Generalization of von Neumann's spectral sets and integral representation of operators

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Abstract

We extend von Neumann's theory of spectral sets, in order to deal with the numerical range of operators. An integral representation for arbitrary operators is given, allowing to extend functional calculus to non-normal operators. We apply our results to the proof of the Burkholder conjecture : let T be an operator consisting in a finite product of conditional expectation, then for any square integrable function f, the iterates $T^n f$ converge almost surely to some limit.

Résumé

Nous modifions la théorie des ensembles spectraux de von Neumann pour l'appliquer à l'image numérique des opérateurs. Nous donnons une représentation intégrale pour des opérateurs bornés quelconques ; ceci étend le calcul fonctionnel aux opérateurs non normaux. Comme application, nous démontrons la conjecture de Burkholder : soit T un opérateur produit d'un nombre fini d'espérances conditionnelles, alors pour toute fonction de carré sommable f, les itérées $T^n f$ convergent presque sûrement.

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1 Introduction

The spectral sets theory was introduced by von Neumann [3] in order to extend functional calculus to non-normal operators on an Hilbert space. Let us recall basic principles of von Neumann's theory as described in [4].

Definition 1 *A set $\sigma \subset \mathbf{C}$ is a spectral set of the operator T on an Hilbert space if it is closed and if for any rational function $u(z)$ one has*

$$\|u(T)\| \leq \sup_{z\in\sigma} |u(z)|. \tag{1}$$

This formula is valid even if $u(z)$ has poles in σ. On the other hand, rational functions are dense, in the sense of Runge Theorem [6], in the set of analytic functions. This usual formulation in terms of rational functions is nothing but a way to avoid technicalities about boundary values and singularities of analytic functions. The following theorem, due to von Neumann [3], gives necessary and sufficient conditions under which a ball, its complement, or a half-plane is a spectral set of T :

Theorem 1 *(von Neumann) A necessary and sufficient condition for one of the domains*

$$|z-\alpha| \leq r, \quad |z-\alpha| \geq r, \quad \mathrm{Re}\,(\alpha z) \geq \beta$$

to be a spectral set of T is that [1]

$$\|T-\alpha\| \leq r, \quad \|(T-\alpha I)^{-1}\| \leq r^{-1}, \quad \mathrm{Re}\,(\alpha T) \geq \beta.$$

This implies obviously that for any T, $\{z : |z| \leq \|T\|\}$ is a spectral set of T.

Notice that theorem 1 implies that for any point α out of the spectrum of T, there exists an r such that $\{z : |z-\alpha| \geq r\}$ is a spectral set and as a consequence, the intersection of the spectral sets of T is its spectrum. This implies that the intersection of two spectral sets is not necessarily a spectral set, what is apparently a serious drawback of the theory.

The concept of numerical range if closely related :

Definition 2 *The numerical range (or field of values) of the operator T is*

$$\Theta(T) = \{z = (f, Tf),\ \|f\| = 1\}.$$

It is well-known that this set is convex [5] p267, and that it is a real interval if and only if T is symetric [5] p269.

Theorem 2 *The intersection of the spectral half-planes is the closure of the numerical range.*

Proof: Denote by $\Sigma(T)$ the intersection of the spectral half-planes. Since both sets are convex and closed, it is enough to prove that for any closed half-plane H, H contains $\Sigma(T)$ iff it contains $\Theta(T)$. If H has the form

$$H = \{z;\ \mathrm{Re}\,(az+b) \geq 0\}$$

we have

$$\begin{aligned} \Sigma(T) \subset H &\iff \mathrm{Re}\,(aT+b) \geq 0 \\ &\iff (f, \mathrm{Re}\,(aT+b)\,f) \geq 0, \quad \|f\| = 1 \\ &\iff \mathrm{Re}\,(a(f,Tf)+b) \geq 0, \quad \|f\| = 1 \\ &\iff \Theta(T) \subset H. \end{aligned}$$

∎

Our main contribution is the following theorem :

[1] With the convention that $\mathrm{Re}\,(T) = (T+T^*)/2$.

Theorem 3 *Let T be an operator on an Hilbert space and σ be a bounded convex subset of* $\mathbf{C}$ *containing* $\Theta(T)$[2]. *There exists a constant C_σ such that for any rational function $u(z)$ one has*

$$\|u(T)\| \leq C_\sigma \sup_{z\in\sigma} |u(z)|. \tag{2}$$

More generally, for any finite sequence of rational functions $u_1, ... u_n$, the following holds

$$\|\sum_i u_i(T)^* u_i(T)\| \leq C_\sigma^2 \sup_{z\in\sigma} \sum_i |u_i(z)|^2. \tag{3}$$

The constant C_σ may be chosen as

$$C_\sigma = \left(\frac{2\pi D^2}{S}\right)^3 + 3$$

where S is the area of σ and D its diameter.

A natural choice for σ is the closure of $\Theta(T)$; however if S is small, one may prefer to choose σ larger.
This theorem is a consequence of the following representation theorem

Theorem 4 *Let σ be as in theorem 3 and assume that is has a piecewise C^1 boundary.*
Denote by $C(\partial\sigma)$ the space of continuous functions on $\partial\sigma$ endowed with the uniform norm. Under the assumptions of theorem 3, there exist a continuous linear operator S_σ on $C(\partial\sigma)$ and a measure $\mu_T(dz)$ on $\partial\sigma$ with values on the set of non-negative self-adjoint operators such that

$$u(T) = \int_{\partial\sigma} (S_\sigma u)(z)\ \mu_T(dz) \tag{4}$$

for any rational function with poles outside of σ, and

$$\int_{\partial\sigma} \mu_T(dz) = I$$
$$\|S_\sigma\| \leq C_\sigma.$$

The operator S_σ and the measure μ_T are detailed in section 3; S_σ is the inverse of the operator R defined in Lemma 6. Let us shortly mention here, that :

$$\mu_T(dz) = \mathrm{Re}\left(\frac{dz}{2i\pi(z-T)}\right)$$

properly defined when $(z-T)$ is not invertible, thus μ_T depends on σ and T, but S_σ only depends on σ. We assume that $\partial\sigma$ is piecewise C^1 for avoiding technicalities in the definition of this measure, however some extra effort would probably lead to the same result without this assumption.
In Section 5, we provide two useful results in order to control spectral sets of product of operators.

Example : the unit circle. Let T an operator such that the set σ of theorem 3 is the unit circle. Let us show the differences between our theorems and the von Neumann result. In this case, the assumption of theorem 3 is :

$$|(f, Tf)| \leq \|f\|^2 \tag{5}$$

which is weaker than von Neumann's condition, since $\|T\|$ can be 2.
Furthermore, if σ is the unit circle, one can compute explicitly the operator S_σ (as the inverse of the operator R of Lemma 6 section 3):

$$S_\sigma(u) = 2u - \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})d\theta.$$

[2]By theorem 2 this condition means that $\mathrm{Re}(\bar{\alpha}(T-z)) \geq 0$ for any point z of $\partial\sigma$ and α the (inwards) normal vector at z.

Thus, we have :

$$\|u(T)\| \quad \le \quad 3 \sup_{\|z\|\le 1} |u(z)|$$

Let us give an explicit example. Consider the nilpotent matrix:

$$T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

This matrix satisfies equation (5) and we can evaluate μ_T on the unit circle:

$$\mathrm{Re}\left(\frac{dz}{2i\pi(z-T)}\right) = \frac{d\theta}{2\pi}\begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix} \tag{6}$$

Thus μ_T is positive on the unit circle and $\|T\| = 2$.

The Burkholder Conjecture. Consider $T = P_1 P_2 \dots P_n$ a finite product of conditional expectations w.r.t σ-fields $\mathcal{F}_1 \dots \mathcal{F}_n$.

It is well-known [8, 9] that for any f in L_2 $T^k f$ converges in L_2 to the conditional expectations w.r.t $\mathcal{F}_1 \cap \mathcal{F}_2 \dots \cap \mathcal{F}_n$ (i.e., T^k converges strongly to the orthogonal projection on the largest invariant space of T); this convergence holds even in L^p [9] for $1 < p < +\infty$.

The conjecture of Burkholder is that the convergence holds almost surely for any function of L_2:

$$\lim_{k \longrightarrow \infty} T^k f = E[f|\mathcal{F}_1 \cap \mathcal{F}_2 \dots \cap \mathcal{F}_n], \quad a.s.$$

Since T is a positive contraction on L_1 and $T1 = 1$, the almost sure convergence of the averages

$$A_n f = \frac{1}{n}\sum_{k=0}^{n-1} T^k f$$

is a consequence Chacon's theorem (th 3.7 in [2]). By using a technique borrowed to Stein [7] one proves that almost sure convergence of $T^n f$ holds if

$$\left\| \sum_{n=0}^{\infty} (n+1)(I-T)^* T^{*n} T^n (I-T) \right\| \quad < \quad \infty.$$

We shall prove this result in section 6: We first use the results on products of operators for proving that an $\Theta(T)$ is included in some sectorial domain (this result has been proved independently in a unpublished paper of A. Brunel [1], proposition 9); we then use equation (3).

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2 The von Neumann theory revisited

The following theorem [3], which is the basis of the proof of theorem 1, is actually the cornerstone of the theory :

Theorem 5 *(von Neumann) If $\|T\| \le 1$, the unit disk U is a spectral set for T.*

We give here a new proof of it which introduces in a simplified context the essential ingredients of the proof of theorem 3.

Proof: Let us assume first that the spectrum of T is contained in the open unit disk U. For any rational function u, analytic on the disk, we have

$$u(T) = \frac{1}{2i\pi}\int_{\partial U} u(z)\frac{dz}{z-T} = \int u(z)\left(\frac{1}{z-T}+\frac{T^*}{1-zT^*}\right)\frac{dz}{2i\pi}$$

since the added term is analytic in the disk; thus

$$u(T) = \int u(z)(z^{-1}-T^*)^{-1}(1-T^*T)(z-T)^{-1}\frac{dz}{2i\pi z}$$

what rewrites

$$u(T) = \int_{\partial U} u(z)\mu_T(dz) \tag{7}$$

for some measure μ_T with values on the set of non-negative self-adjoint operators. Applying this identity to $u(z)=1$, we get that $\mu_T(\partial U)=1$. Since μ_T is self-adjoint, we have

$$u(T)^* = \int_{\partial U} \overline{u(z)}\ \mu_T(dz). \tag{8}$$

The equation

$$\int (u(z)-u(T))^*\ \mu_T(dz)\ (u(z)-u(T)) \geq 0$$

rewrites, thanks to (7) and (8)

$$u(T)^*u(T) \leq \int |u(z)|^2\mu_T(dz).$$

This completes the proof in the case where the spectrum of T is in the open unit disk. If not, we can consider the operators $(1-\varepsilon)T$, apply the result to them and

$$\|u((1-\varepsilon)T)\| \leq \sup_{|z|\leq 1}|u(z)|$$

implying

$$\|v(T)\| \leq \sup_{|z|\leq 1}|v(z/(1-\varepsilon))|$$

for any v rational in the neighborhood of U. We can let ε tend to zero and the result is proved. ∎

3 Proof of theorem 4

The proof of theorem 4 relies on the following Lemma :

Lemma 6 *Let σ be a bounded convex subset of $\mathbf{C}$ and $C(\partial\sigma)$ be the set of complex functions continuous on σ and harmonic on its interior, with the supremum norm. The following equation*

$$(R\varphi)(z_0) = \int_{\partial\sigma}\varphi(z)\ \mathrm{Re}\left(\frac{dz}{2i\pi(z-z_0)}\right) \tag{9}$$

defines a continuous operator on $C(\partial\sigma)$ (the value for $z_0\in\partial\sigma$ is defined by continuity). This operator has a bounded inverse and

$$\|R^{-1}\| \leq \left(\frac{2\pi D^2}{S}\right)^3 + 3$$

Proof of Lemma 6 : For a real continuous function φ, equation 9 rewrites :

$$(R\varphi)(z_0) = \frac{1}{2\pi}\int_{\partial\sigma} \varphi(z)\ d\arg(z - z_0) \tag{10}$$

well defined for $z_0 \in \sigma \setminus \partial\sigma$. First let us show that this equation defines a continuous operator on $C(\partial\sigma)$. Let $z_0 \in \partial\sigma$ and $\{z_n\}$ be a sequence of points of $\sigma \setminus \partial\sigma$ converging to z_0. Let E be a small interval of $\partial\sigma$ containing z_0, with boundary points z_1 and z_2 ($z_0 \neq z_1$ and $z_0 \neq z_2$). On the one hand:

$$\int_E d\arg(z - z_n) = (z_1 - \widehat{z_n, z_2} - z_n) \quad \longrightarrow \quad (z_1 - \widehat{z_0, z_2} - z_0) \tag{11}$$

by continuity of the angle $(z_1 - \widehat{z, z_2} - z)$ in the neighbourhood of z_0. Notice that, by convexity of σ, this limit is at least π ($= \pi$ at a regular point).
On the other hand, it is clear that

$$\int_{\partial\sigma\setminus E} \varphi(z)\ d\arg(z - z_n) \longrightarrow \int_{\partial\sigma\setminus E} \varphi(z)\ d\arg(z - z_0). \tag{12}$$

Let us denote:

$$\mu_{z_0}(dz) = \frac{d\arg(z - z_0)}{2\pi}.$$

Equations (11) and (12) prove that for any continuous function φ on $\partial\sigma$, constant in the neighbourhood of z_0, $\mu_{z_n}(\varphi)$ converges to some limit $\mu_{z_0}(\varphi)$ if $z_n \to z_0$ and this limit does not depend on the particular sequence $z_n \in \sigma \setminus \partial\sigma$. This clearly extends to any continuous function φ, and the measures $\mu_{z_n}(dz)$ converges weakly to $\mu_{z_0}(dz)$. Moreover the function $z \to \mu_z(\varphi)$ is continuous (since its value on the boundary is defined as an existing limit)

Consequently R is well-defined and continuous on $C(\partial\sigma)$.
Let us set

$$R = (I + P)/2.$$

Since the limit (11) is at least π, we have $\mu_{z_0}(\{z_0\}) \geq 1/2$ and $||P|| \leq 1$. This operator rewrites

$$(P\varphi)(z_0) = \int_{\partial\sigma} \varphi(z)\ \nu_{z_0}(dz)$$

where ν_{z_0} is a positive measure of total mass 1 and such that $\nu_{z_0}(\{z_0\}) = 0$ at any regular point.

In order to complete the proof of Lemma 6, we have to prove that R has a continuous inverse. This is a direct consequence of the following Proposition:

Proposition 7 *There exist a positive measure ν^* on $\partial\sigma$ such that for $\varphi \geq 0$ and any $z_0 \in \partial\sigma$:*

$$(P^3\varphi)(z_0) \geq \nu^*(\varphi)$$

and

$$\nu^*(\partial\sigma) = \frac{3}{4}\left(\frac{S}{\pi D^2}\right)^3.$$

Proof of Proposition 7 : Because of the translation invariance of the result we can assume that 0 is the center of gravity of σ :

$$\int_{\partial\sigma} z\ dxdy = 0. \tag{13}$$

The proof is based on a lower bound of P by a non-negative operator of rank two; we have indeed

$$\nu_{z_0}(dz) = \mathrm{Re}\left(\frac{dz}{2i\pi(z - z_0)}\right) \geq \mathrm{Re}\left(\frac{(\bar{z} - \bar{z}_0)dz}{2i\pi D^2}\right) \tag{14}$$

where D is the diameter of σ; this lower bound is non-negative. Now, if we define the operator Q by

$$(Q\varphi)(z_0) = \frac{1}{2S}\int_{\partial\sigma}\varphi(z)\ \mathrm{Im}\,((\bar{z}-\bar{z_0})dz) \tag{15}$$

where S is the total area of σ, equation (14) rewrites , $P\varphi(z_0) \geq (S/\pi D^2)\ (Q\varphi)(z_0)$ for any positive function φ. We obtain the effect of Q on linear functions by use of the Green formula: for any analytical function $f(z)$

$$\begin{aligned}
(Qf)(z_0) &= \frac{1}{2S}\int f(z)\mathrm{Im}\,((\bar{z}-\bar{z_0})dz) \\
&= \frac{1}{2S}\int f(z)((x-x_0)dy-(y-y_0)dx) \\
&= \frac{1}{2S}\int(\partial_x(f(z)(x-x_0))+\partial_y(f(z)(y-y_0))dxdy \\
&= \frac{1}{2S}\int(f'(z)(z-z_0)+2f(z))dxdy.
\end{aligned}$$

Hence using (13):

$$\begin{aligned}
(Q1)(z_0) &= 1 \\
Q(z)(z_0) &= -\frac{z_0}{2} \\
Q(\mathrm{Im}\,(\bar{a}z)+b) &= -\frac{1}{2}\mathrm{Im}\,(\bar{a}z_0)+b.
\end{aligned}$$

By (15), the function $Q\varphi$ is a real linear function of z_0:

$$(Q\varphi)(z_0) = \mathrm{Im}\,(\bar{a}z_0)+b$$

with

$$\begin{aligned}
a &= \frac{1}{2S}\int\varphi(z)dz \\
b &= \frac{1}{2S}\int\varphi(z)\mathrm{Im}\,(\bar{z}dz).
\end{aligned}$$

Since the effect of the kernel Q on a such a function is to multiply a by $-1/2$ and to leave b unchanged, we obtain

$$\begin{aligned}
\left(\frac{\pi D^2}{S}\right)^3 (P^3\varphi)(z_0) &\geq (Q^3\varphi)(z_0) \\
&= Q^2(\mathrm{Im}\,(\bar{a}z)+b) \\
&= b+\frac{\mathrm{Im}\,(\bar{a}z_0)}{4} \\
&\geq \frac{3b}{4}
\end{aligned}$$

since $\mathrm{Im}\,(\bar{a}z_0) \geq -b$ (i.e. $Q\varphi$ is positive). This states the Proposition. ∎

Proof of Lemma 6(cont'd) : Proposition 7 rewrites with $\varepsilon = \nu^*(\partial\sigma)$

$$\begin{aligned}
P^3 &= \varepsilon\int_{\partial\sigma}\varphi(z)\ \frac{\nu^*(dz)}{\varepsilon}+(1-\varepsilon)\int_{\partial\sigma}\varphi(z)\ \nu'_{z_0}(dz) \\
&= \varepsilon L+(1-\varepsilon)Q
\end{aligned}$$

where $\nu'_{z_0}(.)$ is for any z_0 a positive measure, $||Q|| = 1$, $||L|| = 1$, and $PL = L$. Hence

$$(I+P^3)(I-\frac{\varepsilon}{2}L) = I+(1-\varepsilon)Q.$$

Since $I + (1-\varepsilon)Q$ is invertible, this implies that

$$\begin{aligned}(I + P^3)^{-1} &= (I + (1-\varepsilon)Q)^{-1}(I - \frac{\varepsilon}{2}L) \\ \|(I + P^3)^{-1}\| &\leq \frac{2+\varepsilon}{2\varepsilon}\end{aligned}$$

and finally,

$$\|R^{-1}\| = 2\|(I + P^3)^{-1}(I - P + P^2)\| \leq \frac{3(2+\varepsilon)}{\varepsilon} = \left(\frac{2\pi D^2}{S}\right)^3 + 3.$$

∎

Proof of theorem 4 : Set

$$S_\sigma = R^{-1}.$$

Lemma 6 implies that for any analytic function u on σ, with real part φ, one has

$$\begin{aligned}\varphi(z_0) &= \int_{\partial\sigma} S_\sigma \varphi(z) \ \mathrm{Re}\left(\frac{dz}{2i\pi(z-z_0)}\right) \\ &= \mathrm{Re}\left(\int_{\partial\sigma} S_\sigma \varphi(z) \ \frac{dz}{2i\pi(z-z_0)}\right)\end{aligned}$$

hence, the function

$$u(z_0) - \int_{\partial\sigma} S_\sigma \varphi(z) \ \frac{dz}{2i\pi(z-z_0)}$$

is analytic inside σ with zero real part; consequently there exists a real number u_0 such that

$$u(z_0) = iu_0 + \int_{\partial\sigma} S_\sigma \varphi(z) \ \frac{dz}{2i\pi(z-z_0)}.$$

This implies, if the spectrum of T lies in the interior of σ

$$\begin{aligned}u(T) &= iu_0 + \int_{\partial\sigma} S_\sigma \varphi(z) \ \frac{dz}{2i\pi(z-T)} \\ u(T)^* &= -iu_0 + \int_{\partial\sigma} S_\sigma \varphi(z) \ \left(\frac{dz}{2i\pi(z-T)}\right)^*.\end{aligned}$$

Thus

$$\mathrm{Re}\,(u(T)) = \int_{\partial\sigma} S_\sigma \varphi(z) \ \mathrm{Re}\left(\frac{dz}{2i\pi(z-T)}\right) \tag{16}$$

where $\mathrm{Re}\,(U)$ for an operator U is defined as $(U + U^*)/2$. Denoting by ψ the imaginary part of u and applying identity (16) to the function $-iu$ we obtain

$$\mathrm{Im}\,(u(T)) = \int_{\partial\sigma} S_\sigma \psi(z) \ \mathrm{Re}\left(\frac{dz}{2i\pi(z-T)}\right) \tag{17}$$

where $\mathrm{Im}\,(U)$ for an operator U is defined as $(U - U^*)/2i$; putting together equations (16) and (17), we deduce

$$\begin{aligned}u(T) &= \int_{\partial\sigma} S_\sigma u(z) \ \mathrm{Re}\left(\frac{dz}{2i\pi(z-T)}\right) \\ &= \int_{\partial\sigma} S_\sigma u(z) \ \mu_T(dz).\end{aligned}$$

Let us check that the assumptions imply that μ_T is a measure on $\partial\sigma$ with values on the set of non-negative self-adjoint operators. The assumption on σ rewrites, thanks to theorem 1

$$\alpha T + \bar{\alpha} T^* \geq \alpha z + \bar{\alpha}\bar{z}$$

for any $z \in \partial\sigma$ regular and $\bar{\alpha}$ parallel to idz. This implies

$$\begin{aligned} 2\mathrm{Re}\left(\frac{-\bar{\alpha}}{z-T}\right) &= -\bar{\alpha}(z-T)^{-1} - \alpha(z-T)^{-1^*} \\ &= -(z-T)^{-1^*}(\bar{\alpha}(z-T)^* + \alpha(z-T))(z-T)^{-1} \\ &\geq 0. \end{aligned}$$

Thus

$$\mathrm{Re}\left(\frac{dz}{2i\pi(z-T)}\right) \geq 0$$

if z is a regular point; since the spectrum of T is inside σ, the mass of the non-regular points is zero and we conclude that μ_T is a positive measure. Note that equation (16) and $R1 = 1$ imply that $\mu_T(1) = I$. This completes the proof in the case where the spectrum of T lies in the interior of σ. ∎

If not, we have to define the measure μ_T as a suitable limit.
Pick a point z^* in the interior of σ and set

$$T_\varepsilon = (1-\varepsilon)T + \varepsilon z^*.$$

The convexity of σ implies that the spectrum of T_ε is in the interior of σ, and the half-planes containing σ are spectral sets of T_ε (use theorem 1).
For any analytic function u and $\varepsilon' > \varepsilon$:

$$\begin{aligned} \int_{\partial\sigma} S_\sigma(u)(z)(d\mu_{T_\varepsilon} - d\mu_{T_{\varepsilon'}}) &= u(T_\varepsilon) - u(T_{\varepsilon'}) \\ &= u(T_\varepsilon) - v(T_\varepsilon) \end{aligned}$$

where :

$$v(z) = u\left(\frac{1-\varepsilon'}{1-\varepsilon}(z - \varepsilon z^*) + \varepsilon' z^*\right)$$

At this point, Theorem 3 applies to $\{T_\varepsilon\}_{\varepsilon>0}$:

$$\|\int_{\partial\sigma} S_\sigma(u)(z)(d\mu_{T_\varepsilon} - d\mu_{T_{\varepsilon'}})\| \leq C_\sigma \sup_{z\in\sigma} |u(z) - v(z)| \tag{18}$$

As ε' goes to ε, left hand side of (18) goes to 0 (u is uniformly continuous on σ). The same inequality holds for $\bar{u}$, and thus for any continuous function on $\partial\sigma$. Thus, since S_σ is one-to-one, for any continuous function u

$$\lim_{\varepsilon,\varepsilon'\longrightarrow 0} \|\int_{\partial\sigma} u(z)(d\mu_{T_\varepsilon} - d\mu_{T_{\varepsilon'}})\| = 0$$

Thus for any vector f, the positive measures $(f, \mu_{T_\varepsilon} f)$ are weakly continuous on ε and converges to some limit measure $\mu_{T,f}$. For any u, this limit $\mu_{T,f}(u)$ is bilinear on f, thus defining an operator valued measure μ_T.

The theory for these operator valued measures μ_T can be carried out like the projector valued measures of the spectral theory of self-adjoint operators. One easily checks that the following properties, satisfied by T_ε, are also satisfied by T :

$$\begin{aligned} \mu_T(A \cup B) &= \mu_T(A) + \mu_T(B), \quad A \cap B = \emptyset \\ \mu_T(\emptyset) &= 0 \\ \mu_T(\partial\sigma) &= I \\ \mu_T(\cup_{i=1}^{\infty} A_i) &= \text{s-lim}\, \mu_T(A_i), \quad A_i \subset A_{i+1} \end{aligned}$$

and[3] that for any bounded Borel function u on $\partial\sigma$, there exists a unique operator $u(T)$ such that, for any $f \in L_2$:

$$(f, u(T)f) = \int_{\partial\sigma} S_\sigma(u)\ (f, d\mu_T f)$$

∎

4 Proof of theorem 3

Let us assume first that $\partial\sigma$ is piecewise C^1. For any rational function u, one has

$$\int (S_\sigma u(z) - u(T))^*\ \mu_T(dz)\ (S_\sigma u(z) - u(T)) \geq 0$$

what rewrites, thanks to (4)

$$u(T)^* u(T) \leq \int |S_\sigma u(z)|^2 \mu_T(dz). \tag{19}$$

Since for any z the map $u \longrightarrow (Su)(z)$ is continuous on $C(\partial\sigma)$, there exist measures $\nu_z(ds)$ such that

$$S_\sigma u(z) \quad = \quad \int u(s)\nu_z(ds)$$

and

$$||S_\sigma|| = \sup_z ||\nu_z||$$

hence, by the Cauchy-Schwarz inequality

$$\begin{aligned} |S_\sigma u(z)|^2 &\leq ||\nu_z|| \int |u(s)|^2 |\nu_z(ds)| \\ &\leq ||S_\sigma|| \int |u(s)|^2 |\nu_z(ds)|. \end{aligned} \tag{20}$$

Finally equations (19) and (20) imply

$$\begin{aligned} \sum_i u_i(T)^* u_i(T) &\leq \int \sum_i |(S_\sigma u_i)(z)|^2 \mu_T(dz) \\ &\leq ||S_\sigma|| \int \int \sum_i |u_i(s)|^2 |\nu_z(ds)|\ \mu_T(dz) \\ &\leq ||S_\sigma||^2 \sup_{s\in\sigma} \sum_i |u_i(s)|^2. \end{aligned}$$

If now $\partial\sigma$ is not piecewise C^1, we consider the sets

$$\sigma_\varepsilon = \{z;\ d(z,\sigma) \leq \varepsilon\}.$$

These sets are closed and convex and we can take the limit $\varepsilon \to 0$ in the equation

$$\sum_i u_i(T)^* u_i(T) \quad \leq \quad ||C_{\sigma_\varepsilon}||^2 \sup_{s\in\sigma_\varepsilon} \sum_i |u_i(s)|^2.$$

∎

[3] s-lim denotes the strong limit of operators: $A = \text{s-lim}\ A_n$ iff $||A_n f - f||$ converges to 0 for any vector f.

5 Spectral sets of a product of operators

Let us consider a contraction T, i.e. $||T|| \le 1$. The unit circle is a spectral set for T. Obviously the unit circle is a spectral set for products of contractions. In this part we give two examples of properties which remain true through product of operators. These properties will be useful for the proof of Burkholder's conjecture.

Proposition 8 *For any $0 \le \alpha \le 1$, denote by C_α the circle of radius $1-\alpha$ centered at $(\alpha, 0)$.*
Let T_i $(i = 1, 2)$, be operators such that C_{α_i} is a spectral set of T_i.
Then $C_{\alpha_1\alpha_2}$ is spectral set for T_1T_2.

Proof: From Theorem 1 :

$$||T_i - \alpha_i I|| \le 1 - \alpha_i, \quad 1 > \alpha_i > 0$$

then,

$$\begin{aligned} ||T_1T_2 - \alpha_1\alpha_2 I|| &= ||(T_1-\alpha_1)T_2 + \alpha_1(T_2-\alpha_2)|| \\ &\le (1-\alpha_1) + \alpha_1(1-\alpha_2) \\ &= 1 - \alpha_1\alpha_2 \end{aligned}$$

∎

For half-planes, the stability through product is more complicated.

Proposition 9 *For any real ε, denote H_ε the half-plane, containing $(0,0)$, and having $(1,0)$ on its boundary, defined by :*

$$H_\varepsilon = \{z : \ \mathrm{Re}\,((1+i\varepsilon)(1-z)) \ge 0\}.$$

Let T_i $(i = 1, 2)$, be contractions with H_{ε_i} as spectral set. Assume in addition that C_α is spectral set of T_2 for some $\alpha > 0$. If $\varepsilon_1\varepsilon_2 > 0$, there exist $\varepsilon \neq 0$ with the same sign as the ε_i's, such that H_ε is a spectral set of T_1T_2.

Proof: Choose $\gamma < 1$ and set $z_1 = 1 + i\gamma\varepsilon_1$. From the assumptions, setting $S_i = 1 - T_i$, $\{z : \ \mathrm{Re}\,((1+i\varepsilon_i)z) \ge 0\}$ is a spectral set of S_i and one has

$$\begin{aligned} 2\mathrm{Re}\,(z_1S_1) &= 2\gamma\mathrm{Re}\,((1+i\varepsilon_1)S_1) + (1-\gamma)(S_1+S_1^*) \\ &\ge (1-\gamma)(S_1+S_1^*) \\ &= (1-\gamma)\left[I + S_1^*S_1 - (I-S_1)^*(I-S_1)\right] \\ &\ge (1-\gamma)S_1^*S_1. \end{aligned}$$

By the way, this proves that $H_{\gamma\varepsilon}$ is spectral set of a contraction T as soon as H_ε is spectral set of T. On the other hand :

$$\begin{aligned} I - T_1T_2 &= (I-T_1)T_2 + (I-T_2) \\ &= T_2^*S_1T_2 + S_2^*S_1T_2 + S_2 \\ \mathrm{Re}\,(z_1(I-T_1T_2)) &= \mathrm{Re}\,(T_2^*(z_1S_1)T_2) + \mathrm{Re}\,(z_1S_2^*S_1T_2) + \mathrm{Re}\,(z_1S_2)\,. \end{aligned}$$

For the second term, using the inequality $2\mathrm{Re}\,(AB) \ge -AA^* - B^*B$ with $A = z_1\sqrt{1-\gamma}^{-1}S_2^*$ and $B = \sqrt{1-\gamma}S_1T_2$, we obtain :

$$\begin{aligned} 2\mathrm{Re}\,(z_1(I-T_1T_2)) &\ge 2T_2^*\mathrm{Re}\,(z_1S_1)\,T_2 - (1-\gamma)T_2^*S_1^*S_1T_2 \\ &\quad -\frac{|z_1|^2}{1-\gamma}S_2^*S_2 + 2\mathrm{Re}\,(z_1S_2) \\ &\ge -\frac{|z_1|^2}{1-\gamma}S_2^*S_2 + 2\mathrm{Re}\,(z_1S_2)\,. \end{aligned}$$

By the assumption on T_2 and setting $\beta = 1-\alpha$:

$$\begin{aligned} S_2^* S_2 &= (S_2-\beta)^*(S_2-\beta) - \beta^2 + 2\beta \mathrm{Re}\,(S_2) \\ &\leq 2\beta \mathrm{Re}\,(S_2) \end{aligned}$$

consequently,

$$\begin{aligned} \mathrm{Re}\,(z_1(I - T_1T_2)) &\geq -\frac{|z_1|^2}{1-\gamma}\mathrm{Re}\,(\beta S_2) + \mathrm{Re}\,(z_1 S_2) \\ &= 2C\mathrm{Re}\left((1 + i\frac{\varepsilon_1\gamma}{C})S_2\right) \end{aligned}$$

with

$$C = 1 - \frac{(1+\gamma^2\varepsilon_1^2)\beta}{1-\gamma}.$$

Thus, for γ small enough C is positive and $\frac{|\varepsilon_1|\gamma}{C} \leq |\varepsilon_2|$.
Finally, $\mathrm{Re}\,(z_1(I - T_1T_2)) \geq 0$ and the proof is complete. ∎

6 Proof of Burkholder's conjecture

Consider $T = P_1P_2...P_n$ a finite product of projectors. $\{0,1\}$ is spectral set of any self-adjoint projector P. A fortiori, the circle $C_{1/2}$, as well as any half-plane H_ε, are spectral sets of P. From propositions 8 and 9, there exists $\alpha, \varepsilon > 0$ such that C_α, H_ε and $H_{-\varepsilon}$ are spectral sets of T. The set $\sigma = C_\alpha \cap H_\varepsilon \cap H_{-\varepsilon}$ satisfies the assumptions of theorem 3. Notice that :

$$\sup_{z\in\sigma} \frac{|1-z|}{1-|z|} < \infty.$$

Theorem 3 yields :

$$\begin{aligned} \left\| \sum_{n=0}^{p} (n+1)(I-T)^*T^{*n}T^n(I-T) \right\| &\leq C_\sigma \sup_{z\in\sigma} |1-z|^2 \sum_{n=0}^{\infty}(n+1)|z|^{2n} \\ &= C_\sigma \sup_{z\in\sigma} \frac{|1-z|^2}{(1-|z|^2)^2} \\ &\leq C_\sigma \left(\sup_{z\in\sigma} \frac{|1-z|}{1-|z|}\right)^2 \\ &< \infty. \end{aligned} \tag{21}$$

Once this is obtained we conclude by using classical arguments. Define

$$A_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k$$

and let us recall that Chacon's theorem implies the almost sure convergence of $A_n f$ if $f \in L_2$. First we have

$$A_n - T^n = \frac{1}{n}(I-T)(I + 2T + \ldots + nT^{n-1})$$

and setting $g = (I-T)f$

$$\begin{aligned} (A_n - T^n)f &= \frac{1}{n}(I + 2T + \ldots + nT^{n-1})g \\ \sup_n |(A_n - T^n)f| &\leq \sup_n \frac{1}{n}|g + 2Tg + \ldots + nT^{n-1}g| \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_n \frac{1}{n}(1+2+\ldots n)^{1/2} \left(\sum_{k=0}^{n-1} k(T^k g)^2\right)^{1/2} \\
&\leq \left(\sum_{k=0}^{\infty} k(T^k g)^2\right)^{1/2} .
\end{aligned}
$$

Equation (21) implies now the following bound:

$$
\| \sup_n |(A_n - T^n)f| \|_2 \quad \leq \quad C\|f\|_2 \tag{22}
$$

for any $f \in L_2$. This allows us to conclude by a density argument.
Indeed, since T is a contraction, $\{f : f = T^*f\}$ coincides with $\{f : f = Tf\}$. On the other hand, the range of $I-T$ is dense in the orthogonal of $\{f : f = T^*f\}$. Thus, for any $\varepsilon > 0$ and any $f \in L_2$, there exists $g, h, f_1 \in L_2$ such that:

$$
f = (I-T)g + h + f_1, \quad \|h\|_2 < \varepsilon, Tf_1 = f_1
$$

We can assume that f_1 is 0. Notice that equation (21) implies that $T^n(I-T)g$ converges a.s to zero. Furthermore, Chacon's theorem implies the almost sure convergence of $A_n h$, thus:

$$
\limsup_n |T^n f| = \limsup_n |T^n h - A_n h|
$$

Equation (22) applied to h, provides:

$$
\| \limsup_n |T^n f| \|_2 \leq C\varepsilon.
$$

Since ε is arbitrary, $T^n f$ converges a.s, and the conjecture is proved.

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