The Exact Poincaré Lemma in Crystalline Cohomology of Higher Level

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We prove a formal Poincaré Lemma in crystalline cohomology of higher level and we derive a de Rham interpretation of crystalline cohomology of higher level. We also define the Spencer complex of higher level and show that it can be used to compute the cohomology of arithmetic \( \mathbb{D} \)-modules.

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1. The de Rham complex.
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INTRODUCTION

As long as the theory is concerned, crystalline cohomology generalizes well to higher level (see [B3, E-LS, Q-LS, T], for example). Unfortunately,  

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the main tool for computing crystalline cohomology, namely, the usual de Rham complex, cannot be used in higher level. Following Lieberman [L], Berthelot suggests in [B3] to use the so-called Jet Complex. This is what we call in this article the de Rham complex of higher level. In crystalline cohomology, the main ingredient in order to show that the de Rham complex computes de Rham cohomology is the Formal Poincaré Lemma. This is also the case in higher level and we show that crystalline cohomology of higher level can be computed as the cohomology of a complex whose terms are locally free, the de Rham complex of higher level. We are also able to define a Spencer complex in higher level and prove that the cohomology of a $\mathcal{D}^{(m)}_{X/S}$-module agrees with the cohomology of the corresponding $m$-crystal. Finally, using Berthelot’s Frobenius descent, we give an explicit quasi-isomorphism between the de Rham complex of level $m$ of an $m$-crystal and the de Rham complex of level $m + 1$ of its Frobenius pull-back.

In a forthcoming article, we will consider the Filtered Poincaré Lemma and compute the cohomology of $T$-$m$-crystals. We will apply this to the study of the Hodge filtration on Dieudonné modules of level $m$.

Conventions

We let $p$ be a non-zero prime and $m \in \mathbb{N}$. When $m \neq 0$, all schemes are $\mathbb{Z}(p)$-schemes. If $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$ we will write $|I| = i_1 + \cdots + i_n$ and, more generally, we use the classical conventions for multi-indexing.

1. THE DE RHAM COMPLEX

For the notion of $m$-PD-structures the fundamental reference is [B4].

1.1. Definition. We develop an idea introduced by Berthelot in [B3], following Lieberman [L]. More precisely, we explain the construction of the de Rham complex of higher level.

Let $S$ be a scheme and $X$ a smooth $S$-scheme. Everything below will be done relative to $S$ but we will drop the reference to $S$ in order to lighten the notations. We write $P_{X_m}(r)$ for the $m$-PD-envelope of $X$ in $X^{(r+1)}/S$, $d_i:P_{X_m}(r+1) \to P_{X_m}(r)$ for the projection that forgets the $(i+1)$st component, and $s_i:P_{X_m}(r-1) \to P_{X_m}(r)$ for the degeneracy arrow that repeats the $(i+1)$st component. We also write $p_i:P_{X_m}(r) \to X$, for the $i$th projection and $\Delta:X \to P_{X_m}(r)$ for the diagonal embedding.

We consider the simplicial complex of schemes

$$P_{X_m}(r) := \{P_{X_m}(r), d_i, i = 0, \ldots, r; s_i, i = 0, \ldots, r\},$$
and the usual complex $\mathcal{P}_{Xm}(\cdot)$ built out of it. Thus, $\mathcal{P}_{Xm}(r)$ is the structural sheaf of $P_{Xm}(r)$. By definition, the normalized complex associated to $\mathcal{P}_{Xm}(\cdot)$ is $N\mathcal{P}_{Xm}(\cdot) := \cap \ker s^*$. 

Recall that if $A$ is a (not necessarily commutative) ring and $M$ is an $A$-$A$-bimodule, then the tensor algebra $T(M)$ is defined in the obvious way. We will use this vocabulary when $A$ is commutative and $M$ is endowed with two (different) $A$-module structures, called “structure on the left” and “structure on the right.” We might sometimes call this tensor algebra the “non-commutative” tensor algebra.

For example, as a graded algebra, $\mathcal{P}_{Xm}(\cdot)$ is canonically isomorphic to the non-commutative tensor algebra $T_{P_{Xm}}$. If we call $II_{Xm}$ the $m$-PD-ideal of $P_{Xm}$, one easily checks that $N\mathcal{P}_{Xm}(\cdot)$, which is a differential sub-algebra of $\mathcal{P}_{Xm}(\cdot)$, is isomorphic as a graded algebra to $T_{II_{Xm}}(1)$. Finally, if $KK_{Xm}$ denotes the differential ideal in $N\mathcal{P}_{Xm}(\cdot)$ generated by $II_{Xm}$, we call $\Omega_{Xm}$ the jet complex of order $p^m$ or the de Rham complex of level $m$ of $X/S$. Note that, as a graded algebra, $\Omega_{Xm}$ is a quotient of $T(\Omega_{Xm})$. We will make this more precise later.

We can also define the de Rham complex of a $D_{Xm}$-module $F$ as follows. Recall that, for each $k$, $D_{Xm}^{(m)}$ denotes the sheaf of operators of level $m$ and order at most $k$. By definition, this is the dual to the sheaf of principal parts $P_{Xm}^{(m)}$ and $D_{Xm}^{(m)}$ is the union of the $D_{Xm}^{(m)}$ when $k$ varies. The action of $D_{Xm}^{(m)}$ on $F$ induces for all $k$ a pairing $D_{Xm}^{(m)} \times F \to F$ that corresponds to a morphism $d: F \to F \otimes_{\mathcal{O}_X} D_{Xm}^{(m)}(1)$. In the particular case $k = p^m$, we have $\Omega_{Xm}^1 \subset D_{Xm}^{(m)}(1)$ and $d$ takes its values in $F \otimes_{\mathcal{O}_X} \Omega_{Xm}^1$. Thus we get what could be called an $m$-connection $d: F \to F \otimes_{\mathcal{O}_X} \Omega_{Xm}^1$. It is $m$-integrable: using the Leibniz rule, it extends to a differential on $F \otimes_{\mathcal{O}_X} \Omega_{Xm}^1$. This is the de Rham complex of $F$ and we can define the de Rham cohomology of level $m$ of $F$:

\[ H^i_{dR_{Xm}}(X,F) := H^i(X,F \otimes_{\mathcal{O}_X} \Omega_{Xm}^1). \]

1.2. Description of the de Rham Complex. Unless otherwise specified, we will always see $\mathcal{P}_{Xm}(\cdot)$, $N\mathcal{P}_{Xm}(\cdot)$, etc., as left $\mathcal{O}_X$-modules using $p^*\mathcal{O}_X$, and tensor products will be over $\mathcal{O}_X$. Let us now describe the differential on $\mathcal{P}_{Xm}(\cdot)$, $N\mathcal{P}_{Xm}(\cdot)$, and $\Omega_{Xm}$. As usual, the differential on $\mathcal{O}_X$ is induced by the map

\[ \mathcal{O}_X \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X, f \mapsto 1 \otimes f - f \otimes 1. \]
It is also convenient to call $\delta: \mathcal{P}_{Xm}(1) \to \mathcal{P}_{Xm}(2)$ the ring homomorphism $d_1^*$ induced by the map

$$\mathcal{C}_X \otimes \mathcal{C}_X \to \mathcal{C}_X \otimes \mathcal{C}_X, \mathcal{C}_X \otimes f \otimes g \mapsto f \otimes 1 \otimes g.$$ 

The differential on $\mathcal{P}_{Xm}(1)$, $\mathcal{I}_{Xm}(1)$ and $\Omega_{Xm}^1$ is given by

$$d(\alpha) = 1 \otimes \alpha - \delta(\alpha) + \alpha \otimes 1.$$ 

For higher degree, we can use the Leibniz rule:

$$d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^{\text{deg}\alpha} \alpha \otimes d\beta.$$ 

Note also that the non-commutativity of the tensor product means that if $\alpha, \beta \in \mathcal{P}_{Xm}(1)$ and $f \in \mathcal{C}_X$, then $\alpha \otimes f \beta = f \alpha \otimes \beta + d(f).\alpha \otimes \beta$, where the regular product is taken in the monoid $\mathcal{P}_{Xm}(1)$, $\mathcal{I}_{Xm}(1)$ or $\Omega_{Xm}^1$.

We assume now that we have local coordinates $t_1, \ldots, t_n$ on $X$ and write $\tau_i = dt_i$. Then $\mathcal{P}_{Xm}(1)$ (resp. $\mathcal{I}_{Xm}(1)$) is the free module on the generators $\tau^{(j)}_i$ (resp. with $|J| > 0$). It follows that $\mathcal{P}_{Xm}(r)$ (resp. $\mathcal{I}_{Xm}(r)$) is the free module on the generators $\tau^{(j_1)}_1 \otimes \cdots \otimes \tau^{(j_r)}_r$ (resp. with $|J_1|, \ldots, |J_r| > 0$). Note that when $|J| < 2p^m$, we have $\tau^{(j)} = \tau^{j'}$ and we will therefore drop the $\{\}$. We will write $dt_i$ for the image of $\tau_i$ in $\Omega_{Xm}^1$. Then we see that $\Omega_{Xm}^1$ is the free module on the generators $(dt)^j$ with $0 < |J| \leq p^m$. Note that, since $(dt)^{(j)} = 0$ whenever $|J| > p^m$, such terms will never play a role in our calculations. To lighten the notations, we will therefore not make explicit the corresponding conditions of the type $0 < |J| \leq p^m$, in the hope that this will make it easier to read without baffling the reader.

The local description of $\Omega_{Xm}^1$ for $r > 1$ needs some more care. Note first that $\Omega_{Xm}^1$ has a natural structure of graded algebra. We keep using the tensor notation for this structure (using the wedge product, as in the case $m = 0$, would be misleading). Actually, $\Omega_{Xm}^1$ is the quotient of $T(\Omega_{Xm}^1)$ by the ideal generated by the images of the $d[(\tau^{(j_1)}_1) \otimes \cdots \otimes (\tau^{(j_r)}_r)]$ with $0 < |J_1|, \ldots, |J_r| \leq p^m$ and $d$ is given by the classical formula

$$d(\tau^{(j_1)}) = \sum_{0 \leq V < J} \binom{J}{V} \tau^{(j_1 - V)} \otimes \tau^{(V)},$$

we see first that $\Omega_{Xm}^1$ is generated by the $(dt)^j \otimes (dt)^j$ with $0 < |I|$, $|J| \leq p^m$ subject to the relations

$$(*) \quad \sum_{0 < V < J} \binom{J}{V} (dt)^{j-V} \otimes (dt)^{V} = 0 \quad \text{with } p^m < |J| \leq 2p^m.$$
For \( r > 2 \), \( \Omega^r_{Xm} \) is generated by the \((dt)^I \otimes (dt)^{J} \otimes \cdots \otimes (dt)^{J'}\) with 
\( 0 < |I|, \ldots, |J| \leq p^m \) subject to the same relations, tensored on the right and on the left by elements of the form \((dt)^{I} \otimes (dt)^{J} \otimes \cdots \otimes (dt)^{J'}\).

We also have a local description of the \( m \)-connection \( d: \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_x} \Omega^1_{Xm} \)
corresponding to a \( \mathcal{D}^{m\text{-}1}_X \)-module \( \mathcal{F} \); it is given by \( d(x) = \sum \delta^{[U]}(x) \otimes (dt)^{U} \)
where the \( \delta^{[U]} \) are the divided partial derivatives.

In general, the de Rham complex is infinite. However, we will show that it has locally free terms. First, we need a lemma.

1.3. Lemma. Unless \( J = \{0, \ldots, 2p^m, 0, \ldots, 0\} \), in which case there is only one term, there are at least two different terms with invertible coefficients in the formula (\( \ast \)).

Proof. If we write \( J = \{j_1, \ldots, j_n\} \) we may assume by symmetry that 
\( j_1 \geq \cdots \geq j_n \). If \( j_1 > p^m \) we let \( V = (p^m, 0, \ldots, 0) \). Then it follows from
[B4, Lemma 1.1.3] that the coefficient \( \langle \eta \rangle \) is invertible. If not, there exits \( i \) such that 
\( j_1 + \cdots + j_i < p^m \) and that \( j_1 + \cdots + j_{i+1} > p^m \). In this case, we take \( V = (j_1, \ldots, j_i, 0, \ldots, 0) \) and again \( \langle \eta \rangle \) is invertible. By symmetry, unless \( J = \{2p^m, 0, \ldots, 0\} \), in which case our relation simply says that 
\( (dt_1)^{p^m} \otimes (dt_1)^{p^m} = 0 \), there are at least two different terms with invertible coefficients.

1.4. Proposition. The de Rham complex of level \( m \), \( \Omega^r_{Xm} \) is a complex whose terms are locally free of finite rank.

Proof. This is a local question. Thus, we may assume that we have local coordinates and use the notations of Subsection 1.2. For \( r = 0 \) or 1, there is nothing to do. Assume now \( r = 2 \). For different \( J \)'s with \( p^m < |J| \leq 2p^m \),
the relations involve different generators \((dt)^U \otimes (dt)^V \). We know from the
lemma that at least one of the coefficients in each relation is invertible. It follows that 
\( \Omega^2_{Xm} \) is free. More precisely, for each \( I \) with \( p^m < |I| \leq 2p^m \),
we can choose a couple \((A(I), B(I))\) such that \( I = A(I) + B(I) \) and that the set

\[ S = \left\{ (dt)^U \otimes (dt)^V : (U, V) \neq (A(I), B(I)) \right\} \]

is a basis for \( \Omega^2_{Xm} \).

We consider now the case \( r = 3 \). From the case \( r = 2 \) we know that

\[ \left\{ (dt)^U \otimes (dt)^V \otimes (dt)^W : (V, W) \neq (A(I), B(I)) \right\} \]

is a set of generators for \( \Omega^3_{Xm} \), and this module is defined by the following relations for 
\( p^m < |J| \leq 2p^m \) and \( |K| \leq p^m \):

\[ \sum_{0 < V < J} \binom{J}{V} (dt)^{J-V} \otimes (dt)^V \otimes (dt)^K = 0 \quad \text{with} \ K \neq B(I) \]
and
\[
\sum_{0 < V < I, V \neq A(I)} \left\langle J \right\rangle V \frac{(dt)^{J-V} \otimes (dt)^V \otimes (dt)^{B(I)}}{V} - \sum_{0 < V < I, V \neq A(I)} \left\langle A(I) \right\rangle V \frac{(dt)^{J-A(I)} \otimes (dt)^V \otimes (dt)^{I-V}}{V} = 0.
\]

Using the relations with \( K \neq B(I) \), which all involve different generators, we can write \( (dt)^{A(I)} \otimes (dt)^{B(I)} \otimes (dt)^K \) as a linear combination of the other terms. It follows that
\[
\{(dt)^U \otimes (dt)^V \otimes (dt)^W : \text{if } W = B(I) \text{ then } V \neq A(I),
\]
\[
\text{and if } W \neq B(I) \text{ then } (U, V) \neq (A(I'), B(I'))\}
\]
is a set of generators for \( \Omega^3_{X_m} \). We can rewrite the second sum in the remaining relations in terms of these generators. Now we know from the lemma that, in these relations, at least one of the coefficients \( \left\langle J \right\rangle V \) with \( V \neq A(I) \) is invertible and we can write the corresponding term as a linear combination of the other generators, getting a basis of \( \Omega^3_{X_m} \).

Finally, the case \( r > 3 \) easily follows from the above description of \( \Omega^3_{X_m} \) and the case \( r = 3 \).

1.5. Linearization. We consider now a construction analogous to that of Definition 1.1 that will be "linear" in \( \Theta^r \). This is a standard technique: we look at the shifted simplicial complex
\[
P^+_{X_m}(\cdot) = \{ P_{X_m}(r + 1); d_i, i = 1, \ldots, r + 1; s_i, i = 1, \ldots, r + 1 \}
\]
and we consider the associated complex of abelian sheaves \( P^+_{X_m}(\cdot) \) on \( X \). As before, we define the normalized complex \( N\mathcal{P}^+_{X_m}(\cdot) = \cap \text{Ker } s^*_i \).

As a graded algebra, \( P^+_{X_m}(\cdot) \) (resp. \( N\mathcal{P}^+_{X_m}(\cdot) \)) is isomorphic to \( P_{X_m}(1) \otimes_{\mathcal{E}_X} T(\mathcal{P}_{X_m}(1)) \) (resp. \( P_{X_m}(1) \otimes_{\mathcal{E}_X} T(\mathcal{F}_{X_m}(1)) \)). Taking the quotient by \( P_{X_m}(1) \otimes_{\mathcal{E}_X} \mathcal{F}_{X_m} \), we get a differential complex \( P_{X_m}(1) \otimes_{\mathcal{E}_X} \Omega^1_{X_m} \) which is a quotient of \( P_{X_m}(1) \otimes_{\mathcal{E}_X} T\Omega^1_{X_m} \). By analogy with the usual de Rham complex, we will write \( \Omega^+_{p_m} = P_{X_m}(1) \otimes_{\mathcal{E}_X} \Omega^1_{X_m} \). The differential on \( \Omega^+_{p_m} \) is given by the Leibniz rule
\[
d(\alpha \otimes \omega) = d\alpha \otimes \omega + \alpha \otimes d\omega,
\]
where \( d : P_{X_m}(1) \rightarrow P_{X_m}(1) \otimes_{\mathcal{E}_X} \Omega^1_{X_m} \) is the \( m \)-connection for the right structure induced by \( \alpha \mapsto \delta(\alpha) - \alpha \otimes 1 \), and \( d : \Omega^1_{X_m} \rightarrow \Omega^1_{X_m} \) is the differential of the de Rham complex of level \( m \) defined in Subsection 1.2.
As before, we have the classical formula for moving coefficients inside products: if $\omega, \eta \in \Omega^1_{\varphi_m}$ and $\alpha \in \mathcal{P}_{X_m}(1)$, then

$$\omega \otimes \alpha \eta = \alpha \omega \otimes \eta + (d\alpha) \omega \otimes \eta.$$ 

Note that, in local coordinates, $d\tau = 1 \otimes dt$, and that we have

$$d(\tau^{(I)}) = \sum_{0 < U \leq I} \left\langle I \begin{array}{c} U \\ \tau^{(I-U)} \end{array} \right\rangle (d\tau)^U,$$

(where $(d\tau)^V = 0$, when $|V| > p^m$) and

$$d((d\tau)^I) = - \sum_{0 < U < I} \left\langle I \begin{array}{c} U \\ \tau^{I-U} \end{array} \right\rangle (d\tau)^{I-U} \otimes (d\tau)^U.$$

Of course, a set of generators of $\Omega^1_{\varphi_m}$ over $\mathcal{P}_{X_m}(1)$ is given as before by the $(d\tau)^{I_1} \otimes \cdots \otimes (d\tau)^{I_s}$ with $0 < |J_1|, \ldots, |J_s| \leq p^m$ subject to the relations

$$\sum_{0 < V < J} \left\langle J \begin{array}{c} V \\ \tau^{J-V} \end{array} \right\rangle (d\tau)^{J-V} \otimes (d\tau)^V = 0$$

with $p^m < |J| \leq 2p^m$.

2. INTEGRATION OF DIFFERENTIAL FORMS OF HIGHER LEVEL

We assume that we have local coordinates on $X$ as in Subsection 1.2. As before, we write $t = (t_1, \ldots, t_n)$, and we set $\hat{t} = (t_1, \ldots, t_{n-1})$ and $\hat{\tau} = (\tau_1, \ldots, \tau_{n-1})$. Also, given an $n$-uple $I = (i_1, \ldots, i_s)$, we will write $\hat{I} = (i_1, \ldots, i_{n-1})$.

2.1. PROPOSITION AND DEFINITION. There exists a unique linear map $h: \Omega^r_{\varphi_m} \to \Omega^{r-1}_{\varphi_m}$ such that

$$h\left[ \hat{t}^{(I_1)}(d\tau)^{I_1} \otimes \cdots \otimes (d\hat{\tau})^{I_{s-1}} \otimes (d\tau)^{J_s} \right]$$

$$= \begin{cases} 0 & \text{if (1) } s = r \text{ and } j = 0 \text{ or (2) } j \neq 0 \text{ and } J_s \neq 0 \text{ or (3) } p^m + i \\
(1)^{s-1} \binom{i + j}{i}^{-1} \hat{t}^{(I)}(d\hat{\tau})^{I_1} \otimes \cdots \otimes (d\hat{\tau})^{I_{s-1}} \\
\otimes \tau^{(I_1)}(d\tau)^{J_1} \otimes \cdots \otimes (d\tau)^{J_{s-1}} & \text{if } J_s = 0, p^m \mid i \text{ and either (1) } 0 < j < p^m \text{ or (2) } j = p^m \text{ and } s = r. \end{cases}$$
We will show below that this map is well defined and unique. Note that the formula makes sense because the modified binomial coefficient \( \binom{i+j}{i} \) is invertible when \( p^m \mid i \) and \( 0 < j \leq p^m \) [B4, Lemma 1.1.3].

We first want to look at some examples:

2.2. EXAMPLES. (a) For \( r = 1 \) and \( n = 1 \), we get

\[
h(\tau^{(i)}(d\tau)^j) = \begin{cases} \binom{i+j}{i}^{-1} \tau^{(i+j)} & \text{if } p^m \mid i \\ 0 & \text{otherwise.} \end{cases}
\]

This is the formula for integrating differentials of level \( m \) as we can easily verify: we have

\[
(h \circ d)(\tau^{(i)}) = \sum_{0 < u \leq i} \binom{i}{u} h(\tau^{(i-u)}(d\tau)^u).
\]

If \( i = 0 \), this is an empty sum and we get 0. If not, we can write \( i = np^m + t \) with \( 0 < t \leq p^m \) and we see that \( h(\tau^{(i-t)}(d\tau)^t) = 0 \) unless \( p^m \) divides \( i - u \), which means that \( t = u \) since \( 0 < u \leq p^m \). Thus, we see that for \( i > 0 \),

\[
(h \circ d)(\tau^{(i)}) = \binom{i}{i} h(\tau^{(i-t)}(d\tau)^t) = \binom{i}{i}^{-1} \tau^{(i-t+i)} = \tau^{(i)}.
\]

(b) For \( r = 1 \) and \( n = 2 \) the formula is quite similar

\[
h(\tau^{(i)}(d\tau_1)^{i_1}(d\tau_2)^{i_2})
\]

\[
= \begin{cases} \binom{i_1+j_2}{i_2} \tau^{(i_1+j_2)} & \text{if } j_1 = 0 \text{ and } p^m \mid i_2, \\ 0 & \text{otherwise} \end{cases}
\]

It corresponds to integration with respect to the second variable.

(c) For \( r = 2 \) and \( n = 1 \), we get a new situation;

\[
h(\tau^{(i)}(d\tau)^{i} \otimes (d\tau)^{k})
\]

\[
= \begin{cases} \binom{i+j}{i}^{-1} \tau^{(i+j)}(d\tau)^k & \text{if } p^m \mid i \text{ and } j < p^m, \\ 0 & \text{if } p^m \mid i \text{ otherwise} \end{cases}
\]

but our definition does not seem complete because \( h(\tau^{(i)}(d\tau)^p \otimes (d\tau)^i) \)
is not defined yet when $p^m | i$. However, we have the following relations

$$\sum_{j \leq v \leq p^m} \left( p^m + j \right) (d\tau)^p \otimes (d\tau)^v = 0$$

that give the formulas

$$(d\tau)^p \otimes (d\tau)^v = \sum_{j < v \leq p^m} \left( p^m + j \right)^{-1} \left( p^m + j \right)^v \otimes (d\tau)^v$$

and therefore, if $p^m | i$,

$$h(\tau^{(i)}(d\tau)^p \otimes (d\tau)^j) = \sum_{j < v \leq p^m} \left( p^m + j \right)^{-1} \left( p^m + j \right)^v \otimes (d\tau)^v$$

This formula may not look as natural as the one in the definition, but anyway, we see in this case that $h$ is unique and well defined.

Looking at more complicated examples would not be easier than the general case which we now try to explain. Let us first show that, in the formula for $h$, and provided it is well defined, we can “move the $\tau_n^{(i)}$ inside.”

2.3. Lemma. We always have

$$h\left[ \hat{\tau}^{(i)} \tau_n^{(i)} (d\hat{\tau})^j \otimes \cdots \otimes (d\hat{\tau})^{j-l} \otimes (d\tau)^l (d\tau)^j \right]$$

$$\otimes (d\tau)^{j+1} \otimes \cdots \otimes (d\tau)^j$$

$$= h\left[ \hat{\tau}^{(i)} (d\hat{\tau})^j \otimes \cdots \otimes (d\hat{\tau})^{j-l} \otimes \tau_n^{(i)} (d\hat{\tau})^l (d\tau)^j \right]$$

$$\otimes (d\tau)^{j+1} \otimes \cdots \otimes (d\tau)^j.$$

Proof. This is proved by induction on $s$, the case $s = 1$ being trivial. Using the formula for moving coefficients across the tensor products, we get

$$h\left[ \hat{\tau}^{(i)} (d\hat{\tau})^j \otimes \cdots \otimes (d\hat{\tau})^{j-l} \otimes \tau_n^{(i)} (d\hat{\tau})^l (d\tau)^j \otimes \cdots \right]$$

$$= h\left[ \hat{\tau}^{(i)} (d\hat{\tau})^j \otimes \cdots \otimes (d\hat{\tau})^{j-l} \otimes \tau_n^{(i)} (d\hat{\tau})^l (d\tau)^j \otimes \cdots \right]$$

$$+ h\left[ \tau^{(i)} (d\hat{\tau})^j \otimes \cdots \otimes (d\hat{\tau})^{j-l} \otimes d(\tau_n^{(i)}) (d\hat{\tau})^l \right]$$

$$\otimes (d\tau)^{j+1} \otimes \cdots \right].$$

It is therefore sufficient to show that, if $J_{s-1} \neq 0$, then

$$h\left[ \hat{\tau}^{(i)} (d\hat{\tau})^j \otimes \cdots \otimes (d\hat{\tau})^{j-l} \otimes d(\tau_n^{(i)}) (d\hat{\tau})^l \right] = 0.$$
Since
\[ d(\tau_n^{(i)}) = \sum_{0 < u \leq i} \left< \frac{i}{u} \right> \pi_n^{(-u)}(d\tau_n)^u , \]
it is sufficient to show that for \( u \neq 0 \), we have
\[ h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{i-1}} \otimes \pi_n^{(-u)}(d\tau_n)^u \otimes \cdots \right] = 0. \]
By induction, this is
\[ h \left[ \tau_n^{(i)}(d\tau_n)^{J_1} \otimes \cdots \otimes (d\tau_n)^{J_{i-1}} \otimes (d\tau_n)^u \otimes \cdots \right] \]
which is 0 by definition, since \( u \neq 0 \) and \( J_{i-1} \neq 0 \).

2.4. Proposition. If it exists, the map \( h \) is unique. More precisely, for the condition
\[ \sum_{0 < V < J} \left< \frac{J}{V} \right> h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{i-1}} \otimes (d\tau_n)^p \otimes (d\tau_n)^j \otimes \cdots \right] \]
\[ \otimes (d\tau)^{J-V} \otimes (d\tau)^V \otimes \cdots = 0 \]
to be satisfied whenever \( J \neq 0 \) and \( j_n \geq p^n \), it is necessary and sufficient that (writing \( j = j_n \))
\[ h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{i-1}} \otimes (d\tau_n)^p \otimes (d\tau_n)^j \otimes \cdots \right] \]
\[ = - \sum_{j < v < p} \left< \frac{p^m + j}{v} \right> \left< \frac{p^m + j}{v} \right> \]
\[ \times h \left[ \tau_n^{(i)}(d\tau_n)^{J_1} \otimes \cdots \otimes (d\tau_n)^{J_{i-1}} \otimes (d\tau_n)^p \otimes (d\tau_n)^v \otimes \cdots \right] \]
when \( j > 0 \) and that
\[ h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{i-1}} \otimes (d\tau_n)^p \otimes (d\tau_n)^j \otimes \cdots \right] \]
\[ = - h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{i-1}} \otimes (d\tau_n)^p \otimes (d\tau_n)^j \otimes \cdots \right] \]
\[ = \sum_{0 < v < p} \left< \frac{p^m}{v} \right> h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{i-1}} \otimes (d\tau_n)^p \otimes (d\tau_n)^v \otimes \cdots \right]. \]

Proof. First of all, note that the formulas we are going to prove do give a definition of \( h \) for all generators, although for generators of the type
\[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{i-1}} \otimes (d\tau_n)^p \otimes (d\tau_n)^j \otimes \cdots , \]
we only get a definition by descending induction on \( s \).
Now, we look at the above condition
\[
\sum_{0 < v < j} \left\langle \frac{J}{V} \right\rangle h \left[ \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\tau)^{J-V} \otimes (d\tau)^V \otimes \cdots \right] = 0
\]
with \(\hat{J} \neq 0\) and \(j \geq p^m\). Writing \(v = v_n\), note that all the terms in this sum are 0 unless \(v = j\) or \(V = \hat{J}\), and that these two cases are mutually exclusive. Thus, our sum splits into two sums. Actually, if \(j > p^m\), we are only left with one sum
\[
\sum_{0 < v < j} \left\langle \frac{j}{V} \right\rangle h \left[ \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\tau)^{J-V} \otimes (d\tau)^V \otimes \cdots \right] = 0
\]
and it gives us the required formula since all terms are 0 unless \(j - p^m \leq v < j\).

For \(j = p^m\), we get
\[
h \left[ \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau)^{\beta_m} \otimes \cdots \right]
\]
\[+ \sum_{0 \leq v < p^m} \left\langle \frac{p^m}{v} \right\rangle h \left[ \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\tau)^{p^m-v} \otimes (d\hat{\tau})^J \otimes (d\tau)^{\nu} \otimes \cdots \right] = 0
\]
which gives us
\[
h \left[ \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\tau)^{p^m} \otimes (d\hat{\tau})^J \otimes \cdots \right] = -h \left[ \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau)^{\beta_m} \otimes \cdots \right]
\]
\[- \sum_{0 \leq v < p^m} \left\langle \frac{p^m}{v} \right\rangle h \left[ \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\tau)^{p^m-v} \otimes (d\hat{\tau})^J \otimes (d\tau)^{\nu} \otimes \cdots \right].
\]

2.5. Remark. It follows from Lemma 2.3 that, unless \(J_s = 0\), \(p^m \mid i\), \(j = p^m\) and \(s < r\), we have
\[
h \left[ \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau)^J \right]
\]
\[= (-1)^{J_j-1} \hat{\tau}^{(l)}(d\hat{\tau})^{J_j} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \]
\[\otimes (d\tau)^{J_j} \otimes \cdots \otimes (d\tau)^{J_{j-1}} \]
\[ \otimes h \left[ \tau_{\nu}^{(i)}(d\tau)^{\nu J_1} \otimes (d\tau)^{\nu J_2} \otimes \cdots \otimes (d\tau)^{\nu J_L} \right] \otimes (d\tau)^{J_{L+1}} \otimes \cdots \otimes (d\tau)^{J_k} \]

Using the formulas in Proposition 2.4, we also see that we always have
\[ h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_{L+1}} \otimes \cdots \otimes (d\hat{\tau})^{J_k} \right] \]
\[ \otimes (d\tau)^{J_{L+1}} \otimes \cdots \otimes (d\tau)^{J_k} \]
\[ = (-1)^{J_{L+1} - 1} \hat{\tau}^{(i)}(d\hat{\tau})^{J_{L+1}} \otimes \cdots \otimes (d\hat{\tau})^{J_k} \]
\[ \otimes h \left[ \tau_{\nu}^{(i)}(d\tau)^{\nu J_1} \otimes (d\tau)^{\nu J_2} \otimes \cdots \otimes (d\tau)^{\nu J_L} \right]. \]

Moreover, if \( j \neq 0 \), then,
\[ h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{L-1}} \right] \]
\[ \otimes (d\tau)^{J_{L-1}} \otimes \cdots \otimes (d\tau)^{J_k} \]
\[ = (-1)^{-1} \hat{\tau}^{(i)}(d\hat{\tau})^{J_{L-1}} \otimes \cdots \otimes (d\hat{\tau})^{J_k} \]
\[ \otimes h \left[ \tau_{\nu}^{(i)}(d\tau)^{\nu J_1} \otimes (d\tau)^{\nu J_2} \otimes \cdots \otimes (d\tau)^{\nu J_L} \right]. \]

2.6. **Proposition.** The map \( h \) is well defined.

**Proof.** We need to show that our definition is stable under the relations induced by
\[ \sum_{0 < V < J} \left[ \frac{J}{V} \right] (d\tau)^{J-V} \otimes (d\tau)^{V} = 0, \]
for \(|J| > p^m\). We first show that
\[ \sum_{0 < V < J} \left[ \frac{J}{V} \right] h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{L-1}} \otimes (d\hat{\tau})^{J-V} \otimes (d\tau)^{V} \otimes \cdots \right] = 0 \]
whenever \(|J| > p^m\). As in the proof of Proposition 2.4, we write \( j = j_n \) and \( v = v_n \). The case \( j = 0 \) is trivial and the case \( j \geq p^m \) was done in Proposition 2.4. Thus, we are left with the case \( 0 < j < p^m \). As usual, our sum splits into two sums:
\[ \sum_{0 \leq V < J} \left[ \frac{J}{V} \right] h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{L-1}} \otimes (d\hat{\tau})^{J-V} \right] \]
\[ \otimes (d\tau)^{V} \otimes (d\tau)^{J_1} \otimes \cdots \]
\[ + \sum_{0 \leq V < j} \left[ \frac{J}{V} \right] h \left[ \hat{\tau}^{(i)}(d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{L-1}} \right] \]
\[ \otimes (d\tau)^{J-V} \otimes (d\tau)^{V} \otimes (d\hat{\tau})^{J} \otimes \cdots \].
Only the case $p^m | i$ has to be considered. The first sum reduces to
\[
\begin{align*}
&h \left[ \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau_i)^{j} \otimes \cdots \right] \\
&= (-1)^{i} \left\langle \frac{i + j}{i} \right\rangle^{i-1} \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau_i)^{j} \otimes \cdots \\
&= (-1)^{i} \left\langle \frac{i + j}{i} \right\rangle^{i-1} \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau_i)^{j} \otimes \cdots \\
&= (-1)^{i} \left\langle \frac{i + j}{i} \right\rangle^{i-1} \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau_i)^{j} \otimes \cdots.
\end{align*}
\]
Now, we need to compute the second sum. We get
\[
(-1)^{s-1} \sum_{0 \leq v < j} \left\langle \frac{j}{v} \right\rangle \left\langle \frac{i + j - v}{i} \right\rangle^{i-1} \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\tau_i)^{j} \otimes \cdots
\]
\[
\otimes (d\tau_v)^{\nu} (d\hat{\tau})^J \otimes \cdots
\]
\[
= (-1)^{s-1} \left\langle \frac{i + j}{i} \right\rangle^{i-1} \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau_i)^{j} \otimes \cdots
\]
\[
+ (-1)^{s-1} \sum_{0 \leq v \leq i + j} \left\langle \frac{i + j}{v} \right\rangle \cdots \otimes (d\tau_v)^{\nu} (d\hat{\tau})^J \otimes \cdots
\]
\[
= (-1)^{s-1} \left\langle \frac{i + j}{i} \right\rangle^{i-1} \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau_i)^{j} \otimes \cdots
\]
\[
+ (-1)^{s-1} \left\langle \frac{i + j}{i} \right\rangle^{i-1} \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\hat{\tau})^J \otimes (d\tau_i)^{j} \otimes \cdots.
\]
Adding both formulas, we do get zero as expected. To finish the proof, we also need to make sure that
\[
\sum_{0 < V < j} \left\langle \frac{J}{V} \right\rangle h \left[ \hat{\tau}_{\nu}^{(i)} (d\hat{\tau})^{J_1} \otimes \cdots \otimes (d\hat{\tau})^{J_{j-1}} \otimes (d\tau_i)^{j} \otimes (d\tau)^{j-V} \otimes (d\tau)^{V} \otimes \cdots \right] = 0
\]
whenever $|J| > p^m$. We consider the $n$-uple $K = (\hat{J}, j_n + p^m)$. Since the differential satisfies $d^2 = 0$, we always have
\[
\sum_{0 < W < K} \left\langle \frac{K}{W} \right\rangle d \left[ (d\tau)^{K-W} \right] \otimes (d\tau)^{W} = \sum_{0 < W < K} \left\langle \frac{K}{W} \right\rangle (d\tau)^{K-W} \otimes d \left[ (d\tau)^{W} \right],
\]
and it follows that
\[
\sum_{0 < W < K} \left( K \left( W \right) h \left[ \hat{\tau}^{(I)} \right] \left( d \hat{\tau} \right)^{J_1} \otimes \cdots \otimes (d \hat{\tau})^{J_{-1}} \otimes d \left[ (d \tau)^{K-W} \right] \otimes (d \tau)^W \otimes \cdots \right)
\]
\[
= \sum_{0 < W < K} \left( K \left( W \right) h \left[ \hat{\tau}^{(I)} \right] \left( d \hat{\tau} \right)^{J_1} \otimes \cdots \otimes (d \hat{\tau})^{J_{-1}} \otimes (d \tau)^{K-W} \right)
\]
\[
\otimes d \left[ (d \tau)^W \right] \otimes \cdots \right].
\]

In the first sum, if \(|W| > p^m\), we obviously get 0. If not, then \(|K - W| > p^m\) and we can use the first part of the proof to conclude that the corresponding term is 0. Thus, we see that
\[
\sum_{0 < W < K} \left( K \left( W \right) h \left[ \hat{\tau}^{(I)} \right] \left( d \hat{\tau} \right)^{J_1} \otimes \cdots \otimes (d \hat{\tau})^{J_{-1}} \otimes (d \tau)^{K-W} \right)
\]
\[
\otimes d \left[ (d \tau)^W \right] \otimes \cdots \right) = 0.
\]

Writing \(k = k_n\) and \(w = w_n\), it is clear that all the terms are zero unless \(w = k\) or \(W = K\). Therefore, we have
\[
\sum_{0 \leq W < K} \left( K \left( W \right) h \left[ \hat{\tau}^{(I)} \right] \left( d \hat{\tau} \right)^{J_1} \otimes \cdots \otimes (d \hat{\tau})^{J_{-1}} \otimes (d \hat{\tau})^{K-W} \right)
\]
\[
\otimes d \left[ (d \hat{\tau})^W \right] \otimes \cdots \right]
\]
\[
+ \sum_{0 \leq w < k} \left( K \left( W \right) h \left[ \hat{\tau}^{(I)} \right] \left( d \hat{\tau} \right)^{J_1} \otimes \cdots \otimes (d \hat{\tau})^{J_{-1}} \right)
\]
\[
\otimes (d \tau)^k \otimes d \left[ (d \tau)^{p^m} \right] \otimes \cdots \right) = 0.
\]

Using the first part of the proof, we see that all the terms in the first sum are zero unless \(W = 0\) and \(k = p^m\). But in this case, \(|\hat{K} - \hat{W}| > p^m\) and we also get 0. Finally using Definition 2.1, we easily see that all the terms in the second sum are 0 unless \(k - w = p^m\). This gives us exactly what we want.

3. THE FORMAL POINCARÉ LEMMA

NIN HIGHER LEVEL

We are going to prove in full generality a result which extends Example 2.2(a). First, we need a lemma.
3.1. Lemma. With the notations of Section 2, we have, for $0 < j < p^m$,
\[ h \left[ d \left( \tau_n^{(i)}(d\tilde{\tau})^j(d\tau_n)^j \right) \otimes P \right] = h d \left[ \tau_n^{(i)}(d\tilde{\tau})^j(d\tau_n)^j \right] \otimes P. \]

Proof. Since $h$ is linear and
\[ d \left( \tau_n^{(i)}(d\tilde{\tau})^j(d\tau_n)^j \right) = d \left( \tau_n^{(i)} \otimes (d\tilde{\tau})^j(d\tau_n)^j \right) + \tau_n^{(i)} d \left[ (d\tilde{\tau})^j(d\tau_n)^j \right], \]
it is enough to show that we can move the $P$ outside both in $h[d(\tau_n^{(i)}) \otimes (d\tilde{\tau})^j(d\tau_n)^j \otimes P]$ and in $h[\tau_n^{(i)}d((d\tilde{\tau})^j)(d\tau_n)^j \otimes P]$. We first compute
\[ d \left( \tau_n^{(i)} \right) \otimes (d\tilde{\tau})^j(d\tau_n)^j = \sum_{0 < u \leq i} \left( i \mu \right) \tau_n^{(i-u)}(d\tau_n)^u \otimes (d\tilde{\tau})^j(d\tau_n)^j. \]
Since $j \neq 0$ and $j \neq p^m$, it follows from Remark 2.5 that we can move the $P$ outside in $h[d(\tau_n^{(i)}) \otimes (d\tilde{\tau})^j(d\tau_n)^j \otimes P]$. Now, we compute
\[ d \left[ (d\tilde{\tau})^j(d\tau_n)^j \right] = - \sum_{0 < (V; e) < (J; j)} \binom{J}{V} \binom{j}{V} (d\tilde{\tau})^{j-V} (d\tau_n)^{j-v} \otimes (d\tilde{\tau})^V (d\tau_n)^v. \]
Multiplying by $\tau_n^{(i)}$ on the left, tensoring with $P$ on the right, and applying $h$, this expression splits into two sums as usual
\[ - \sum_{0 \leq V < j} \binom{J}{V} h \left[ \tau_n^{(i)}(d\tilde{\tau})^{j-V} \otimes (d\tilde{\tau})^V (d\tau_n)^j \otimes P \right] \]
\[ - \sum_{0 \leq V < j} \binom{j}{V} h \left[ \tau_n^{(i)}(d\tau_n)^{j-v} \otimes (d\tilde{\tau})^j (d\tau_n)^v \otimes P \right] \]
(actually, if $J = 0$, then $v$ cannot be 0 in the second sum, but it does not matter). Thanks to Remark 2.5 again, we see that we can move the $P$ outside in $h[\tau_n^{(i)}d((d\tilde{\tau})^j)(d\tau_n)^j \otimes P]$ since $j < p^m$ in the first sum and $j - v < p^m$ in the second.

3.2. Proposition. With the notations of Section 2, we have $h \circ d + d \circ h = \text{Id} - \pi$ where $\pi$ is the projector that sends $\tau_i$ and $(d\tau_i)^j$ to itself if $i < n$ and to 0 otherwise.

Proof. We only need to check this identity on a set of generators of $\Omega^r_{p^m}$, and we take the elements of the form
\[ Q \otimes \tau_n^{(i)}(d\tilde{\tau})^j(d\tau_n)^j \otimes P \]
with $Q := \tilde{\tau}^{(i)}(d\tilde{\tau})^1 \otimes \cdots \otimes (d\tilde{\tau})^{j-1}$, $P := (d\tau)^{j+1} \otimes \cdots \otimes (d\tau)^j$ and, e
ther $s = r$ or $0 < j < p^n$. We have

$$(hd + dh) \left[ Q \otimes \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \otimes P \right]$$

$$= hd \left[ Q \otimes \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \otimes P \right] + dh \left[ Q \otimes \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \otimes P \right]$$

$$= h \left[ dQ \otimes \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \otimes P \right]$$

$$+ (-1)^{s-1} h \left[ Q \otimes d \left( \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right) \otimes P \right]$$

$$+ (-1)^s h \left[ Q \otimes \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \otimes dP \right]$$

$$+ (-1)^s h \left[ dQ \otimes h \left( \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right) \otimes P \right]$$

$$= (-1)^s dQ \otimes h \left[ \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right] P + Q \otimes h \left[ d \left( \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right) \otimes P \right]$$

$$- Q \otimes h \left[ \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right] dP + (-1)^{s-1} dQ \otimes h \left[ \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right] P$$

$$+ Q \otimes dh \left( \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right) \otimes P + Q \otimes h \left[ \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right] dP$$

$$= Q \otimes h \left[ d \left( \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right) \otimes P \right] + Q \otimes dh \left( \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right) \otimes P,$$

and, using Lemma 3.1 we get

$$(hd + dh) \left[ Q \otimes \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \otimes P \right]$$

$$= Q \otimes (hd + dh) \left[ \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right] \otimes P.$$

Thus, we may assume that $r \leq 1$. In other words, we are left with showing that we always have

$$(hd + dh) \left[ \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j \right] = \tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j$$

unless $i = j = 0$ in which case we get 0. We write $\omega := \tau_n^{(0)} (d\tau_n)^j$ and we compute $(hd + dh)[\omega (d\hat{\tau})^j]$. We assume first $j \neq 0$ and $J \neq 0$. The same computation as in Lemma 3.1 gives us

$$hd[\omega (d\hat{\tau})^j] = \sum_{0 \leq u \leq i} \left\langle i \atop u \right\rangle h [\tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j]$$

$$- \sum_{0 \leq v < J} \left\langle J \atop V \right\rangle h [\tau_n^{(0)} (d\hat{\tau})^j (d\tau_n)^j]$$

$$- \sum_{0 \leq v < j} \left\langle J \atop v \right\rangle h [\tau_n^{(0)} (d\tau_n)^j (d\tau_n)^j].$$
\[
\begin{aligned}
&= \sum_{0 < u \leq l} \left\langle i_{\mu} \right\rangle h \left[ \tau_n^{l-u}(d\tau_n)^u \otimes (d\tau_n)^i \right] (d\hat{\tau})^j \\
&\quad + (d\hat{\tau})^j h \left[ \tau_n^{l-u}(d\tau_n)^i \right] \\
&\quad - \sum_{0 < u < l} \left\langle j \right\rangle h \left[ \tau_n^{l-u}(d\tau_n)^{i-j} \otimes (d\tau_n)^j \right] (d\hat{\tau})^j \\
&\quad - h \left[ \tau_n^{l-u}(d\tau_n)^i \right] \otimes (d\hat{\tau})^j \\
&= hd(\omega) \otimes (d\hat{\tau})^j - h(\omega)(d\hat{\tau})^j + (d\hat{\tau})^j h(w) \\
&= hd(\omega) \otimes (d\hat{\tau})^j - dh(w)(d\hat{\tau})^j.
\end{aligned}
\]

Adding to
\[
dh \left[ \omega(d\hat{\tau})^j \right] = dh \left[ (\omega)(d\hat{\tau})^j \right] = dh(\omega) \otimes (d\hat{\tau})^j + h(\omega)dh \left[ (d\hat{\tau})^j \right],
\]
we see that
\[
(hd + dh) \left[ \omega(d\hat{\tau})^j \right] = (hd + dh)(\omega) \otimes (d\hat{\tau})^j.
\]

Obviously, this formula still holds when \( J = 0 \). What happens if \( j = 0 \) and \( J \neq 0 \)? We have
\[
hd \left[ \tau_n^{l-u}(d\hat{\tau})^j \right] = \sum_{0 < u \leq l} \left\langle i_{\mu} \right\rangle h \left[ \tau_n^{l-u}(d\tau_n)^u \otimes (d\hat{\tau})^i \right] \\
- \sum_{0 < u < l} \left\langle j \right\rangle h \left[ \tau_n^{l-u}(d\tau_n)^{i-j} \otimes (d\tau_n)^j \right] = hd(\tau_n^{l-u}) \otimes (d\hat{\tau})^i.
\]

On the other hand, we have \( dh[\tau_n^{l-u}(d\hat{\tau})^j] = 0 = dh(\tau_n^{l-u})(d\hat{\tau})^i \) since, by definition, \( h = 0 \) when \( r \leq 0 \). Thus, we see that
\[
(hd + dh) \left[ \tau_n^{l-u}(d\hat{\tau})^j \right] = (hd + dh)(\tau_n^{l-u})(d\hat{\tau})^i.
\]

In other words, we may assume that \( n = 1 \) (and still \( r \leq 1 \)). Thus, we want to show that we always have
\[
(hd + dh) \left[ \tau^{l-u}(d\hat{\tau})^j \right] = \tau^{l-u}(d\hat{\tau})^j
\]
unless \( i = j = 0 \), in which case we get 0. For \( j = 0 \), this is done in Example 2.2(a) so we assume now \( j \neq 0 \). If \( i = 0 \), then we have
\[
(hd + dh) \left[ \tau^{l-u}(d\hat{\tau})^j \right] = hd \left[ \tau^{l-u}(d\hat{\tau})^j \right] + dh \left[ (d\hat{\tau})^j \right] \\
= - \sum_{0 < u < l} \left\langle j \right\rangle h \left[ \tau^{l-u}(d\hat{\tau})^{i-j} \otimes (d\tau_n)^i \right] + d(\tau_n^{l-u}).
\[
= - \sum_{0 < u < j} \left\langle \frac{i + j}{i} \right\rangle \tau^{(u)}(d\tau)^r \\
+ \sum_{0 < v < j} \left\langle \frac{j}{v} \right\rangle \tau^{(u)}(d\tau)^r = (d\tau)^t.
\]

Thus, we are left with the case \(i \neq 0\) and \(j \neq 0\). We have
\[
hd\left[\tau^{(0)}(d\tau)^t\right] = h[d\tau^{(0)} \otimes (d\tau)^t] + h[\tau^{(0)}d((d\tau)^t)]
\]
\[
= \sum_{0 < u \leq i} \left\langle \frac{i}{u} \right\rangle h\left[\tau^{(u)}(d\tau)^u \otimes (d\tau)^t\right]
\]
\[
- \sum_{0 < v < j} \left\langle \frac{j}{v} \right\rangle h\left[\tau^{(0)}(d\tau)^j \otimes (d\tau)^t\right].
\]

Assume first that \(p^m \nmid i\), then the second sum is 0. Moreover, in the first sum, if we write \(i = qp^m + t\) with \(0 < t < p^m\), the only \(u\) for which we do not get a 0 term is \(u = t\). Thus, we see that
\[
hd\left[\tau^{(0)}(d\tau)^t\right] = \left\langle \frac{i}{t} \right\rangle \tau^{(0)}(d\tau)^t = \tau^{(0)}(d\tau)^t.
\]

Since in this case \(dh[\tau^{(0)}(d\tau)^t] = 0\), we are done.

Finally, we are left with the case \(p^m \mid i\) and \(i \neq 0\). We have
\[
dh\left[\tau^{(0)}(d\tau)^t\right] = d\left[\left\langle \frac{i + j}{i} \right\rangle^{-1} \tau^{(i+j)}\right]
\]
\[
= \left\langle \frac{i + j}{i} \right\rangle^{-1} \sum_{0 < u \leq p^m} \left\langle \frac{i + j}{u} \right\rangle \tau^{(i+j-u)}(d\tau)^u
\]
and
\[
hd\left[\tau^{(0)}(d\tau)^t\right] = \sum_{0 < u \leq p^m} \left\langle \frac{i}{u} \right\rangle h\left[\tau^{(i+u)}(d\tau)^u \otimes (d\tau)^t\right]
\]
\[
- \sum_{0 < v < j} \left\langle \frac{j}{v} \right\rangle h\left[\tau^{(0)}(d\tau)^j \otimes (d\tau)^t\right]
\]
\[
= \left\langle \frac{i}{p^m} \right\rangle h\left[\tau^{(i-p^m)}(d\tau)^{p^m} \otimes (d\tau)^t\right]
\]
\[
- \sum_{0 < v < j} \left\langle \frac{j}{v} \right\rangle \left\langle \frac{i + j - v}{i} \right\rangle^{-1} \tau^{(i+j-v)}(d\tau)^v
\]

Now, we use the formula of Proposition 2.3, or more precisely in this case,
the formula of Example 2.2(c):

$$h \left[ \tau^{(i-p_m)}(d\tau)^{p_m} \otimes (d\tau)^i \right]$$

\[= - \sum_{j < v \leq p_m} \left\langle \frac{p^m + j}{j} \right\rangle^{-1} \left\langle \frac{p^m + j}{v} \right\rangle \left[ \tau^{(i-p_m)}(d\tau)^{p_m+j-v} \otimes (d\tau)^v \right] \]

\[= - \sum_{j < v \leq p_m} \left\langle \frac{p^m + j}{j} \right\rangle^{-1} \left\langle \frac{p^m + j}{i - p^m} \right\rangle^{-1} \tau^{(i+j-v)}(d\tau)^v. \]

Now, we have

$$\left\langle \frac{i}{p^m} \right\rangle \left\langle \frac{p^m + j}{j} \right\rangle^{-1} \left\langle \frac{p^m + j}{v} \right\rangle \left\langle \frac{i + j - v}{i - p^m} \right\rangle^{-1} = \left\langle \frac{i + j}{i} \right\rangle^{-1} \left\langle \frac{i + j}{v} \right\rangle$$

and it follows that

$$\left\langle \frac{i}{p^m} \right\rangle h \left[ \tau^{(i-p_m)}(d\tau)^{p_m} \otimes (d\tau)^i \right]$$

\[= - \left\langle \frac{i + j}{i} \right\rangle^{-1} \sum_{j < v \leq p_m} \left\langle \frac{i + j}{v} \right\rangle \tau^{(i+j-v)}(d\tau)^v. \]

Since we also have

$$\left\langle \frac{j}{v} \right\rangle \left\langle \frac{i + j - v}{i} \right\rangle^{-1} = \left\langle \frac{i + j}{i} \right\rangle^{-1} \left\langle \frac{i + j}{v} \right\rangle$$

we get that

$$h d \left[ \tau^{(i)}(d\tau)^i \right] = - \left\langle \frac{i + j}{i} \right\rangle^{-1} \sum_{0 < v \leq p_m} \left\langle \frac{i + j}{v} \right\rangle \tau^{(i+j-v)}(d\tau)^v. \]

Adding both formulas, we get

$$\left[ h d + dh \right] \left[ \tau^{(i)}(d\tau)^i \right] = \tau^{(i)}(d\tau)^i$$

and the proposition is proved.

As a consequence, we get the Formal Poincaré Lemma in higher level:

3.3. Theorem. The de Rham complex $\Omega_{p_m}$ is a resolution of $\mathcal{O}_X$.

Proof. We proceed by induction on the dimension on $X$. The assertion being local in nature, it is sufficient to show that if $f: X \to X'$ is a smooth morphism of relative dimension one of smooth schemes over $S$, then the map of complexes on $X, f^* f^* \Omega_{p_m} \to \Omega_{p_m}$ is a quasi-isomorphism. More precisely, locally, there are coordinates $t_1, \ldots, t_n$ on $X$ such that $t_1, \ldots, t_{n-1}$
are coordinates on $X'$ and $f^*$ is a homotopy equivalence. Actually, since this map, which is just the inclusion $\tau_i \mapsto \tau_i$ and $(d\tau_i)^j \mapsto (d\tau_i)^j$ for $i = 1, \ldots, n - 1$, has an obvious retraction given by $\tau_n \mapsto 0$ and $(d\tau_n)^j \mapsto 0$, it is sufficient to show that the projector $\pi: \tau_n \mapsto 0$ and $(d\tau_n)^j \mapsto 0$ on $\Omega^0_{P_m}$ is homotopic to the identity. In other words, we need to find a homotopy $h$ on $\Omega^0_{P_m}$ such that $h \circ d + d \circ h = \text{Id} - \pi$. But this is just the content of Proposition 3.2.

4. THE POINCARÉ LEMMA IN
CRYSTALLINE COHOMOLOGY

We show that, as in the case $m = 0$, one can derive a de Rham interpretation of crystalline cohomology from the Formal Poincaré Lemma. What follows is a straightforward generalization of the classical proof (see [B1, B-O]).

For the basic notions concerning the crystalline site and topos of level $m$, the reader can look at LS-Q, 4.1.

We let $(S, \alpha, b)$ be an $m$-PD-scheme with $p$ locally nilpotent such that $p \in \alpha$ and let $X$ be a smooth $S$-scheme to which the $m$-PD-structure of $S$ extends.

4.1. Crystals and the de Rham Complex. If $E$ is an abelian sheaf on $\text{Cris}^{(m)}(X/S)$, the Čech–Alexander complex $E_{P(r)}$ of $E$ is the complex of abelian sheaves on $X$ obtained by applying $E$ to the simplicial complex $P_{X_m}(\cdot)$. The normalized Čech–Alexander complex is the complex $NE_{P(r)} = \cap \ker s^i$.

Assume $E$ is an $m$-crystal. Then for each $r$, the first projection $p_1: P_{X_{X_m}}^{(r)}(r) \to X$ gives us an isomorphism $E_{P(r)} \cong E_X \otimes_{\mathcal{O}_X} \mathcal{P}_{X_m}(r)$ and we will sometimes write $E_X \otimes_{\mathcal{O}_X} \mathcal{P}_{X_m}(\cdot)$ instead of $E_{P(r)}$. However, the reader should keep in mind that, at this point, the differential is not linear in $E_X$.

We also have $NE_{P(r)} = E_X \otimes_{\mathcal{O}_S} N\mathcal{P}_{X_m}(r)$ and we will likewise sometimes write $NE_{P(r)} = E_X \otimes_{\mathcal{O}_S} N\mathcal{P}_{X_m}(\cdot)$. Recall that $\mathcal{X}_{X_m}$ denotes the differential ideal in $N\mathcal{P}_{X_m}(\cdot)$ generated by $\mathcal{Y}_{X_m}(1)^{p_{X_m}+1}$. Then $E_X \otimes_{\mathcal{O}_S} \mathcal{X}_{X_m}$ is stable under the differential of $E_X \otimes_{\mathcal{O}_S} N\mathcal{P}_{X_m}(\cdot)$ and we get another description of the de Rham complex of the $\mathcal{X}_{X_m}$-module $E_X$:

$$E_X \otimes \Omega^\cdot_{X_m} = (E_X \otimes N\mathcal{P}_{X_m}(\cdot)) / (E_X \otimes \mathcal{X}_{X_m}).$$

We can also consider the linearized construction: we build the complex of abelian sheaves $E_{P(r)}$ on $X$ obtained by applying the sheaf $E$ to the simplicial complex $P_{X_m}(\cdot)$. The map of simplicial complexes $d_0: P_{X_m}(\cdot) \to P_{X_m}(\cdot)$ induces a morphism of complexes $d_0^* : E_{P(r)} \to E_{P(r)}$.

If we consider $P_{X_m}(\cdot)$ as a simplicial complex over $X$ by using the first projection $p_1$, if $E$ is an $m$-crystal and if we identify $E_{P(r)}$ with $E_X \otimes_{\mathcal{O}_S}$
$P_{m}(r)$ as before, then $E_{m}^{+} = E_{X} \otimes_{\mathcal{O}_{X}} P_{m}(-1)$ with a differential that is linear in $E_{X}$. In other words, we see that $E_{m}^{+} = E_{X} \otimes_{\mathcal{O}_{X}} P_{m}(-1)$. With these notations, the map $d_{n}^{m}: \frac{\mathcal{P}_{m}(1)}{H_{n}^{2}} \to E_{X} \otimes_{\mathcal{O}_{X}} P_{m}(n)$ is given for each $r$ by the composition of the scalar extension

$$E_{X} \otimes P_{m}(r) \to P_{m}(1) \otimes E_{X} \otimes P_{m}(r)$$

and the map

$$e \otimes \text{Id}: P_{m}(1) \otimes E_{X} \otimes P_{m}(r) \to E_{X} \otimes P_{m}(1) \otimes P_{m}(r) = E_{X} \otimes P_{m}(r + 1),$$

where $e: P_{m}(1) \otimes E_{X} \to E_{X} \otimes P_{m}(1)$ is the Taylor isomorphism.

4.2. Crystalline Interpretation of Linearization. We consider the crystalline site of level $m$, $\text{Cris}^{(m)}(X/S)$ and the associated topos $(X/S)^{(m)}_{\text{cris}}$. We have a canonical morphism of topoi

$$u = u_{X/S}^{(m)}: (X/S)^{(m)}_{\text{cris}} \to X_{\text{Zar}}$$

and a localization morphism

$$j_{X}:(X/S)^{(m)}_{\text{cris}} \to (X/S)^{(m)}_{\text{cris}}.$$

Note that $j_{X}$ is exact. If we compose $u$ with $j_{X}$ we get a morphism of topoi

$$u_{|X} = (X/S)^{(m)}_{\text{cris}} \to X_{\text{Zar}}.$$

It is actually a morphism of ringed topoi. To see this, note that the structural sheaf of $X_{\text{Zar}}$ is $\mathcal{O}_{X}$ and the structural sheaf of $(X/S)^{(m)}_{\text{cris}}$ is the restriction of $\mathcal{O}_{X/S}^{(m)}$. Thus, we need to have a morphism of rings $u_{|X}^{-1} \mathcal{O}_{X} \to \mathcal{O}_{X/S}^{(m)}$. In other words, given an open subset $U$ of $X$, an $m$-PD-thickening $U \to Y$, and a morphism $p: Y \to X$ extending the inclusion of $U$ in $X$, we need a morphism $\mathcal{O}_{U} \to \mathcal{O}_{Y}$ and we just take $p^{-1}$. One can check that $u_{|X}$ is exact and that the adjunction map $u_{|X} u_{|X}^{-1} $ is an isomorphism.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X}$-modules, we let $L^{(m)}(\mathcal{F}) := j_{X}^{*} u_{|X}^{*} \mathcal{F}$ and $L^{(m)}(\mathcal{F}) := L^{(m)}(\mathcal{F})_{X}$.

4.3. Proposition. (1) If $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X}$-modules, then $L^{(m)}(\mathcal{F})$ is an $m$-crystal and we have $L^{(m)}(\mathcal{F}) = P_{X}(1) \otimes_{\mathcal{O}_{X}} \mathcal{F}$.

(2) We have $u_{|X}^{(m)} \mathcal{F} = \mathcal{F}$ and $R^{i} u_{|X}^{(m)} \mathcal{F} = 0$ for $i > 0$.

(3) If $E$ is an $m$-crystal, there is a canonical isomorphism $E \otimes L^{(m)}(\mathcal{F}) \cong L^{(m)}(E_{X})$.

Proof. Note first of all that, by definition, given any open subset $U$ of $X$ and any $m$-PD-thickening $U \to Y$, we have $L^{(m)}(\mathcal{F})_{Y} = p_{1}^{*} p_{2}^{*} \mathcal{F}$ where $p_{1}: U_{m}(Y \times_{S} X) \to Y$ and $p_{2}: P_{m}(Y \times_{S} X) \to X$ denote the projections. In particular, $L^{(m)}(\mathcal{F}) = P_{X}(1) \otimes_{\mathcal{O}_{X}} \mathcal{F}$. Now, in order to prove that
$L^{(m)}(\mathcal{F})$ is an $m$-crystal, we need to check that, given a morphism of $m$-PD-thickenings $f: Y' \to Y$, we have $f^*p_1^* + p_2^* = p_1^* + p_2^*$ (where $p_1$ and $p_2$ are defined in the same way as $p_1$ and $p_2$). We may clearly assume that $U = X$. Moreover, the question is local. Thus, since $X$ is smooth, we may assume that $X \to Y$ has a retraction. It is therefore sufficient to consider the case $Y = X$. Writing $Y$ instead of $Y'$, $g$ for the canonical map $P_{X,m}(Y \times_S X) \to P_{X,m}(1)$, $p_1: P_{X,m}(1) \to X$ and $q_1: P_{X,m}(Y \times_S X) \to Y$ for the first projection, we are led to show that $f^*p_1^* = q_1^*g^*$. But this follows from the fact that, $P_{X,m}(1)$ being flat over $X$, the canonical map $P_{X,m}(Y \times_S X) \to Y \times_S P_{X,m}(1)$ is an isomorphism.

We now prove the second assertion. Since the adjunction map $u_{|X} \circ u^*_{|X} \circ E \to E$ is an isomorphism, we have $u_{|X} \circ L^{(m)}(\mathcal{F}) = u_{|X} \circ u^*_{|X} \circ E = E$. Now, since $u_{|X}$ and $j_{X}$ are exact, we obtain

$$R u_{|X} \circ L^{(m)}(\mathcal{F}) = R u_{|X} \circ u^*_{|X} \circ E = R u_{|X} \circ u^*_{|X} \circ E = R u_{|X} \circ L^{(m)}(\mathcal{F}).$$

For the last assertion, we use the same notations as in the beginning of the proof. We need a compatible family of isomorphisms $E_Y \otimes p_1^* + p_2^*$, that is,

$$E_Y \otimes \mathcal{P}_{X,m}(Y \times X) \otimes_{\mathcal{E}_X} \mathcal{F} \sim \mathcal{P}_{X,m}(Y \times X) \otimes E_X \otimes \mathcal{F}.$$ 

But, $E$ being a crystal, we have

$$E_Y \otimes_{\mathcal{E}_X} \mathcal{P}_{X,m}(Y \times X) = E_{P(Y \times U)} = \mathcal{P}_{X,m}(Y \times X) \otimes_{\mathcal{E}_X} E_X.$$ 

4.4. Remark. Before going any further, we would like to describe the Taylor isomorphism of $L^{(m)}(\mathcal{F}) := \mathcal{P}_{X,m}(1) \otimes \mathcal{F}$. By definition, it is obtained by composing the isomorphism

$$\mathcal{P}_{X,m}(1) \otimes L_X^{(m)}(\mathcal{F}) \to \mathcal{P}_{X,m}(1) \otimes (\mathcal{P}_{X,m}(1) \otimes \mathcal{F}) \sim \mathcal{P}_{X,m}(2) \otimes \mathcal{F}$$ 

with the inverse of

$$L_X^{(m)}(\mathcal{F}) \otimes \mathcal{P}_{X,m}(1) = (\mathcal{P}_{X,m}(1) \otimes \mathcal{F}) \otimes \mathcal{P}_{X,m}(1) \sim \mathcal{P}_{X,m}(2) \otimes \mathcal{F}$$ 

$$[(f \otimes g) \otimes (h \otimes k) \otimes x] \mapsto fh \otimes k \otimes g \otimes x,$$

where $\otimes'$ means that the tensor product uses both left structures. Thus, the Taylor isomorphism is given by

$$\varepsilon: \mathcal{P}_{X,m}(1) \otimes L_X^{(m)}(\mathcal{F}) \sim L_X^{(m)}(\mathcal{F}) \otimes \mathcal{P}_{X,m}(1)$$ 

$$(f \otimes g) \otimes [(h \otimes k) \otimes x] \mapsto [(f \otimes k) \otimes x] \otimes (1 \otimes gh).$$

4.5. Proposition. (1) There exists a unique $\theta^{(m)}_{X/S}$-linear differential on $L^{(m)}(\Omega_{X,m}^*)$ such that $L^{(m)}(\Omega_{X,m}^*) = \Omega_{X,m}^*$. 

(2) If $E$ is an $m$-crystal, we have
\[ Ru_{X/S}^{m}(E \otimes L^{(m)}(\Omega_{Xm})) = E_{X} \otimes \Omega_{Xm}. \]

Proof. We saw in Proposition 4.3 that $L_{X}^{(m)}(\Omega_{Xm}^{r})$ is an $m$-crystal and that, for all $r$, we have $L_{X}^{(m)}(\Omega_{Xm}^{r}) = \Omega_{Xm}^{r}$. To prove the first assertion it is sufficient to check that the differential of $\Omega_{Xm}^{r}$ is compatible with the Taylor isomorphisms. In order to do so, we describe the Taylor isomorphism of $L_{X}^{(m)}(\mathcal{F})$ in the particular case $\mathcal{F} = \mathcal{P}_{Xm}(k)$. We have canonical isomorphisms $\mathcal{P}_{Xm}(1) \otimes L^{(m)}(\mathcal{F}) = \mathcal{P}_{Xm}(k + 2)$ and $L_{X}^{(m)}(\mathcal{F}) \otimes \mathcal{P}_{Xm}(1) = \mathcal{P}_{Xm}(k + 2)$. From Remark 4.4, we see that the Taylor isomorphism is induced by the automorphism
\[ f \otimes g \circ h_{1} \otimes \cdots \otimes h_{k} \rightarrow f \otimes h_{1} \otimes \cdots \otimes h_{k} \otimes g \]
of $\mathcal{P}_{Xm}(k + 2)$. In particular, we see that the maps
\[ d : L^{(m)}(\mathcal{P}_{Xm}(k)) = \mathcal{P}_{Xm}(k + 1) \rightarrow L^{(m)}(\mathcal{P}_{Xm}(k + 1)) = \mathcal{P}_{Xm}(k + 2) \]
are compatible with the Taylor isomorphisms, for $i > 0$. It follows that the same is true for the differential of the complex $\mathcal{P}_{Xm}^{(m)}(\cdot)$ and also for the differential of the complex $\Omega_{Xm}^{r}$, which is a subquotient of $\mathcal{P}_{Xm}^{(m)}(\cdot)$.

The second assertion will follow from Proposition 4.2 once we know that $u_{*}$ sends the differential of $E_{X} \otimes \Omega_{Xm}^{r}$ to the differential of $E_{X} \otimes \Omega_{Xm}^{r}$. Note that, in general, the map
\[ d^{u}_{0} \mathcal{F} = u_{*} L^{(m)}(\mathcal{F}) \rightarrow L_{X}^{(m)}(\mathcal{F}) = \mathcal{P}_{Xm}(1) \otimes \mathcal{F} \]
is just the adjunction map for $u_{X}$. In particular, it is functorial in $L^{(m)}(\mathcal{F})$ and injective. In order to show that $u_{*}$ sends the differential of $E_{X} \otimes \Omega_{Xm}^{r}$ to the differential of $E_{X} \otimes \Omega_{Xm}^{r}$, it is therefore sufficient to check that $d^{u}_{0}$ preserves differentials. But as we mentioned in Subsection 4.1, $d^{u}_{0}$ comes from a morphism of simplicial complexes.

4.6. Remark. It is not difficult to describe the $\mathcal{P}_{X}^{(m)}$-structure on $L_{X}^{(m)}(\mathcal{F})$. Let us first consider the $m$-connection
\[ \nabla : L_{X}^{(m)}(\mathcal{F}) \rightarrow L_{X}^{(m)}(\mathcal{F}) \otimes \mathcal{P}_{Xm}(1) \]
induced by the Taylor isomorphism (we write $\nabla$ and not $d$, so as to avoid confusion with the differential of the de Rham complex). We have
\[
\nabla[(f \otimes g) \otimes x] = [(1 \otimes g) \otimes x] \otimes (1 \otimes f) - [(f \otimes g) \otimes x] \otimes (1 \otimes 1) = [(1 \otimes g) \otimes x] \otimes (1 \otimes f) - [(1 \otimes g) \otimes x] \otimes (f \otimes 1) = [(1 \otimes g) \otimes x] \otimes df.
\]
If we have local coordinates on $X$, this last formula gives
\[ \nabla(\tau \otimes x) = \nabla[(1 \otimes t) \otimes x] - \nabla[(t \otimes 1) \otimes x] = -[(1 \otimes 1) \otimes x] \otimes \tau \]
from which we derive
\[ \nabla(\tau^{(I)} \otimes x) = \sum_{0 < V \subseteq J} (-1)^{|V|} \left[ I \atop V \right] \tau^{(J-V)} \otimes x \otimes \tau^{(I)}. \]

At this point, it is convenient to use partial derivatives. Thus, we see that
\[ \nabla^{(I)}[(f \otimes 1) \otimes x] = \partial^{(I)}(f) \otimes x \]
and
\[ \nabla^{(I)}(\tau^{(J)} \otimes x) = (-1)^{|J|} \left[ I \atop J \right] \tau^{(J-I)} \otimes x. \]

Finally, we get
\[ \nabla^{(I)}[(f \tau^{(J)} \otimes x) = \sum_{0 \leq U \subseteq I} (-1)^{|U|} \left[ I \atop U \right] \partial^{(I-U)}(f) \tau^{(J-U)} \otimes x. \]

Using this formula, it is not hard to check directly that
\[ (\nabla^{(I)} \circ d)(f \tau^{(J)} \otimes (d\tau)^{I_1} \otimes \cdots \otimes (d\tau)^{I_n}) \]
\[ = (d \circ \nabla^{(I)})(f \tau^{(J)} \otimes (d\tau)^{I_1} \otimes \cdots \otimes (d\tau)^{I_n}). \]

This gives another proof that the differential of the complex \( L^{(m)}(\Omega^{\pm}_X) \) is \( \mathcal{D}_X^{(m)} \)-linear.

We prove now the Poincaré Lemma for crystals of level \( m \) and show that de Rham cohomology of level \( m \) can be used to compute crystalline cohomology of level \( m \).

4.7. Theorem. If \( E \) is an \( m \)-crystal on \( X/S \) then the complex \( E \otimes L^{(m)}(\Omega^{\pm}_X) \) is a resolution of \( E \), we have an isomorphism in the derived category
\[ R\underline{\text{H}}^{(m)}_{X/S}(E) \sim E_X \otimes \Omega^{\pm}_X, \]
and, for all \( i \), an isomorphism
\[ H^{(m)}_{\text{cris}, i}(X/S, E) \sim H^{(m)}_{dR, i}(X, E_X). \]

Proof. The first assertion is not completely formal since the functor from the category of locally quasi-nilpotent \( \mathcal{D}_X^{(m)} \)-modules to the category of sheaves on \( \text{Cris}^{(m)}(X/S) \) is not exact in general [B1, IV, 1.7.8]. We need to show that the obvious map \( E \to E \otimes L^{(m)}(\Omega^{\pm}_X) \) is a quasi-isomorphism. It is sufficient to check that if \( U \to T \) is a sufficiently small \( m \)-PD-thickening, the map \( E_U \to E_T \otimes L^{(m)}(\Omega^{\pm}_X)_T \) is a quasi-isomorphism. We may assume that \( U = X \) and that \( X \to T \) has a restriction. Moreover, as we saw in Proposition 1.4, \( \Omega^{\pm}_X \) is a complex of locally free sheaves, and we may therefore assume that \( T = X \) and \( E = \mathcal{O}^{(m)}_{X/S}. \) Then, our assertion follows from the Formal Poincaré Lemma (Theorem 3.3).
Applying $Ru^{(m)}_{X/S,\gamma}$ gives the second assertion thanks to Proposition 4.5. Finally, we take cohomology on both sides.

4.8. Generalization to Singular Schemes. Even if we no longer assume that $X$ is smooth over $S$, we can still consider the crystalline topos $(X/S)^{(m)}_{\text{cris}}$ of level $m$ and the canonical morphism of topoi

$$u = u^{(m)}_{X/S,\gamma}(X/S)^{(m)}_{\text{cris}} \to X_{\text{Zar}}.$$ 

Let $E$ be an $m$-crystal on $X/S$. If $i:X \hookrightarrow Y$ is an embedding into a smooth $S$-scheme, we know that $i_\ast E$ is a crystal on $Y/S$ and that $R^qi_\ast E = 0$ for $q > 0$ (see [B-O, 7.1.2] in the case $m = 0$). We write $E_Y := (i_\ast E)_Y$. We can form the de Rham complex $E_Y \otimes \Omega_{\gamma}^m$ (which is actually supported on $X$) and it follows from Theorem 4.7 that

$$RH(X, Ru^{(m)}_{X/S,\gamma} E) = RH(Y, Ri_\ast Ru^{(m)}_{Y/S,\gamma} E) = RH(Y, Ri^{(m)}_{Y/S,i_\ast E}) = RH(Y, (i_\ast E)_Y \otimes \Omega_{\gamma}^m).$$

Therefore, we have

$$H^1_{\text{cris},m}(X/S, E) \sim H^i_{dR,m}(Y, E_Y).$$

5. COMPLEMENTS AND APPLICATIONS

All formal schemes are $p$-adic formal schemes over a fixed complete discrete valuation ring $\mathcal{O}$ of characteristic 0 with residue field of characteristic $p$.

5.1. The de Rham Complex of Finite Order. Let $S$ be a scheme or a formal scheme and $X$ be a smooth (formal) $S$-scheme.

In the formal scheme case, we cannot use divided power envelopes of infinite order. However, for each integer $k$, we can consider the Čech–Alexander complex $\mathcal{P}^k_{X,m}(\cdot)$ of order $k$ [B4, Sect. 2.1], and just as we did in Subsection 1.1, the normalized complex $N\mathcal{P}^k_{X,m}(\cdot)$ associated to it, and the quotient $\Omega^k_{X,m}$ of $N\mathcal{P}^k_{X,m}(\cdot)$ by the differential ideal generated by $\mathcal{P}^k_{X,m}(1)(p^{m+1})$. Note that $\mathcal{P}^k_{X,m}(\cdot)$ is not in general the tensor algebra on $\mathcal{P}^k_{X,m}(1)$ and that $\Omega^k_{X,m} = N\mathcal{P}^k_{X,m}(\cdot)$ when $k \leq p^m$. Moreover, since as in Subsection 1.2 all the generators and all the elements occurring in the relations are of order at most $p^m$, $\Omega^k_{X,m}$ does not depend on $k$ when $k \geq 2p^m$. It is important to remark that for schemes we have $\Omega^k_{X,m} = \Omega_{X,m}$ when $k \geq 2p^m$ and, taking into account the independence, it will do no harm to write in general $\Omega^k_{X,m} = \Omega^k_{X,m}$ for $k \geq 2p^m$. This complex has locally free terms.

We can define the de Rham complex of a $\mathcal{D}^{(m)}_{X}$-module $\mathcal{F}$ as before: the action of $\mathcal{D}^{(m)}_{X}$ on $\mathcal{F}$ induces, for all $k$, a morphism $d: \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{D}^{(m)}_{X}} \mathcal{P}^k_{X,m}$ that gives rise for $k \geq p^m$, and in particular for $k \geq 2p^m$, to an $m$-connec-
tion $d: \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X,m}$. Using the Leibniz rule, it extends to a differential on $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^*_{X,m}$. This is the de Rham complex of $\mathcal{F}$ and we can define the de Rham cohomology of level $m$ of $\mathcal{F}$:

$$H^i_{dR,m}(X, \mathcal{F}) := H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^*_{X,m}).$$

We now want to describe a construction analogous to that of Subsection 1.5 that works over formal schemes and has also some intrinsic interest for schemes.

An integer $k$ being fixed as before, we consider the shifted simplicial complex $P^k_{X,m} = \{P^k_{X,m}(r + 1); d_i, i = 1, \ldots, r + 1; s_i, i = 1, \ldots, r + 1\}$, the associated complex of abelian sheaves $\mathcal{P}_{X,m}^k(\cdot)$, and the normalized complex $N\mathcal{P}_{X,m}^k(\cdot) := \cap \ker s_i^*$. We let $\mathcal{P}_{X,m}^k$ be the differential ideal of $N\mathcal{P}_{X,m}^k(\cdot)$ generated by $d_0^{-1}(\mathcal{P}_{X,m}^k(1)^{[p^n+1]} \subset N\mathcal{P}_{X,m}^k(2)$ and we write $\Omega^{k\cdot}_{X,m}$ for the quotient complex, called the linearized de Rham complex of order $k$.

Only the case $k \geq 2p^m$ is of interest to us. By construction, the morphism of complexes $d^+_r: \mathcal{P}_{X,m}^k(\cdot) \to \mathcal{P}_{X,m}^{k+1}(\cdot)$ induces a morphism of complexes $\Omega^r_{X,m} \to \Omega^r_{X,m}$. For each $r$, the map $\Omega^r_{X,m} \to \Omega^r_{X,m}$ extends by linearity to an isomorphism $\mathcal{P}_{X,m}^k(1) \otimes_{\mathcal{O}_X} \Omega^r_{X,m} \to \Omega^r_{X,m}$ (as one easily checks by a local computation). Thus, there should not be any confusion in writing $\mathcal{P}_{X,m}^k(1) \otimes_{\mathcal{O}_X} \Omega^r_{X,m}$ for the linearized de Rham complex of order $k \geq 2p^m$.

As in Section 2, we can define a homotopy on this complex and deduce that it is a resolution of $\mathcal{O}_X$.

By duality, the linearized de Rham complex $\mathcal{P}_{X,m}^k(1) \otimes_{\mathcal{O}_X} \Omega^r_{X,m}$ of order $k \geq 2p^m$ gives rise to a complex $\mathcal{RHom}^{\mathcal{O}_X}(\mathcal{O}^r_{X,m}, \mathcal{D}^{(m)}_X)$ which is a right resolution of $\mathcal{O}_X$. Since the de Rham complex has locally free terms, going to the limit on $k$ gives rise to a complex $\mathcal{RHom}^{\mathcal{O}_X}(\mathcal{O}^r_{X,m}, \mathcal{D}^{(m)}_X)$ of $\mathcal{D}^{(m)}_X$-modules called the Spencer complex of level $m$.

5.2. **Proposition.** Let $S$ be a scheme or a formal scheme and $X$ be a smooth (formal) $S$-scheme. Then,

1. The Spencer complex of level $m$ over $X/S$ is a right resolution of $\mathcal{O}_X$ by locally free $\mathcal{D}^{(m)}_X$-modules.

2. For any $\mathcal{D}^{(m)}_X$-module $\mathcal{F}$, there is an isomorphism

$$\mathcal{RHom}^{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^r_{X,m}.$$

3. For any $\mathcal{D}^{(m)}_X$-module $\mathcal{F}$, we have

$$\mathcal{RHom}^{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = H^1_{dR,m}(X, \mathcal{F}).$$
Proof. (1) Since, by construction, the Spencer complex of level $m$ is a right resolution of $\mathcal{O}_X$, there only remains to check that the differential in the Spencer complex is $D_m$-linear. Fix two integers $k$ and $l$ with $k \geq 2p^m$. It follows from the universal property of the divided power envelope that, for each $r$, the map $X^{(r+2)}/S \to X^{(r+1)}/S$ that forgets the second component induces a map

$$P^{l}_{Xm}(1) \times_X P^{k+}_{Xm}(r) \to P^{(k+1)+}_{Xm}(r).$$

By functoriality, we actually get a morphism of simplicial complexes

$$P^{k+1}_{Xm} \to P^l_{Xm} \otimes_{\mathcal{E}_X} P^k_{Xm}.$$

From this, we deduce a morphism of complexes

$$P^{k+1}_{Xm} \to P^l_{Xm} \otimes_{\mathcal{E}_X} P^k_{Xm}$$

from which we derive a morphism of complexes

$$P^{k+1}_{Xm} \otimes_{\mathcal{E}_X} \Omega_X \to P^l_{Xm} \otimes_{\mathcal{E}_X} P^k_{Xm} \otimes_{\mathcal{E}_X} \Omega_X.$$

Now we apply duality in order to get a pairing

$$\mathcal{D}_X^{(m)} \times \mathcal{Rm}_{\mathcal{E}_X}(\Omega^*_X, \mathcal{D}_X^{(m)}) \to \mathcal{Rm}_{\mathcal{E}_X}(\Omega^*_X, \mathcal{D}_X^{(m)}).$$

Taking the limit on $k$ and $l$ gives a pairing

$$\mathcal{D}_X^{(m)} \times \mathcal{Rm}_{\mathcal{E}_X}(\Omega^*_X, \mathcal{D}_X^{(m)}) \to \mathcal{Rm}_{\mathcal{E}_X}(\Omega^*_X, \mathcal{D}_X^{(m)}),$$

which, by construction, is nothing but the canonical action of $\mathcal{D}_X^{(m)}$ on the Spencer complex.

(2) It follows from the first part that

$$\mathcal{Rm}_{\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{F}) = \mathcal{Rm}_{\mathcal{D}_X^{(m)}}(\mathcal{Rm}_{\mathcal{E}_X}(\Omega^*_X, \mathcal{D}_X^{(m)}), \mathcal{F}) = \mathcal{F} \otimes_{\mathcal{E}_X} \Omega^*_X,$$

where the left hand side is the de Rham complex of $\mathcal{F}$.

(3) Take cohomology.

5.3. Crystalline Cohomology over a Formal Base. Let $S$ be a formal scheme. If $X$ is a smooth formal $S$-scheme, completing the Spencer complex $\mathcal{Rm}_{\mathcal{E}_X}(\Omega^*_X, \mathcal{D}_X^{(m)})$ gives a right resolution $\mathcal{Rm}_{\mathcal{E}_X}(\Omega^*_X, \mathcal{D}_X^{(m)})$ of $\mathcal{O}_X$ because the de Rham complex is made of locally free terms. Thus, if $\mathcal{F}$ is a $\mathcal{D}_X^{(m)}$-module, we have as before

$$\mathcal{Rm}_{\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{F}) = \mathcal{Rm}_{\mathcal{D}_X^{(m)}}(\mathcal{Rm}_{\mathcal{E}_X}(\Omega^*_X, \mathcal{D}_X^{(m)}), \mathcal{F}) = \mathcal{F} \otimes_{\mathcal{E}_X} \Omega^*_X,$$

and

$$\mathcal{RHom}_{\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{F}) = \mathcal{H}^m_{dR,m}(X, \mathcal{F}).$$

Assume now that the ramification index of $\mathcal{V}$ is $\leq (p - 1)p^m$, that $S$ is endowed with an $m$-PD-structure $(\alpha, \beta)$ compatible with $p$, and let $X$ be
an $S$-scheme to which the $m$-PD-structure of $S$ extends. Let $E$ be a quasi-coherent crystal on $X/S$.

Assume $i: X \hookrightarrow Y$ is an embedding into a smooth formal $S$-scheme and, for all $n \gg 0$, let $i_n: X \rightarrow Y_n$ be the map induced mod $\alpha^{n+1}$. We know that $E_Y := (i_n E)_Y$ is a $E(m)_X$-module and it follows that $E_Y := \lim_{n} E_Y$ has a natural structure of $E(m)_X$-module. We can therefore form the de Rham complex $E_Y \otimes \Omega_{Y/m}$.

5.4. Proposition. We have

$$H^{i}_{\text{cris}, m}(X/S, E) \sim H^{i}_{dR, m}(Y, E_Y).$$

Proof. Using Subsection 4.8, one proceeds as in [B-O, Theorem 7.23]. To finish this article, we want to explain how Berthelot’s Frobenius descent [B5] applies in our context. So, we let $S$ and complex and Proposition 5.4 gives the announced isomorphism.

Frobenius is an isomorphism of $B5$, combined with the identifications of Proposition 5.3 tells us that

$$\text{an} \text{ lifting of } F \text{ gi} X \text{ Frobenius of } F_{2.2.3} \text{ and Sect. 4.1 that if }$$

$Rham complex $E$ has a natural structure of $DD$ $E$ that

$$\text{for all } n \text{ } X \text{1 This is a local question and we may assume that }$$

Proof. 5.5. Proposition. $1$

We ha

$$5.5. \text{PROPOSITION.} \quad \text{1) The morphism } F_0 \text{ induces an isomorphism}$$

$$F^*_0: H^{i}_{\text{cris}, m}(X'_0/S, E') \sim H^{i}_{\text{cris}, m+s}(X'_0/S, E).$$

2) Assume $S$ is scheme killed by $p^{N+1}$. If $m \geq N$ and if $F: X \rightarrow X'$ is a lifting of $F_0$ given in local coordinates by $F^*(t_i) = t_i^{p^m}$, then $F$ induces a quasi-isomorphism

$$F^*: E'_X \otimes_{E'_S} \Omega^{1}_{X'} \rightarrow E'_X \otimes_{E'_S} \Omega^{1}_{X(m+s)}.$$
Since $S$ is killed by $p^{N+1}$, it follows from Lemma 5.6 below that, for $m \geq N$, $F^*$ sends the differential ideal $\mathcal{I}_{X,m}$ into $\mathcal{I}_{X(m+s)}$. Thus, we get a morphism of complexes $F^*: E_X \otimes_{\mathcal{O}_X} \Omega_{X,m} \to E_X \otimes_{\mathcal{O}_X} \Omega_{X(m+s)}$. Thanks to Proposition 5.2, it corresponds to the canonical map

$$F^*: \mathcal{RHom}_{\mathcal{X}^n}(\mathcal{O}_X, E_X) \to \mathcal{RHom}_{\mathcal{X}^{n+s}}(\mathcal{O}_X, E_X).$$

We know from [B5, Proposition 3.3.8] that this is an isomorphism.

There remains to prove the following lemma that generalizes [B5, 2.2.4].

5.6. LEMMA. (1) For $u \neq 0$, the integer $(p^m_u)$ has $p$-adic order $\geq m - u + 1$.

(2) For $m \geq N$, we have $F^*(\tau_{p^m}) = \tau_{p^{m+s}} \mod p^{N+1}$.

(3) Let $N$ be such that $X$ is killed by $p^{N+1}$. Then, for $m \geq N$, the image of $\mathcal{I}_{X,m}(1)^{p^{n+1}}$ by the Frobenius map is inside $\mathcal{I}_{X(m+s)}(1)^{p^{n+s+1}}$.

Proof. (1) We can write $u = qp + r$ with $r < p$. In case $q = 0$, we have

$$\text{ord}\left(\frac{p^m}{u}\right) = m \geq m - u + 1.$$ 

If $q > 0$, we proceed by induction on $m$, the case $m = 1$ being trivial. So let $m > 1$. Then, we have

$$\text{ord}\left(\frac{p^m}{u}\right) \geq \text{ord}\left(\frac{p^m}{qp}\right) = \text{ord}\left(\frac{p^{m-1}}{q}\right),$$

and by induction, this is $\geq m - q \geq m - u + 1$.

(2) We have $F^*(\tau_i) = (t^r_i + \tau_i)p^r - t^p_i + p\lambda$ for some $\lambda \in \mathcal{R}_{\mathcal{X}m}(1)$ and it follows that

$$F^*(\tau_{p^m}) = (\tau_{t^r} + p\lambda)p^m = \sum_{0 \leq u < p^m} \left(\frac{p^m}{u}\right)(\tau_{t^r})^{p^m-u}(p\lambda)^u$$

$$= \sum_{0 \leq u < p^m} \left(\frac{p^m}{u}\right)p^{ap^m_u}t^r(p^{m-u}).$$
If $u > 0$, then $(u^n)p^u = 0 \mod p^{N+1}$ thanks to the first part, and we get $F^*(\tau_p^u) = \tau_p^{p^{u+1}} \mod p^{N+1}$.

(3) It is sufficient to show that $\tau_p^{(q_p^m+k)} \in \mathcal{R}_{X_n}((1)$ is sent into $\mathcal{K}_{X(m+1)(q_p^m+k)}$. We already know by [B5, Proposition 2.2.2] that, for all $l$, the image of $\mathcal{K}_{X_n(l)}$ under $F$ is inside $\mathcal{K}_{X(m+1)(l)}$. Since $F^*(\tau_p^{(q_p^m+k)}) = F^*(\tau_p^m)F^*(\tau_l^m)$, our assertion follows from the second part of this lemma.

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