcal{O}_\mathcal{X}Q \to \mathcal{O}_X(\tilde{1}Z)Q] \simeq H^1_{\tilde{Z}Q}[-1].
$$

(3.17)

Recall that we can always lift a finite closed subset $Z$ to a smooth relative divisor $D \subset X$: if $Z = \{x_1, \ldots, x_n\}$, we choose an unramified lifting $a_i \in X_K$ for each $i$ and let $D = \text{Spec } \prod \mathcal{V}(a_i)$. Note then that a (coherent) $\mathcal{O}_{D_K}$-module $H$ is simply a finite collection of (finite dimensional) $(a_i)$-vector spaces $H_a$ for $a \in D_K$. Note also that the canonical inclusion $D_K = D_K^p = \tilde{D}_K \hookrightarrow |Z|$ has a unique retraction contracting each disc onto its “center”. It also follows from lemma 3.4 that $H^1_{\tilde{Z}Q}$ is an $\mathcal{O}_{D_K}$-module.

Proposition 3.6 Let $D \subset X$ be a smooth divisor with reduction $Z$. If $E$ is a locally free module on $|Z|$, there exists a coherent $\mathcal{O}_{D_K}$-module $H$ such that $E \simeq \mathcal{O}_Z \otimes_{\mathcal{O}_{D_K}} H$ and we have

$$
\text{RSp}_x^*i_Z!E \simeq (H^1_{\tilde{Z}Q} \otimes_{\mathcal{O}_{D_K}} H)[-1].
$$

(3.18)

Proof: The first assertion follows from the remark before and the second one then follows from lemma 3.5 since $\text{RSp}_x^*$ is additive. □
Definition 3.7 A perverse sheaf on $\hat{X}$ is a complex of $O_{\hat{X}}\mathbb{Q}$-modules $E$ such that $H^0(E)$ is $O_{\hat{X}}\mathbb{Q}$-flat, $H^1(E)$ has finite support on $\hat{X}$ and $H^i(E) = 0$ otherwise.

It is sometimes convenient to split this definition in two (in order to see perverse sheaves as the heart of a $t$-structure). We can denote by $D_{\geq 0}(\hat{X})_\mathbb{Q}$ the category of bounded complexes of $O_{\hat{X}}\mathbb{Q}$-modules $E$ where $H^0(E)$ is $O_{\hat{X}}\mathbb{Q}$-flat and $H^1(E)$ has finite support and $H^i(E) = 0$ for $i > 1$. Then the category of perverse sheaves is $D_{\geq 0}(\hat{X})_\mathbb{Q} \cap D_{\leq 0}(\hat{X})_\mathbb{Q}$.

Proposition 3.8 If $E$ is a constructible-free module on $X^\text{an}_K$, then $\mathbb{R}\mathfrak{sp}_\ast E$ is a perverse sheaf on $\hat{X}$. More precisely, we have

$$\mathbb{R}\mathfrak{sp}_\ast E \simeq [E \to H^1_{Z} \otimes_{O_{D_K}} H]$$

(3.19)

where $Z \subset X_k$ is a finite closed subset, $E$ is a coherent locally free $O_{\hat{X}}(\{Z\})\mathbb{Q}$-module, $D \subset X$ is a smooth divisor with reduction $Z$, and $H$ is a coherent $O_{D_K}$-module.

Proof: We showed in corollary 1.12 that there exists an exact sequence

$$0 \to i_{Z!}E_Z \to E \to i_{U!}E_U \to 0$$

(3.20)

where $Z \subset X_k$ is a finite closed subset with affine open complement $U$, $E_U$ is a coherent locally free $i_{U!}O_{X^\text{an}}$-module and $E_Z$ is a locally free $O_{\{Z\}}$-module.

There exists a smooth divisor $D \subset X$ with reduction $Z$ and a coherent $O_{D_K}$-module $H$ such that $E \simeq O_{\{Z\}} \otimes_{O_{D_K}} H$. We saw in proposition 3.6 that

$$\mathbb{R}\mathfrak{sp}_\ast i_{Z!}E \simeq (H^1_{Z} \otimes_{O_{D_K}} H)[-1].$$

(3.21)

On the other hand, we know from proposition 3.3 that

$$\mathbb{R}\mathfrak{sp}_\ast i_{U!}E_U = E$$

(3.22)

where $E$ is a coherent locally free $O_{\hat{X}}(\{Z\})\mathbb{Q}$-module. From the above exact sequence, we obtain the following exact triangle

$$(H^1_{Z} \otimes_{K_Z} H)[-1] \to \mathbb{R}\mathfrak{sp}_\ast E \to E \to$$

(3.23)

and we are done. □

In practice, a constructible module $E$ is given by some set $\mathcal{R}(M) \to \mathcal{R}(\mathcal{F})$ where $M$ is a finite $A_k^\text{an}$-module ($\text{Spec } A$ being an algebraic lifting of the open complement $U$ of $Z$ in $X_k$) and $\mathcal{F}$ is a coherent module on $\{Z\}$. Now, let $U'$ be any affine open subset of $X$ with formal lifting $\mathcal{U}' = \text{Spf } A'$ and $f$ an equation for $Z$ in $\mathcal{U}'$. Let $Z' := U' \cap Z$ and $\mathcal{F}' := i_{Z'!}E$. There are canonical maps $A^\dagger \to A^\dagger[f] \otimes_{A_k^\dagger} M$ as well as $A^\dagger[f] \otimes_{A_k^\dagger} M \to \mathcal{R}(\mathcal{F})$ when $x \in U'$. We can compose the map

$$A^\dagger[f] \otimes_{A_k^\dagger} M \to \mathcal{R}(\mathcal{F})$$

(3.24)

obtained by extension of scalars from $M \to \mathcal{R}(M) \to \mathcal{R}(\mathcal{F}) \to \mathcal{R}(\mathcal{F}')$ with the projection $\mathcal{R}(\mathcal{F}') \to \delta(\mathcal{F}')$. When $E$ is constructible-free (the other cases are not interesting for us), we get

$$\Gamma(U', \mathbb{R}\mathfrak{sp}_\ast E) = \left[A^\dagger[f] \otimes_{A_k^\dagger} M \to \delta(\mathcal{F}')\right].$$

(3.25)
4 Connections on constructible modules

We start with the very general following definition.

Definition 4.1 If \( Y \subset X_k \) is any locally closed subset and \( i_Y : [Y[ \to X^n_K \) denotes the inclusion map, a connection on an \( i_Y^{-1}\mathcal{O}^{an}_X \)-module \( E \) is a \( K \)-linear map
\[
\nabla : E \to E \otimes i_{Y^{-1}}\mathcal{O}^{an}_X i_Y^{-1}\Omega^{1}_X
\]
(4.1)
satisfying the Leibnitz rule. A \( \nabla \)-module on \( \lfloor Y \rfloor \) is an \( i_Y^{-1}\mathcal{O}_X^{an} \)-module \( E \) endowed with a connection. A horizontal map between \( \nabla \)-modules on \( Y \) is a \( i_Y^{-1}\mathcal{O}_X^{an} \)-linear map that commutes with the connection.

Of course, if \( Y' \subset Y \), any \( \nabla \)-module \( E \) on \( \lfloor Y \rfloor \) restricts to a \( \nabla \)-module on \( \lfloor Y' \rfloor \). Note also that if \( Z \subset X_k \) is a closed subset, meaning either a finite subset or \( X_k \) itself, then \( \lfloor Z \rfloor \) is open in \( X^n_K \) and we fall back onto the usual notion of connection in analytic geometry. Finally, note - and this is important - that there is no finiteness condition at this point.

Proposition 4.2 A constructible \( \nabla \)-module \( E \) on \( X^n_K \) is constructible-free.

Proof: Since a coherent module with a connection on a disc is necessarily free when the valuation is discrete, we can apply corollary [1,8].

It is very convenient to be able to use the description of connections in terms of stratifications. We recall the definition of the first infinitesimal neighborhood \( \mathcal{P} \) of the diagonal. If \( X \) is defined by some ideal \( \mathcal{I} \) into \( X \times X \), then \( \mathcal{P} \) is the closed subscheme of \( X \times X \) defined by \( \mathcal{I}^2 \). Recall that \( X \) and \( \mathcal{P} \) have the same underlying space and that there is a short exact sequence
\[
0 \to \Omega^1_X \to \mathcal{O}_P \to \mathcal{O}_X \to 0.
\]
(4.2)
Actually, we will only need the analogous construction on \( X^n_K \) which can also be deduced from this one by functoriality. We will consider the maps \( q_1, q_2 = P^n_K \to X^n_K \) and \( \Delta : X^n_K \to P^n_K \) induced by the projections and the diagonal embedding. We insist on the fact that we identify the underlying spaces of \( X^n_K \) and \( P^n_K \). In particular, if \( Y \) is a subset of \( X_k \), we may consider \( \lfloor Y \rfloor \) as a subset of \( P^n_K \) and we will still denote by \( i_Y \) the embedding into \( P^n_K \). This should not lead to any confusion as long as we only consider \( i_Y \) as a continuous map.

Definition 4.3 If \( Y \subset X_k \) is a locally closed subset, a 1-stratification on an \( i_Y^{-1}\mathcal{O}^{an}_X \)-module \( E \) is an isomorphism
\[
\epsilon : i_Y^{-1}q_2^*i_Y* E \simeq i_Y^{-1}q_1^*i_Y* E
\]
(4.3)
such that \( (i_Y^{-1}\Delta^*i_Y*)(\epsilon) \) is the identity of \( E \). A morphism of such is an \( i_Y^{-1}\mathcal{O}^{an}_X \)-linear map that commutes with the 1-stratifications.

Alternatively, we can write this isomorphism as
\[
i_Y^{-1}\mathcal{O}_{P^n_K} \otimes i_Y^{-1}\mathcal{O}^{an}_X \otimes i_Y^{-1}\mathcal{O}_{P^n_K} E \simeq E \otimes i_Y^{-1}\mathcal{O}^{an}_X \otimes i_Y^{-1}\mathcal{O}_{P^n_K}.
\]
(4.4)
The induced map $E \to E \otimes_{i_Y^{-1}\mathcal{O}_{X_K^\text{an}}} i_Y^{-1}\mathcal{O}_{P_K^\text{an}}$ coincides with the inclusion mod $i_Y^{-1}\mathcal{I}_K^\text{an}$ and the difference is a map
\[ \nabla : E \to E \otimes_{i_Y^{-1}\mathcal{O}_{X_K^\text{an}}} i_Y^{-1}\Omega_{X_K^\text{an}}^1. \] (4.5)

**Proposition 4.4** This construction establishes an equivalence between the category of $i_Y^{-1}\mathcal{O}_{X_K^\text{an}}$-modules endowed with a 1-stratification and the category of $\nabla$-modules on $Y$.

**Proof:** Standard (see proposition 2.9 of [11] for example). \(\square\)

Note that the category of $\nabla$-modules on $Y$ is also equivalent to the category of left $i_Y^{-1}\mathcal{D}_{X_K^\text{an}}$-modules if $\mathcal{D}_{X_K^\text{an}}$ denotes the sheaf of (algebraic) differential operators on $X_K^\text{an}$.

Let $U \subset X_k$ be an affine open subset and $\text{Spec } A \subset X$ an algebraic lifting. We will now recall the description of a connection on an $A_K^1$-module in term of stratification as we just did above for $i_Y^{-1}\mathcal{O}_{X_K^\text{an}}$-modules. We set $P_A := (A \otimes A)/I^2$, where $I$ is the ideal of multiplication $A \otimes A \to A$, and consider its weak completion $P_A^\dagger$ in the sense of Monsky and Washnitzer. Then a 1-stratification on an $A_K^1$-module $M$ a $P_A^\dagger$-linear isomorphism
\[ P_A^\dagger \otimes A_K^1 M \simeq M \otimes A_K^1 P_A^\dagger. \] (4.6)

Again, this is equivalent to a connection on $M$.

**Proposition 4.5** Let $U \subset X_k$ be an affine open subset of $X_k$ with algebraic lifting $\text{Spec } A$. Then, the functor $\Gamma([U], -)$ induces an equivalence between coherent (necessarily locally free) $\nabla$-modules on $U$ and finite (necessarily projective) $\nabla$-$A_K^1$-modules.

**Proof:** We showed in proposition [13] that $\Gamma([U], -)$ induces an equivalence between coherent modules $E$ on $U$ and finite $A_K^1$-modules $M$. It follows that there is an equivalence between 1-stratifications on $E$ and 1-stratifications on $M$ because, obviously, with the above notations, we have
\[ \Gamma([U], i_U^{-1}\mathcal{O}_{P_K^\text{an}}) = P_A^\dagger. \] (4.7)

**Proposition 4.6** If $Y$ is a locally closed subset of $X_k$ and $E$ is a $\nabla$-module on $[Y]$, then $i_Y^*E$ has a natural connection. If $Z$ is a closed subset of $X_k$ and $E$ is a $\nabla$-module on $[Z]$, then $i_Z^*E$ has a natural connection. These functors are fully faithful.

**Proof:** Recall that we identify the underlying spaces of $X_K^\text{an}$ and $P_K^\text{an}$. We have for $j = 1, 2$,
\[ q_j^*i_Y^*E \simeq i_Y^*q_j^*i_Y^*E. \] (4.8)

Therefore, the 1-stratification extends canonically. The partial inverse is as usual induced by $i_Y^{-1}$. The proof follows the same lines for $i_Z$ since we will have
\[ q_j^*i_Z^*E \simeq i_Z^*q_j^*i_Z^*E. \] (4.9)

This last isomorphism can be checked on stalks and we see that they are both equal to $(q_j^*i_Z^*E)_a$ if $a \in [Z]$ and zero otherwise. \(\square\)
Proposition 4.7 A ∇-module $E$ on $X_K^{an}$ is constructible if and only if there exists an exact sequence
\[ 0 \to i_Z!E_Z \to E \to i_{U*}E_U \to 0 \] (4.10)
where $U$ is a non-empty affine open subset of $X_k$ with closed complement $Z$, $E_U$ is a coherent (necessarily locally free) ∇-module on $|U|$ and $E_Z$ is a coherent (necessarily locally free) ∇-module on $|Z|$. We can even assume that $E_U$ is free.

**Proof:** Using 1-stratifications, it follows from proposition 1.12. □

Recall that we defined above, when $T \subset X_k$ is a non empty finite closed subset and $\mathcal{F}$ is a coherent $\mathcal{O}_{|T|}$-module, the Robba module of $\mathcal{F}$ as
\[ \mathcal{R}(\mathcal{F}) := (i_{T*}\mathcal{F})_{|\xi} \]
where $i_T :|T| \to X_K^{an}$ is the inclusion map and $|\xi|$ is the “generic point” of $X_K^{an}$. In particular, we will consider the Robba ring $\mathcal{R}_T := \mathcal{R}(\mathcal{O}_T)$. If for each $x \in T$, we choose an unramified lifting $a$ of $x$, then $\mathcal{R}_T$ is the direct sum of the usual Robba rings over $K(a)$. We will denote by $\Omega^1_{K,x}$ the module of finite differentials over $\mathcal{R}_T$.

**Lemma 4.8** If $T$ is a non empty finite closed subset of $X_k$, then $\Omega^1_{\mathcal{R}_T}$ is canonically isomorphic to $\mathcal{R}(\Omega^1_{|T|})$. Moreover, any connection on a coherent $\mathcal{O}_{|T|}$-module $\mathcal{F}$ induces an connection $\mathcal{R}(\mathcal{F})$.

**Proof:** We may assume that $T$ is reduced to one closed point $x$. Then, we know that $|x|$ is a disc with some parameter $t$, that $\mathcal{R}_x$ is a usual Robba ring and we have
\[ \mathcal{R}(\Omega^1_{|x|}) = \mathcal{R}(\mathcal{O}_x|dt) = \mathcal{R}_x|dt = \Omega^1_{R_x} \]
(4.12)

Also, we have
\[ \mathcal{R}(i^{-1}_{x*}\mathcal{O}_K^{an}) \simeq \mathcal{R}_x[\tau]/\tau^2 \] (4.13)
where $\tau = p_2^{-1}(t) - p_1^{-1}(t)$. Therefore a 1-stratification on an $\mathcal{O}_{|x|}$-module $\mathcal{F}$ will induce an isomorphism
\[ \mathcal{R}_x[\tau]/\tau^2 \otimes_{\mathcal{R}_x} \mathcal{R}(\mathcal{F}) \simeq \mathcal{R}(\mathcal{F}) \otimes_{\mathcal{R}_x} \mathcal{R}_x[\tau]/\tau^2. \] (4.14)
This is a 1-stratification on $\mathcal{R}(\mathcal{F})$ that corresponds to a connection. □

Recall that if $T \subset X_k$ is a non empty finite closed subset and $U \subset X_k$ is a non empty affine open subset with algebraic lifting $\text{Spec} \; A$, there is a canonical morphism $A^1_K \to \mathcal{R}_T$ and that the Robba fiber of an $A^1_K$-module $M$ at $T$ is
\[ \mathcal{R}_T(M) := \mathcal{R}_T \otimes_{A^1_K} M. \] (4.15)

Note that if $M$ is endowed with a connection, then $\mathcal{R}_T(M)$ inherits automatically a connection.

**Lemma 4.9** Let $U$ be a non empty affine open subset of $X_k$ with algebraic lifting $\text{Spec} \; A$ and $T$ be a finite closed subset of $U$. Let $M$ be a finite $\nabla$-$A^1_K$-module and $M_T$ the restriction to $|T|$ of the corresponding $i_U^1\mathcal{O}_{X_K^{an}}$-module. Then, there is a horizontal isomorphism
\[ \mathcal{R}_T(M) \simeq \mathcal{R}(M_T). \] (4.16)
**Proof:** Deduced from lemma 2.4 using 1-stratifications. □

**Definition 4.10** Let $U$ be a non empty affine open subset of $X_k$ with closed complement $Z$ and Spec $A$ an algebraic lifting of $U$.

A $\nabla$-kit is a triple made of a finite $\nabla$-$A^1_K$-module $M$, a coherent $\nabla$-module $\mathcal{F}$ on $|Z|$ and a horizontal map $R_Z(M) \to R(\mathcal{F})$.

A morphism of $\nabla$-kits is a morphism of kits given by horizontal maps.

Two $\nabla$-kits $R_Z(M) \to R(\mathcal{F})$ on $U$ and $R_Z(M') \to R(\mathcal{F})$ on $U'$ are equivalent if their restrictions to $U \cap U'$ are isomorphic as $\nabla$-kits.

**Proposition 4.11** The category of constructible $\nabla$-modules on $X^\text{an}_K$ is equivalent to the category of $\nabla$-kits modulo equivalence.

**Proof:** Using proposition 2.7 it is essentially sufficient to notice that if $E$ is a constructible module, then $q^*_2 E$ is also constructible. Actually, if $E$ is given by some kit $R_Z(M) \to R(\mathcal{F})$, then $q^*_2 E$ will be given by

$$R_Z(P_{\text{Ak}} \otimes A^1_K M) \to R(i^{-1}_Z q^*_2 \mathcal{F}).$$

(4.17)

And the analogous results holds for $q^*_1 E$. Therefore, a 1-stratification on $E$ is equivalent to a 1-stratification on $M$ and a compatible 1-stratification on $\mathcal{F}$. □

We denote by $\mathcal{D}_X$ the sheaf of (algebraic) differential operators on $\hat{X}$. We call a $\mathcal{D}_X \otimes \mathcal{O}_X$-module *perverse* if the underlying $\mathcal{O}_X \otimes \mathcal{O}_X$-module is perverse (see definition 3.7).

**Proposition 4.12** $R\flat \mathcal{O}_X$ induces a functor from the category of constructible $\nabla$-modules on $X^\text{an}_K$ to the category of perverse $\mathcal{D}_X \otimes \mathcal{O}_X$-modules.

**Proof:** It is essentially sufficient to recall the construction of $R\flat \mathcal{O}_X E$ from the corresponding kit $R(M) \to R(\mathcal{F})$. More precisely, let $U'$ be an affine open subset of $X$ with formal lifting $\mathcal{U}' = \text{Spf } \mathcal{A}'$ and $f$ an equation for $Z$ in $\mathcal{U}'$. Let $Z' := U' \cap Z$ and $\mathcal{F}' := i^{-1}_Z \mathcal{F}$. Then, we have

$$\Gamma(U', R\flat \mathcal{O}_X E) = \left[ \mathcal{A}'[1/f][1]_Q \otimes A^1_K M \to \delta(\mathcal{F}) \right].$$

(4.18)

where the non-trivial map in this complex is obtained by scalar extension from the composite $M \to R(M) \to R(\mathcal{F})$, restriction of the image to the direct factor $R(\mathcal{F}')$, and then, projection onto $\delta(\mathcal{F}')$. All those maps are horizontal. □

We also want to state another result that we will need later on. If $Z$ is a finite closed subset of $\hat{X}$, we set

$$\mathcal{D}_X(1Z) = \mathcal{O}_X(1Z) \otimes \mathcal{O}_X \mathcal{D}_X.$$

(4.19)

**Proposition 4.13** If $U$ is a non empty affine open subset of $X_k$ with closed complement $Z$, then $\mathcal{O}_X(1Z)$ establishes an equivalence between the category of coherent $\nabla$-modules on $|U|$ and $\mathcal{D}_X(1Z)$ modules that are coherent on $\mathcal{O}_X(1Z)\mathcal{Q}$.

**Proof:** Use 1-stratifications again and proposition 3.3 □
5 Overconvergent connections

Now, we embed $X$ diagonally into $X \times X$. If $Y \subset X_k$ is any subset, we will denote by $][Y][]$ the tube of $Y$ into $X_k^\an \times X_k^\an$. We call
\[ p_1, p_2 : ][Y][] \to X_k^\an \] (5.1)
the maps induced by the projections. We will still denote by $i_Y : ][Y][] \to X_k[[]$ the embedding, hoping that this will not create any confusion. Note that
\[ ][Y][] = p_j^{-1}(][Y][] \subset ][X_k][[] \quad \text{for} \quad j = 1, 2. \] (5.2)

Sometimes, we may also denote by $p_1, p_2 : ][Y][] \to ][Y][]$ the maps induced by the projections (at least when $Y$ is a closed subset).

Locally, the geometry of $][X_k][]$ is not too bad (this is the strong fibration theorem of Berthelot) as we can see right now. Since $X$ is smooth of relative dimension 1, there exists locally an étale map $t$ to the affine line $A_1$. We will then say that $t$ is a local parameter on $X$. Assume that $t$ is defined on some open subset $V \subset X$ with reduction $U \subset X_k$.

Then $\tau := p_2^*(t) - p_1^*(t)$ defines an étale map $V \times V \to V \times A_1$. More precisely, there is a commutative diagram
\[ \begin{array}{ccc}
V \times V & \xrightarrow{p_1} & V \times A_1 \\
\Delta & \searrow & 0 \\
& V & \\
\end{array} \] (5.3)

As in proposition 1.1, it follows from lemma 4.4 of [5] that this map induces an isomorphism $][U][] \simeq ][U][] \times D(0, 1^-)$. Moreover, the morphism
\[ V_1^\an \times V_1^\an \to V_1^\an \times A_1 \] (5.4)
is étale (formally étale and boundaryless). Thus, Proposition 4.3.4 of [2] implies that it induces an isomorphism between a neighborhood $V'$ of $][U][]$ in $][X_k][]$ and a neighborhood $V''$ of $][U][] \times D(0, 1^-)$ into $X_k^\an \times D(0, 1^-)$. If we denote by $Z$ the closed complement of $U$ and let $V_\lambda = X_k^\an \setminus Z[\lambda$ as usual, we may assume that $V'' = \cup_{\lambda, \eta}(V_\lambda \times D(0, \eta^+))$ with $\lambda, \eta \to 1$. We can summarize the situation in the following commutative diagram
\[ \begin{array}{ccc}
V_k^\an \times V_k^\an & \xrightarrow{\simeq} & V_k^\an \times A_k^\an \\
\tilde{V}' & \xrightarrow{\simeq} & \cup_{\lambda, \eta}(V_\lambda \times D(0, \eta^+)) \\
][\tilde{U}[] & \xrightarrow{\simeq} & ][U[] \times D(0, 1^-). \\
\end{array} \] (5.5)

Of course, if $T \subset X_k$ is a finite closed subset, we may always assume that $T \subset U$, and we obtain $][T][[] \simeq ][T[] \times D(0, 1^-)$. If $x$ is a closed point with unramified lifting $a$, we even get
\[ ][x[] \simeq D_K(a)(0, 1^-) \times D(0, 1^-) \simeq B^2_K(a)(0, 1^-) \] (5.6)
which can also be proved directly as in proposition 1.1. Finally, we can describe the inverse image by specialization $][\xi][]$ of the generic point $\xi$ of $X_k$: we have
\[ ][\xi][] \simeq D[\xi](0, 1^-) \] (5.7)
where the disc is over the completion \( \mathcal{H}([\xi]) \) of the field \( \mathcal{O}_{X_K^\text{an}}[\xi] \).

Following Berthelot, we will define now the overconvergence condition. Note before that the natural inclusion of the first infinitesimal neighborhood \( P \hookrightarrow X \times X \), induces by functoriality a morphism \( P_K^\text{an} \hookrightarrow [\![X_k[\![ \) and the projections \( q_1,q_2 : P_K^\text{an} \to X_K^\text{an} \) introduced in the previous section are induced by \( p_1 \) and \( p_2 \). Recall also that we denote by \( i_Y \) both embeddings \( Y \hookrightarrow X_K^\text{an} \) and \( Y \hookrightarrow X_k \) when \( Y \) is a locally closed subset of \( X \) (which one is meant should be clear from the context).

**Definition 5.1** Let \( Y \) be a locally closed subset of \( X_k \). A connection on an \( i_Y^{-1}\mathcal{O}_{X_K^\text{an}} \)-module \( E \) is overconvergent if the corresponding 1-stratification is induced by a Taylor isomorphism

\[
\epsilon : i_Y^{-1}p_i^*i_Y^*E \simeq i_Y^{-1}p_i^*i_Y^*E \tag{5.8}
\]

on \( \mathcal{O}_{X_k} \) which satisfies the cocycle condition (see below) on triple products. If \( Y \) is a closed subset, the connection is just said to be convergent. We will also say that the \( \nabla \)-module \( E \) is (over-) convergent.

Again, this definition applies in very general geometric situations. It is also important to insist on the fact that there are no finiteness conditions at that point. Note that the convergence condition is a lot simpler to describe: if \( Z \) is a closed subset in \( X_k \) (either finite or \( X_k \) itself), and if we still denote by \( p_1,p_2 \) \( ]Z[[\to Z] \), the maps induced by the projections, we simply have \( i_2^{-1}p_j^*i_Z^*E = p_j^*E \) for \( i = 1,2 \) and the Taylor isomorphism reads

\[
\epsilon : p_j^*E \simeq p_j^*E \tag{5.9}
\]

on \( \mathcal{O}_{X_k} \).

It is quite easy to make the *cocycle condition* precise. We embed diagonally \( X \) into the triple product \( X \times X \times X \). We denote by \( ]Y[[ \) the tube of \( Y \subset X_k \) inside \( X_K^\text{an} \times X_K^\text{an} \times X_K^\text{an} \) and by \( p_{12},p_{23},p_{13} \) \( ]Y[[\to X_k \) \( \to X_k \) \( \to X_k \) the maps induced by the projections. Then the cocycle condition reads

\[
(i_Y^{-1}p_{12}^*i_Y^*)\circ (i_Y^{-1}p_{23}^*i_Y^*) = (i_Y^{-1}p_{13}^*i_Y^*) \tag{5.10}
\]

**Lemma 5.2** Overconvergence is preserved under restriction to a locally closed subset \( Y' \subset Y \).

**Proof:** We denote by the same letter \( i \) both inclusion maps \( ]Y'[[\to ]Y[ \) and \( ]Y'[[\to ]Y[ \). Then we have for \( j = 1,2 \),

\[
i_Y^{-1}p_j^*i_Y^*i^{-1}E = i_Y^{-1}(\mathcal{O}_{X_k} \otimes p_j^{-1}\mathcal{O}_{X_K^\text{an}} p_j^{-1}i_Y^*i^{-1}E) \tag{5.11}
\]

\[
= i_Y^{-1}\mathcal{O}_{X_k} \otimes i_Y^{-1}p_j^{-1}\mathcal{O}_{X_K^\text{an}} i_Y^{-1}p_j^{-1}i_Y^*i^{-1}E = i_Y^{-1}\mathcal{O}_{X_k} \otimes i_Y^{-1}p_j^{-1}\mathcal{O}_{X_K^\text{an}} i^{-1}p_j^{-1}i_Y^*E \tag{5.12}
\]

\[
= i^{-1}(i_Y^{-1}\mathcal{O}_{X_k} \otimes i_Y^{-1}p_j^{-1}\mathcal{O}_{X_K^\text{an}} p_j^{-1}i_Y^*E) = i^{-1}(i_Y^{-1}p_j^*i_Y^*E). \tag{5.13}
\]

We will use this explicit isomorphism below for the restriction from \( X_k \) to some \( Y \) in which case it says that the adjunction map is an isomorphism

\[
i_Y^{-1}p_j^*E \simeq i_Y^{-1}p_j^*i_Y^*i^{-1}E. \tag{5.14}
\]
Proposition 5.3 If $U$ is an open subset of $X_k$ and $E$ is an overconvergent $\nabla$-module on $|U|$, then $i_{U*}E$ is convergent. If $Z$ is a closed subset of $X_k$ and $E$ is a convergent $\nabla$-module on $|Z|$, then $i_{Z!}E$ is convergent.

Proof: We want to show that for $j = 1, 2$, we have

\[ p_j^* i_{U*}E \simeq i_{U*}i_{U}^{-1} p_j^* i_{U*}E. \]  
(5.15)

It is actually sufficient to show that

\[ p_j^{-1} i_{U*}E \simeq i_{U*}i_{U}^{-1} p_j^{-1} i_{U*}E. \]  
(5.16)

And this follows from the fact that there is a cartesian diagram

\[
\begin{array}{ccc}

|U| & \xrightarrow{p} & |X_k| \\
\downarrow & & \downarrow \\
|U| & \xrightarrow{p^2} & X_K^an
\end{array}
\]

where the horizontal arrows are closed immersions. Therefore, the 1-stratification extends canonically.

The same type of argument shows that

\[ p_j^* i_{Z!}E \simeq i_{Z!}i_{Z}^{-1} p_j^* i_{Z!}E \]  
(5.18)

in the second case. □

The next proposition will be the first illustration of the power of the overconvergence condition. Recall that a $\nabla$-module on an analytic variety is said to be (locally) trivial if it is (locally) generated by a finite set of horizontal sections. Note that a locally trivial $\nabla$-module on a disc is always trivial. For further use, we also state a lemma.

Lemma 5.4 If $D \subset X$ is a smooth divisor with reduction $Z$, then the inclusion $D_K \hookrightarrow |Z|$ and its retraction induce an equivalence between coherent $O_{D_K}$-modules and locally trivial $\nabla$-modules on $|Z|$.

Proof: We may assume that $Z$ is reduced to one point $x$ with unramified lifting $a$, in which case we are simply considering the inclusion of the origin $\{0\} \hookrightarrow D_K(a)(0,1^-)$ into the open disc. This is a section of the structural map from the disk to the point which induces by definition an equivalence between finite dimensional vector spaces and trivial $\nabla$-modules. □

Proposition 5.5 If $Z \subset X_k$ is a finite closed subset, then any coherent convergent $\nabla$-module on $|Z|$ is locally trivial.

Proof: We may assume that $Z$ is reduced to a rational point $x$ and lift it to a rational point $a \in X_K$. Think of $a$ as the “center” of $|x|$. We set $V := a^* E$ which is a finite dimensional vector space. Next, we consider the composition $X \to \text{Spec } V \to X$ of the projection $p$ and the section $a$. Its graph $X \to X \times X$ induces a morphism $\gamma : |x| \to |x| \times |x|$ along which we can pull back the Taylor isomorphism. We obtain an isomorphism

\[ \phi : O_{|x|} \otimes_K V = p^* a^* E = \gamma^* p_2^* E \simeq \gamma^* p_1^* E = E. \]  
(5.19)
Using the cocycle condition, one sees that the 1-stratification of $E$ is compatible with the trivial one on the left hand side. More precisely, we use the map $X \times X \to X \times X \times X$ obtained by tensoring with the identity with the above graph. We may then pull the cocycle condition back along the induced map $]x[\to ]x[\to ]x[\to and get $\epsilon \circ p_2^* (\phi) = p_1^* (\phi)$. □

Actually, the convergence condition on $]x[$ is also called the Robba condition and it is a classical result that a finite $\nabla$-module that satisfies the Robba condition on an open disc is automatically trivial.

**Corollary 5.6** If $E$ is a constructible convergent $\nabla$-module on $X_K^{an}$ and $x \in X_k$, then the restriction of $E$ to $]x[$ is trivial. □

We now turn to the description of overconvergence on open subsets. Before doing anything else and although we will continue to prove most statements without referring to rigid cohomology, we should mention the following comparison result (recall that we denote with a subscript 0 the rigid object associated to a Berkovich one):

**Proposition 5.7** If $U$ is an open subset of $X_k$, the functor $E \mapsto (i_U)_* E$ induces an equivalence between the category of coherent overconvergent $\nabla$-modules on $U$ and overconvergent isocrystals on $U$.

**Proof:** Using proposition 7.2.13 (and definition 7.2.10) of [22], this is simply a translation of the above overconvergence condition into the language of rigid geometry as in proposition 1.4. □

Assume that $t$ is a local parameter defined on some affine open subscheme $\text{Spec } A \subset X$ with reduction $U$. Write as usual $V_\lambda = X_K^{an} \setminus Z_\lambda = \mathcal{M}(A_\lambda)$ where $Z$ is the closed complement of $U$. If $M$ is a finite $\nabla A_\lambda^\$-module, it extends as usual to some finite $\nabla A_\lambda$ module $M_\lambda$ with $\lambda < 1$ and we may define the $\lambda$-radius of convergence of $M$ as

$$R(M, \lambda) = \inf_{s \in M_\lambda} \inf \left\{ \lambda, \lim_{k \to 0} \left\| \frac{\partial^k}{\partial t^k} (s) \right\|_\lambda^{\frac{1}{k}} \right\}$$

(5.20)

where $\| - \|$ is a Banach norm on $M_\lambda$. We say that $M$ is overconvergent if

$$\lim_{\lambda \to 1} R(M, \lambda) = 1.$$  

(5.21)

Those who are interested in differential equations should notice that this condition is equivalent to requiring all the Robba fibers $\mathcal{R}_x(M)$ to be solvable whenever $x \in Z$ (use the “isometry” $A_K^\dag \to \mathcal{R}_Z$).

In general, it is always possible to find a finite open covering $\text{Spec } A = \cup \text{Spec } A_i$ and for each $i$, a local parameter $t_i$ defined on $\text{Spec } A_i$. Then a finite $\nabla A_k^\$-module $M$ is said to be over overconvergent if there exists such a covering where each $M_i := A_i^{\dag K} \otimes A_K^\$ $M$ is overconvergent with respect to $t_i$.

If one is willing to use rigid analytic geometry, the next proposition is a particular case of proposition 7.2.15 of [22].

**Proposition 5.8** Let $U \subset X_k$ be an affine open subset of $X_k$. Then, the functor $\Gamma(]U[, -)$ induces an equivalence between coherent overconvergent $\nabla$-modules on $U$ and finite overconvergent $\nabla A_k^\$-modules.
Proof: Due to the local nature of the question, we may assume that there exists a local parameter $t$ defined on an algebraic lifting $\text{Spec } A$. Let $E$ be a $\nabla$-module on $|U|$ and $M$ the corresponding $\nabla_A^1$-module. One can check that the Taylor isomorphism is necessarily induced by the Taylor series

$$ s \mapsto \sum_{k \geq 0} \frac{1}{k!} \frac{\partial^k}{\partial t^k} (s)^{\tau_k} $$

(5.22)

(see the remark after proposition 5.10 below). The overconvergence condition on $|U|$ means that this series converges on some neighborhood of $|U|$ inside $|X_k|$. And we may assume that this neighborhood has the form

$$ \bigcup_{\lambda, \eta} (V_\lambda \times D(0, \eta^+)) \quad \text{with} \quad \lambda, \eta \to 1 \quad (5.23) $$

We will need below general stratifications using all infinitesimal neighborhoods of $X$. We denote by $P^{(n)}$ the $n$-th infinitesimal neighborhood of $X$ in $X \times X$ defined by $I^{n+1}$ if $X$ is defined by $I$ into $X \times X$, and by $p_1^{(n)}, p_2^{(n)} = P_K^{(n), an} \to X_K^m$ the maps induced by the projections. With our former notations, we have $P = P^{(1)}$ and $q_j = p_j^{(1)}$.

Definition 5.9 If $Y \subset X_k$ is a locally closed subset, a stratification on an $i_Y^{-1}O_{X_K^n}$-module $E$ is a compatible family of isomorphisms

$$ \epsilon_n : i_Y^{-1}p_2^{(n)*} i_Y E \simeq i_Y^{-1}p_1^{(n)*} i_Y E \quad (5.24) $$

such that the cocycle condition holds on triple products and $\epsilon_0$ is the identity. A morphism of such is an $i_Y^{-1}O_{X_K^n}$-linear map that commutes with the stratifications.

Proposition 5.10 The category of stratified $i_Y^{-1}O_{X_K^n}$-modules is equivalent to the category of $\nabla$-modules on $Y$.

Proof: Standard (see proposition 2.11 of [11] for example) since the base has dimension one, and thus, any connection is integrable. □

It should also be mentioned that the stratification is automatically induced by the Taylor isomorphism when $E$ is overconvergent.

Proposition 5.11 Let $E$ be a $\nabla$-module on $X_K^{an}$, $U$ an affine open subset of $X$ and $Z$ its closed complement. Assume that $i_U^{-1}E$ is a coherent overconvergent $\nabla$-module and that $i_Z^{-1}E$ is a locally trivial $\nabla$-module. Then, $E$ is convergent.

Proof: The point is to show that the stratification is induced by a Taylor isomorphism $\epsilon : p_2^*E \simeq p_1^*E$ on $|]X_k[|$. We already have isomorphisms

$$ \epsilon_U : i_U^{-1}p_2^*E \simeq i_U^{-1}p_1^*E \quad \text{and} \quad \epsilon_Z : i_Z^{-1}p_2^*E \simeq i_Z^{-1}p_1^*E $$

(5.25)

coming from the overconvergence of $E$ on $U$ and $Z$ (see the remark following lemma 5.2). And we can use again the fact that a sheaf on a topological space is uniquely determined
by its restriction to an open subset, its restriction to the closed complement and the
adjunction map. It is therefore sufficient to show that the diagram

\[
\begin{array}{c}
i_U^{-1}p_2^*E \\ i_U^{-1}i_Z^{-1}p_2^*E \end{array} \quad \xrightarrow{\epsilon_U} \quad \begin{array}{c}
i_U^{-1}p_1^*E \\ i_U^{-1}i_Z^{-1}p_1^*E \end{array}
\]

(5.26)
is commutative. Since the analogous diagram with stratifications is commutative by hy-
pothesis (we have a stratification on $E$ thanks to proposition 5.10), it is sufficient to prove
that the canonical map

\[
\text{Hom}_{i_U^{-1}O \mid X_k} (i_U^{-1}p_2^*E, i_U^{-1}i_Z^{-1}p_1^*E) \to \lim \text{Hom}_{i_U^{-1}O \mid X_k} (i_U^{-1}p_2^{(n)*}E, i_U^{-1}i_Z^{-1}p_1^{(n)*}E)
\]

(5.27)
is injective. Since $i_U^{-1}E$ is finitely presented and $i_Z^{-1}E$ is locally trivial, it is sufficient to
prove that the canonical maps

\[
\Gamma(\Gamma([U][i_U^{-1}i_x^{-1}p_1^*O\mid X_k]]) \to \lim \Gamma(\Gamma([U][i_U^{-1}i_x^{-1}p_1^{(n)*}O\mid X_k])
\]

(5.28)
are injective whenever $x \in Z$. By considering the stalks and since $\Gamma([x]$ is closed in $\Gamma[X_k]$, one
sees that

\[
i_U^{-1}i_x^{-1}p_1^*O\mid X_k] = i_U^{-1}i_x^{-1}O\mid x].
\]

(5.29)
Also, if we fix a local parameter $t$ around $x$, we can easily identify the right hand side of
(5.28) with $\mathcal{R}_x\{t\}$. We are led to check that the canonical map

\[
\Gamma(\Gamma([x], i^{-1}_x t^{-1}i_x^{-1}O\mid x]) \to \mathcal{R}_x\{t\}
\]

(5.30)
is injective. Since $\Gamma([x]$ is closed in $\Gamma[X_k]$ is paracompact, it follows from proposition 2.5.1 iii)
of [20] that

\[
\Gamma(\Gamma([x], i^{-1}_x t^{-1}i_x^{-1}O\mid x]) = \lim \Gamma(V')\mid x]\{t\} \mathcal{O}_{V'}
\]

(5.31)
where $V'$ runs through the open neighborhoods of $\Gamma([x$ inside $\Gamma[X_k]$. It is therefore sufficient
to show that if $V'$ is such a neighborhood, the map

\[
\Gamma(V')\mid x]\{t\} \mathcal{O}_{V'} \to \mathcal{R}_x\{t\}
\]

(5.32)
is injective. The local parameter $t$ is defined on $\text{Spec } A$. As usual, if $T$ denotes the closed com-
plement of $U$, we let $V_\lambda := X_k^\mu \setminus T|_\lambda$. After removing some points in $U$ if necessary, we may assume that

\[
V' = \bigcup_{\lambda, \eta} (V_\lambda \times D(0, \eta^+))
\]

(5.33)
with $\lambda, \eta \to 1$. We are then reduced to showing that the map

\[
\Gamma([x]\backslash x|_{\lambda} \times D(0, \eta^+), \mathcal{O}) \to \mathcal{R}_x\{t\}
\]

(5.34)
is injective. An it is sufficient to consider the obvious injective map

\[
\Gamma([x]\backslash x|_{\lambda} \times D(0, \eta^+), \mathcal{O}) \hookrightarrow \Gamma([x]\backslash x|_{\lambda}, \mathcal{O})\{t\}
\]

(5.35)
and take inverse limit when $\mu \to 1$. \qed
Corollary 5.12 Let $E$ be a constructible $\nabla$-module on $X_K^{an}$. The connection on $E$ is convergent if and only if there exists a finite covering of $X_k$ by locally closed subsets $Y$ such that the connection is overconvergent on each $Y$.

Proof: Follows from proposition 5.11.

Corollary 5.13 A $\nabla$-module $E$ on $X_K^{an}$ is constructible convergent if and only if there exists an exact sequence

$$0 \to i_Z! E_Z \to E \to i_U^* E_U \to 0$$

(5.36)

where $U$ is a non-empty affine open subset of $X_k$ with closed complement $Z$, $E_U$ is a coherent overconvergent $\nabla$-module on $|U|$ and $E_Z$ is a locally trivial $\nabla$-module on $|Z|$. We can even assume that $E_U$ is free.

Proof: Follows from propositions 4.7 and 5.11.

Recall that the notion of $\nabla$-kit was introduced in definition 4.10.

Definition 5.14 A $\nabla$-kit $R(M) \to R(F)$ is said to be convergent if $M$ is overconvergent and $F$ is locally trivial.

In other words, a convergent $\nabla$-kit is given by a an overconvergent $\nabla$-$A_K^{\dagger}$-module $M$ where $\text{Spec} \ A \subset X$ is non empty affine open subset with reduction $U$, a collection of finite dimensional $K(a)$-vector spaces $H_a$ for each point $x$ not in $U$, where $a$ is an unramified lifting of $x$, and horizontal $A_K^{\dagger}$-linear maps $M \to R_a \otimes_{K(a)} H_a$.

Theorem 5.15 The category of constructible convergent $\nabla$-modules on $X_K^{an}$ is equivalent to the category of convergent $\nabla$-kits modulo equivalence.

Proof: Follows from propositions 4.11 and 5.11.

6 Constructibility and $\mathcal{D}$-modules

We introduce now the ring of arithmetic differential operators on $\hat{X}$. It can be described as follows. First of all, if $D_{\hat{X}}$ denotes the sheaf of algebraic differential operators on $\hat{X}$ as before, and $\hat{D}_{\hat{X}}$ is its $p$-adic completion, we have

$$D_{\hat{X}} \subset D_{\hat{X}}^{\dagger} \subset \hat{D}_{\hat{X}}.$$  

(6.1)

Moreover, as shown in proposition 2.4.4 of [6], if $t$ is a local parameter on $X$ defined on some subset $U = \text{Spf} \ A \subset \hat{X}$, then

$$\Gamma(U, D_{\hat{X}}^{\dagger}) = \left\{ \sum_{k=0}^{\infty} \frac{1}{k!} f_k \frac{\partial^k}{\partial t^k} \mid \exists c > 0, \eta < 1, \ \| f_k \| \leq c \eta^k \right\}.$$  

(6.2)
We are interested in $\mathcal{D}^b_{\mathcal{X}^Q}$-modules. The case of coherent $\mathcal{D}^b_{\mathcal{X}^Q}$-modules with finite support is easy to describe: if $D \subset X$ is a smooth divisor with reduction $Z$, then the direct image functor

$$
\begin{array}{ccc}
\mathcal{D}^b_{\text{coh}}(\mathcal{O}_{\mathcal{X}^Q}) & \xrightarrow{i_Z^*} & \mathcal{D}^b_{\text{coh}}(\mathcal{D}_X^!]Q) \\
H & \xrightarrow{i_Z^* i_Z^* \text{Hom}_{\mathcal{O}_X}(\Omega^1_{\mathcal{X}^Q}, \mathcal{D}_X^!]Q \otimes_{\mathcal{O}_{\mathcal{X}^Q}} H)} & H \\
\end{array}
$$

induces an equivalence between coherent $\mathcal{O}_{\mathcal{X}^Q}$-modules and coherent $\mathcal{D}^b_{\mathcal{X}^Q}$-modules with support in $Z$ (see section 5.3.3 of [8] for the general statement). Its inverse is induced by the exceptional inverse image $i_Z^!$.

For $Z$, a finite closed subset of $X$, we introduced $\mathcal{H}^Z_\mathbb{X}$ in (3.11). In fact, there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{X}} \to \mathcal{O}_{\mathbb{X}}(\mathbb{X}) \to \mathcal{H}^Z_{\mathbb{X}} \to 0$$

of $\mathcal{D}^b_{\mathbb{X}}$-modules. Moreover, $\mathcal{O}_{\mathbb{X}}$, $\mathcal{O}_{\mathbb{X}}(\mathbb{X})$ and $\mathcal{H}^Z_{\mathbb{X}}$ are all $\mathcal{D}^b_{\mathbb{X}}$-coherent.

The next result is also well-known (formula 4.4.5.2 of [8]) but rather easy to check (and instructive) in our situation.

**Proposition 6.1** If $D \subset X$ is a smooth divisor with reduction $Z$, and $H$ is a coherent $\mathcal{O}_{\mathcal{D}^!]Q}$-module, there is a canonical isomorphism of $\mathcal{D}^b_{\mathcal{X}^Q}$-modules

$$i_Z^* H \simeq \mathcal{H}^Z_{\mathbb{X}} \otimes_{\mathcal{O}_{\mathcal{D}^!]Q}} H.$$ (6.5)

**Proof:** We may assume that $Z = \{x\}$ where $x$ is the only zero of a local parameter $t$ defined on some formal affine open subset $\mathcal{U} = \text{Spf} \mathbb{A}$, and that $H = K(a)$ with $a$ an unramified lifting of $x$. We have a commutative diagram with exact rows (for multiplication on the right)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\mathcal{U}, \mathcal{D}^b_{\mathbb{X}}) & \xrightarrow{\partial} & \Gamma(\mathcal{U}, \mathcal{D}^b_{\mathbb{X}}) & \longrightarrow & A & \longrightarrow & 0 \\
0 & \longrightarrow & \Gamma(\mathcal{U}, \mathcal{D}^b_{\mathbb{X}}) & \xrightarrow{\partial t} & \Gamma(\mathcal{U}, \mathcal{D}^b_{\mathbb{X}}) & \longrightarrow & A[1/t]^\dagger & \longrightarrow & 0 \\
\end{array}
$$

from which we deduce an isomorphism

$$\Gamma(\mathcal{U}, \mathcal{H}^Z_\mathbb{X}) / (\Gamma(\mathcal{U}, \mathcal{D}^b_{\mathbb{X}}) t) \simeq A[1/t]^\dagger / A.$$ (6.7)

The next proposition is a very fancy way of stating an almost trivial result. However, it may be seen as an analog of theorem 6.5 of [19] in a very simple situation. This may also be seen as a special case of theorem 2.5.10 of [13].
Proposition 6.2 If $Z$ is a finite closed subset of $X_k$, then $\mathcal{R}\tilde{\mathcal{P}}_* i_{Z!}$ induces an equivalence between the category of coherent convergent $\nabla$-modules on $|Z|$ and the category complexes of $\mathcal{D}_X^{\dagger}$-modules with support in $Z$ and coherent cohomology concentrated in degree 1.

Proof: If $D \subset X$ is a smooth divisor with reduction $Z$, and $H$ is a coherent $\mathcal{O}_{\hat{D}_Q}$-module, it follows from proposition 6.1 and proposition 3.6 that

$$\mathcal{R}\tilde{\mathcal{P}}_* i_{Z!} (\mathcal{O}_{|Z|} \otimes \mathcal{O}_{\hat{D}_Q} H) = i_{Z+} H[1].$$

Moreover, we know that $\mathcal{O}_{|Z|} \otimes \mathcal{O}_{\hat{D}_Q}$ is an equivalence of categories between coherent $\mathcal{O}_{\hat{D}_K}$-modules and locally trivial $\nabla$-modules on $|Z|$, and that $i_{Z+}$ is an equivalence of categories between coherent $\mathcal{O}_{\hat{D}_K}$-modules and coherent $\mathcal{D}_X^{\dagger}$-modules with support in $Z$. Since specialization $\hat{D}_K \to Z$ is bijective, we can identify $\mathcal{O}_{\hat{D}_Q}$ with $\mathcal{O}_{\hat{D}_K}$ and consequently $\mathcal{O}_{\hat{D}_Q}$-modules with $\mathcal{O}_{\hat{D}_K}$-modules. Finally, we know from proposition 6.5 that coherent convergent $\nabla$-modules on $|Z|$ are locally trivial (and conversely). □

We now turn to the case of an affine open subset of $X_k$. It is necessary to introduce the ring of arithmetic differential operators $\mathcal{D}_X^{\dagger} (\hat{1} Z)$ with overconvergent poles along a non empty finite closed subset $Z$. Since we are only interested in this ring modulo torsion, it is more convenient to use the modified definition given in 2.6.2 of [24].

We introduced in (6.10) the ring $\mathcal{D}_X^{\dagger} (\hat{1} Z)$ of algebraic differential operators with overconvergent poles. If $j : X_k \setminus Z \to X_k$ denotes the inclusion map, we will have

$$\text{both } \mathcal{D}_X^{\dagger} \text{ and } \mathcal{D}_X^{\dagger} (\hat{1} Z) \subset \mathcal{D}_X^{\dagger} (\hat{1} Z) \subset j_* j^{-1} \mathcal{D}_X^{\dagger}. \quad (6.10)$$

If there exists a local parameter $t$ defined on some open subset $U = \text{Spf } \mathcal{A}$ such that $Z \cap U = \{x\}$ where $x$ is the only zero of $t$, then

$$\Gamma(U, \mathcal{D}_X^{\dagger} (\hat{1} Z)) = \{ \sum_{k,l=0}^{\infty} \frac{1}{k!} t^k \frac{\partial^k}{\partial t^k}, \exists c > 0, \eta < 1, \|f_{k,l}\| \leq c \eta^{k+l} \}. \quad (6.11)$$

Note that $\mathcal{D}_X^{\dagger} (\hat{1} Z)_{\mathbb{Q}}$-modules that are $\mathcal{O}_X^{\dagger} (\hat{1} Z)_{\mathbb{Q}}$-coherent form a full subcategory of the category of $\mathcal{D}_X^{\dagger} (\hat{1} Z)_{\mathbb{Q}}$ modules. In other words, the forgetful functor is fully faithful. More generally, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are coherent $\mathcal{O}_X^{\dagger} (\hat{1} Z)_{\mathbb{Q}}$-modules, we have

$$\mathcal{R}\text{Hom}_{\mathcal{D}_X^{\dagger} (\hat{1} Z)_{\mathbb{Q}}} (\mathcal{E}_1, \mathcal{E}_2) = \mathcal{R}\text{Hom}_{\nabla} (\mathcal{E}_1, \mathcal{E}_2) = \mathcal{R}\text{Hom}_{\mathcal{D}_X^{\dagger} (\hat{1} Z)_{\mathbb{Q}}} (\mathcal{E}_1, \mathcal{E}_2) \quad (6.12)$$

where the middle term denotes the de Rham complex of $\mathcal{H}\text{om}_{\mathcal{O}_X^{\dagger} (\hat{1} Z)_{\mathbb{Q}}} (\mathcal{E}_1, \mathcal{E}_2)$ (use corollary 3.2.3 of [10]).

We will have to consider scalar extension and we will write when $\mathcal{E}$ is a coherent $\mathcal{D}_X^{\dagger} Q$-module,

$$\mathcal{E}^{\dagger} (\hat{1} Z) := \mathcal{D}_X^{\dagger} (\hat{1} Z)_{\mathbb{Q}} \otimes_{\mathcal{D}_X^{\dagger} Q} \mathcal{E}. \quad (6.13)$$

Then there exists an exact triangle

$$i_{Z+} i_{\hat{1} Z}^{\dagger} \mathcal{E} \to \mathcal{E} \to \mathcal{E}^{\dagger} (\hat{1} Z) \to. \quad (6.14)$$

We want to explain now, as in proposition 4.4.3 of [3], that if $U = X \setminus Z$, and $E$ is a coherent overconvergent $\nabla$-module on $\mathcal{U}$, then the action of $\mathcal{D}_X^{\dagger} (\hat{1} Z)$ on $\mathcal{R}\tilde{\mathcal{P}}_* i_{U*} E$ extends to an action of $\mathcal{D}_X^{\dagger} (\hat{1} Z)$.  

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Let $t$ be a local parameter on $X$ defined in the neighborhood of some $x \in X_k$. Shrinking $U$ if necessary (and adding the corresponding finite set of points to $Z$), we may assume that $t$ is defined on some lifting $\text{Spec} \, A$ of $U$ and that $x \in Z$. Also, there exists an open neighborhood $U' = \text{Spf} \, A'$ of $x$ in $\hat{X}$ such that $Z \cap U' = \{x\}$ and $x$ is the only zero of $t$ on $U'$.

The coherent $\nabla$-module $E$ is given by a finite $\nabla$-$A_K^{\dagger}$-module $M$ and

$$\Gamma(U', \hat{\mathcal{P}}_* i_{U*} E) = A'[1/t]_Q \otimes_{A_K^{\dagger}} M =: M'. \quad (6.15)$$

Also, we can write $M = \lim_{\lambda < 1} M_\lambda$ where $M_\lambda$ is a finite $\nabla$-$A_\lambda$-module so that

$$M' = \lim_{\lambda < 1} M_\lambda' \quad \text{with} \quad M_\lambda' := A_\lambda' \{\lambda/t\} \otimes_{A_\lambda} M. \quad (6.16)$$

If $P \in \Gamma(U', D^a_X(\hat{\mathcal{Z}}))$, then there exists $c > 0$ and $\eta < 1$ such that

$$P = \sum_{k,l=0}^\infty \frac{1}{k! t^l} \frac{\partial^k}{\partial t^k} \quad \text{with} \quad \|f_{k,l}\| \leq c \eta^{k+l}. \quad (6.17)$$

If $M$ is overconvergent and $\lambda$ close enough to 1, we have $\|\frac{1}{k!} \frac{\partial^k(s)}{\partial t^k}\| \eta^k \to 0$ for $s \in M_\lambda$. Of course, we will also have $\lambda > \eta$ for $\lambda$ close enough to one. Then, we see that if $s'$ denotes the restriction of $s$ to $U_K \cap V_\lambda$, we have

$$\left\| \frac{1}{k!} \frac{\partial^k(s')}{\partial t^k} \right\| \leq c \eta^{k+l} \left\| \frac{1}{k!} \frac{\partial^k(s)}{\partial t^k} \right\| \leq c \eta^k \to 0 \quad (6.18)$$

and we can set

$$P(s') = \sum_{k,l=0}^\infty \frac{1}{k! t^l} \frac{\partial^k(s')}{\partial t^k}. \quad (6.19)$$

In other words, we can extend the action by continuity on $s'$. Since the question is local on $\hat{X}$ (and also, that it is sufficient to define the action on a finite set of generators), we see that the action of $D^a_X(\hat{\mathcal{Z}})$ on $\hat{\mathcal{P}}_* i_{U*} E$ does extend to an action of $D^a_X(\hat{\mathcal{Z}})$.

Of course, by continuity, any horizontal map will give a $D^a_X(\hat{\mathcal{Z}})$-linear map and we have proven the following:

**Proposition 6.3** If $U$ is a non empty affine open subset of $X_k$ and $Z$ denotes its closed complement, then the functor $\hat{\mathcal{P}}_* i_{U*}$ induces a fully faithful functor form the category of coherent overconvergent $\nabla$-modules $E$ on $|U|$ to the category of $D^a_X(\hat{\mathcal{Z}})\mathcal{Q}$-modules. $\square$

Actually, it is a lot better:

**Theorem 6.4 (Berthelot)** The essential image of this functor is the category of coherent $D^a_X(\hat{\mathcal{Z}})\mathcal{Q}$-modules that are $\mathcal{O}_X(\hat{\mathcal{Z}})\mathcal{Q}$-coherent.

**Proof:** This is exactly the content of a letter from Berthelot to Caro ([9]). Actually, the condition is even weaker since we can simply assume that the restriction to the formal lifting $U$ of $U$ is $\mathcal{O}_U$-coherent. $\square$

We want to extend these results to constructible convergent $\nabla$-modules on $X_{an}^p$. For the moment, we can prove the following:
Proposition 6.5 \( \mathcal{R}\tilde{p}_a \) induces a functor from the category of constructible convergent \( \nabla \)-modules on \( X_{\alpha}^p \) to the category of perverse \( \mathcal{D}^!_{X,\mathbb{Q}} \)-modules.

Proof: We know from proposition 4.12 that \( \mathcal{R}\tilde{p}_a E \) is a perverse \( \mathcal{D}^!_{X,\mathbb{Q}} \)-module. Moreover, we gave in proposition 3.8 a description of the terms of this complex. Since \( H^1_Z \) is a \( \mathcal{D}^!_{X,\mathbb{Q}} \)-module, it follows from proposition 6.4 that both terms of \( \mathcal{R}\tilde{p}_a E \) are \( \mathcal{D}^!_{X,\mathbb{Q}} \)-modules. It only remains to check that the non-trivial map in this complex is \( \mathcal{D}^!_{X,\mathbb{Q}} \)-linear.

We showed in theorem 5.15 that a constructible convergent \( \nabla \)-module \( E \) on \( X_{\alpha}^an \) is given by an overconvergent \( \nabla \)-modules on \( A_{K}^! \)-module \( M \) where \( \text{Spec } A \subset X \) is non empty affine open subset with reduction \( U \), a collection of finite dimensional \( K(a) \)-vector spaces \( H_a \) for each point \( x \notin U \), where \( a \) is an unramified lifting of \( x \), and horizontal \( A_{K}^! \)-linear maps

\[
M \to R_a \otimes_{K(a)} H_a. \tag{6.20}
\]

We explained at the end of section 4 how to construct \( \mathcal{R}\tilde{p}_a E \) from these data. If \( t \) is a local parameter defined on some open subset \( U' = \text{Spf } A' \) with \( Z \cap U' = \{x\} \) where \( x \) is the only zero of \( t \), and \( t \) is actually defined on a lifting \( \text{Spec } A \) of \( U \), we have

\[
\Gamma(U', \mathcal{R}\tilde{p}_a E) = \left[ A'[1/t]_{\mathbb{Q}} \otimes_{A_{K}^!} M \to \delta_a \otimes_{K(a)} H_a \right]. \tag{6.21}
\]

Thus, we must show that the morphism

\[
A'[1/t]_{\mathbb{Q}} \otimes_{A_{K}^!} M \to \delta_a \otimes_{K(a)} H_a \tag{6.22}
\]

is \( \Gamma(U', \mathcal{D}^!_{X}) \)-linear. This follows from the fact that it is horizontal (because the original map (6.20) is so) and continuous (recall that the action is defined on both sides by continuity).

\( \Box \)

7 Formal fibers

If \( x \subset X_k \) is a closed point, we can consider the completion \( \hat{x} \) of \( X \) along \( x \) (usually written \( \hat{X} \)). Note that if \( a \) is an unramified point over \( x \), then \( \hat{x} \cong \text{Spf } V(a)[[t]] \). As explained in [14] and [15], although \( \hat{x} \) is not a \( p \)-adic formal scheme, there exists a beautiful theory of arithmetic \( \mathcal{D} \)-modules on \( \hat{x} \) which is completely analogous to the theory for \( X \). The ring \( \mathcal{O}_{\hat{x}} \) is isomorphic to the set of bounded function on the open unit disc over \( K(a) \). The ring \( \mathcal{R}^{\text{bd}}_{x} := \mathcal{O}_{\hat{x}}(t_x)_{\mathbb{Q}} \) is isomorphic to the bounded Robba ring \( \mathcal{R}_{a}^{\text{bd}} \) over \( K(a) \). And we have

\[
\delta_x = \mathcal{R}_x / \mathcal{O}_{\hat{x}} \simeq \mathcal{R}^{\text{bd}}_{x} / \mathcal{O}_{\hat{x}}. \tag{7.1}
\]

The rings \( \mathcal{D}^!_{\hat{x}} \) and \( \mathcal{D}^!_{x}(t_x) \) have exactly the same description as in (6.2) and (6.10) respectively (using the Gauss norm).

There is a canonical morphism of formal schemes \( i_{\hat{x}} : \hat{x} \to \hat{X} \) which is formally étale. One can define for a coherent \( \mathcal{D}^!_{X,\mathbb{Q}} \)-module (or perfect complex) \( \mathcal{E} \), its exceptional inverse image

\[
i_{\hat{x}}^! \mathcal{E} := \mathcal{D}^!_{\hat{x}} \otimes_{\mathcal{D}^!_{X,\mathbb{Q}}} \mathcal{E} \tag{7.2}
\]

which is a coherent \( \mathcal{D}^!_{\hat{x}} \)-module, and we have the following adjunction formula:
Lemma 7.1 If $\mathcal{E}$ is a coherent $\mathcal{D}^1_{\hat{X}^!}$-module (or perfect complex) and $x \in X_k$ is a closed point, then
\[
\mathrm{RHom}_{\mathcal{D}^1_{\hat{X}^!}}(\mathcal{E}, \mathcal{H}^!_{x}) \simeq i_x \ast \mathrm{RHom}_{\mathcal{D}^1_{\hat{X}^!}}(i^!_{\hat{X}^!} \mathcal{E}, \delta_x).
\] (7.3)

Proof: We may clearly assume that $\mathcal{E} = \mathcal{D}^1_{\hat{X}^!}$ and we fall back on lemma 3.4. \qed

Definition 7.2 Let $\mathcal{E}$ be a coherent $\mathcal{D}^1_{\hat{X}^!}$-module (or perfect complex) and $x \in X_k$, a closed point. Then, the bounded Robba fiber of $\mathcal{E}$ at $x$ is
\[
\mathcal{R}^\text{bd}_x(\mathcal{E}) := i^!_{\hat{X}^!} \mathcal{E}^!(\hat{X}^!).
\] (7.4)

If $Z \subset X_k$ is a finite closed subset and $x \in Z$, we can also define for a coherent $\mathcal{D}^1_{\hat{X}^!}(\hat{Z})^!_Q$-module $\mathcal{E}$, its exceptional inverse image
\[
i^!_{\hat{X}^!} \mathcal{E} := \mathcal{D}^1_{\hat{X}^!}(\hat{X}^!) \otimes_{\mathcal{D}^1_{\hat{X}^!}(\hat{Z})^!_Q} \mathcal{E}.
\] (7.5)

Note that if $\mathcal{E}$ is a coherent $\mathcal{D}^1_{\hat{X}^!}$-module, we have
\[
\mathcal{R}^\text{bd}_x(\mathcal{E}) \simeq \mathcal{R}^\text{bd}_x(\mathcal{E}(\hat{Z}))
\] whenever $Z$ is a finite closed subset of $X_k$ (and in particular, $\mathcal{R}^\text{bd}_x(\mathcal{E}) = 0$ when $\mathcal{E}$ has finite support). Finally, if $\mathcal{E}$ is a $\mathcal{D}^1_{\hat{X}^!}(\hat{Z})^!_Q$-module which is coherent both as $\mathcal{D}^1_{\hat{X}^!}$-module and as $\mathcal{O}_{\hat{X}^!}(\hat{Z})^!_Q$-module, we have
\[
\mathcal{R}^\text{bd}_x(\mathcal{E}) \simeq \mathcal{R}^\text{bd}_x \otimes_{\mathcal{O}_{\hat{X}^!}(\hat{Z})^!_Q} \mathcal{E}
\] (7.6)

because the canonical map $\hat{x} \rightarrow \hat{X}$ is formally étale. In particular, this is a finite $\nabla$-module on $\mathcal{R}^\text{bd}_x$.

Definition 7.3 Let $\mathcal{E}$ be a coherent $\mathcal{D}^1_{\hat{X}^!}$-module (or perfect complex). Then, $\mathcal{E}$ has Frobenius type at a closed point $x$ (resp. has Frobenius type) if $\mathcal{R}^\text{bd}_x(\mathcal{E})$ has a Frobenius structure (resp. a Frobenius structure at all closed points $x \in X_k$).

Proposition 7.4 Let $Z \subset X_k$ be a finite closed subset and $x \in Z$. Let $\mathcal{E}$ be a $\mathcal{D}^1_{\hat{X}^!}(\hat{Z})^!_Q$-module which is coherent both as $\mathcal{D}^1_{\hat{X}^!}$-module and as $\mathcal{O}_{\hat{X}^!}(\hat{Z})^!_Q$-module. Assume that $\mathcal{E}$ has Frobenius type at $x$. Then, we have
\[
\mathrm{RHom}_{\mathcal{D}^1_{\hat{X}^!}}(\mathcal{E}, \mathcal{H}^!_{x}) \simeq i_x \ast \mathrm{RHom}_{\mathcal{D}^1_{\hat{X}^!}}(\mathcal{R}^\text{bd}_x(\mathcal{E}), \delta_x).
\] (7.9)

Proof: Note first that, in our situation, we have $i^!_{\hat{X}^!} \mathcal{E} = \mathcal{R}^\text{bd}_x(\mathcal{E})$. Thus, it follows from lemma 7.1 that
\[
\mathrm{RHom}_{\mathcal{D}^1_{\hat{X}^!}}(\mathcal{E}, \mathcal{H}^!_{x}) \simeq i_x \ast \mathrm{RHom}_{\mathcal{D}^1_{\hat{X}^!}}(\mathcal{R}^\text{bd}_x(\mathcal{E}), \delta_x).
\] (7.10)
Since $R_{bd}(E)$ is a finite $F$-$\nabla$-module on $R_{bd}$, if follows from theorem 3.3 of [14] and proposition 5.6 of [15] that

$$\text{RHom}_{D^!_\mathbb{Q}}(R_{bd}(E), O^\text{an}_x) = 0.$$  \hfill (7.11)

But there is an exact sequence

$$0 \to O^\text{an}_x \to R_x \to \delta_x \to 0,$$  \hfill (7.12)

and we obtain

$$\text{RHom}_{D^!_\mathbb{Q}}(R_{bd}(E), \delta_x) = \text{RHom}_{D^!_\mathbb{Q}}(R_{bd}(E), R_x)$$  \hfill (7.13)

$$= \text{RHom}_{D^!_\mathbb{Q}}(R_{bd}(E), R_x) = \text{RHom}_\nabla(R_{bd}(E), R_x)$$  \hfill (7.14)

$$= \text{RHom}_\nabla(R_x(E), R_x)$$  \hfill (7.15)

where the equality (7.14) is analogous to (6.12). \hfill \Box

We do now the same kind of construction on the constructible side. Recall that if $x \in X_k$ is a closed point, we introduced in definition 2.1, the Robba ring at $x$ as

$$R_x := (i_x \ast O \mid x \mid) \mid x \mid.$$  \hfill (7.16)

Since $O_{X^K,\mid x \mid}$ is a field, the adjunction map $O_{X^K,\mid x \mid} \to R_x$ takes values into the maximal subfield of $R$ which is exactly $R_{bd}$. We globalize now the definition:

**Definition 7.5** Let $x \in X_k$ be a closed point. If $E$ be a constructible $O_{X^K}$-module, the bounded Robba fiber of $E$ at $x$ is

$$R_{bd}(E) := R_{bd} \otimes O_{X^K,\mid x \mid} E \mid x \mid.$$  \hfill (7.17)

A constructible $\nabla$-module $E$ has Frobenius type at $x$ if $R_{bd}(E)$ has a Frobenius structure.

Note that this definition is generic in the sense that if $U \subset X_k$ is any open subset, then $R_{bd}(i_U \ast i_U^{-1} E) = R_{bd}(E)$. We globalize now the definition:

**Definition 7.6** A constructible convergent $\nabla$-module $E$ has Frobenius type if

1. $E$ has Frobenius type at all closed points $x \in X_k$.
2. $R\mathfrak{sp}_s E$ is a perfect complex of $D^!_{X \mathbb{Q}}$-module.

It is very likely that the second condition is not necessary and it is also possible that the convergence condition is automatic.

For computations, it is convenient to have also at our disposal a more algebraic approach. If Spec $A \subset X$ is an algebraic lifting of the complement $U$ of $Z$, taking global sections of the map $O_X(\mid Z \mid) \to i_x \ast O \mid x \mid \mid x \mid$ will give a morphism $A^!_K \to R_{bd}$. This is the composite map

$$A^!_K \to O_{X^K,\mid x \mid} \to R_{bd}.$$  \hfill (7.18)
Definition 7.7 Let $x \in X_k$ be a closed point. Let $U$ be a non empty affine open subset of $X_k$ with algebraic lifting Spec $A$ and $M$ a finite $A_K^{\dagger}$-module. Then, the bounded Robba fiber of $M$ at $x$ is

$$R_{x}^{\text{bd}}(M) := R_{x} \otimes_{A^{\dagger}_K} M.$$  \hfill (7.19)

A finite $\nabla$-$A_K^{\dagger}$-module $M$ has Frobenius type at a closed point $x$ (resp. has Frobenius type) if the bounded Robba fiber at $x$ (resp. at all closed points $x \in X_k$) has a Frobenius structure.

Recall that we also introduced in definition 2.3 the Robba fiber $R_x(M)$ of $M$ at $x$ and that, clearly, $R_x(M) = R_x \otimes_{R_x^{\text{bd}}} R_x^{\text{bd}}(M)$.

Lemma 7.8 Let $x \in X_k$ be a closed point.

1. Let $Z \subset X_k$ be a finite closed subset and $\mathcal{E}$ be a $D_X(\mathcal{O}_X Z)_{\mathbb{Q}}$-module which is coherent both as $D_X(\mathcal{O}_X Z)_{\mathbb{Q}}$-module and as $\mathcal{O}_X(\mathcal{O}_X Z)_{\mathbb{Q}}$-module. If $M := \Gamma(\mathcal{X}, \mathcal{E})$, we have

$$R_{x}^{\text{bd}}(\mathcal{E}) = R_{x}^{\text{bd}}(M).$$ \hfill (7.20)

2. Let $U$ be an open subset of $X_k$ and $E$ a coherent $i_U^{-1}\mathcal{O}_{X_{\text{an}}}^{\mathbb{A}}$-module. If $M := \Gamma([U], E)$, we have

$$R_{x}^{\text{bd}}(E) = R_{x}^{\text{bd}}(M).$$ \hfill (7.21)

Proof: Since all functors are right exact, the assertions follow from the cases $\mathcal{E} = \mathcal{O}_X(\mathcal{O}_X Z)_{\mathbb{Q}}$ and $E = i_U^{-1}\mathcal{O}_{X_{\text{an}}}^{\mathbb{A}}$. \hfill $\square$

Proposition 7.9 Let $E$ be a constructible convergent $\nabla$-module and $\mathcal{E} := \mathcal{R}\mathcal{S}\mathcal{P}_{\ast}^\dagger E$. Assume that $\mathcal{E}$ is a perfect complex of $D_X^{\dagger}_{\mathbb{Q}}$-module. Then, if $x \in X_k$ is any closed point, we have

$$R_{x}^{\text{bd}}(\mathcal{E}) = R_{x}^{\text{bd}}(E).$$ \hfill (7.22)

In particular $E$ has Frobenius type (at $x$) if and only if $\mathcal{R}\mathcal{S}\mathcal{P}_{\ast}^\dagger E$ has Frobenius type (at $x$).

Proof: If $U$ is an open subset of $X_k$ with closed complement $Z$, we know that

$$R_{x}^{\text{bd}}(\mathcal{E}) = R_{x}^{\text{bd}}(\mathcal{E}(\mathcal{Z})) \quad \text{and} \quad R_{x}^{\text{bd}}(E) = R_{x}^{\text{bd}}(i_U i_U^{-1} E).$$ \hfill (7.23)

We may therefore assume that $E = i_U^\ast E_U$ with $E_U$ coherent as $i_U^{-1}\mathcal{O}_{X_{\text{an}}}^{\mathbb{A}}$-module. And then use lemma 7.8. \hfill $\square$

8 Constructible modules versus $\mathcal{D}$-modules

On the analytic side, we will stick to the “connection” vocabulary and write $R\mathcal{H}om_{\nabla}(E', E'')$ when $E'$ and $E''$ are two $\nabla$-modules on $X_{\text{an}}^{\mathbb{A}}$ for example, but we will systematically identify this space with $R\mathcal{H}om_{\mathcal{D}_{X_{\text{an}}}^{\mathbb{A}}}(E', E'')$. 

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**Proposition 8.1** If $E$ is a $\nabla$-module on $X_K^{an}$, then

$$\text{R} \tilde{\text{p}}_* \text{R} \text{Hom}_\nabla(\mathcal{O}_{X_K^{an}}, E) \simeq \text{R} \text{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \text{R} \tilde{\text{p}}_* E). \quad (8.1)$$

In fact, if $E$ is a constructible convergent $\nabla$-module, we have

$$\text{R} \tilde{\text{p}}_* \text{R} \text{Hom}_\nabla(\mathcal{O}_{X_K^{an}}, E) \simeq \text{R} \text{Hom}_{\mathcal{D}_X^t}(\mathcal{O}_X, \text{R} \tilde{\text{p}}_* E). \quad (8.2)$$

**Proof:** The analytic Spencer complex

$$[\text{Hom}_{\mathcal{O}_X}(\Omega^1_{X_K^{an}}, \mathcal{D}_{X_K^{an}}) \to \mathcal{D}_{X_K^{an}}], \quad (8.3)$$

which is locally given by $u \mapsto u(dt)$, is a locally free resolution of $\mathcal{O}_{X_K^{an}}$. It follows that

$$\text{R} \text{Hom}_{\mathcal{D}_{X_K^{an}}}(\mathcal{O}_{X_K^{an}}, E) = E \otimes \mathcal{O}_{X_K^{an}} \Omega^\bullet_{X_K^{an}}. \quad (8.4)$$

But it is also true that the algebraic Spencer complex

$$[\text{Hom}_{\mathcal{O}_X}(\Omega^1_{X^{an}}, \mathcal{D}_X) \to \mathcal{D}_X], \quad (8.5)$$

is a resolution of $\mathcal{O}_X$ so that

$$\text{R} \text{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \text{R} \tilde{\text{p}}_* E) = [\text{R} \tilde{\text{p}}_* E \otimes \mathcal{O}_X \Omega^\bullet_X]. \quad (8.6)$$

The first result follows since

$$\text{R} \tilde{\text{p}}_* (E \otimes \mathcal{O}_{X_K^{an}} \Omega^\bullet_{X_K^{an}}) = [\text{R} \tilde{\text{p}}_* E \otimes \mathcal{O}_X \Omega^\bullet_X]. \quad (8.7)$$

Now, if $E$ is a constructible convergent $\nabla$-module, we can consider the arithmetic Spencer complex

$$[\text{Hom}_{\mathcal{O}_{X^{an}}}(\Omega^1_{X^{an}}, \mathcal{D}_X) \to \mathcal{D}_X], \quad (8.8)$$

which is also a resolution of $\mathcal{O}_X$ (proposition 4.3.3 of [7]) and get

$$\text{R} \text{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \text{R} \tilde{\text{p}}_* E) = [\text{R} \tilde{\text{p}}_* E \otimes \mathcal{O}_X \Omega^\bullet_X]. \quad (8.9)$$

**Corollary 8.2** If $U$ is an open subset of $X_K$ and $E'$ and $E''$ are two coherent $\nabla$-modules on $[U]$, then

$$\text{R} \tilde{\text{p}}_* \text{R} \text{Hom}_\nabla(i_{U*}E', i_{U*}E'') \simeq \text{R} \text{Hom}_{\mathcal{D}_X}(\text{R} \tilde{\text{p}}_* i_{U*}E', \text{R} \tilde{\text{p}}_* i_{U*}E''). \quad (8.10)$$

Actually, if $E'$ and $E''$ are overconvergent, we have

$$\text{R} \tilde{\text{p}}_* \text{R} \text{Hom}_\nabla(i_{U*}E', i_{U*}E'') \simeq \text{R} \text{Hom}_{\mathcal{D}_X}(\text{R} \tilde{\text{p}}_* i_{U*}E', \text{R} \tilde{\text{p}}_* i_{U*}E'') \quad (8.11)$$

**Proof:** It follows from proposition 8.3 that if

$$E := \text{Hom}_{\mathcal{O}_{X_K^{an}}}(E', E'') \quad (8.12)$$

then we have

$$\text{R} \tilde{\text{p}}_* i_{U*} E \simeq \text{Hom}_{\mathcal{O}_X}(\text{R} \tilde{\text{p}}_* i_{U*} E', \text{R} \tilde{\text{p}}_* i_{U*} E''). \quad (8.13)$$

One can use the 1-stratifications in order to show that this isomorphism is horizontal. Moreover, since $E'$ is coherent, $E$ will be overconvergent when $E'$ and $E''$ are. And we can then apply the proposition to $i_{U*} E$. \hfill \Box
Lemma 8.3 Let $U$ be an open subset of $X_k$ with closed complement $Z$. If $E'$ is a locally trivial $\nabla$-module on $|Z|$ and $E''$ is a coherent overconvergent $\nabla$-module on $|U|$, we have
\[ \mathcal{R}\text{Hom}_\nabla(i_{Z!}E', i_{U!*}E'') = 0 \quad \text{and} \quad \mathcal{R}\text{Hom}_{\mathcal{D}^b_X}^\dagger (\mathcal{R}s\tilde{p}_* i_{Z!}E', \tilde{s}\tilde{p}_* i_{U!*}E'') = 0. \] (8.14)

Proof: First of all, we have $i_{U!}^{-1} i_{Z!} E' = 0$ and therefore, by adjunction,
\[ \mathcal{R}\text{Hom}_{\mathcal{O}_{X_K}^\nabla}(i_{Z!}E', i_{U!*}E'') = \mathcal{R}\text{Hom}_{i_{U!}^{-1}\mathcal{O}_{X_K}^\nabla}(i_{U!}^{-1} i_{Z!} E', E'') = 0. \] (8.15)

The first assertion follows. For the second one, we can write
\[ \mathcal{R}s\tilde{p}_* i_{Z!} E' = i_{Z+} H[-1] \quad \text{and} \quad \tilde{s}\tilde{p}_* i_{U!*} E'' = \mathcal{E} \] (8.16)
where $H$ is a coherent $\mathcal{O}_{D_K}$-module, $D$ being a smooth lifting of $Z$, and and $\mathcal{E}$ is a $\mathcal{D}^b_X(1|\mathcal{Q})$-module. Note that
\[ (i_{Z+} H)(1|\mathcal{Z}) := \mathcal{D}^b_X(1|\mathcal{Z}) \otimes_{\mathcal{D}^b_X} i_{Z+} H = 0. \] (8.17)
But then, by adjunction, we have
\[ \mathcal{R}\text{Hom}_{\mathcal{D}^b_X(1|\mathcal{Q})}(i_{Z+} H, \mathcal{E}) = \mathcal{R}\text{Hom}_{\mathcal{D}^b_X(1|\mathcal{Z})}(i_{Z+} H)(1|\mathcal{Z}), \mathcal{E}) = 0. \] (8.18)

Recall that we defined for a constructible module $E$, its Robba fiber (resp. bounded Robba fiber) at a closed point $x \in X_k$, as
\[ \mathcal{R}_x(E) = \mathcal{R}_x \otimes_{\mathcal{O}_{X_K}^\nabla, |\xi|} E[\xi] \quad \text{(resp.} \quad \mathcal{R}_x^{\text{bd}}(E) = \mathcal{R}_x^{\text{bd}} \otimes_{\mathcal{O}_{X_K}^\nabla, |\xi|} E[\xi]). \] (8.19)

Lemma 8.4 Let $U$ be an open subset of $X_k$ and $x \notin U$. If $E$ is a coherent $i_{U!}^{-1}\mathcal{O}_{X_K}^\nabla$-module, then
\[ \mathcal{R}\text{Hom}_{\mathcal{O}_{X_K}^\nabla}(i_{U!*} E, i_{x!*} \mathcal{O}_{|x|}) = i_{\xi*} \mathcal{H}\text{om}_{\mathcal{O}_{x}}(\mathcal{R}_x(E), \mathcal{R}_x)[-1]. \] (8.20)
If $E$ is a coherent $\nabla$-module on $|U|$, we have
\[ \mathcal{R}\text{Hom}_\nabla(i_{U!*} E, i_{x!*} \mathcal{O}_{|x|}) = i_{\xi*} \mathcal{H}\text{om}_\nabla(\mathcal{R}_x(E), \mathcal{R}_x)[-1]. \] (8.21)

Proof: By adjunction, we have
\[ \mathcal{R}\text{Hom}_{\mathcal{O}_{X_K}^\nabla}(i_{U!*} E, i_{x!*} \mathcal{O}_{|x|}) = 0. \] (8.22)
Using the exact sequence
\[ 0 \rightarrow i_{x!} \mathcal{O}_{|x|} \rightarrow i_{x!*} \mathcal{O}_{|x|} \rightarrow i_{\xi*} \mathcal{R}_x \rightarrow 0 \] (8.23)
and adjunction, we get
\[ \mathcal{R}\text{Hom}_{\mathcal{O}_{X_K}^\nabla}(i_{U!*} E, i_{x!*} \mathcal{O}_{|x|}) = i_{\xi*} \mathcal{H}\text{om}_{\mathcal{O}_{x}}(\mathcal{R}_x(E), \mathcal{R}_x)[-1] \] (8.24)
and the first assertion follows by extending scalars. Then, we can write
\[ \mathcal{R}\text{Hom}_\nabla(i_{U!*} E, i_{x!*} \mathcal{O}_{|x|}) = \mathcal{R}\text{Hom}_{\mathcal{D}^b_X(1|\mathcal{Q})}(\mathcal{O}_{X_K}^\nabla, \mathcal{R}\text{Hom}_{\mathcal{O}_{X_K}^\nabla}(i_{U!*} E, i_{x!*} \mathcal{O}_{|x|})) \] (8.25)
Recall that a constructible ∇-module $E$ is said to have Frobenius type at a closed point $x \in X_k$ if its bounded Robba Fiber at this point has a Frobenius structure.

**Proposition 8.5** Let $E$ be a coherent overconvergent ∇-module on $|U|$ and $x \not\in U$. Assume $E$ has Frobenius type at $x$ and that $\tilde{s}_{p,x} E$ is $D^1_{\hat{X}Q}$-coherent. Then, we have

$$R\text{Hom}_\nabla(i_U E, i_x \xi) \simeq R\text{Hom}_{D^1_{\hat{X}Q}}(\tilde{s}_{p,x} E, \mathcal{H}_x^1)[-1].$$ \hspace{1cm} (8.29)

**Proof:** Follows from lemma 8.4 and proposition 7.4. □

Recall from definition 7.6, that a constructible convergent ∇-module $E$ has Frobenius type if it has Frobenius type at all closed points and $R\tilde{s}_{p,x} E$ is a perfect complex of $D^1_{\hat{X}Q}$-modules.

**Theorem 8.6** Let $E'$ and $E''$ be two constructible convergent ∇-modules on $X^\text{an}_K$. Assume that $E'$ has Frobenius type. Then, we have

$$R\text{Hom}_\nabla(E', E'') \simeq R\text{Hom}_{D^1_{\hat{X}Q}}(\tilde{s}_{p,x} E', \tilde{s}_{p,x} E'').$$ \hspace{1cm} (8.30)

**Proof:** Since $E'$ and $E''$ are constructible, there exists an open subset $U$ of $X_k$ such that both $i^{-1}_U E'$ and $i^{-1}_U E''$ are coherent. If we use corollary 5.13 for $E''$, it follows from corollary 8.2 and proposition 8.5 that

$$R\text{Hom}_\nabla(i_U E', i^{-1}_U \xi) \simeq R\text{Hom}_{D^1_{\hat{X}Q}}(\tilde{s}_{p,x} i_U E', \mathcal{H}_x^1)[-1].$$ \hspace{1cm} (8.31)

If we apply this result to the case $E' = \mathcal{O}_X$, and use corollary 5.13 for $\mathcal{O}_X$, then proposition 8.1 gives

$$R\text{Hom}_\nabla(i_x \xi \mathcal{O}_X, E'') \simeq R\text{Hom}_{D^1_{\hat{X}Q}}(\mathcal{H}_x^1[-1], \tilde{s}_{p,x} E'').$$ \hspace{1cm} (8.32)

And we can apply again corollary 5.13 but to $E'$ this time. □

**Corollary 8.7** The functor $R\tilde{s}_{p,x}$ induces a fully faithful functor from the category of constructible convergent ∇-modules of Frobenius type to the category of $D^1_{\hat{X}Q}$-modules. □

We want now to understand its essential image. Since we are only interested in curves, we can make the following ad hoc definition:
Definition 8.8 A coherent $\mathcal{D}^!_{\mathcal{X}^{an}_{\mathbb{Q}}}$-module $\mathcal{E}$ is holonomic if there exists a finite closed subset $Z$ of $X_k$ such that $\mathcal{E}(\mathcal{I}Z)$ is $\mathcal{O}_{\mathcal{X}^{an}_{\mathbb{Q}}}$-coherent. A bounded complex of $\mathcal{D}^!_{\mathcal{X}^{an}_{\mathbb{Q}}}$-modules is holonomic if it has holonomic cohomology.

Note that, since $X$ is a curve, an $F-\mathcal{D}^!_{\mathcal{X}^{an}_{\mathbb{Q}}}$-module is holonomic in the usual sense if and only if the underlying module is holonomic in this sense.

Proposition 8.9 If $\mathcal{E}$ is a perverse holonomic $\mathcal{D}^!_{\mathcal{X}^{an}_{\mathbb{Q}}}$-module of Frobenius type, there exists a constructible convergent $\nabla$-module $E$ on $X^{an}_K$ such that $R\hat{\mathcal{S}}\hat{\mathcal{P}}_*E = \mathcal{E}$.

Proof: Let $\mathcal{E}$ be a perverse holonomic $\mathcal{D}^!_{\mathcal{X}^{an}_{\mathbb{Q}}}$-module. By definition, it is a complex with holonomic cohomology concentrated in degree 0 and 1, $\mathcal{O}_{\mathcal{X}^{an}_{\mathbb{Q}}}$-flat in degree 0 and finitely supported in degree 1. Let $Z$ be a finite closed subset of $X_k$ such that $\mathcal{H}^q(\mathcal{E})$ is supported in $Z$ and $\mathcal{H}^q(\mathcal{E})|_{\mathcal{I}Z} = \mathcal{O}_{\mathcal{X}^{an}_{\mathbb{Q}}}(-1)$ in the usual sense if and only if the underlying module is holonomic in this sense.

First of all, we see that $\mathcal{H}^q(\mathcal{E}(\mathcal{I}Z)) = \mathcal{H}^q(\mathcal{E})|_{\mathcal{I}Z} = 0$ when $q \neq 0$. In other words, $\mathcal{E}(\mathcal{I}Z)$ is a coherent $\mathcal{D}^!_{\mathcal{X}^{an}_{\mathbb{Q}}}$-module which is also $\mathcal{O}_{\mathcal{X}^{an}_{\mathbb{Q}}}$-coherent. Thanks to theorem 6.4, we can write $\mathcal{E}(\mathcal{I}Z) = \hat{s}\hat{\mathcal{P}}_*\iota_U E_U$ with $E_U$ a coherent overconvergent $\nabla$-module on $|U|$.

By definition, if $x$ is a point in $Z$ and $a$ a non ramified lifting of $x$, we have

$$i_x^! \mathcal{E} = L\iota_x^! \mathcal{E}[-1] = K(a) \otimes_{\mathcal{O}_{\mathcal{X}^{an}_{\mathbb{Q}},x}} \mathcal{E}_x[-1]$$

If $x$ is defined as the zero of a local parameter $t$ we can use the free left resolution

$$\mathcal{O}_{\mathcal{X}^{an}_{\mathbb{Q}},x} \xrightarrow{t} \mathcal{O}_{\mathcal{X}^{an}_{\mathbb{Q}},x}$$

of $K(a)$ to compute $i_x^! \mathcal{E}$. Then, there is a spectral sequence (because the simple complex associated to

$$\mathcal{E}_x \xrightarrow{t} \mathcal{E}_x$$

is isomorphic to $i_x^! \mathcal{E}$)

$$E_2^{p,q} := \mathcal{H}^p\left(\mathcal{H}^q(\mathcal{E}_x) \xrightarrow{i_x^!} \mathcal{H}^q(\mathcal{E}_x)\right) \Rightarrow i_x^! \mathcal{E}$$

with $H^q(\mathcal{E}_x)$ flat in degree 0, $t$-torsion in degree 1 and 0 otherwise. It follows that $i_x^! \mathcal{E}$ is just a finite dimensional vector space placed in degree 1. Thanks to proposition 6.2, we can therefore write $i_{Z+}i_Z^! \mathcal{E} = R\hat{\mathcal{S}}\hat{\mathcal{P}}_*\iota_{Z+} E_Z$ with $E_Z$ a locally trivial $\nabla$-module on $|Z|$. And the assertion then follows from theorem 8.6.

Corollary 8.10 The functor $R\hat{\mathcal{S}}\hat{\mathcal{P}}_*$ induces an equivalence between the category of constructible convergent $\nabla$-modules of Frobenius type on $X^{an}_K$ and perverse holonomic $\mathcal{D}^!_{\mathcal{X}^{an}_{\mathbb{Q}}}$-modules of Frobenius type.

Proof: Follows from proposition 6.3, corollary 8.7 and proposition 8.9.
9 $F\nabla$-modules

There exists no global lifting of Frobenius on the curve $X$ in general and this is indeed never the case when the genus of $X$ is at least 2. It will therefore be necessary to work locally.

Recall that the field $K$ is endowed with a Frobenius (a field isometry $\sigma$ that lifts the Frobenius of $k$). If $\mathcal{U}$ is a formal $\mathcal{V}$-scheme, any $\sigma$-linear lifting of the Frobenius of the special fiber $\mathcal{U}_k := \mathcal{U} \otimes_Y k$ will be called a Frobenius on $\mathcal{U}$. If $\mathcal{U}$ is formally of finite type, we will consider its generic fiber $\mathcal{U}_K$ in Berthelot’s sense (see section 1 of [3] for example) which is an analytic variety on $K$. Any Frobenius on $\mathcal{U}$ will induce a $\sigma$-linear endomorphism of $\mathcal{U}_K$ that we will still call a Frobenius on $\mathcal{U}_K$. Note that when $\mathcal{U}$ is formally smooth and affine, there always exists such a Frobenius on $\mathcal{U}$ (not unique, though).

This remark may be applied to any affine open subset $U \subset X_k$: if $\mathcal{U}$ is the formal lifting of $U$, there exists a Frobenius on $\mathcal{U}$ that induces a Frobenius on the generic fiber $|U| = \mathcal{U}_K$. More generally, we may consider the completion $\hat{Y}$ of $X$ along some locally closed subset $Y$ of $X_k$. One may then consider the generic fiber of $\hat{Y}$ and we have $|Y| = \hat{Y}_K$. Again, any Frobenius on $\hat{Y}$ will induce a Frobenius on $|Y|$. Of course, beside open subsets $U$ of $X_k$ in which case $\hat{U} = \mathcal{U}$, only finite closed subsets $Z$ of $X_k$ will arise, and in case of a single closed point $x$, we already met $\hat{x}$ in section 7.

Since the tube $|U|$ of a non empty affine open subset of $X_k$ is closed (and not open) in the Berkovich topology, it will be necessary to extend the Frobenius to a neighborhood of this tube. This cannot be done algebraically: when $A$ is a smooth $\mathcal{V}$-algebra, it is not often the case that the Frobenius of $A_k := k \otimes_Y A$ lifts to $A$. However, it always lifts to its completion $\hat{A}$ and even to its weak completion $A^\dagger$ (Artin approximation). In particular, if $\text{Spec } A \subset X$ is a lifting of some open subset $U$ of $X_k$, there exists a $\sigma$-linear lifting $\varphi$ of the Frobenius on $A^\dagger$. It induces a $\sigma$-linear map $\varphi : V_{\lambda} \to V_{\mu}$ between affinoid neighborhoods of $|U|$ in $X_k^{an}$. And this map extends a lifting $\hat{\varphi}$ of some Frobenius of $U$.

**Definition 9.1** Let $Y$ be a locally closed subset of $X_k$. A strong Frobenius on $|Y|$ is a $\sigma$-linear morphism $F : V \to V'$ between neighborhoods of $|Y|$ in $X_k^{an}$ whose restriction to the tube of $|Y|$ is the generic fiber of a Frobenius $\hat{F}$ of $\hat{Y}$.

Of course, if $Y' \subset Y$, any strong Frobenius on $|Y'|$ will induce a strong Frobenius on $|Y'|$. If $Z$ is a closed subset of $X_k$, then $|Z|$ is open in $X_k^{an}$ and any Frobenius $\hat{F}$ on $\hat{Z}$ will induce a strong Frobenius $\hat{F}_K$ on $|Z|$. Also, when $U$ is an affine open subset of $X_k$ with algebraic lifting $\text{Spec } A$, it is not difficult to see that any strong Frobenius on $|U|$ comes from a unique Frobenius $\varphi$ on $A^\dagger$.

In general, since the Frobenius of $Y$ is finite and flat, the morphism $\hat{F}$ is necessarily finite and flat and we may also assume that $F$ itself is finite and flat after shrinking a little bit the neighborhoods.

If we are given any $\mathcal{O}_{X_k^{an}}$-module $E$ and a strong Frobenius $F : V \to V'$ on $|Y|$, we can consider the $i_Y^{-1}\mathcal{O}_{X_k^{an}}$-module $i^{-1}F^*i_*E$ where $i : |Y| \to V$ and $i' : |Y'| \to V$ denote the inclusion maps. We shall simply denote it by $F^*E$ if there is no risk of confusion.

Since we need to work locally, we will extend a little bit definition [1.2] and call an $i_Y^{-1}\mathcal{O}_{X_k^{an}}$-module $E$ constructible if it becomes coherent after restriction to a finite locally closed covering of $Y$. In practice, if $Z \subset X_k$ is a finite closed subset, a constructible module on
\[ Z \] is simply a coherent \( \mathcal{O}_{|Z|} \)-module. On the other side, if \( U \subset X_k \) is an open subset, then \( i^*_U \) and \( i^{U,1}_U \) induce an equivalence between constructible modules on \( |U| \) and constructible modules on \( X^\text{an}_K \) with support on \( |U| \) (which is a closed subset for the Berkovich topology).

**Proposition 9.2** Let \( Y \) be a locally closed subset of \( X_k \) and \( F \) a strong Frobenius on \( |Y| \). If \( E \) is a constructible module on \( |Y| \), then the corresponding \( i_Y^1 \mathcal{O}_{X^\text{an}_K} \)-module \( F^* E \) is also constructible.

**Proof:** Follows from the fact that inverse image of coherent is coherent. \( \square \)

**Proposition 9.3** Let \( Y \) be a locally closed subset of \( X_k \) and \( F \) a strong Frobenius on \( |Y| \). If \( E \) a \( \nabla \)-module on \( |Y| \), then \( F^* E \) is naturally a \( \nabla \)-module. If \( E \) is overconvergent, then \( F^* E \) is also overconvergent.

**Proof:** This is easily seen using 1-stratifications and the fact that Frobenius is finite flat. \( \square \)

**Proposition 9.4** Let \( U \) be an open subset of \( X_k \) and \( F \) a strong Frobenius on \( |U| \). If \( E \) is an \( i^U_1 \mathcal{O}_{X^\text{an}_K} \)-module, there is a natural isomorphism

\[
F^* i^{-1}_U \mathcal{R}\mathcal{S}\rho_* i^*_U E \simeq i^{-1}_U \mathcal{R}\mathcal{S}\rho_* i^*_U F^* E \tag{9.1}
\]

If \( E \) is a \( \nabla \)-module on \( |U| \), the isomorphism is even \( \mathcal{D}^+_U \mathbb{Q} \)-linear.

**Proof:** Note first that \( i^*_U \) is exact because \( |U| \) is closed in \( X^\text{an}_K \). Also, we have to be a little careful because \( \mathcal{S}\rho_* \) is not a direct image. But we may use an injective resolution of \( (i^*_U, E)_G \), and it will then be sufficient to check that

\[
F^* i^{-1}_U \mathcal{S}\rho_* i^*_U E \simeq i^{-1}_U \mathcal{S}\rho_* i^*_U F^* E. \tag{9.2}
\]

This easily follows from the fact that \( F \) is finite and flat (up to \( \sigma \)-linearity). For the second assertion, one uses 1-stratifications and the fact that \( F \) is finite flat again.

The last one can be split in two as usual. One may first assume that there exists a finite closed subset \( Z \subset X_k \) such that \( E = i^{-1}_U i_{U \setminus Z} E' \) where \( E' \) is a coherent \( \nabla \)-module on \( |U \setminus Z| \). Then both sides of (9.2) are \( \mathcal{D}^+_U \mathbb{Q} \)-modules which are \( \mathcal{O}_{U \mathbb{Q}}(\{Z\}) \)-coherent and any horizontal map between such is automatically \( \mathcal{D}^+_U \mathbb{Q} \)-linear. On the other hand, one may assume that \( E = i^{-1}_U i_Z E' \) where \( E' \) is a locally trivial \( \nabla \)-module on \( |Z| \). We may even assume that \( Z = \{x\} \) is reduced to one closed point and that \( E' = \mathcal{O}_{|x|} \). But this last case reduces to the first one because then, in the derived category, we will have

\[
E \simeq \left[ i^{-1}_U \mathcal{O}_{X^\text{an}_K} \rightarrow i^{-1}_U i_{U \setminus x} \mathcal{O}_{X^\text{an}_K} \right]. \tag{9.3}
\]

**Definition 9.5** Let \( Y \) be a locally closed subset of \( X_k \) and \( F \) a strong Frobenius on \( |Y| \). An \( F \)-\( \nabla \)-module on \( |Y| \) is a \( \nabla \)-module \( E \) on \( |Y| \) endowed with a horizontal Frobenius isomorphism \( \Phi : F^* E \simeq E \).
Of course, if \( Y' \subset Y \), any \( F-\nabla \)-module on \( |Y| \) will restrict to a \( F-\nabla \)-module on \( |Y'| \).

If \( Z \) is a finite closed subset of \( X_k \), it follows from corollary 17.2.2 of [21] for example, that any coherent (or, equivalently, constructible) \( F-\nabla \)-module on \( |Z| \) is locally trivial (or, equivalently, convergent). Actually, if \( a \) is an unramified lifting of a closed point \( x \) and we still denote by \( \sigma \) the Frobenius of \( K(a) \), we have an equivalence of category

\[
\{ \text{coherent } F-\nabla \text{-modules on } |x| \} \leftrightarrow \{ \text{finite } \sigma \text{-modules on } K(a) \}
\]

\[
\begin{array}{ccc}
E & \overset{\nabla}{\longrightarrow} & E^\nabla \\
\mathcal{O}_{|x|} \otimes_{K(a)} H & \overset{i}{\longleftarrow} & H.
\end{array}
\]

In particular, we see that the category of coherent \( F-\nabla \)-module on \( |Z| \) is essentially independent of the choice of \( F \).

Now, if \( U \) is an affine open subset of \( X_k \), it follows from proposition 5.8 and corollary 8.3.10 of [22] for example, that a coherent \( F-\nabla \)-module on \( |U| \) is automatically overconvergent and that they form a category which is essentially independent of the choice of \( F \).

We shall now prove that, more generally, a constructible \( F-\nabla \)-module \( E \) is always overconvergent and that the category of such is essentially independent of the choice of Frobenius.

**Proposition 9.6** Let \( Y \) be a locally closed subset of \( X_k \) and \( F \) a strong Frobenius on \( |Y| \). Then any constructible \( F-\nabla \)-module \( E \) on \( |Y| \) is overconvergent.

**Proof:** Assume first that \( Y \) is not open or empty. Then, \( Y \) is a finite subset of \( X_k \) and the case as been dealt with just before. Otherwise, \( Y \) is a non empty open subset of \( X_k \) and it is equivalent to prove that the constructible \( \nabla \)-module \( i_Y^* E \) is convergent on \( X^\text{nr} \).

There exists a non empty affine open subset \( U \subset Y \) with closed complement \( Z \) such that both \( i_U^{-1} i_Y^* E \) and \( i_Z^{-1} i_Y^* E \) are coherent. Using corollary 5.12, it is sufficient to show that they are both overconvergent. Only the case of \( U \) is new and it will also follow from the discussion before the statement. \( \square \)

**Lemma 9.7** Let \( Y \) be a locally closed subset of \( X_k \) and \( F_1, F_2 \) be two strong Frobenius on \( |Y| \). If \( E \) is an overconvergent \( \nabla \)-module on \( |Y| \), there is a canonical isomorphism \( F_2^* E \simeq F_1^* E \).

**Proof:** Thus, we are given two strong Frobenius \( F_1 : V_1 \to V'_1 \) and \( F_2 : V_2 \to V'_2 \) on \( |Y| \).

We may consider the map

\[
F_1 \times F_2 : (V_1 \times V_2) \cap |X| \to (V'_1 \times V'_2) \cap |X|
\]

\[
(9.5)
\]

and pull back the Taylor isomorphism of \( E \) in order to get

\[
(F_1 \times F_2)^* (\epsilon) : F_2^* E \simeq F_1^* E.
\]

\[
(9.6)
\]

The cocycle condition implies that it is canonical. \( \square \)

It is now straightforward to define overconvergent or constructible \( F-\nabla \)-modules even when the Frobenius does not lift globally.
Definition 9.8 Let $E$ be a convergent (resp. constructible) $\nabla$-module on $X^\an_K$. A Frobenius structure on $E$ is given by a finite open covering $X_k = \cup U_n$, and for each $n$, a strong Frobenius $F_n$ on $|U_n|$ and a horizontal isomorphism $F_n^* i_U^{-1} E \simeq i_U^{-1} E$. Moreover, we require that they are compatible with the canonical isomorphism $F_m^* i_U^{-1} E \simeq F_m^* i_U^{-1} E$ when restricted to $U_n \cap U_m$. Such an object will be called a convergent (resp. constructible) $F$-$\nabla$-module on $X^\an_K$.

In this definition, we implicitly use proposition 9.6. In particular, a constructible $F$-$\nabla$-module on $X^\an_K$ is always convergent.

It follows from propositions 9.6 and 9.7 that the category of convergent (resp. constructible) $F$-$\nabla$-modules on $X^\an_K$ is essentially independent of the choice of the covering as well as of the choice of the strong Frobenius.

One could also define $F$-$\nabla$-kits and show that the category of constructible $F$-$\nabla$-modules is equivalent to the category of $F$-$\nabla$-kits modulo equivalence.

The following result is a direct consequence of the local monodromy theorem:

Theorem 9.9 If $E$ is a constructible $F$-$\nabla$-module on $X^\an_K$, then $R\tilde{sp}_* E$ is a holonomic $F$-$\mathcal{D}^\dagger_{\hat{X}Q}$-module.

Proof: First of all, it follows from proposition 9.4 that $R\tilde{sp}_* E$ has a canonical structure of $F$-$\mathcal{D}^\dagger_{\hat{X}Q}$-module. Now, the question is local. More precisely, thanks to propositions 5.13 and 6.2, we may assume that $E = i_U^* E_U$ with $E_U$ coherent. We showed in proposition 9.2 that $sp_*(i_U^* E_U) = sp_*(i_U^* E_U)_0$ and in proposition 5.7 that $(i_U^* E_U)_0$ is an overconvergent $F$-isocrystal. Then, it is shown in theorem 4.3.4 of [12] (see also [25]) that $sp_*(i_U^* E_U)_0$ is holonomic.

We also have the following:

Corollary 9.10 Any constructible $F$-$\nabla$-module on $X^\an_K$ has Frobenius type.

Proof: The Frobenius structure on $E$ will induce a Frobenius structure on all the bounded Robba fibers. Moreover, we saw in theorem 9.9 that $R\tilde{sp}_* E$ is holonomic and in particular, it is a perfect complex.

We can at last state and prove the overconvergent Deligne-Kashiwara correspondence for curves.

Theorem 9.11 The functor $R\tilde{sp}_*$ induces an equivalence between constructible $F$-$\nabla$-modules on $X^\an_K$ and perverse holonomic $F$-$\mathcal{D}^\dagger_{\hat{X}Q}$-modules on $\hat{X}$.

Proof: Follows from corollary 8.10 and corollary 9.10

References


