Abstract. After recalling basic definitions and results from the theory of overconvergent isocrystals, we describe the action of Frobenius on such objects and introduce the notion of $F$-isocrystal. Then we study the behavior of the Frobenius action and the Frobenius structure under cohomological operations. Finally, we give theorems concerning the slopes (ultrametric measure) of Frobenius on rigid cohomology. We also recall the main result concerning the weights (archimedean measure) of Frobenius.

2000 Mathematics Subject Classification: 14F30

Introduction

This is the text of a lecture given at the university of Padova during the Dwork Trimester in Italy. I was asked by Pierre Berthelot to talk about Frobenius action, $F$-isocrystals and slope filtration in rigid cohomology.

The first lecture was given by Kiran Kedlaya who introduced the notions of convergent and overconvergent isocrystals. I could therefore focus on Frobenius structures. However, for the convenience of the reader, I will recall here in the first section, basic results and constructions that do not involve Frobenius structures. We will try to stay as general as possible.

0. Isocrystals and their cohomology

Let $K$ be a complete ultrametric field of characteristic zero with ring of integers $V$ and residue field $k$. Let $S$ be a locally topologically finitely presented $p$-adic formal $V$-scheme.

If $X \hookrightarrow Y$ is an open immersion of $S_k$-schemes locally of finite type, one can define the notion of overconvergent isocrystals on $X,Y/S$ as in [2]. They form...
an abelian category $\text{Isoc}^{\dagger}(X,Y/S)$ with internal $\text{Hom}$ and $\otimes$. This category is functorial in $X,Y/S$ as well as in $K$.

In the case $Y = X$, then $E$ is called a *convergent isocrystal on $X$* and the category is written $\text{Isoc}(X/S)$. On the other hand, when $Y$ is proper, the category is essentially independent of $Y$ and $E$ is called an *overconvergent isocrystal on $X$*. The category is then written $\text{Isoc}^{\dagger}(X/S)$.

Also, in the case $S = \text{Spec} \mathbb{V}$, the notations are slightly different since we systematically write $K$ instead of $S$. Finally, an *isocrystal* $H$ on $K$ is simply a finite dimensional vector space. The category of (over) convergent isocrystals on the point $\text{Spec}(k)/\text{Spec}(K)$ can be identified with the category of isocrystals on $K$.

If $E \in \text{Isoc}^{\dagger}(X,Y/S)$ and $g : Y \to S$ denotes the canonical map, one can define the *relative rigid cohomology* of $E$:

$$Rg_{\text{rig}}E \in D^b(S_K, \mathcal{O}_{S_K}).$$

With this generality, there is yet no reference for this construction. This was planned to appear in the second part of [2]. Anyway, one can show that rigid cohomology is functorial in $E$ and also in $X,Y/S$.

One can also define the *relative rigid cohomology with proper support* of $E$:

$$Rg_{\text{rig},c}E \in D^b(S_K, \mathcal{O}_{S_K}).$$

This is again functorial in $E$ but functoriality in $X,Y/S$ is not always true : if we denote by $\psi : Y' \to Y$ the morphism, we want $\psi^{-1}(X) = X'$.

Let $f : X \to S$ denote the canonical map. If $Y = X$, then both cohomologies coincide and we actually write $Rf_{\text{conv}}E$. On the other hand, when $Y$ is proper, $Rg_{\text{rig}}E$ is essentially independent of $Y$ and we will write $Rf_{\text{rig}}E$ and $Rf_{\text{rig},c}E$.

In the case $S = \text{Spf} \mathbb{V}$, we will write $R\Gamma_{\text{rig}}(X,Y/K,E)$ and $R\Gamma_{\text{rig},c}(X,Y/K,E)$ with cohomology spaces $H^i_{\text{rig}}(X,Y/K)$ or $H^i_{\text{rig},c}(X,Y/K)$. Finally, when $Y = X$ (resp. $Y$ is proper), we will write $H^i_{\text{conv}}(X/K)$ (resp. $H^i_{\text{rig}}(X/K)$ or $H^i_{\text{rig},c}(X/K)$).

### 1. Frobenius action on isocrystals

Let $p$ be a prime and $q = p^f$ with $f \in \mathbb{N} \setminus \{0\}$. If $X$ is a scheme of characteristic $p$, we let $F_X : X \to X$ be the $f$-iterated of Frobenius given by $x \mapsto x^q$ and call it the *absolute frobenius* of $X$. If we want to emphasize the role of $q$, we can write $F_X^f$.

Let $S$ be a scheme of characteristic $p$. If $X$ is an $S$-scheme, we write $X^{(q/S)}$ or simply $X^{(q)}$ if no confusion should arise, for the pull-back of $X$ along $F_S$. We call the map $F_{X/S} : X \to X^{(q/S)}$ induced by $F_X$ the *relative frobenius* of $X/S$. Thus,
we have a commutative diagram with cartesian square

\[
\begin{array}{ccc}
F_X : & X & \xrightarrow{F_{X/S}} & X^{(q)} \rightarrow X \\
\downarrow & \square & \downarrow \\
S & F_S \rightarrow & S
\end{array}
\]

As before, if we want to emphasize the role of \( q \), we can write \( F_{X/S}^q \).

Let \( K \) be a complete ultrametric field of mixed characteristic \( p \) with ring of integers \( V \) and residue field \( k \). Let \( S \) be a locally topologically finitely presented \( p \)-adic formal \( V \)-scheme and \( F_S \) a lifting (if there is one) of \( F_S \).

Let \( X \hookrightarrow Y \) be an open immersion of \( S_k \)-schemes locally of finite type over \( k \).

By functoriality, there is an inverse image functor

\[
F_{X,Y/S}^* : \text{Isoc}^{\dagger}(X,Y/S) \rightarrow \text{Isoc}^{\dagger}(X,Y/S)
\]

that depends on the choice of \( F_S \). Actually, this functor splits into the base extension functor through \( F_S \),

\[
F_S^* : \text{Isoc}^{\dagger}(X,Y/S) \rightarrow \text{Isoc}^{\dagger}(X^{(q)},Y^{(q)}/S), \quad E \mapsto E^{(q)}
\]

followed by the pull back by the relative Frobenius (which is \( S \) linear)

\[
F_{X,Y/S}^* : \text{Isoc}^{\dagger}(X^{(q)},Y^{(q)}/S) \rightarrow \text{Isoc}^{\dagger}(X,Y/S).
\]

If \( E \in \text{Isoc}^{\dagger}(X,Y/S) \), a (strong) Frobenius on \( E \) is an isomorphism \( \Phi : F^* E \simeq E \). An overconvergent \( F \)-isocrystal on \( X,Y/S \) is such a pair \( (E,\Phi) \). They form an abelian category \( F-\text{Isoc}^{\dagger}(X,Y/S) \) with internal \( \text{Hom} \) and \( \otimes \). The Tate twist \( E(n) \) of \( E \) is defined by multiplying \( \Phi \) by \( q^{-n} \). Note also that this category is functorial in \( X,Y/S \).

In the case \( X = Y \) (resp. \( Y \) proper), then \( E \) is called a convergent (resp. overconvergent) \( F \)-isocrystal on \( X \) and the category is written \( F-\text{Isoc}(X/S) \) (resp. \( F-\text{Isoc}^{\dagger}(X/S) \)).

Again, in the case \( S = \text{Spec} V \), the notations are slightly different since we systematically write \( K \) instead of \( S \). We should also note that, in this case, the choice of \( F_S \) corresponds to the choice of a Frobenius endomorphism \( \sigma \) of \( K \), that is, an isometry that lifts the Frobenius of \( k \).

An isocrystal \( H \) on \( K \) is simply a finite dimensional vector space, a Frobenius on \( H \) is just a \( \sigma \)-linear automorphisms \( \phi \) of \( H \) and such a couple is called an \( F \)-isocrystal on \( K \). Clearly, the category of (over) convergent \( F \)-isocrystals on the point \( \text{Spec}(k)/\text{Spec}(K) \) can be identified with the category of \( F \)-isocrystals on \( K \).
2. Frobenius action on rigid cohomology

By functoriality, if $E \in I soc^\dagger(X, Y/\mathcal{S})$ and $g : Y \to S$ denotes the canonical map, there is a morphism

$$F^*_{X,Y} : LF^*_S R_{g_{rig}} E \to R_{g_{rig}} F^* E.$$  

Of course, it splits into

$$F^*_S : LF^*_S R_{g_{rig}} E \to R_{g_{rig}}^{(q)} E^{(q)}$$

followed by

$$F^*_{X,Y/S} : R_{g_{rig}}^{(q)} E^{(q)} \to R_{g_{rig}} F^* E.$$  

If $\Phi$ is a Frobenius on $E$, we also get by functoriality an isomorphism

$$R_{g_{rig}} \Phi : R_{g_{rig}} F^* E \cong R_{g_{rig}} E.$$  

Composition gives the Frobenius morphism $\phi : LF^*_S R_{g_{rig}} E \to R_{g_{rig}} E$ on cohomology.

By functoriality, we also have morphisms

$$F^* : LF^*_S R_{g_{rig,c}} E \to R_{g_{rig,c}} F^* E.$$  

and

$$R_{g_{rig,c}} \Phi : R_{g_{rig,c}} F^* E \cong R_{g_{rig,c}} E.$$  

and by composition the Frobenius morphism $\phi_c : LF^*_S R_{g_{rig,c}} E \to R_{g_{rig,c}} E$.

When $X = Y$ (resp. $Y$ is proper), if $f : X \to S$ denotes the canonical map, we get the Frobenius morphism

$$\phi : LF^*_S R_{f_{conv}} E \to R_{f_{conv}} E$$

(resp. morphisms

$$\phi : LF^*_S R_{f_{rig}} E \to R_{f_{rig}} E$$

and

$$\phi_c : LF^*_S R_{f_{rig,c}} E \to R_{f_{rig,c}} E).$$

In the case $S = \text{Spec} \mathcal{V}$, we get $\sigma$-linear maps

$$\phi : R \Gamma_{rig}(X, Y/K, E) \to R \Gamma_{rig}(X, Y/K, E)$$

and

$$\phi_c : R \Gamma_{rig,c}(X, Y/K, E) \to R \Gamma_{rig,c}(X, Y/K, E),$$

or, at the cohomological level,

$$\phi^i : H^i_{rig}(X, Y/K, E) \to H^i_{rig}(X, Y/K, E)$$
and
\[ \phi_i^c : H^{i}_{\text{rig},c}(X,Y/K,E) \to H^{i}_{\text{rig},c}(X,Y/K,E). \]

Using de Jong alterations method ([9]), one can prove the following (we call algebraic variety any separated scheme of finite type over a field):

**Theorem 2.1.** ([6], 2.1) Assume \( k \) perfect and the valuation discrete, and fix a Frobenius on \( K \).

If \( X \) is an algebraic variety over \( k \), then \( H^{i}_{\text{rig},c}(X/K) \) is an \( F \)-isocrystal on \( K \).

If \( X \) has dimension \( d \), then the trace map
\[ H^{2d}_{\text{rig},c}(X/K) \to K(-d) \]
is a morphism of \( F \)-isocrystals.

If \( X \) is smooth, then \( H^{i}_{\text{rig}}(X/K) \) is an \( F \)-isocrystal on \( K \).

If \( X \) is smooth and \( Z \) is a smooth closed subvariety of codimension \( r \), the Gysin map
\[ H^{i-2r}_{\text{rig}}(Z/K)(-r) \to H^{i}_{\text{rig}}(X/K) \]
is a morphism of \( F \)-isocrystals.

Finally, if \( X \) is smooth of pure dimension \( d \), then the Poincaré pairing
\[ H^{i}_{\text{rig},c}(X/K) \times H^{2d-i}_{\text{rig}}(X/K)(d) \to K \]
is a perfect pairing of \( F \)-isocrystals.

The same questions arise with more general coefficients. Unfortunately, we cannot do much since we do not even know finite dimensionality (see however the recent announcement by Kiran Kedlaya). Anyway, using Dwork operator, one can show the following

**Theorem 2.2.** (see [8], 2.1) If \( X \) is an algebraic variety (resp. a smooth algebraic variety) over \( k \) and \( E \) is an overconvergent isocrystal on \( X/K \), then
\[ F^*_X : H^{i}_{\text{rig}}(X/K,E) \to H^{i}_{\text{rig}}(X/K,E^{(q)}) \]
(resp. \( F^*_X : H^{i}_{\text{rig}}(X/K,E) \to H^{i}_{\text{rig}}(X/K,E^{(q)}) \)
is bijective.

Finally, a careful study of extensions of overconvergent isocrystals gives the following (recall that an object in a Tannakian category is unipotent if it is an iterated extension of 1):

**Theorem 2.3.** ([6], 2.2.2) Assume \( k \) perfect and the valuation discrete. If \( X \) is a smooth algebraic variety over \( k \), then, \( F^*_X \) induces an auto-equivalence of the category of unipotent overconvergent isocrystals on \( X/K \).
3. Slopes of Frobenius

We let $\mathcal{V}$ be a discrete valuation ring of mixed characteristic $p$ with ramification index $e$, fraction field $K$ and perfect residue field $k$. We fix a uniformiser $\pi$ and a Frobenius endomorphism $\sigma$ such that $\sigma(\pi) = \pi$. This can always be done after a finite extension of $K$.

If $\lambda \in \mathbb{Q}$, then we can write in a unique way $ef\lambda = \frac{r}{s}$ with $r, s \in \mathbb{Z}$ coprime and $s > 0$ (recall that $q = p^e$). The $\sigma$-linear multiplication by $T$ on $K[T]$ (it is the composition of pull-back by $\sigma$ and multiplication by $T$) induces a semi-linear endomorphism of $K(-\lambda) := K[T]/(T^s - \pi^r)$. Manin’s theorem says that, if $k$ is algebraically closed, then any $F$-isocrystal $H$ is a direct sum of $K(-\lambda)$ for various $\lambda$’s called the slopes of $H$.

In general, if $K^{un} = W(k) \otimes_{W(k)} K$ denotes the maximal unramified extension of $K$ and $(H, \phi)$ is an $F$-isocrystal on $K$, we say that $\lambda$ is a slope for $H$ if it is a slope of $K^{un} \otimes_K H$.

Let $X \hookrightarrow Y$ be an open immersion of algebraic varieties over $k$ and $E \in F-\text{Isoc}^\dagger(X,Y/K)$. We say that $\lambda \in \mathbb{Q}$ is a slope for $E$ if there exists a closed point $x \in X$ such that $E_x$ has slope $\lambda$. We say that $E$ is pure if it has exactly one slope and that it is unit-root if this slope is 1. Then, we have

**Theorem 3.1.** ([10]) Theorem 2.1 is still valid for unit-root $F$-isocrystals.

The next theorem generalizes Manin’s theorem and can be derived from it by the formalism of tannakian categories (an overconvergent $F$-isocrystal is unipotent if the underlying overconvergent isocrystal is unipotent):

**Theorem 3.2.** ([6], 3.2.3) If $E$ is a unipotent overconvergent $F$-isocrystal on a smooth algebraic variety $X$ over $k$ algebraically closed, then $E$ has a filtration $\text{Fil}_\lambda$ with $\text{Gr}_\lambda$ pure of slope $\lambda$.

**Theorem 3.3.** ([5], 3.2.3) When $X$ is an open subset of the affine line, then $E$ is a direct sum of $F$-isocrystals whose set of slopes is of the form $\{\lambda, \lambda + 1, \ldots, \lambda + r\}$.

It is a general question to determine the slopes of $H^i_{\text{rig}}(X,Y/K,E)$ and $H^i_{\text{rig},c}(X,Y/K,E)$.

Using de Jong’s alteration techniques, the following result is a direct consequence of the corresponding fact in crystalline cohomology:

**Theorem 3.4.** ([6], 3.1.2) Let $X$ be an algebraic variety (resp. a smooth algebraic variety) of dimension $d$ over $k$. Then, the slopes of $H^i_{\text{rig},c}(X/K)$ (resp. $H^i_{\text{rig}}(X/K)$) are integers between $\max(0, i - d)$ and $\min(i, d)$.
4. Weights of Frobenius

A *Weil number of weight* $i \in \mathbb{Z}$ (relatively to $q$) is (at least) an algebraic number having all its archimedean absolute values equal to $q^{i/2}$. A *polynomial* over $K$ is *mixed* (resp. *pure of weight* $i$) if all its roots are Weil numbers (resp. of weight $i$). An $F$-isocrystal $(H, \phi)$ is *mixed* (resp. *pure of weight* $i$) according to the nature of the characteristic polynomial of $\phi$.

The following theorem can be proved as theorem 3.4.

**Theorem 4.1.** ([4], 2.4 and [3], 2.2) If $X$ is an algebraic variety (resp. a smooth algebraic variety) over $\mathbb{F}_q$, then $H^i_{rig,c}(X/K)$ (resp. $H^i_{rig}(X/K)$) is mixed with weights between $\max\{0, 2(i−n)\}$ and $i$ (resp. $i$ and $\min\{2i, 2n\}$).

References


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