The solution to the BSDEs with terminal condition in L_1

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1 Introduction

A BSDE (Backward Stochastic Differential Equation) consists in the following problem: find two processes $(y_s, z_s)_{0\leq s\leq T}$ such that

$$\begin{aligned}
&dy_s = -f(\omega, t, y_s, z_s)ds + z_s dw_s, \quad s \in [0,T]\\
&y_T = \xi,\\
&\int_0^t z_s dw_s \quad L_1 - \text{martingale}
\end{aligned}$$

given

$$\begin{aligned}
&w_s \text{ an } \mathcal{F}_s - \text{adapted Brownian motion on } \mathbb{R}^d,\\
&\xi \in \mathcal{F}_T,\\
&f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^d \longrightarrow \mathbb{R}^k \quad \text{progressively measurable.}
\end{aligned}$$

If $\xi \in L_p$ and f is Lipschitz w.r.t. (y,z) this problem has been solved essentially by Pardoux and Peng [2] for $p = 2$, and El Karoui, Peng and Quenez [1] for $1 < p \leq 2$. The Lipschitz condition on f can be replaced by the weaker so-called "monotonicity" condition.

We consider here the case where we assume only that $E[|\xi|] < \infty$. This case has been considered by Peng [3] under the following assumptions:

- $k = 1$ (i.e. $y \in \mathbb{R}$)
- f Lipschitz,
- $f(t,y,z) = f(t,y) + f(t,z)$,
- $\xi > 0$.

A specific motivation for the case $E[|\xi|] < \infty$ is the following:

If $f = 0$, then $y_0 = E[\xi]$. If $f \neq 0$ Peng [3] considers y_0 as a non-linear expectation: $y_0 = \mathcal{E}_f[\xi]$. Similarly y_t is viewed as a non-linear martingale: $y_t = \mathcal{E}_f[\xi|\mathcal{F}_t]$. Hence the problem of L_1-continuity arises quite naturally.

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2 Results

The main result of this paper is the following theorem:

Theorem 1 *Assume that for some constant C_f and any $y, y', z, z' \in \mathbb{R}^d$ the following hold w.p.1*

$$\begin{aligned}&|f(\omega,t,y',z') - f(\omega,t,y,z)| \leq C_f|y'-y| + C_f\varphi(|z'-z|), \qquad \varphi(x) = \min(x, x^\alpha)\\ &E[|\xi|] < \infty.\end{aligned}$$

Then there exists a unique process (y_s, z_s) satisfying

$$\begin{aligned}&dy_s = -f(\omega,t,y_s,z_s)ds + z_s dw_s, \quad s \in [0,T]\\ &y_T = \xi,\\ &\int_0^t z_s dw_s \text{ martingale}\\ &\sup_{0\leq t\leq T} E[|y_t|] + E\left[|\int_0^T z_t dt|\right] < \infty\end{aligned}$$

up to a zero process for this last norm.

For simplicity, in what follows we do not write explicitely the dependence of f on t or ω.

The following lemma (see also [4] Prop. 4.7 of Chap. IV for a similar result) will play a key role in the proof of the theorem:

Lemma 1 *Let X_t be a martingale. For any $0 < \alpha < 1$:*

$$E[X_*^\alpha] \leq \frac{1}{1-\alpha} E[|X_T|]^\alpha, \qquad X_* = \sup_{0\leq t\leq T} |X_t|$$

As a consequence, for any previsible process z_s such that both sides are finite

$$E\int_0^T |z_s|^\alpha ds \leq C_{T,\alpha} E\left[|\int_0^T z_s dw_s|\right]^\alpha.$$

Proof. Set $c = E[|X_T|]$ and $\psi(x) = x^\alpha$. Using Doob inequality:

$$E[\psi(X_*)] \;=\; = E\int_0^\infty 1_{X_*>y}\psi'(y)dy \leq \int_0^\infty \min(cy^{-1}, 1)\psi'(y)dy = \frac{c^\alpha}{1-\alpha}.$$

Second point

$$\begin{aligned}E[\int_0^T |z_s|^\alpha ds] &\leq T^{1-\alpha/2} E\left[(\int_0^T |z_s|^2 ds)^{\alpha/2}\right]\\ &\leq CT^{1-\alpha/2} E\left[\sup_{0\leq t\leq T} |\int_0^T z_s dw_s|^\alpha\right] \quad \text{(BDG)}\\ &\leq \frac{CT^{1-\alpha/2}}{1-\alpha} E\left[|\int_0^T z_s dw_s|\right]^\alpha \quad \text{(1st point)}\end{aligned}$$

□

In the following lemma we give an explicit L_1 bound for the solution:

Lemma 2 *Assume that $\|\xi\|_1 < \infty$ and that for some $\alpha < 1$*

$$|f(y,z)| \leq C_f + C_f|y| + C_f^{1-\alpha}|z|^\alpha$$

and consider a solution to the BSDE:

$$y_t = \xi + \int_t^T f(y_s, z_s)ds - \int_t^T z_s dw_s.$$

Then

$$E|y_t| + E|\int_t^T z_s dw_s| \leq C_{\alpha,T} e^{TC_f}(C_f + \|\xi\|_1).$$

Proof. Set $u_t = E|y_t|, \quad v_t = E|\int_t^T z_s dw_s|$. From

$$y_t = E[\xi + \int_t^T f(y_s, z_s)ds|\mathcal{F}_t]$$

we get

$$\begin{aligned}
u_t &\leq E[|\xi|] + E[\int_t^T |f(y_s, z_s)|ds] \\
&\leq \|\xi\|_1 + TC_f + C_f \int_t^T u_s ds + C_f^{1-\alpha} E \int_t^T |z_s|^\alpha ds \\
&\leq \|\xi\|_1 + TC_f + C_f \int_t^T u_s ds + C_f^{1-\alpha} C_{\alpha,T} v_t^\alpha \quad \text{(prev. lemma)} \\
&\leq \|\xi\|_1 + TC_f + C_f \int_t^T u_s ds + v_t/3 + C_f C'_{T,\alpha}
\end{aligned}$$

From

$$\int_t^T z_s dw_s = \xi + \int_t^T f(y_s, z_s)ds - y_t$$

we get:

$$\begin{aligned}
v_t &\leq E[|\xi|] + u_t + E[\int_t^T |f(y_s, z_s)|ds] \\
&\leq 2\|\xi\|_1 + 2TC_f + 2C_f \int_t^T u_s ds + 2v_t/3 + 2C_f C'_{T,\alpha} \\
v_t/3 &\leq 2\|\xi\|_1 + 2TC_f + 2C_f \int_t^T u_s ds + 2C_f C'_{T,\alpha}
\end{aligned}$$

Putting this bound v_t back in the bound for u_t, we get

$$u_t \leq 3\|\xi\|_1 + 3TC_f + 3C_f \int_t^T u_s ds + 3C_f C'_{T,\alpha}$$

We conclude with Gronwall lemma that $u_t + v_t \leq C''_{\alpha,T} e^{3TC_f}(C_f + \|\xi\|_1)$. $\square$

The proof of existence and unicity relies on bounds on the difference between two solutions:

Theorem 2 *Consider two solutions (y_t, z_t) and (y'_t, z'_t) to BSDEs with same f but different terminal conditions ξ and ξ'. Assume that for some $\alpha < 1$*

$$|f(y', z') - f(y, z)| \leq C_f |y' - y| + C_f \varphi(|z' - z|), \qquad \varphi(x) = \min(x, x^\alpha)$$
$$\|\xi\|_1 + \|\xi'\|_1 < \infty.$$

Then

$$\sup_{0 \leq t \leq T} E|y'_t - y_t| + E|\int_0^T (z'_s - z_s) dw_s| \leq C(\|\xi' - \xi\|_1^\alpha + \|\xi' - \xi\|_1)$$

for some $C = C(\alpha, T, C_f)$.

Proof. Define

$$\begin{aligned} \tilde{y} &= y' - y, \\ \tilde{z} &= z' - z, \\ \tilde{f}_t &= f(y'_t, z'_t) - f(y_t, z_t), \end{aligned}$$

and the matrix e_t as the solution to

$$de_t = -C_f e_t \frac{\tilde{f}_t \tilde{y}_t^T}{|\tilde{f}_t| \, |\tilde{y}_t|} dt, \quad e_0 = I.$$

e_t is deterministically bounded. The processes

$$\tilde{y}'_t = e_t \tilde{y}_t, \quad \tilde{z}'_t = e_t \tilde{z}_t$$

satisfy

$$\begin{aligned} d\tilde{y}'_t &= e_t \tilde{f}_t \left(1 - C_f \frac{|\tilde{y}_t|}{|\tilde{f}_t|}\right) dt + e_t \tilde{z}_t dw_t \\ &= v'_t \varphi(\tilde{z}'_t) dt + \tilde{z}'_t dw_t, \qquad |v'_t| \leq C(f, T). \end{aligned}$$

$\tilde{z}'_t dw_t$ is an L_1-martingale increment since $M_t = \int_0^t \tilde{z}_s dw_s$ satisfies

$$\int_0^t \tilde{z}'_s dw_s = e_t M_t + \int_0^t e_s M_s u_s ds.$$

Hence $(\tilde{y}'_t, \tilde{z}'_t)$ appears as a solution to a (simple) BDSE. In order to use standard results on BDSEs, we will now perform a second reduction which will vanish the terminal condition. Define now η_s as

$$e_T(\xi' - \xi) = E[e_T(\xi' - \xi)] + \int_0^T \eta_t dw_t.$$

Set

$$y''_t = \tilde{y}'_t - E[e_T(\xi' - \xi)|\mathcal{F}_t], \quad z''_t = \tilde{z}'_t - \eta_t$$

we have

$$dy''_t = v'_t\varphi(z''_t + \eta_t)dt + z''_t dw_t$$

with vanishing terminal condition. Let us show that this solution is necessarily in L_p if $p > 1$ satisfies $p\alpha < 1$; we have

$$\begin{aligned}
y''_t &= -E\left[\int_t^T v'_t\varphi(z''_s + \eta_s)ds|\mathcal{F}_t\right] \\
|y''_t| &\leq CE\left[\int_0^T (|z''_s|^\alpha + |\eta_s|^\alpha)ds|\mathcal{F}_t\right] \\
E\sup_t |y''_t|^p &\leq C_pE\left[\left(\int_0^T (|z''_s|^\alpha + |\eta_s|^\alpha)ds\right)^p\right] \quad \text{(BDG)} \\
&\leq C'_pE\left[\int_0^T (|z''_s|^{p\alpha} + |\eta_s|^{p\alpha})ds\right] \\
&\leq C_{p,T,\alpha}E\left[\left|\int_0^T z''_s dw_s\right|\right]^{p\alpha} + C_{p,T,\alpha}E\left[\left|\int_0^T \eta''_s dw_s\right|\right]^{p\alpha} \quad \text{(Lemma 1)} \\
&< +\infty.
\end{aligned}$$

It is now easy to bound the martingale term

$$\begin{aligned}
\|\int_0^T z''_s dw_s\|_p &\leq \|y''_T - y''_0\|_p + \|\int_0^T v'_t\varphi(z''_s + \eta_s)ds\|_p \\
&\leq \|y''_T - y''_0\|_p + C\|\int_0^T (|z''_s|^\alpha + |\eta_s|^\alpha)ds\|_p
\end{aligned}$$

and both terms have been bounded just above. From the classical results on BSDEs, we obtain, since the terminal condition is 0:

$$\|\sup_{0\leq t\leq T} |y''_t|\,\|_p^p + \|\sup_{0\leq t\leq T} |\int_0^t z''_s dw_s|\,\|_p^p \leq CE\int_0^T \varphi(\eta_t)^p dt$$

Since $p\alpha < 1$, Lemma 1 gives

$$E\int_0^T |\eta_t|^{\alpha p} dt \quad\leq\quad C_{\alpha,T}\|\xi' - \xi\|_1^{\alpha p}.$$

Finally with this p

$$\|\sup_{0\leq t\leq T} |y''_t|\,\|_p + \|\sup_{0\leq t\leq T} |\int_0^t z''_s dw_s|\,\|_p \leq C''\|\xi' - \xi\|_1^\alpha$$

This remains true if $p = 1$. Hence

$$\sup_{0\leq t\leq T} E|y'_t - y_t| + E|\int_0^T (z'_s - z_s)dw_s| \leq C(\|\xi' - \xi\|_1^\alpha + \|\xi' - \xi\|_1)$$

for some $C = C(\alpha, T, C_f)$. $\square$

Notice that the function $\xi \longrightarrow (y, z)$ is continuous but apparently not Lipschitz.

Proof of theorem 1. For the unicity, set $\xi' = \xi$ in Theorem 2.

For the existence, choose $\xi_n \in L_2$ such that $\|\xi - \xi_n\|_1 \leq 2^{-n}$. The sequence of solutions (y_n, z_n) converges exponentially fast for the norm:

$$N(y,z) = \sup_{0 \leq t \leq T} E|y_t| + E|\int_0^T z_s dw_s|$$

hence for the distance

$$d((y,z),(y',z')) = \sup_{0 \leq t \leq T} E|y_t - y'_t| + E\left[\left(\int_0^T (z_s - z'_s)^2 ds\right)^{\alpha/2}\right]$$

and we can pass to the limit in the equation

$$y_t^n = \xi_n + \int_t^T f(y_s^n, z_s^n) ds - \int_t^T z_s^n dw_s$$

$\square$

3 Counter-example: $\alpha = 1$ is impossible

Consider the equation with $f(y,z) = z$:

$$\begin{aligned} dy_s &= -z_s ds + z_s dw_s \\ y_1 &= \xi \end{aligned}$$

then $y_s e^{w_s - s/2}$ is a martingale (Itô), hence

$$y_s = e^{-w_s - (1-s)/2} E[\xi e^{w_1} | \mathcal{F}_s].$$

Choose $\xi^* \geq 0$, $\xi^* \in L_1$ and $\xi^* e^{w_1} \notin L_1$

$$\xi^k = \min(\xi^*, k) \to \xi^* \text{ in } L_1$$

$$y_s^k = e^{-w_s - (1-s)/2} E[\xi^k e^{w_1} | \mathcal{F}_s] \ \nearrow \ y_s^*$$

But $E[y_1^*] = +\infty$.

4 Conclusion

One may ask if the Lipschitz condition on f can be weakened. The answer is yes: The classical monotonicity condition can be used instead (second condition below). This a joint work (submitted to *Stoch Proc. and Appl.)* with Philippe Briand, Ying Hu & E. Pardoux. However we do not seem to obtain the continuity w.r.t. ξ. The precise assumptions are

$$\begin{aligned}
&|f(t,y,z)-f(t,y,z')| \le \lambda|z-z'| \\
&\langle y-y', f(t,y,z)-f(t,y',z)\rangle \le \lambda|y-y'|^2 \\
&y \to f(t,y,z) \quad \text{continuous} \\
&|f(t,y,0)| \le f_t(\omega)+\varphi(|y|), \quad \varphi \nearrow \ \text{continuous}, \varphi(0)=0 \\
&|f(t,y,z)-f(t,y,0)| \le \lambda(f_t(\omega)+|y|+|z|^\alpha), \quad \alpha<1 \\
&E\left[|\xi| + \int f_t dt\right] < \infty.
\end{aligned}$$

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