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A Note on Uniform Observability

Bernard Delyon

***Abstract*—We prove in this note that the classical inequality $P_t \leq \mathcal{O}_t^{-1} + \mathcal{C}_t$ relating the variance of the Kalman filter estimate, the observability matrix, and the controllability matrix is not true. This inequality is the cornerstone of the asymptotic stability theory of the Kalman filter for time-varying systems. We provide another inequality of the same type.**

***Index Terms*—Kalman filter, time-varying, uniform observability.**

I. Introduction

We consider the following system:

$$\dot{x}_t = A_t x_t + v_t$$
$$\dot{y}_t = C_t x_t + w_t$$
$$E\begin{pmatrix} v_s v_t^T & v_s w_t^T \\ w_s v_t^T & w_s w_t^T \end{pmatrix} = \begin{pmatrix} Q_t & R_t \\ R_t^T & S_t \end{pmatrix} \delta(t-s).$$

with an initial value with Gaussian distribution $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$. The corresponding Kalman filter is

$$\dot{\hat{x}}_t = A_t \hat{x}_t + \left(P_t C_t^T + R_t\right) S_t^{-1} (\dot{y}_t - C_t \hat{x}_t)$$
$$\dot{P}_t = A_t P_t + P_t A_t^T + Q_t - \left(P_t C_t^T + R_t\right) S_t^{-1} \left(C_t P_t + R_t^T\right).$$

The matrix P_t is the variance of the estimation error $\hat{x}_t - x_t$. Bounding P_t is, for obvious reasons, an important issue. In [2, p. 359], R. E. Kalman considers the case where $R_t = 0$ and states the following lemma (we set $W_t = C_t^T S_t^{-1} C_t$).

Lemma 1 (Case $R_t = 0$): Let P_t, $\mathcal{O}_t$, $\mathcal{C}_t$ be the solutions to

$$\dot{P}_t = A_t P_t + P_t A_t^T + Q_t - P_t W_t P_t \quad P_0 = P_0^T \geq 0$$
$$\dot{\mathcal{O}}_t = -\mathcal{O}_t A_t - A_t^T \mathcal{O}_t + W_t \quad \mathcal{O}_0 = 0$$
$$\dot{\mathcal{C}}_t = A_t \mathcal{C}_t + \mathcal{C}_t A_t^T + Q_t \quad \mathcal{C}_0 = 0$$

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then

$$P_t \leq \mathcal{O}_t^{-1} + \mathcal{C}_t$$

as soon as $\mathcal{O}_t^{-1}$ exists.

This lemma is more explicitly stated and "proved" in [1, p. 234, 243]. The flaw in the proof is apparent in the last correlation inequality at the end of [1, p. 234].

We show here in Section II that this lemma is untrue and prove in Section III the following modified version of it.

Lemma 2 (Case $R_t = 0$): Let A_t, Q_t, W_t be arbitrary square matrices with same dimensions, piecewise continuous w.r.t. t, such that Q_t and W_t are symmetric and W_t is nonnegative for all t. Let P_t, $\mathcal{O}_t$, $\mathcal{D}_t$ be the solutions to

$$\dot{P}_t = A_t P_t + P_t A_t^T + Q_t - P_t W_t P_t \quad P_0 = P_0^T \geq 0$$
$$\dot{\mathcal{O}}_t = -\mathcal{O}_t A_t - A_t^T \mathcal{O}_t + W_t \quad \mathcal{O}_0 = 0 \tag{1}$$
$$\dot{\mathcal{D}}_t = -\mathcal{D}_t A_t - A_t^T \mathcal{D}_t + \mathcal{O}_t Q_t \mathcal{O}_t \quad \mathcal{D}_0 = 0 \tag{2}$$

then

$$P_t \leq \mathcal{O}_t^{-1} + \mathcal{O}_t^{-1} \mathcal{D}_t \mathcal{O}_t^{-1} \tag{3}$$

as soon as $\mathcal{O}_t^{-1}$ exists. Furthermore, one has

$$\mathcal{O}_t \leq \int_0^t e^{2\alpha(t-s)} \|W_s\| \, ds \quad \alpha = \sup_{0 \leq s \leq t} \|A_s\|$$
$$\mathcal{D}_t \leq \int_0^t e^{2\alpha(t-s)} \|\mathcal{O}_s\|^2 \|Q_s\| \, ds.$$

Some classical comments, which are shared by both lemmas, are in order.

2) The important point here is that the bound is independent of P_0. This allows indeed to get bounds for P_t, $t \geq 0$, by considering the system on a finite-time interval $(t-\sigma, t)$:
   P_t remains bounded if for all t the solutions to (1) and (2) over $(t-\sigma, t)$ with initial condition $\mathcal{O}_{t-\sigma} = \mathcal{D}_{t-\sigma} = 0$ satisfy $\|\mathcal{O}_t^{-1} + \mathcal{O}_t^{-1}\mathcal{D}_t\mathcal{O}_t^{-1}\| < C$.
3) A bound on $\mathcal{D}_t$ is easily obtained assuming boundedness of A_t and integrability of $\|W_t\| + \|Q_t\|$ over finite intervals. The main condition is the invertibility of $\mathcal{O}_t$.
4) The case $R_t \neq 0$ is actually covered via simple changes in A_t and Q_t.
5) Since P_t^{-1} satisfies
   $$\dot{P}_t^{-1} = -P_t^{-1} A_t - A_t^T P_t^{-1} + W_t - P_t^{-1} Q_t P_t^{-1}$$
   another application of this theorem leads to a lower bound on P_t based on the invertibility of the controllability matrix $\mathcal{C}_t$.
6) The solution to the equation for P_t^{-1} with initial value $P_0^{-1} = 0$ is smaller than $\mathcal{O}_t$; this implies that in the limit $P_0 \to \infty$, one has $P_t \geq \mathcal{O}_t^{-1}$. This is why the term O_t^{-1} cannot be avoided.

II. Counterexample

This example is made with $A_t = 0$. Consider

$$P_t = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{t+1} \end{pmatrix} \quad Q_t = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$W_t = \begin{pmatrix} 1 & t+1 \\ t+1 & (t+1)^2 + 1 \end{pmatrix}$$

then we have

$$P_t W_t P_t = \begin{pmatrix} 1 & 1 \\ 1 & 1 + (t+1)^{-2} \end{pmatrix} = Q_t - \dot{P}_t.$$

Hence, P_t is the solution to the equation. On the other hand

$$\mathcal{C}_t = tQ_0 = \begin{pmatrix} t & t \\ t & t \end{pmatrix}$$

and

$$\mathcal{O}_t = t\begin{pmatrix} 1 & t/2+1 \\ t/2+1 & t^2/3+t+2 \end{pmatrix}$$

$$\mathcal{O}_t^{-1} = \frac{1}{t}\,\frac{1}{t^2/12+1}\begin{pmatrix} t^2/3+t+2 & -t/2-1 \\ -t/2-1 & 1 \end{pmatrix}.$$

Take $x = (1, -1)^T$. We have $x^T\mathcal{C}_t x = 0$ and $x^T\mathcal{O}_t^{-1}x$ converges to zero, but $x^T P_t x$ converges to 1. Hence, there exists a time T such that $P_T \le \mathcal{O}_T^{-1} + \mathcal{C}_T$ is untrue.

III. Proof of Lemma 2

Proof: In order to limit the number of terms in the forthcoming equations, we first reduce the proof to the case $A_t = 0$. We consider the transition matrix

$$\dot{\Phi}_t = A_t\Phi_t \quad \Phi_0 = I$$

and set

$$\tilde{P}_t = \Phi_t^{-1}P_t\Phi_t^{-T}$$
$$\tilde{\mathcal{O}}_t = \Phi_t^T\mathcal{O}_t\Phi_t$$
$$\tilde{\mathcal{D}}_t = \Phi_t^T\mathcal{D}_t\Phi_t$$
$$\tilde{Q}_t = \Phi_t^{-1}Q_t\Phi_t^{-T}$$
$$\tilde{W}_t = \Phi_t^T W_t\Phi_t.$$

Notice that $\tilde{P}_t$ is the estimation variance of the state $\tilde{x}_t = \Phi_t^{-1}x_t$, and more generally, the tilded quantities correspond to the untilded ones, up to a time-varying change of coordinates in the state space. One easily obtains

$$\dot{\tilde{P}}_t = \tilde{Q}_t - \tilde{P}_t\tilde{W}_t\tilde{P}_t \quad \tilde{P}_0 = P_0$$
$$\dot{\tilde{\mathcal{O}}}_t = \tilde{W}_t \quad \tilde{\mathcal{O}}_0 = 0$$
$$\dot{\tilde{\mathcal{D}}}_t = \tilde{\mathcal{O}}_t\tilde{Q}_t\tilde{\mathcal{O}}_t \quad \tilde{\mathcal{D}}_0 = 0$$

and (3) rewrites

$$\tilde{P}_t \le \tilde{\mathcal{O}}_t^{-1} + \tilde{\mathcal{O}}_t^{-1}\tilde{\mathcal{D}}_t\tilde{\mathcal{O}}_t^{-1}.$$

We shall prove the equivalent inequality

$$\tilde{\mathcal{O}}_t\tilde{P}_t\tilde{\mathcal{O}}_t \le \tilde{\mathcal{O}}_t + \tilde{\mathcal{D}}_t. \tag{4}$$

The derivative of the left-hand side is

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal{O}}_t\tilde{P}_t\tilde{\mathcal{O}}_t &= \tilde{W}_t\tilde{P}_t\tilde{\mathcal{O}}_t + \tilde{\mathcal{O}}_t\left(\tilde{Q}_t - \tilde{P}_t\tilde{W}_t\tilde{P}_t\right)\tilde{\mathcal{O}}_t + \tilde{\mathcal{O}}_t\tilde{P}_t\tilde{W}_t \\ &= \tilde{\mathcal{O}}_t\tilde{Q}_t\tilde{\mathcal{O}}_t - \left(I - \tilde{\mathcal{O}}_t\tilde{P}_t\right)\tilde{W}_t\left(I - \tilde{P}_t\tilde{\mathcal{O}}_t\right) + \tilde{W}_t \\ &= \dot{\tilde{\mathcal{O}}}_t + \dot{\tilde{\mathcal{D}}}_t - \left(I - \tilde{\mathcal{O}}_t\tilde{P}_t\right)\tilde{W}_t\left(I - \tilde{P}_t\tilde{\mathcal{O}}_t\right)\end{aligned}$$

which implies (4) by integration. For the bound on $\mathcal{O}_t$, notice that

$$\mathcal{O}_t = \Phi_t^{-T}\tilde{\mathcal{O}}_t\Phi_t^{-1} = \int_0^t \Phi_t^{-T}\Phi_s^{-T}W_s\Phi_s^{-1}\Phi_t^{-1}\,ds.$$

Since the matrix $M_t = \Phi_s^{-1}\Phi_t^{-1}$ satisfies (s is fixed) $\dot{M}_t = -M_tA_t$, $M_s = I$, one has

$$\|M_t\| = \left\|I - \int_s^t M_uA_u\,du\right\| \le 1 + \alpha\int_s^t \|M_u\|\,du$$

and by Gronwall's lemma $\|M_t\| \le e^{\alpha(t-s)}$. Hence

$$\|\mathcal{O}_t\| \le \int_0^t e^{2\alpha(t-s)}\|W_s\|\,ds.$$

The bound on $\mathcal{D}_t$ is now immediate since this matrix satisfies the same equation as $\mathcal{O}_t$, with only a change on W_t. ■

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Time Maximum Disturbance Design for Stable Linear Systems: A Model Predictive Scheme

K. H. You and E. B. Lee

***Abstract*—It is known that the most stressful bounded (time maximum) disturbance for stabilized linear systems is of bang-bang type. This bang-bang disturbance can often be implemented with a switch set of current states for second-order systems. The isochrones, as the level sets, determine the value of the disturbance index in a state space setting. In this note, we suggest an efficient way to construct the time maximum disturbance from the information of isochronal wave front using it in a model predictive scheme. This overcomes the shortcomings of the original switch set which are constructed through time backward computation and only available for first and second-order systems. Simulation results show how the isochrones evolve and can be utilized in synthesizing the time maximum disturbance for linear systems of second and then higher order. For third-order systems, the associated L_∞-gain of the time maximizing disturbance is found.**

***Index Terms*—Bang-bang, isochrones, L_∞-gain model predictive disturbance, time maximum disturbance.**

I. Introduction

Testing robustness of stable systems by using a degradation index (measure of disturbance severity) is being developed. The index could be one of time, of fuel, or a general quadratic criterion. When the severity index to be maximized is the time distance from the equilibrium point (delay tactic), it is now known that the disturbance is of bang-bang type and can be implemented with a switch curve for second-order systems [4]. The switch curve is a useful method of storing the information concerning the optimal disturbance selection, and it can often be given in closed analytic form for the second-order systems [5].

A further fact is that for the damped harmonic oscillator given as a linear second-order system, the limit of the reachable sets boundaries from the origin when $T \to \infty$ is a maximum limit cycle when using

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