Some estimates for analytic functions of bounded operators with given numerical range

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Abstract

Let S be a bounded region of the complex plane. Being given a linear operator T on a Hilbert space whose numerical range is contained in S and a holomorphic function f uniformly bounded in S, we provide an estimate of the norm of $f(T)$ with respect to this uniform bound. We significantly improve previous estimates, especially in the case where S is flat.

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1 Introduction

We consider an Hilbert space H and operator T on H with numerical range W :

$$W = W(T) = \{z = (u, Tu),\ u \in H,\ \|u\| = 1\}.$$

It is classical that W is convex, contains the spectrum of H and that for $z \in \mathbb{C}$ one has

$$\|(T - zI)^{-1}\| \le \frac{1}{d(z, W)} = \sup_{a \in W} |(a - z)^{-1}|. \tag{1}$$

See for instance [1].

Consider convex region S of the complex plane. We are interested in inequalities of the type

$$\|f(T)\| \le C_S \sup_{z \in S} |f(z)|$$

for any rational function f, and any T such that $W(T) \subset S$. The constant C_S should depend only on S. This is a well known result of von Neumann [5, 6] in the case S is a half-plane (and $C_S = 1$) or a circle. In the case S is a bounded region of the plane, this inequality is proved in [3] ; however, one can see that the constant C_S given there is very high, especially if S is flat. In the case S is a strip or a sector of angle α, this inequality is proved in [2] with $C_S = \min(\frac{\pi - \alpha}{\alpha}, 2 + \frac{2}{\sqrt{3}})$.

In this paper we will consider a bounded region S of the complex plane, and improve significantly the value of C_S given in [3]; for instance if S is an ellipse, we get $C_S \le 2 + \pi + 4\log(a/b)$ (a and b are the major and minor axes), while [3] obtains $C_S \le 512(a/b)^3 + 3$. Unfortunatelly, the existence of a bound independent of S remains an open question.

In [3] and [2] the computation of C_S relies on the following decomposition of the Cauchy integral (Γ is the boundary of S) :

$$f(T) \;=\; \frac{1}{2i\pi} \int_\Gamma f(z) \left(\frac{dz}{z - T} - \frac{d\bar{z}}{\bar{z} - T^*} \right) + \frac{1}{2i\pi} \int_\Gamma f(z) \frac{d\bar{z}}{\bar{z} - T^*} \tag{2}$$

The point of this decomposition is that the (infinitesimal) operator factor of f in the first term is self-adjoint non-negative and integrates to 2, implying that the norm of the first term is bounded by $2\|f\|_\infty$ (see [3] for the details), and the second term is bounded by using "standard" techniques. We will give here a better bound to this second term, leading to a smaller value of C_S.

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2 The theorem

As we have briefly shown above, the first step in the proof consists in using the representation of f as a Cauchy integral, and the second step is to split this integral into two terms, the first one being two times its real part and the second one being its conjugate. We add here a third step which is an integral representation of the conjugate term. Everything is thus based on the following lemma:

Lemma 1 *Let S be a convex bounded set of the complex plane, $\Gamma = \partial S$. We assume that 0 is an interior point of S and that Γ is C^1. For $\sigma \in \Gamma$, and denote by θ the argument of the tangent vector to Γ at σ. σ as a function of θ is defined $d\theta$ a.s.*

Let u be a point interior to S and f be analytic in the neighborhood of S, then

$$\int_\Gamma f(\sigma)\frac{d\bar{\sigma}}{\bar{\sigma}-\bar{u}} = -2i\int_0^{2\pi}\int_0^1 f(\rho\sigma)\frac{\sigma e^{-2i\theta}(\bar{\sigma}-\bar{u})}{(\bar{\sigma}-\sigma(1-\rho)e^{-2i\theta}-\bar{u})^2}d\rho d\theta. \tag{3}$$

Proof. We can assume that the boundary is smooth, so that $\sigma(\theta)$ is a C^1 function with derivative σ_θ.

Let $\varphi(\theta, z)$ be a function C^1 2π-periodic w.r.t. θ and analytic in the neighborhood of S w.r.t. z. We denote by φ_θ and φ_z the partial derivatives w.r.t. θ and z. Then

$$\begin{aligned}
\int_\Gamma \varphi(\theta,\sigma)d\sigma &= \int_\Gamma\int_0^1 \frac{\partial}{\partial\rho}(\rho\varphi(\theta,\rho\sigma))d\rho d\sigma \\
&= \int_\Gamma\int_0^1 (\varphi(\theta,\rho\sigma)+\rho\sigma\varphi_z(\theta,\rho\sigma))\sigma_\theta d\rho d\theta \\
&= \int_\Gamma\int_0^1 \left[\frac{\partial}{\partial\theta}(\sigma\varphi(\theta,\rho\sigma)) - \sigma\varphi_\theta(\theta,\rho\sigma))\right] d\rho d\theta \\
&= -\int_0^{2\pi}\int_0^1 \varphi_\theta(\theta,\rho\sigma)\sigma d\rho d\theta. \qquad (4)
\end{aligned}$$

The function

$$\varphi(\theta, z) = \frac{f(z)}{z-\sigma(\theta)+e^{2i\theta}(\bar{\sigma}(\theta)-\bar{u})}$$

is analytic w.r.t. z since any point $z \in \text{int}(S)$ satisfies $\text{Im}((z-\sigma)e^{-i\theta}) > 0$. Notice that $d\bar{\sigma} = e^{-2i\theta}d\sigma$. Hence

$$\int_\Gamma f(\sigma)\frac{d\bar{\sigma}}{\bar{\sigma}-\bar{u}} = \int \varphi(\theta,\sigma)d\sigma$$

The relation $d\bar{\sigma} = e^{-2i\theta}d\sigma$ implies also that

$$\varphi_\theta(\theta, z) = 2i\frac{f(z)e^{2i\theta}(\bar{\sigma}(\theta)-\bar{u})}{(z-\sigma(\theta)+e^{2i\theta}(\bar{\sigma}(\theta)-\bar{u}))^2}.$$

Hence (3) is exactly (4). ∎

Lemma 2 *We keep the notations of Lemma 1. Let T be a bounded operator such that $W(T)$ is contained in the interior of S, then*

$$\left\|\int_\Gamma f(\sigma)\frac{d\bar{\sigma}}{\bar{\sigma}-T^*}\right\| \le 2\pi(\pi + TV_\Gamma(\log(r))\sup_{z\in S}|f(z)| \tag{5}$$

where TV_Γ is the total variation of a function defined on Γ and $r = |\sigma|$.

Proof. Equation (1) guaranties the convergence of

$$\varphi(T^*) = \int_\Gamma f(\sigma)\frac{d\bar{\sigma}}{\bar{\sigma} - T^*}$$
$$\psi(T^*) = -2i\int_0^{2\pi}\int_0^1 f(\rho\sigma)\frac{\sigma e^{-2i\theta}(\bar{\sigma} - T^*)}{(\bar{\sigma} - \sigma(1-\rho)e^{-2i\theta} - T^*)^2}d\rho d\theta.$$

Since the analytic functions φ and ψ coincide on the spectrum of T^*, we have $\varphi(T^*) = \psi(T^*)$. Hence

$$\left\|\int_\Gamma f(\sigma)\frac{d\bar{\sigma}}{\bar{\sigma} - T^*}\right\| \leq 2\int_0^{2\pi}\left\|\int_0^1 f(\rho\sigma)\frac{\sigma e^{-2i\theta}(\bar{\sigma} - T^*)}{(\bar{\sigma} - \sigma(1-\rho)e^{-2i\theta} - T^*)^2}d\rho\right\| d\theta$$

Since $W(T)$ is contained in the tangent hyperplane $P_\sigma = \{z = \sigma + ie^{i\theta}x, Re(x) \geq 0\}$, we have by the previously mentionned von Neumann's Theorem

$$\left\|\int_\Gamma f(\sigma)\frac{d\bar{\sigma}}{\bar{\sigma} - T^*}\right\| \leq 2\int_0^{2\pi}\sup_{z\in P_\sigma}\left|\int_0^1 f(\rho\sigma)\frac{\sigma e^{-2i\theta}(\bar{\sigma} - \bar{z})}{(\bar{\sigma} - \sigma(1-\rho)e^{-2i\theta} - \bar{z})^2}d\rho\right| d\theta$$

By the maximum principle, the supremum is attained on the boundary:

$$\begin{aligned}
\left\|\int_\Gamma f(\sigma)\frac{d\bar{\sigma}}{\bar{\sigma} - T^*}\right\| &\leq 2\int_0^{2\pi}\sup_{x\in\mathbb{R}}\left|\int_0^1 f(\rho\sigma)\frac{x\sigma e^{-3i\theta}}{(\sigma(1-\rho)e^{-2i\theta} - xe^{-i\theta})^2}d\rho\right| d\theta \\
&= 2\int_0^{2\pi}|\sigma|\sup_x\left|\int_0^1 f(\rho\sigma)\frac{x}{(\sigma(1-\rho)e^{-i\theta} - x)^2}d\rho\right| d\theta \\
&\leq 2\|f\|_\infty\int_0^{2\pi}|\sigma|\sup_x\int_0^1\frac{x}{|\sigma\rho e^{-i\theta} - x|^2}d\rho d\theta \\
&= 2\|f\|_\infty\int_0^{2\pi}|\sigma|\sup_{\varepsilon\in\{-1,1\}}\int_0^\infty|\varepsilon\sigma\rho e^{-i\theta} - 1|^{-2}d\rho d\theta \\
&= 2\|f\|_\infty\int_0^{2\pi}\sup_{\varepsilon\in\{-1,1\}}\int_0^\infty|\varepsilon xe^{i(\zeta-\theta)} - 1|^{-2}dx d\theta
\end{aligned}$$

where ζ is the argument of σ. It is worth noticing that if $r(\zeta) = |\sigma(\theta)|$, the relations

$$\sigma = re^{i\zeta}$$
$$\frac{d\sigma}{d\zeta} = \alpha(\zeta)e^{i\theta}, \alpha \geq 0$$

imply by differentiation of the first identity that

$$r' + ir = \alpha(\zeta)e^{i(\theta-\zeta)}, \quad r' = \frac{dr}{d\zeta}.$$

Hence

$$\frac{r}{r'} = \tan(\theta - \zeta).$$

Notice also that since $\sin(\theta - \zeta) > 0$ we can choose the determination in such a way that $\theta - \zeta \in (0, \pi)$ with continuity along Γ, and

$$\theta - \zeta = \frac{\pi}{2} - \operatorname{atan}\left(\frac{r'}{r}\right).$$

In the case $\varepsilon = 1$ we get

$$\begin{aligned}
\int_0^\infty|xe^{i(\zeta-\theta)} - 1|^{-2}dx &= \int_0^\infty(x^2 - 2cx + 1)^{-1}dx, \quad c = \cos(\theta - \zeta) \\
&= \left[\frac{1}{s}\operatorname{atan}\left(\frac{s}{c - x}\right)\right]_0^\infty, \quad s = \sin(\theta - \zeta) \\
&= \frac{\pi - (\theta - \zeta)}{\sin(\theta - \zeta)}.
\end{aligned}$$

The case $\varepsilon = -1$ reduces to the same equation with replacement of $\theta - \zeta$ with $\pi - \theta + \zeta$. Hence

$$\left\| \int_\Gamma f(\sigma) \frac{d\bar\sigma}{\bar\sigma - T^*} \right\| \leq 2\|f\|_\infty \int_0^{2\pi} \frac{\max(\pi - (\theta - \zeta), \theta - \zeta)}{\sin(\theta - \zeta)} d\theta$$

For any function g, one has

$$\int_\Gamma g(\theta - \zeta) d(\theta - \zeta) = \int_\Gamma dG(\theta - \zeta) = 0$$

(G is any primitive of g). Hence

$$\begin{aligned} \left\| \int_\Gamma f(\sigma) \frac{d\bar\sigma}{\bar\sigma - T^*} \right\| &\leq 2\|f\|_\infty \int_0^{2\pi} \frac{\max(\pi - (\theta - \zeta), \theta - \zeta)}{\sin(\theta - \zeta)} d\zeta \\ &\leq 2\|f\|_\infty \int_0^{2\pi} (\pi/2 + \operatorname{atan}(\lambda(\zeta)))\sqrt{1 + \lambda(\zeta)^2} d\zeta, \quad \lambda(\zeta) = \frac{|r'|}{r}. \end{aligned}$$

Using that for $x > 0$ one has

$$(\pi/2 + \operatorname{atan}(x))\sqrt{1 + x^2} \leq \pi/2 + \pi x$$

we get

$$\left\| \int_\Gamma f(\sigma) \frac{d\bar\sigma}{\bar\sigma - T^*} \right\| \leq 2\pi\|f\|_\infty \int_0^{2\pi} \left(\frac{1}{2} + \frac{|r'|}{r} \right) d\zeta = 2\pi\|f\|_\infty (\pi + TV_\Gamma(\log(r)).$$

■

Theorem 1 *Let T be a bounded operator such that $W(T) \subset S$, f be a function analytic in the neighborhood of S, and z_0 be an interior point of S, then*

$$\|f(T)\| \leq (2 + \pi + TV_\Gamma(\log |\ . - z_0|)) \sup_{z \in S} |f(z)|$$

where TV_Γ is the total variation of a function defined on Γ.

Proof. Notice that we can assume that $W(T)$ is contained in the interior of S and that Γ is smooth (if not apply the result with $S_\varepsilon = \{z : d(z, S) \leq \varepsilon\}$ and let $\varepsilon \to 0$).

Because of (2), the result is now an immediate consequence of the previous lemma and of

$$\left\| \frac{1}{2i\pi} \int_\Gamma f(z) \left(\frac{dz}{z - T} - \frac{d\bar z}{\bar z - T^*} \right) \right\| \leq 2\|f\|$$

proved in [3]. ■

3 Extension

For any $p > 0$, H^p can be endowed with the inner product

$$\langle u, v \rangle = \sum_{i=1}^p \langle u_i, v_i \rangle.$$

A matrix of operators on H, $A = (A_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$ will be considered as an operator from H^p to H^q with the norm

$$\|A\| = \sup_{\|u\| = \|v\| = 1} |\langle u, Av \rangle| = \sup_{\|u\| = \|v\| = 1} |\sum_{i,j} \langle u_i, A_{ij} v_j \rangle|.$$

We shall be interested in bounding the norm of $F(T)$ where $F = (f_{ij})$ and each f_{ij} is an analytic function on S. For instance, if $F = (f_1, ..f_n)$ (here $q = 1$), one obtains easily

$$\|F(T)\| = \| \sum_i f_i(T)^* f_i(T) \|.$$

Bounds on such expressions have been used in [3] for estimating infinite sums.

Theorem 2 *(Completely bounded formulation) Let T be a bounded operator such that $W(T) \subset S$. Let $F = (f_{ij})_{1 \le i \le p, 1 \le j \le q}$ be a matrix of analytic functions in the neighborhood of S, then*

$$\|F(T)\| \le (2 + \pi + TV_\Gamma(\log|\ . - z_0|)) \sup_{z \in S} \|F(z)\|$$

where TV_Γ is the total variation of a function defined on Γ, and z_0 is any interior point of S.

In particular, by Paulsen's theorem [4], there exist an operator Q such that

$$\begin{aligned} &\|Q\|\ \|Q^{-1}\| \le 2 + \pi + TV_\Gamma(\log|\ . - z_0|) \\ &\|F(QTQ^{-1})\| \le \sup_{z \in S} \|F(z)\| \end{aligned}$$

for any F as above.

Proof. As in the proof of Theorem 1 we can assume that $W(T)$ is contained in the interior of S and that Γ is smooth.

$$\begin{aligned} F(T) &= \frac{1}{2i\pi} \int_\Gamma F(z) \left(\frac{dz}{z - T} - \frac{d\bar{z}}{\bar{z} - T^*} \right) + \frac{1}{2i\pi} \int_\Gamma F(z) \frac{d\bar{z}}{\bar{z} - T^*} \\ &= A_1 + A_2 \end{aligned}$$

For the first term

$$\begin{aligned} \langle u, A_1 v \rangle &= \frac{1}{2i\pi} \sum_{i,j} \int_\Gamma f_{ij}(z) \left\langle u_i, \left(\frac{dz}{z - T} - \frac{d\bar{z}}{\bar{z} - T^*} \right) v_j \right\rangle \\ &= \frac{1}{2i\pi} \sum_{i,j} \int_0^{2\pi} f_{ij}(z) \langle u_i, p(\zeta, T) v_j \rangle d\zeta \end{aligned}$$

and $p(\zeta, T)$ is a non-negative self-adoint operator with $\frac{1}{2i\pi} \int_\Gamma p(\zeta, T) d\zeta = 2I$. Hence

$$\begin{aligned} |\langle u, A_1 v \rangle| &\le \frac{1}{2\pi} \int_\Gamma \|F(z)\|\ \|(\langle u_i, p(\zeta, T) v_j \rangle)_{ij}\| d\zeta \\ &\le \frac{1}{2\pi} \int_\Gamma \|F(z)\|\ \|p(\zeta, T) u\|\ \|p(\zeta, T) v\| d\zeta \\ &\le \frac{1}{2\pi} \sup_{z \in S} \|F(z)\| \left(\int_\Gamma \|F(z)\|\ \|p(\zeta, T) u\|^2 d\zeta \right)^{1/2} \left(\int_\Gamma \|F(z)\|\ \|p(\zeta, T) v\|^2 d\zeta \right)^{1/2} \\ &\le 2\|u\|\ \|v\| \sup_{z \in \mathbb{C}} \|F(z)\|. \end{aligned}$$

The bound on $\|A_2\|$ is obtained exactly as in lemma 2, using the fact that the von Neuman Theorem remains true in this case [2]: If $W(T)$ is contained in a half plane P and if $g_{ij}(z)$ are bounded analytic functions on P, then

$$\|G(T)\| \le \sup_{z \in P} \|G(z)\|$$

where G is the matrix (g_{ij}) (since in lemma 2 we have $|g_{ij}(z)| \le c/|z|$ and T bounded, the result in this case is also a straightforward application of (2), where the second term is always 0, and an estimation of A_1 as above). ■

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