THE FUNDAMENTAL GROUP OF COMPACT KÄHLER THREEFOLDS

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ABSTRACT. Let X be a compact Kähler manifold of dimension three. We prove that there exists a projective manifold Y such that $\pi_1(X) \simeq \pi_1(Y)$. We also prove the bimeromorphic existence of algebraic approximations for compact Kähler manifolds of algebraic dimension $\dim X - 1$. Together with the work of Graf and the third author, this settles in particular the bimeromorphic Kodaira problem for compact Kähler threefolds.

1. INTRODUCTION

1.A. **Main result.** Compact Kähler manifolds arise naturally as generalisations of complex projective manifolds, and Kodaira's problem asked if every compact Kähler manifold is deformation equivalent to a projective manifold. A positive answer to this problem trivially implies that the larger class of Kähler manifolds realises the same topological invariants. The classification of analytic surfaces [Kod63] implies a positive answer to Kodaira's problem in this case (cf. also [Buc08] for a different approach). However, Voisin's counterexamples [Voi04, Voi06] show that there exist compact Kähler manifolds of dimension at least four that do not deform to projective ones. Nevertheless it is interesting to study Kodaira's problem at the level of some specific topological invariants, like the fundamental group:

1.1. Conjecture. *Let X be a compact Kähler manifold. Then the fundamental group $\pi_1(X)$ is projective, i.e. there exists a projective manifold M such that $\pi_1(X) \simeq \pi_1(M)$.*

Note that unlike other problems on fundamental groups, this conjecture does not reduce to the case of surfaces: while, by the Lefschetz hyperplane theorem, the fundamental group of any projective manifold is realised by a projective surface, it is a priori not clear if the same holds in the Kähler category. Several partial results on Conjecture 1.1 have been obtained in the last years (cf. [CCE15], [CCE14, Théorème 0.2], and [Cla16, Corollary 1.3]). In this paper we give a complete answer in dimension three:

1.2. Theorem. *Let X be a smooth compact Kähler threefold. Then $\pi_1(X)$ is projective.*

The proof of the result comes in several steps: if X is covered by rational curves, then its MRC-fibration $X \dashrightarrow Z$ induces an isomorphism $\pi_1(X) \simeq \pi_1(Z)$ [Kol93, Theorem 5.2] [BC15, Corollary 1], so we are done. If X is not covered by rational curves, we make a case distinction based on the algebraic dimension, i.e. the

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transcendence degree of the field of meromorphic functions on X. The case $a(X) = 0$ has been solved in [CC14] (see also [Gra16, Corollary 1.8]), and for the case $a(X) = 1$ we can describe in detail the structure of the fundamental group using the algebraic reduction $X \dashrightarrow C$ dominating a curve[1]. The most difficult case is when $a(X) = 2$, where the resolution of the algebraic reduction defines an elliptic fibration $X' \to S$ over a surface. Here the structure of the fundamental group is not known, even for a projective threefold. The main contribution of this paper is to use the theory of elliptic fibrations developed by Nakayama [Nak88, Nak02a, Nak02b] to show the existence of a smooth bimeromorphic model X' of X which admits algebraic approximations. The arguments work in fact without the assumption that $\dim X = 3$. We will then derive the projectivity of Kähler groups in this case as a corollary.

1.B. **Algebraic approximation.** Voisin's examples [Voi06] show that there exist compact Kähler manifolds of dimension at least ten such that none of their *smooth* bimeromorphic models deform to a projective manifold. Her examples are uniruled and up to now, no non-uniruled manifold has been discovered which satisfies the same property. In higher dimension, mild singularities occur naturally in the bimeromorphic models considered by the minimal model program. In this spirit, Peternell and independently Campana proposed a more flexible, bimeromorphic version of Kodaira's problem:

1.3. Conjecture. *Let X be a compact Kähler manifold that is not uniruled. Then there exists a bimeromorphic map $X \dashrightarrow X'$ to a normal compact Kähler space X' with terminal singularities that admits an algebraic approximation (cf. Definition 2.3).*

Algebraic approximation provides an explicit way to prove that a Kähler group is projective: since X' has terminal singularities, the fundamental group is invariant under the bimeromorphic map $X \dashrightarrow X'$ [Tak03]. If we can always choose the algebraic approximation $\mathcal{X} \to \Delta$ to be a locally trivial deformation in the sense of [FK87, p.627] [Ser06], then Conjecture 1.3 implies Conjecture 1.1.

Very recently Graf [Gra16] and the third author [Lin16, Lin17a, Lin17b] have made progress on the Kodaira problem, by proving the existence of algebraic approximations for all smooth compact Kähler threefolds of Kodaira dimension κ at most one (including the case $\kappa = -\infty$, namely uniruled threefolds [HP16, Cor.1.4]). We prove Conjecture 1.3 for non-algebraic manifolds of the highest algebraic dimension:

1.4. Theorem. *Let X be a compact Kähler manifold of algebraic dimension $a(X) = \dim X - 1$. Then there exists a bimeromorphic map $X' \dashrightarrow X$ such that X' is a compact Kähler manifold admitting an algebraic approximation.*

As $\kappa(X) \leq a(X)$, the conclusion of Theorem 1.4 holds in particular when $\kappa(X) = \dim X - 1$. Thus Theorem 1.4 together with [Gra16, Lin16, Lin17b] establishes Conjecture 1.3 for all compact Kähler threefolds. As far as we can see, our techniques do not imply the existence of algebraic approximations for threefolds whose algebraic reduction $X' \to S$ is over a surface. In fact while resolving the bimeromorphic map

[1] After the submission of the first version of this paper to the arxiv, the third author posted his preprint [Lin16] on algebraic approximation which implies this case. Our proof is completely different and should be useful for generalisations to higher dimension.

$X' \dashrightarrow X$ appearing in our statement, one might blow up some curves that are not contracted by $X' \to S$. In this case it is difficult to relate the deformation theories of X and $X' \to S$. Thus the original Kodaira problem is still open for threefolds with $a(X) = 2$.

Finally as we already mentioned above, Theorem 1.4 has the following immediate corollary on the projectivity of Kähler fundamental groups.

1.5. Corollary. *Let X be a smooth compact Kähler manifold of dimension n and algebraic dimension $a(X) = n - 1$. Then $\pi_1(X)$ is projective.*

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2. Notation and basic definitions

All complex spaces are supposed to be of finite dimension, a complex manifold is a smooth Hausdorff irreducible complex space. A fibration is a proper surjective morphism with connected fibres between complex spaces. A fibration $\varphi : X \to Y$ is locally projective if there exists an open covering $U_i \subset Y$ such that $\varphi^{-1}(U_i) \to U_i$ is projective, i.e. admits a relatively ample line bundle.

We refer to [Gra62, Fuj79, Dem85] for basic definitions about (p, q)-forms and Kähler forms in the singular case.

2.1. Definition. *Let $\varphi : X \to Y$ be a holomorphic map. A relative Kähler form is a* smooth *real closed $(1, 1)$-form ω on X such that for every φ-fibre F, the restriction $\omega|_F$ is a Kähler form. We say that φ is Kähler if such a relative Kähler form exists.*

Projective morphisms (*e.g.* finite morphisms) are examples of Kähler morphisms.

2.2. Remark. If φ is a Kähler morphism over a Kähler base Y, then $X_U := \varphi^{-1}(U)$ is Kähler for every relatively compact open set $U \subset Y$. In fact if ω_X is a relative Kähler form on X and ω_Y is a Kähler form on Y, then for all $m \gg 0$ the form $\omega_X + m\varphi^*\omega_Y$ is Kähler [Bin83, Proposition 4.6 (2)], [Fuj79].

2.3. Definition. *Let X be a normal compact Kähler space. We say that X admits an algebraic approximation if there exist a flat morphism $\pi : \mathcal{X} \to \Delta$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in Δ converging to 0 such that $\pi^{-1}(0)$ is isomorphic to X and $\pi^{-1}(t_n)$ is a projective variety for all n.*

In general a flat deformation does not preserve the fundamental group, this holds however for deformations that are locally trivial in the sense of [FK87, p.627]. For the case of fibrations we will work with an even more restricted class of deformations:

2.4. Definition. *Let $\varphi : X \to S$ be a fibration between normal compact complex spaces. A locally trivial deformation of (X, φ) is a pair of fibrations*

$$\pi : \mathcal{X} \to \Delta, \qquad \Phi : \mathcal{X} \to S \times \Delta$$

such that $\pi = p_S \circ \Phi$, where $p_S : S \times \Delta \to \Delta$ is the projection onto the first factor and the following holds:

- $X \simeq \pi^{-1}(0)$ *and* $\varphi = \Phi|_{\pi^{-1}(0)}$*;*
- *There exists an open cover* $(U_i)_{i\in I}$ *of* S *such that (up to replacing* Δ *by a smaller polydisc containing* 0*) we have*
$$\Phi^{-1}(U_i \times \Delta) \simeq \varphi^{-1}(U_i) \times \Delta$$
for all $i \in I$.

Let us recall some basic definitions on geometric orbifolds introduced in [Cam04]. They are pairs (X, Δ) where X is a complex manifold and Δ a Weil $\mathbb{Q}$-divisor; they appear naturally as bases of fibrations to describe their multiple fibres: let $\varphi : X \to Y$ be a fibration between compact Kähler manifolds and consider $|\Delta| \subset Y$ the union of the codimension one components of the φ-singular locus. If $D \subset |\Delta|$, we can write
$$\varphi^*(D) = \sum_j m_j D_j + R,$$
where D_j is mapped onto D and $\varphi(R)$ has codimension at least 2 in Y.

The integer $m(\varphi, D) = \mathrm{gcd}_j(m_j)$ is called the classical multiplicity of φ above D and we can consider the $\mathbb{Q}$-divisor

$$\Delta = \sum_{D\subset|\Delta|} (1 - \frac{1}{m(\varphi, D)})D. \tag{1}$$

The pair (Y, Δ) is called the *orbifold base* of φ.

2.5. Remark. In Campana's work [Cam04] both the classical and the non-classical multiplicities $\inf_j(m_j)$ play an important role. Let us note that for elliptic fibrations, these multiplicities coincide: the problem is local on the base, moreover we can reduce to the case of a relatively minimal elliptic fibration. Then it is sufficient to observe that in Kodaira's classification of singular fibres which are not multiple [Kod60], [BHPVdV04, Chapter V, Table 3], there is always at least one irreducible component of multiplicity one.

Let us recall what smoothness means for a geometric orbifold.

2.6. Definition. *A geometric orbifold* (X/Δ) *is said to be smooth if the underlying variety* X *is a smooth manifold and if the* $\mathbb{Q}$*-divisor* Δ *has only normal crossings. If in a coordinate patch, the support of* Δ *can be defined by an equation*
$$\prod_{j=1}^{r} z_j = 0,$$
we will say that these coordinates are adapted to Δ.

In the category of smooth orbifolds, there is a good notion of fundamental group. It is defined in the following way : if $\Delta = \sum_{j\in J}(1 - \frac{1}{m_j})\Delta_j$, choose a small loop γ_j around each component Δ_j of the support of Δ. Consider now the fundamental group of $X^\star = X\backslash\mathrm{Supp}(\Delta)$ and its normal subgroup generated by the loops $\gamma_j^{m_j}$:
$$\langle\langle\gamma_j^{m_j}, j \in J\rangle\rangle \leq \pi_1(X^\star).$$

2.7. Definition. *The fundamental group of* (X/Δ) *is defined to be:*
$$\pi_1(X/\Delta) := \pi_1(X^\star)/\langle\langle\gamma_j^{m_j}, j \in J\rangle\rangle.$$

2.8. Remark. By definition the loops γ_j define torsion elements in $\pi_1(X/\Delta)$. Thus we see that if $\pi_1(X/\Delta)$ is torsion-free, the natural surjection $\pi_1(X/\Delta) \twoheadrightarrow \pi_1(X)$ is an isomorphism.

3. Elliptic fibrations

The structure of elliptic fibrations and their deformation theory has been described in detail in the landmark paper of Kodaira [Kod60] for surfaces and its generalisation to higher dimension by Nakayama [Nak02b, Nak02a]. For the convenience of the reader we review this theory and explain some additional properties that will be important in the proof of Theorem 1.4.

3.A. **Smooth fibrations and their deformations.**

Let $S^\star$ be a complex manifold, and let $f^\star : X^\star \to S^\star$ be a smooth elliptic fibration. We associate a variation of Hodge structures H (VHS for short in the sequel) of weight 1 over $S^\star$, the underlying local system being given by the first cohomology group of the fibers $H^1(X_s, \mathbb{Z}) \simeq \mathbb{Z}^2$. A rank 2 and weight 1 VHS over $S^\star$ is equivalent to the following data: a holomorphic function to the upper half plane

$$\tau_H : \widetilde{S^\star} \to \mathbb{H}$$

defined on the universal cover $\widetilde{S^\star} \to S^\star$ which is equivariant under a representation

$$\rho_H : \pi_1(S^\star) \longrightarrow \mathrm{SL}_2(\mathbb{Z}).$$

Now if $\gamma \in \pi_1(S^\star)$, let us write

$$\rho_H(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

the image of γ under ρ_H. It is then straightforward to check that the following formula

$$((m,n),\gamma)\cdot(x,z) = \left(\gamma(x), \frac{z + m\tau_H(x) + n}{c_\gamma\tau_H(x) + d_\gamma}\right)$$

defines an action of the semi-direct product $\mathbb{Z}^2 \rtimes \pi_1(S^\star)$ on $\tilde{S^\star} \times \mathbb{C}$ which is fixed point free and properly discontinuous. We can then form the quotient to get a smooth elliptic fibration

$$p : \mathbf{J}(H) \longrightarrow S^\star$$

which is endowed with a canonical section $\sigma : S^\star \to \mathbf{J}(H)$. Following the terminology of [Kod63, Nak02a] we call p the *basic elliptic fibration* (associated to H). Note that $f^\star$ and p are locally isomorphic over $S^\star$, but their global structure can be quite different.

Since p has a global section, its sheaf of holomorphic sections $\mathcal{J}(H)$ is a well-defined sheaf of abelian groups and we have an exact sequence of sheaves

$$0 \longrightarrow H \longrightarrow \mathcal{L}_H \longrightarrow \mathcal{J}(H) \longrightarrow 0 \tag{2}$$

where

$$\mathcal{L}_H := R^1p_*\mathcal{O}_{\mathbf{J}(H)} \simeq R^1f^\star_*\mathcal{O}_{X^\star}.$$

Let us note that $\mathcal{L}_H$ can also be interpreted as the zeroth graded piece $\mathcal{H}/F^1\mathcal{H}$ of the Hodge filtration on $\mathcal{H} := H \otimes \mathcal{O}_{S^\star}$ and thus depends only on the VHS H. Since $f^\star$ is smooth, it has local sections over every point of $S^\star$ and the difference of two such sections on the intersection of their sets of definition can be seen as a section of $\mathcal{J}(H)$. In this way we have just associated a cohomology class

$$\eta(f^\star) \in H^1(S^\star, \mathcal{J}(H))$$

to $f^\star$, which is independent of the choices of local sections. The class $\eta(f^\star)$ can also be constructed in a more conceptual way. Pushing forward the exponential sequence on $X^\star$ by $f^\star$ yields a long exact sequence on $S^\star$:

$$0 \longrightarrow R^1 f^\star_* \mathbb{Z}_X \longrightarrow R^1 f^\star_* \mathcal{O}_X \longrightarrow R^1 f^\star_* \mathcal{O}^*_X \longrightarrow R^2 f^\star_* \mathbb{Z}_X \simeq \mathbb{Z}_{S^\star} \longrightarrow 0 \tag{3}$$

Recall that $\mathcal{J}(H) = R^1 f^\star_* \mathcal{O}_X / R^1 f^\star_* \mathbb{Z}_X$, the class $\eta(f^\star)$ is the image of $1 \in H^0(S^\star, \mathbb{Z}_{S^\star})$ under the connecting morphism

$$\delta : H^0(S^\star, \mathbb{Z}_{S^\star}) \longrightarrow H^1(S^\star, \mathcal{J}(H)).$$

The map η is in fact a bijection.

3.1. Theorem. [Kod63, Theorem 10.1, Theorem 11.5] [Nak02b, Proposition 1.3.1, Proposition 1.3.3] *Let $S^\star$ be a complex manifold, and let H be a* VHS *of rank 2 and weight 1 over $S^\star$.*

(a) The map $f^\star \mapsto \eta(f^\star)$ defines a one-to-one correspondence between the isomorphism classes of smooth elliptic fibrations over $S^\star$ inducing H and the elements of the cohomology group $H^1(S^\star, \mathcal{J}(H))$.
(b) A smooth elliptic fibration $f^\star : X^\star \to S^\star$ is a projective morphism if and only if $\eta(f^\star)$ is a torsion class.

The short exact sequence (2) induces an exact sequence

$$H^1(S^\star, \mathcal{L}_H) \xrightarrow{\exp} H^1(S^\star, \mathcal{J}(H)) \xrightarrow{\mathsf{c}} H^2(S^\star, H). \tag{4}$$

The vector space $V := H^1(S^\star, \mathcal{L}_H)$ appears as a deformation space of smooth elliptic fibrations over $S^\star$. More precisely, given an elliptic fibration $f^\star : X^\star \to S^\star$ inducing H, there exists a family of elliptic fibrations $\Pi : \mathcal{X} \to S^\star \times V$ over $S^\star$ parameterized by V such that the fiber over $t \in H^1(S^\star, \mathcal{L}_H)$ is an elliptic fibration whose associated element in $H^1(S^\star, \mathcal{J}(H))$ is $\exp(t) + \eta(f^\star)$. Viewing Π as a smooth elliptic fibration, the cohomology class $\eta(\Pi) \in H^1(S^\star \times V, \mathcal{J}(\mathrm{pr}^{-1}H))$ associated to Π is equal to $\exp(\xi) + \mathrm{pr}^*\eta(f^\star)$, where $\mathrm{pr} : S^\star \times V \to S^\star$ denotes the projection onto the first factor and

$$\xi \in H^1(S^\star, \mathcal{L}_H) \otimes H^0(V, \mathcal{O}_V) \subset H^1(S^\star \times V, \mathcal{L}_{\mathrm{pr}^{-1}H})$$

the element which corresponds to the identity map $V \to H^1(S^\star, \mathcal{L}_H)$.

3.2. Theorem. [Cla16, Proposition 2.4 and 2.5]

(a) Let $f^\star_1 : X^\star_1 \to S^\star$ and $f^\star_2 : X^\star_2 \to S^\star$ be two smooth elliptic fibrations over $S^\star$ inducing H. Then $f^\star_1$ can be deformed into $f^\star_2$ in the family described above if and only if $\mathsf{c}(\eta(f^\star_1)) = \mathsf{c}(\eta(f^\star_2))$
(b) Let $f^\star : X^\star \to S^\star$ be a smooth elliptic fibration such that $X^\star$ is Kähler. Then $\mathsf{c}(\eta(f^\star))$ is torsion and $f^\star$ can be deformed to a projective fibration.

3.B. **Local structures and Weierstraß models.**

3.3. Definition. *An elliptic fibration $f : X \to S$ is a fibration whose general fibre X_s is isomorphic to an elliptic curve. We say that f has a meromorphic section in a point $s \in S$ if there exists an analytic neighbourhood $s \in U \subset S$ and a meromorphic map $s : U \dashrightarrow X$ such that $f \circ s$ is the inclusion $U \hookrightarrow S$.*

Apart from smooth elliptic fibrations discussed in the last section, the second simplest examples of elliptic fibrations are Weierstraß fibrations. These fibrations turn out to be crucial in the study of elliptic fibrations.

3.4. Definition. *Let S be a complex manifold.*

(a) A Weierstraß fibration over S consists of a line bundle $\mathcal{L}$ on S and two sections $\alpha \in H^0(S, \mathcal{L}^{(-4)})$ and $\beta \in H^0(S, \mathcal{L}^{(-6)})$ such that $4\alpha^3 + 27\beta^2$ is a non zero section of $H^0(S, \mathcal{L}^{(-12)})$. With these data, we can associate a projective family of elliptic curves:

$$\mathbb{W} := \mathbb{W}(\mathcal{L}, \alpha, \beta) = \left\{Y^2Z = X^3 + \alpha XZ^2 + \beta Z^3\right\} \subset \mathbb{P}$$

where

$$\mathbb{P} := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$$

and X, Y and Z are canonical sections of $\mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{L}^{(-2)}$, $\mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{L}^{(-3)}$ and $\mathcal{O}_{\mathbb{P}}(1)$ respectively. The restriction of the natural projection $\mathbb{P} \to S$ to $\mathbb{W}$ gives rise to a flat morphism $p_{\mathbb{W}} : \mathbb{W} \to S$ whose fibres are irreducible cubic plane curves. This elliptic fibration is endowed with a distinguished section $\{X = Z = 0\}$.

(b) A Weierstraß fibration $\mathbb{W}(\mathcal{L}, \alpha, \beta)$ is said to be minimal if there is no prime divisor $\Delta \subset S$ such that $\mathrm{div}(\alpha) \geq 4\Delta$ *and* $\mathrm{div}(\beta) \geq 6\Delta$.

(c) A locally (minimal) Weierstraß fibration is an elliptic fibration $f : X \to S$ such that there exists an open covering $(U_i)_{i \in I}$ of S such that the restriction of f to $X_i := f^{-1}(U_i)$ is a (minimal) Weierstraß fibration.

3.5. Remark. Since the total space of the Weierstraß fibration $p_{\mathbb{W}} : \mathbb{W} \to S$ is by definition a hypersurface in a manifold, the complex space $\mathbb{W}$ is Gorenstein, so the canonical sheaf is locally free. In the case where $\mathbb{W}(\mathcal{L}, \alpha, \beta)$ is minimal and the discriminant divisor $\mathrm{div}(4\alpha^3 + 27\beta^2)$ has normal crossings, we know by [Nak88, Corollary 2.4] that $\mathbb{W}$ has rational, hence canonical, singularities (and it holds of course for the total space of a locally Weierstraß fibration). Since $p_{\mathbb{W}}$ is flat and the restriction of $K_{\mathbb{W}}$ to every fibre is trivial, we have $K_{\mathbb{W}} \simeq p_{\mathbb{W}}^* L$ for some line bundle L on S. Thus a locally Weierstraß fibration is relatively minimal.

It is well-known that smooth elliptic fibrations with a section are always Weierstraß:

3.6. Theorem. [Kod63] [Nak02b, Proposition 1.2.4] *Let S be a complex manifold, and let $f : X \to S$ be a smooth elliptic fibration admitting a section $s : S \to X$. Then there exists a canonically defined isomorphism $X \to \mathbb{W}$ over S to some Weierstraß fibration $\mathbb{W} \to S$ sending s onto the distinguished section.*

For non-smooth fibrations we can only hope to work with bimeromorphic models:

3.7. Definition. *Let $f : X \to S$ be an elliptic fibration and let $f^\star : X^\star \to S^\star$ denote the restriction to a non-empty Zariski open subset $S^\star \subset S$ such that f is smooth. A (locally) Weierstraß model of f is a (locally) Weierstraß fibration $p : \mathbb{W} \to S$ such that $X^\star$ is isomorphic to $\mathbb{W}^\star := p^{-1}(S^\star)$ over S.*

The existence of meromorphic sections is an obvious necessary condition for the existence of a Weierstraß model. When the base S is smooth, it is also sufficient:

3.8. Theorem. [Nak88, Theorem 2.5] *Let S be a complex manifold, and let $f : X \to S$ be an elliptic fibration. If f admits a meromorphic section, then f has a unique minimal Weierstraß model.*

The following vanishing result will be useful.

3.9. Theorem. [Nak02b, Theorem 3.2.3] *Let $f : X \to S$ be an elliptic fibration such that both X and S are smooth and that f is smooth over the complement of a normal crossing divisor in S. Then we have*

$$R^j f_* \mathcal{O}_X = 0 \qquad \forall\ j \geq 2.$$

The following example explains the importance of the normal crossing condition for the theory of elliptic fibrations:

3.10. Example. Let S be a smooth non-algebraic compact Kähler surface that admits an elliptic fibration $g : S \to \mathbb{P}^1$. Let $\mathbb{F}_1 \to \mathbb{P}^1$ be the first Hirzebruch surface, and set $X := \mathbb{F}_1 \times_{\mathbb{P}^1} S$. Then X is a smooth compact Kähler threefold, and we denote by $f : X \to \mathbb{P}^2$ the composition of the elliptic fibration $X \to \mathbb{F}_1$ with the blowdown $\mathbb{F}_1 \to \mathbb{P}^2$.
Then f is not locally projective since it has a two-dimensional fibre isomorphic to the non-projective surface S. Note however that g has at least 3 singular fibres ([Bea81, Proposition 1], cf. Proposition A.1 for a detailed proof in the Kähler case). Thus the discriminant locus of f consists of at least 3 lines meeting in one point. In particular it is not a normal crossing divisor and it is quite easy to see that $R^2 f_* \mathcal{O}_X \neq 0$ in this case.

A general elliptic fibration does not admit local meromorphic sections at every point, a fact that is the starting point of Nakayama's global theory of elliptic fibration using the ∂-étale cohomology. For our needs we can use the strategy of Kodaira [Kod63] to reduce to this case via base change:

3.11. Proposition. *Let $f : X \to S$ be an elliptic fibration such that both X and S are smooth and that f is smooth over the complement of an SNC divisor in S. Suppose that f is locally projective (e.g. when X is a Kähler manifold* [Nak02b, Theorem 3.3.3]*) and S is projective, then there exists a finite Galois cover $\tilde{S} \to S$ by some projective manifold $\tilde{S}$ such that*

$$X \times_S \tilde{S} \to \tilde{S}$$

has local meromorphic sections over every point of $\tilde{S}$. The elliptic fibration $X \times_S \tilde{S} \to \tilde{S}$ is smooth over the complement of an SNC divisor in $\tilde{S}$.

This statement is a variant of [Nak02b, Corollary 4.3.3]: in our case S is projective, but we lose the control over the branch locus.

Proof. For every irreducible component D_i of D we denote by $m_i \in \mathbb{N}$ the multiplicity of the generic fibre over D_i. By [Laz04, Proposition 4.1.12] we can choose a covering $\tilde{S} \to S$ ramifying with multiplicity exactly m_i over D_i and the ramification divisor is SNC. By construction the elliptic fibration $X \times_S \tilde{S} \to \tilde{S}$ has no multiple fibre in codimension one. Up to taking another finite cover and the Galois closure we can suppose that $\tilde{S} \to S$ is Galois and the local monodromies are unipotent. Since the elliptic fibration is locally projective, we can now apply [Nak02b, Theorem 4.3.1 and 4.3.2] to conclude that it has local meromorphic sections over every point of $\tilde{S}$. $\square$

3.C. **Elliptic fibrations with local meromorphic sections.** In this subsection we always work under the following

3.12. Assumption. Let S be a complex manifold, and let $f : X \to S$ be an elliptic fibration having local meromorphic sections over every point of S. We denote by $j : S^\star \subset S$ a Zariski open subset such that $X^\star := f^{-1}(S^\star) \to S^\star$ is smooth and assume that the complement $S \setminus S^\star$ is a normal crossing divisor.

Denote by H the VHS on $S^\star$ induced by the smooth elliptic fibration $X^\star \to S^\star$. Let

$$\mathcal{L} := \mathcal{L}_{H/S} := R^1 f_* \mathcal{O}_X.$$

Let $p^\star : \mathbf{J}(H) \to S^\star$ be the basic elliptic fibration associated with H. By Theorem 3.8, we can extend $p^\star$ to a Weierstraß model

$$p : \mathbb{W} \to S.$$

When X is smooth, $\mathcal{L}_{H/S}$ is isomorphic to the zeroth graded piece of the Hodge filtration of the lower canonical extension of $\mathcal{H} = H \otimes \mathcal{O}_{S^\star}$ to S by [Nak02b, Lemma 3.2.3]. This induces a natural map $j_* H \to \mathcal{L}_{H/S}$, which is injective by [Nak02b, Lemma 3.1.3]. Let $\mathcal{J}(H)^{\mathbb{W}}$ denote the quotient $\mathcal{L}_{H/S}/j_*H$. The exact sequence

$$0 \to j_* H \to \mathcal{L}_{H/S} \xrightarrow{\exp} \mathcal{J}(H)^{\mathbb{W}} \to 0 \tag{5}$$

extends the exact sequence (2) defined on $S^\star \subset S$.

Let $\mathbb{W}^\# \subset \mathbb{W}$ denote the Zariski open of $\mathbb{W}$ consisting of points $x \in \mathbb{W}$ where $p : \mathbb{W} \to S$ is smooth. The variety $\mathbb{W}^\#$ is a complex analytic group variety over S, where over a point $t \in S$ which parameterises a nodal (resp. cuspidal) rational curve in $p : \mathbb{W} \to S$, the fiber is the multiplicative group $\mathbb{C}^\times$ (resp. additive group $\mathbb{C}$). For each integer m, the multiplication-by-m $\mathbb{W}^\# \to \mathbb{W}^\#$ extends to a meromorphic map $\mathbf{m} : \mathbb{W} \dashrightarrow \mathbb{W}$, which is generically finite when $m \neq 0$.

3.13. Remark. In [Nak02a, p. 550], the sheaf $\mathcal{J}(H)^{\mathbb{W}}$ is first *defined* to be the germs of holomorphic sections of $p : \mathbb{W} \to S$, then one proves that $\mathcal{J}(H)^{\mathbb{W}}$ sits inside the exact sequence (5). However with this definition of $\mathcal{J}(H)^{\mathbb{W}}$, the exactness of (5) fails as it follows from the false claim that local sections of p are contained in $\mathbb{W}^\#$. Indeed, the Weierstraß fibration parameterized by $\alpha \in \mathbb{C}$ defined by $Y^2 Z = X^3 + \alpha X$ has a section $\alpha \mapsto (X(\alpha) = 0, Y(\alpha) = 0)$ which passes through the cusp of the singular central fiber.

In order to keep the sequence (5) exact, the correct definition of $\mathcal{J}(H)^{\mathbb{W}}$ should be the sheaf of germs of holomorphic sections of $\mathbb{W} \to S$ whose image is *contained in* $\mathbb{W}^\#$. In this way, as already mentioned in [Nak02a, p. 550] since $\mathbb{W}^\#$ acts on $\mathbb{W}$ by translations [Nak02a, Lemma 5.1.1 (7)], a local section of $\mathcal{J}(H)^{\mathbb{W}}$ gives rise to a local automorphism of $\mathbb{W}$.

We can associate an elliptic fibration to a cohomology class $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ [Nak02a, p.550]:

3.14. Construction. *Fix an open cover* $(U_j)_{j\in\mathbb{N}}$ *of* S *such that the class* η *is represented by a cocycle* $(\eta_{ij})_{i<j}$ *where* $\eta_{ij} \in H^0(U_i \cap U_j, \mathcal{J}(H)^{\mathbb{W}})$*. By the remark above, with the choice of a zero-section* $U_i \to \mathbb{W}^{\#}|_{U_i}$ *for each* i*, we can identify the* η_{ij} *to automorphisms of* $\mathbb{W}_{ij} := \mathbb{W}|_{U_i\cap U_j}$ *over* S*. The cocycle condition assures that the condition of the gluing lemma* [Har77, Chapter II, Exercise 2.12] *is satisfied in our situation, so we can glue the elliptic fibrations* $\mathbb{W}_i := \mathbb{W}_{|U_i} \to U_i$ *to an elliptic fibration* $p^\eta : \mathbb{W}^\eta \to S$*. Since the gluing morphisms are translations so act as the identity on the* VHS*, the* VHS *induced by* p^η *on* $S^\star$ *is* H*. This construction is independent of the choices of* (U_i) *and the zero-sections* $U_i \to \mathbb{W}^{\#}|_{U_i}$

According to the above construction, given $p^\eta : \mathbb{W}^\eta \to S$ and an open cover (U_i) of S as above, the multiplication-by-m's on $\mathbb{W}|_{U_i} \to U_i$ glue together to a meromorphic map $\mathbf{m} : \mathbb{W}^\eta \dashrightarrow \mathbb{W}^{m\eta}$ over S, which up to isomorphisms is independent of the choices of (U_i) and the zero-sections $U_i \to \mathbb{W}^{\#}|_{U_i}$.

Now given a locally minimal Weierstrass fibration $f : X \to S$ constructed by 3.14, we shall explain how to find η to which f associates. Consider the long exact sequence

$$\cdots \longrightarrow R^1 f_*\mathbb{Z}_X \longrightarrow R^1 f_*\mathcal{O}_X \longrightarrow R^1 f_*\mathcal{O}_X^* \longrightarrow R^2 f_*\mathbb{Z}_X \longrightarrow \cdots . \tag{6}$$

Since $X \to S$ is obtained by gluing the pieces $\mathbb{W}_i \to U_i$ by translation maps $\tau_{ij} : \mathbb{W}_{ij} \to \mathbb{W}_{ij}$, which act trivially on $H^1(\mathbb{W}_{ij}, \mathbb{Z})$, we have $R^1 f_*\mathbb{Z}_X = R^1 p_*\mathbb{Z}_{\mathbb{W}}$. The translations τ_{ij} also act trivially on $H^1(\mathbb{W}_s, \mathcal{O}_{\mathbb{W}_s})$ where $\mathbb{W}_s := p^{-1}(s)$ for any $s \in U_i \cap U_j$. As $p : \mathbb{W} \to S$ is flat and $H^1(\mathbb{W}_s, \mathcal{O}_{\mathbb{W}_s}) \simeq \mathbb{C}$, by Grauert's base change theorem we deduce that $R^1 f_*\mathcal{O}_X = R^1 p_*\mathcal{O}_{\mathbb{W}}$.

3.15. Lemma. *The map* $R^1 p_*\mathbb{Z} \to R^1 p_*\mathcal{O}_{\mathbb{W}}$ *induced by* $\mathbb{Z} \hookrightarrow \mathcal{O}_{\mathbb{W}}$ *is isomorphic to* $\varphi : j_*H \to \mathcal{L}_{H/S}$ *in* (5).

Proof. Let $\tau : Y \to \mathbb{W}$ be a minimal desingularization of $\mathbb{W}$ and $g := p \circ \tau : Y \to S$. First we have $\tau_*\mathcal{O}_Y = \mathcal{O}_{\mathbb{W}}$ and $\tau_*\mathbb{Z} = \mathbb{Z}$. Since $\mathbb{W}$ has at worst rational singularities, the sheaves $R^1\tau_*\mathcal{O}_Y$ and $R^1\tau_*\mathbb{Z}$ vanish. Applying the Grothendieck spectral sequence to the composition $g = p \circ \tau$ yields $R^1 g_*\mathcal{O}_Y = R^1 p_*\mathcal{O}_{\mathbb{W}}$ and $R^1 g_*\mathbb{Z} = R^1 p_*\mathbb{Z}$. As we know that the map $R^1 g_*\mathbb{Z} \to R^1 g_*\mathcal{O}_Y$ induced by $\mathbb{Z} \hookrightarrow \mathcal{O}_Y$ is isomorphic to φ [Nak02a, Theorem 5.4.9], Lemma 3.15 follows. ☐

Therefore $R^1 f_*\mathbb{Z}_X \to R^1 f_*\mathcal{O}_X$ is isomorphic to the morphism $j_*H \to \mathcal{L}_{H/S}$ in (5). Finally since the fibers of f are of dimension 1, we have $R^2 f_*\mathcal{O}_X = 0$. Thus (6) becomes

$$j_*H \longrightarrow \mathcal{L}_{H/S} \longrightarrow R^1 f_*\mathcal{O}_X^* \longrightarrow R^2 f_*\mathbb{Z}_X \longrightarrow 0. \tag{7}$$

Recall that $j_*H \to \mathcal{L}_{H/S}$ is injective and $\mathcal{J}(H)^{\mathbb{W}}$ sits inside the short exact sequence

$$0 \longrightarrow j_*H \longrightarrow \mathcal{L}_{H/S} \longrightarrow \mathcal{J}(H)^{\mathbb{W}} \longrightarrow 0. \tag{8}$$

As a fiber F of f is either an elliptic curve, a nodal rational curve, or a rational curve with a cusp, we have $H^2(F, \mathbb{Z}) = \mathbb{Z}$. Since p is proper, by [Ive82, Theorem III.6.2] $R^2 f_*\mathbb{Z}_X \simeq \mathbb{Z}_S$. Hence we have a second short exact sequence

$$0 \longrightarrow \mathcal{J}(H)^{\mathbb{W}} \longrightarrow R^1 f_*\mathcal{O}_X^* \longrightarrow \mathbb{Z}_S \longrightarrow 0. \tag{9}$$

If $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ denotes the element which defines (9), then f will be the elliptic fibration associated to η.

3.16. Remark. It is important that f is a locally Weierstraß fibration *constructed from some element* $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ in order to identify $R^1 f_*\mathbb{Z}_X \to R^1 f_*\mathcal{O}_X$ with $j_*H \to \mathcal{L}_{H/S}$. There are examples of elliptic fibrations due to N. Nakayama [Nak18] showing that not every locally Weierstraß fibration can be constructed in this way.

Each class $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ comes equipped with a *tautological family*:

3.17. Proposition. *Assuming 3.12. Given $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$, there exists a locally trivial (cf. Definition 2.4) family of elliptic fibrations $\pi : \mathcal{X} \longrightarrow S \times V$ over S parameterised by $V := H^1(S, \mathcal{L})$ satisfies the following property: an elliptic fibration $X \to S$ is a member of π if and only if X is isomorphic to $W^\theta \to S$ over S for some $\theta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ such that $\mathsf{c}(\eta) = \mathsf{c}(\theta)$.*

Proof. As in the smooth case, let

$$\xi \in H^1(S, \mathcal{L}) \otimes H^0(V, \mathcal{O}_V) \subset H^1(S \times V, \mathcal{L}_{\mathrm{pr}^{-1}H/S\times V})$$

be the element which corresponds to the identity map $V \to H^1(S, \mathcal{L}_{H/S})$ where $\mathrm{pr} : S \times V \to S$ denotes the projection onto the first factor. Let $\pi : \mathcal{X} \to S \times V$ be the elliptic fibration obtained by Construction 3.14 from $\exp(\xi) + \mathrm{pr}^*\eta \in H^1(S \times V, \mathcal{J}(\mathrm{pr}^{-1}H)^{\mathbb{W}})$. Then considering π as a family of elliptic fibrations over S parameterised by V, the fiber over $t \in H^1(S, \mathcal{L}_{H/S})$ is the elliptic fibration constructed by 3.14 from $\exp(t) + \eta \in H^1(S, \mathcal{J}(H))$. Thus π satisfies the desiring property. As V is contractible, in order to construct $\pi : \mathcal{X} \to S \times V$, it is possible to take the open cover of $S \times V$ in Construction 3.14 to be $\{U_i \times V\}$ for some open cover $\{U_i\}$ of S. Thus $\pi : \mathcal{X} \to S \times V$ is locally trivial. ☐

The cohomology group $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ is a parameter set of elliptic fibrations over S with VHS H, but for classification purposes it is too small. We denote by $\mathcal{J}(H)_{mer}$ the sheaf of *meromorphic* sections of $p : \mathbb{W} \to S$. Since p has a global meromorphic section, we see that $\mathcal{J}(H)_{mer}$ has a group structure [Nak02b, p. 243-244]. There is a trivial inclusion of sheaves of abelian groups

$$\mathcal{J}(H)^{\mathbb{W}} \subset \mathcal{J}(H)_{mer} \tag{10}$$

which is an isomorphism on $S^\star$: since $\mathbb{W}$ is smooth over $S^\star$ we have $\mathbf{J}(H) \simeq \mathbb{W}|_{S^\star}$, moreover any meromorphic section is holomorphic over $S^\star$ [Nak02b, Lemma 1.3.5]. In particular the quotient sheaf

$$\mathcal{Q}_H := \mathcal{J}(H)_{mer}/\mathcal{J}(H)^{\mathbb{W}}$$

is supported on $D = S \setminus S^\star$. By [Nak02a, Theorem 5.4.9] we have a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & & \\
0 \arrow[r] & j_*H \arrow[r] & \mathcal{L}_{H/S} \arrow[r] & \mathcal{J}(H)^{\mathbb{W}} \arrow[rr] \arrow[d] & & 0 & \\
 & & 0 \arrow[r] & \mathcal{J}(H)_{mer} \arrow[r, "\Psi_f"] \arrow[d] & R^1 f_*\mathcal{O}_X^*/\mathcal{V}_X \arrow[r] \arrow[d] & \mathbb{Z}_S \arrow[r] \arrow[d] & 0 \\
 & & 0 \arrow[r] & \mathcal{Q}_H \arrow[r] \arrow[d] & R^2 f_*\mathbb{Z}_X/\mathcal{V}_X \arrow[r] & \mathbb{Z}_S \arrow[r] & 0 \\
 & & & 0 & & &
\end{tikzcd} \tag{11}$$

where
$$\mathcal{V}_X := \operatorname{Ker}\left(R^1 f_* \mathcal{O}_X^* \longrightarrow j_*((R^1 f_* \mathcal{O}_X^*)_{|S^\star})\right)$$
and Ψ_f is constructed from the local meromorphic sections of f.

3.18. Definition. *We define*
$$\eta(f) \in H^1(S, \mathcal{J}(H)_{mer})$$
to be the image of $1 \in H^0(S, \mathbb{Z}_S)$ under the connecting morphism of the long exact sequence associated to the second line of Diagram (11).

By [Nak02a, Proposition 5.5.1] we have an injection
$$\mathcal{E}_0(S, D, H) \hookrightarrow H^1(S, \mathcal{J}(H)_{mer}),$$
where $\mathcal{E}_0(S, D, H)$ is the set of bimeromorphic equivalence classes of elliptic fibrations $f : X \to S$ having meromorphic sections over every point of S and such that $f^{-1}(S^\star) \to S^\star$ is bimeromorphic to a smooth elliptic fibration over $S^\star$ inducing the VHS H. By Construction 3.14 we have
$$\imath^W : H^1(S, \mathcal{J}(H)^{\mathbb{W}}) \to \mathcal{E}_0(S, D, H) \hookrightarrow H^1(S, \mathcal{J}(H)_{mer})$$
but contrary to the smooth case it is not clear if the images coincide. If S is a curve, the skyscraper sheaf $\mathcal{Q}_H$ has no higher cohomology so the map
$$H^1(S, \mathcal{J}(H)^{\mathbb{W}}) \longrightarrow H^1(S, \mathcal{J}(H)_{mer})$$
is surjective.

If $\eta(f) \in H^1(S, \mathcal{J}(H)_{mer})$ is the image of some $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$, then there is a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J}(H)^{\mathbb{W}} \arrow[r] \arrow[d] & R^1 p_* \mathcal{O}_{\mathbb{W}}^* \arrow[r] \arrow[d] & \mathbb{Z}_S \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \mathcal{J}(H)_{mer} \arrow[r] & R^1 f_* \mathcal{O}_X^* / \mathcal{V}_X \arrow[r] & \mathbb{Z}_S \arrow[r] & 0 \end{tikzcd} \tag{12}$$

where the first row is the short exact sequence (9) defined by $p^\eta : \mathbb{W}^\eta \to S$.

3.D. **The Kähler case.** From now on we will focus on the case where the total space of the elliptic fibration f is compact Kähler. In that case, the element $\eta(f) \in H^1(S, \mathcal{J}(H)_{mer})$ represented by f lies in the image of $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$, up to replacing $\eta(f)$ by a larger multiple (cf. [Nak02a, Proposition 7.4.2] for a more general statement).

3.19. Lemma. *In the situation of Assumption 3.12, suppose also that X is bimeromorphic to a compact Kähler manifold. Then the image of the class $\eta(f) \in H^1(S, \mathcal{J}(H)_{mer})$ is torsion in $H^1(S, \mathcal{Q}_H)$. In particular there exists an integer $m \geq 1$ such that $m \cdot \eta(f)$ can be lifted to some element in $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$.*

Proof. Since the class $\eta(f)$ depends only on the bimeromorphic equivalence class of $X \to S$ we can suppose that X is a compact Kähler manifold. Indeed, let $X' \dashrightarrow X$ be a bimeromorphic map from a compact Kähler manifold X' and let $\tilde{X}' \to X$ be a resolution of $X' \dashrightarrow X$ by successively blowing-up X' along smooth subvarieties. Then $\tilde{X}'$ is a Kähler manifold and $\tilde{X}' \to S$, which is the composition of $\tilde{X}' \to X$ with $X \to S$, is an elliptic fibration bimeromorphic to X. So we may replace X by $\tilde{X}'$ for instance.

Let $\omega \in H^2(X, \mathbb{R})$ be a Kähler class on X. By density, there exists a class $\alpha \in H^2(X, \mathbb{Q})$ (in general not of type $(1,1)$) such that $\alpha \cdot F \neq 0$ where F is a general f-fibre. The class α defines a global section of $R^2 f_* \mathbb{Q}_X$ and we can cancel the denominators in such a way that α defines a section of $R^2 f_* \mathbb{Z}_X$ and thus a non-zero element $\bar{\alpha} \in H^0(S, R^2 f_* \mathbb{Z}_X / \mathcal{V}_X)$. The third line of (11) induces an exact sequence

$$H^0(S, R^2 f_* \mathbb{Z}_X / \mathcal{V}_X) \xrightarrow{\tau} H^0(S, \mathbb{Z}_S) \xrightarrow{\delta} H^1(S, \mathcal{Q}_H)$$

and is straightforward to check that $\tau(\bar{\alpha}) = F \cdot \alpha$. It is then a positive multiple of the class $1 \in H^0(S, \mathbb{Z}_S)$ and it follows that $\delta(1)$ is a torsion class in $H^1(S, \mathcal{Q}_H)$. A diagram chase in (11) shows that $\delta(1)$ is the image of $\eta(f)$ in $H^1(S, \mathcal{Q}_H)$. $\square$

We can now generalise Theorem 3.2 (cf. also [Nak02a, Proposition 7.4.2]):

3.20. Theorem. *In the situation of Assumption 3.12, let us also assume that X is bimeromorphic to a compact Kähler manifold. Suppose also that $\eta(f)$ is in the image of $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$. Denote by*

$$\mathsf{c} : H^1(S, \mathcal{J}(H)^{\mathbb{W}}) \to H^2(S, j_* H)$$

the morphism defined by the first line of the exact sequence (11). *Then the class $\mathsf{c}(\eta(f))$ is torsion in $H^2(S, j_* H)$.*

Proof. Let $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ be an element which maps to $\eta(f) \in H^1(S, \mathcal{J}(H)_{mer})$ and let $p^\eta : W := \mathbb{W}^\eta \to S$ be the minimal locally Weierstrass fibration which represents $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$. By [Lin16, Lemma 8.1], the following diagram commutes

$$\begin{tikzcd} H^0(S, R^2 p^\eta_* \mathbb{Z}) \simeq H^0(S, \mathbb{Z}) \arrow[r, "\alpha"] \arrow[rr, bend left, "d_2"] & H^1(S, \mathcal{J}(H)^{\mathbb{W}}) \arrow[r, "\mathsf{c}"] & H^2(S, R^1 p^\eta_* \mathbb{Z}) \simeq H^2(S, j_* H) \end{tikzcd}$$

where c and α are the connecting morphisms in the long exact sequences induced by (8) and (9) respectively. As $\alpha(1) = \eta$, it suffices to prove the following lemma, which implies that $d_2 \otimes \mathbb{R} = 0$. $\square$

3.21. Lemma. $H^2(W, \mathbb{R}) \to H^0(S, R^2 p^\eta_* \mathbb{R})$ *is surjective.*

Proof. Since W is normal and has at worst rational singularities, by [HP16, Injection (3)] we have an injection

$$H^{1,1}_{BC}(W) \hookrightarrow H^2(W, \mathbb{R}).$$

Assume to the contrary that $H^2(W, \mathbb{R}) \to H^0(S, R^2 p^\eta_* \mathbb{R})$ is not surjective, so in particular its restriction to $H^{1,1}_{BC}(W)$ is not surjective. Let $\tau : \tilde{W} \to W$ be a Kähler desingularization of W. By the projection formula, given an element $\omega \in H^2(W, \mathbb{R})$, its image in $H^0(S, R^2 p^\eta_* \mathbb{R}) \simeq H^0(S, \mathbb{R}) \simeq \mathbb{R}$ equals $\int_F \tau^* \omega$ where F is a smooth fiber of $p^\eta \circ \tau : \tilde{W} \to S$. Let $n := \dim W$. The non-surjectivity assumption implies that $\tau^* H^{1,1}_{BC}(W) \subset [F]^\perp$ where the orthogonal is with respect to the Poincaré duality pairing

$$H^{n-1,n-1}(\tilde{W})_{\mathbb{R}} \times H^{1,1}(\tilde{W})_{\mathbb{R}} \to H^{n,n}(\tilde{W})_{\mathbb{R}} \simeq \mathbb{R}.$$

However since $\ker(\tau_*)^\perp \subset \tau^* H^{1,1}_{BC}(W)$ by [HP16, Lemma 3.3], we deduce that $\tau_*[F] = 0$, which is not possible. $\square$

3.22. Remark. The last result gives a direct proof of a phenomenon which was observed by Kodaira in the case $\dim S = 1$: he first proved that the cohomology group $H^2(S, j_*H)$ is finite if the VHS is not trivial. He then computed the first Betti number of an elliptic surface when H is trivial and obtained in [Kod63, Theorem 11.9] that this quantity is even if $\mathsf{c}(\eta(f)) = 0$ and odd otherwise. *A posteriori* we can conclude that an elliptic surface $f : X \to S$ (without multiple fibres) is Kähler if and only if $\mathsf{c}(\eta(f))$ is torsion in $H^2(S, j_*H)$. We will now prove that this equivalence also holds in our setting:

3.23. Proposition. *In the situation of Assumption 3.12, suppose also that the base S is a compact Kähler manifold. Assume that the Weierstraß fibration $\mathbb{W} \to S$ is minimal. Let $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ be a class such that $\mathsf{c}(\eta)$ is torsion in $H^2(S, j_*H)$. Then the total space of $\mathbb{W}^\eta \to S$ is bimeromorphic to a compact Kähler manifold.*

Proof. Recall that the Weierstraß model $p : \mathbb{W} \to S$ associated to H is a projective morphism. Since S is compact Kähler, the total space $\mathbb{W}$ is Kähler by Remark 2.2. As in the case of smooth elliptic fibrations, the Weierstraß fibration comes equipped with a family of of elliptic fibrations (over S) $\mathcal{W} \longrightarrow S \times H^1(S, \mathcal{L})$ parametrised by the vector space $H^1(S, \mathcal{L})$ such that

$$\eta(\mathcal{W}_t \to S) = \exp(t) \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$$

for any $t \in H^1(S, \mathcal{L})$. By Remark 3.5 the complex spaces $\mathbb{W}^\eta$ have at most canonical, hence rational, singularities. From [Nam01, Proposition 5] we know that any small flat deformation of compact Kähler space having rational singularities remains Kähler. Thus $\mathbb{W}^{\exp(t)}$ is Kähler for t in a neighborhood U of $0 \in H^1(S, \mathcal{L})$. Now if t is given in $H^1(S, \mathcal{L})$ let us consider a positive integer m such $t/m \in U$. The multiplication-by-m

$$\mathbf{m} : \mathbb{W}^{\exp(t/m)} \dashrightarrow \mathbb{W}^{m \cdot \exp(t/m)} = \mathbb{W}^{\exp(t)}.$$

is generically finite. Since $\mathbb{W}^{\exp(t/m)}$ is Kähler, $\mathbb{W}^{\exp(t)}$ is bimeromorphic to a compact Kähler manifold.

Since $\mathsf{c}(\eta)$ is torsion by assumption, there exists a positive integer k and an element $t \in H^1(S, \mathcal{L})$ such that $k \cdot \eta = \exp(t)$. As the multiplication-by-k

$$\mathbb{W}^\eta \dashrightarrow \mathbb{W}^{k \cdot \eta} = \mathbb{W}^{\exp(t)}$$

is generically finite and the target is bimeromorphic to a compact Kähler manifold, we conclude that $\mathbb{W}^\eta$ is also bimeromorphic to a compact Kähler manifold. $\square$

3.E. **A G-equivariant version of Construction 3.14.** Let $p : \mathbb{W} \to S$ be a Weierstraß fibration satisfying Assumption 3.12. In 3.14, we have constructed for each $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ an η-twisted locally Weierstraß fibration $p^\eta : \mathbb{W}^\eta \to S$. Now let G be a finite group acting on both $\mathbb{W}$ and S such that p is G-equivariant and the zero-section $\Sigma \subset \mathbb{W}$ is G-stable. In this paragraph, we will generalize Construction 3.14 by associating a G-equivariant locally Weierstraß fibration $p^\eta : \mathbb{W}^\eta \to S$ to each element $\eta_G \in H^1_G(S, \mathcal{J}(H)^{\mathbb{W}})$ and *vice versa* (here the notation $H^1_G(S, \mathcal{J}(H)^{\mathbb{W}})$ stands for the G-equivariant cohomology). This construction is due to Kodaira and can be found in [Kod63, Section 14] for the case $\dim S = 1$. The same argument therein also works in higher dimension and we will only describe the constructions following [Kod63, Section 14] and refer to *loc. cit.* for verifications of the details (see also [Cla16, §2.3] for a quick review of equivariant cohomology in the context of equivariant smooth families of tori).

Given an element $\eta_G \in H^1_G(S, \mathcal{J}(H)^{\mathbb{W}})$. Let $\{U_i\}_{i\in I}$ be a G-invariant good open cover of S and let G act on I such that $g^{-1}(U_i) = U_{gi}$. The element η_G can be represented by a 1-cocycle $\{(\eta_{ij})_{i,j\in I}, (\eta_i^g)_{i\in I, g\in G}\}$ where (η_{ij}) is a 1-cocycle with coefficients in $\mathcal{J}(H)^{\mathbb{W}}$ and η_i^g are local sections of $\mathcal{J}(H)^{\mathbb{W}}$ defined over U_i satisfying some cocycle conditions. The 1-cocycle (η_{ij}) represents the image η of η_G in $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ and let $p^\eta : \mathbb{W}^\eta \to S$ be the associated locally Weierstraß fibration. Fix biholomorphic maps

$$\eta_i : \mathbb{W}^\eta_i := (p^\eta)^{-1}(U_i) \to \mathbb{W}_i := p^{-1}(U_i)$$

such that $\eta_i \circ \eta_j^{-1} = \mathrm{tr}(\eta_{ij})$ where $\mathrm{tr}(\eta_{ij})$ denotes the translation by the holomorphic section η_{ij}. For each $g \in G$, the cocycle conditions allow us to patch together

$$\psi_g^i := \eta_i^{-1} \circ \mathrm{tr}(\eta_i^g) \circ g \circ \eta_{gi} : \mathbb{W}^\eta_{gi} \to \mathbb{W}^\eta_i$$

and obtain an automorphism $\psi_g : \mathbb{W}^\eta \to \mathbb{W}^\eta$, which defines a G-action on $\mathbb{W}^\eta$ such that p^η is G-equivariant. Up to isomorphism, the above construction does not depend on the choice of $\{(\eta_{ij}), (\eta_i^g)\}$ representing η_G. We called $p^\eta : \mathbb{W}^\eta \to S$ together with the thus defined G-action the *G-equivariant locally Weierstraß fibration associated to η_G*.

Given a G-equivariant locally Weierstraß fibration $p^\eta : \mathbb{W}^\eta \to S$ twisted by $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$, we can also reconstruct the (unique) element $\eta_G \in H^1_G(S, \mathcal{J}(H)^{\mathbb{W}})$ starting with which p^η is constructed. First of all, the G-action on p^η induces a G-action on $H := (R^1 p^\eta_* \mathbb{Z})_{|S^\star}$ where $S^\star \subset S$ is a Zariski open over which p^η is smooth. By [Nak88, Corollary 2.6], the G-action on H extends to a G-action on $\mathbb{W}$ such that p is G-equivariant, and it is for this G-action we define the G-equivariant cohomology group $H^1_G(S, \mathcal{J}(H)^{\mathbb{W}})$. Now let $\{U_i\}$ be a G-invariant good open cover of S and fix biholomorphic maps $\eta_i : \mathbb{W}^\eta_i \to \mathbb{W}_i$ such that $\eta_i \circ \eta_j^{-1} = \mathrm{tr}(\eta_{ij})$ for some 1-cocycle $\{\eta_{ij}\}$ representing $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$. Let $\psi_g : \mathbb{W}^\eta \to \mathbb{W}^\eta$ be the action of $g \in G$ on $\mathbb{W}^\eta$. If we define

$$\mathrm{tr}(\eta_i^g) := \eta_i \circ \psi_g \circ \eta_{gi}^{-1} \circ g^{-1}$$

then $\{(\eta_{ij}), (\eta_i^g)\}$ represents an element $\eta_G \in H^1_G(S, \mathcal{J}(H)^{\mathbb{W}})$. The class η_G depends only on the G-equivariant locally Weierstraß fibration $p^\eta : \mathbb{W}^\eta \to S$ and $p^\eta \mapsto \eta_G$ is the converse of the construction in the last paragraph.

Our discussion of $\mathcal{E}_0(S, \Delta, H)$ right after Definition 3.18 can also be generalized to the G-equivariant setting. Let $f : X \to S$ be an elliptic fibration satisfying Assumption 3.12. Let G be a finite group acting on S and on X such that f is G-equivariant. Let H be a local system of rank 2 over $S\backslash\Delta$ endowed with a G-action compatible with the G-action on S. Let $\mathcal{E}_0^G(S, \Delta, H)$ denote the set of bimeromorphic classes of all such G-equivariant elliptic fibrations f such that $(R^1 f_* \mathbb{Z})_{|S\backslash\Delta}$ is G-equivariantly isomorphic to H. To each G-equivariant elliptic fibration $f \in \mathcal{E}_0^G(S, \Delta, H)$, we can associate an element $\eta_G(f) \in H^1_G(S, \mathcal{J}(H)_{mer})$ similar to the above construction in an injective manner. According to the above, there exists a map $H^1_G(S, \mathcal{J}(H)^{\mathbb{W}}) \to \mathcal{E}_0^G(S, \Delta, H)$ which associates $\eta_G \in H^1_G(S, \mathcal{J}(H)^{\mathbb{W}})$ to the bimeromorphic class of the G-equivariant elliptic fibration $p^\eta : \mathbb{W}^\eta \to S$ and the composition

$$\imath_G^W : H^1_G(S, \mathcal{J}(H)^{\mathbb{W}}) \to \mathcal{E}_0^G(S, \Delta, H) \hookrightarrow H^1_G(S, \mathcal{J}(H)_{mer})$$

equals the map induced by the natural injective map $\mathcal{J}(H)^{\mathbb{W}} \hookrightarrow \mathcal{J}(H)_{mer}$.

At the end of this sub-section, we show that in the situation where $f : X \to S$ is a G-equivariant elliptic fibration for some finite group G, if the conclusion of Lemma 3.19 holds, then it also holds G-equivariantly.

3.24. Lemma. *Let $\eta_G \in H^1_G(S, \mathcal{J}(H)_{mer})$ and let η denote its image in $H^1(S, \mathcal{J}(H)_{mer})^G$. Assume that there exist $m \in \mathbb{Z}_{>0}$ and $\eta' \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ such that $m\eta = \imath^W(\eta')$, then up to replacing m with a larger multiple, $m\eta_G$ can be lifted to an element in $H^1_G(S, \mathcal{J}(H)^{\mathbb{W}})$.*

Proof. As G is finite, up to replacing m with a larger multiple, we can assume that $\eta' \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})^G$. The Grothendieck spectral sequence induces a commutative diagram

$$\begin{tikzcd}
H^1_G(S, \mathcal{J}(H)^{\mathbb{W}}) \arrow[r] \arrow[dr, "\imath^W_G"] & H^1(S, \mathcal{J}(H)^{\mathbb{W}})^G \arrow[r] \arrow[dr, "\imath^W"] & H^2\left(G, H^0(S, \mathcal{J}(H)^{\mathbb{W}})\right) \\
H^1\left(G, H^0(S, \mathcal{J}(H)_{mer})\right) \arrow[r] & H^1_G(S, \mathcal{J}(H)_{mer}) \arrow[r] & H^1(S, \mathcal{J}(H)_{mer})^G
\end{tikzcd}$$

with exact rows. As G is finite, $H^2\left(G, H^0(S, \mathcal{J}(H)^{\mathbb{W}})\right)$ is a torsion group. So up to replacing m with a larger multiple, there exists $\eta'_G \in H^1_G(S, \mathcal{J}(H)_{mer})$ which maps to $\eta' \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$. Again since G is finite, $H^1\left(G, H^0(S, \mathcal{J}(H)_{mer})\right)$ is also a torsion group. So up to replacing m with a larger multiple, we deduce that $m\eta_G = \imath^W_G(\eta'_G)$. $\square$

3.F. **Hodge theory of Weierstraß models.** The main purpose of this paragraph is to establish the following result.

3.25. Theorem. *Let $p : \mathbb{W} \to S$ be the minimal Weierstraß fibration associated to the* VHS *H over a compact Kähler manifold S satisfying Assumption 3.12 and let*

$$\mathcal{W} \longrightarrow S \times H^1(S, \mathcal{L}) \longrightarrow H^1(S, \mathcal{L})$$

be the tautological family associated to p (i.e. the family constructed in Proposition 3.17 for $\eta = 0$). Then the subset of $H^1(S, \mathcal{L})$ parameterizing projective fibrations $\mathcal{W}_t \to S$ is dense.

3.26. Remark. Before giving the proof of Theorem 3.25, let us remark that Theorem 3.25 is equivalent to the surjectivity of the canonical map

$$H^1(S, j_*H_{\mathbb{R}}) \longrightarrow H^1(S, \mathcal{L}). \tag{13}$$

Indeed, as S is assumed to be compact Kähler, by Remark 2.2, Theorem 3.20, and Proposition 3.23 the total space $\mathbb{W}$ is Kähler. So each fiber of the tautological family $\mathcal{W} \longrightarrow S \times H^1(S, \mathcal{L}) \longrightarrow H^1(S, \mathcal{L})$ is also Kähler. The elliptic fibration $\mathcal{W}_t \to S$ is projective if and only if its cohomology class $\eta(\mathcal{W}_t \to S)$ is torsion in $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ (see [Nak02a, Theorem 6.3.8]). Using the first line of (11), the exact sequence

$$H^1(S, j_*H) \longrightarrow H^1(S, \mathcal{L}) \longrightarrow H^1(S, \mathcal{J}(H)^{\mathbb{W}})$$

shows that this happens exactly when t lies in the range of the map $H^1(S, j_*H_{\mathbb{Q}}) \longrightarrow H^1(S, \mathcal{L})$. Hence the density of projective elliptic fibrations is equivalent to the surjectivity of the map (13).

When $S^* = S$, the surjectivity of (13) is a straightforward consequence of the existence of a pure Hodge structure of weight 2 on the lattice $H^1(S, H)$ as constructed by Deligne (see [Zuc79, Theorem 2.9]).

Theorem 3.25 will serve as a crucial ingredient in the proof of Theorem 1.4. More precisely, it will be the following corollary that we use in the proof.

3.27. Corollary. *Let G be a finite group and $f : X \to S$ a G-equivariant elliptic fibration satisfying Assumption 3.12 over a compact Kähler manifold. Then the image of $H^1(S, j_*H_{\mathbb{Q}})^G$ in $H^1(S, \mathcal{L})^G$ under the map $H^1(S, j_*H_{\mathbb{R}})^G \longrightarrow H^1(S, \mathcal{L})^G$ is dense.*

Proof. By Remark 3.26, Theorem 3.25 implies that (13) is surjective As G is a finite group, the G-invariant part of (13) is also surjective. Corollary 3.27 thus follows from the density of $H^1(S, j_*H_{\mathbb{Q}})^G$ in $H^1(S, j_*H_{\mathbb{R}})^G$. $\square$

To prove the density of projective fibrations in the tautological family we will use the following criterion which is reminiscent from Buchdahl's works [Buc06, Buc08]:

3.28. Lemma. *Let $\pi : \mathcal{X} \longrightarrow B$ be a smooth family of compact Kähler manifolds, and let $\Phi : \mathcal{X} \longrightarrow S \times B \longrightarrow B$ be a fibration such that $\pi = pr_B \circ \Phi$. Consider the following* VHS *over B:*

$$\mathbb{V} := R^2\pi_*\mathbb{Q}/H^2(S, \mathbb{Q}).$$

Let $b \in B$ be a point, and $[\omega]$ a Kähler class defined on $X := \mathcal{X}_b$. If the composition of the maps

$$T_{B,b} \xrightarrow{\kappa\varsigma} H^1(X, T_X) \xrightarrow{\bullet\wedge[\omega]} H^2(X, \mathcal{O}_X) \longrightarrow \mathbb{V}_b^{0,2}$$

is surjective, then the set of parameters $u \in B$ such that the morphism $\mathcal{X}_u \to S$ is projective is dense near b.

In the statement above, the first arrow is the Kodaira-Spencer map associated to π, and the second one is induced by the contraction with the class $\omega \in H^1(X, \Omega_X^1)$.

Proof. This is nothing but [Voi02, Proposition 17.20, p.410] applied to the VHS $\mathbb{V}$. $\square$

The deformation families provided by Nakayama's theory are not smooth, so in order to apply the relative Buchdahl criterion we have to pass to a smooth model. Kollár's theory of strong resolutions [Kol07, Chapter 3] gives a resolution in families:

3.29. Lemma. *Let $p_0 : \mathbb{W}_0 \to S$ be a fibration from a normal compact complex space onto a compact complex manifold S, and $p : \mathcal{W} \to S \times B$ be a locally trivial deformation of $(\mathbb{W}_0, p_0)$ (cf. Definition 2.4). Then (up to replacing B by a smaller open set) there exists a resolution of singularities $\mu : \mathcal{X} \to \mathcal{W}$ such that the family*

$$\pi := pr_B \circ p \circ \mu : \mathcal{X} \to B$$

is a family of compact complex manifolds, and for every $b \in B$ the map $\mu_b : \mathcal{X}_b \to \mathbb{W}_b$ is a resolution of singularities that is functorial.

Proof. By [Kol07, Theorem 3.35] (the analytic situation is dealt with in [Wło09]) there exists a functorial resolution $\mu : \mathcal{X} \to \mathcal{W}$. This resolution commutes with smooth maps, so if $(U_i)_{i\in I}$ is a finite open cover[2] of S such that $p^{-1}(U_i \times B) \simeq p_0^{-1}(U_i) \times B$, then the resolution is the product of the functorial resolution of $p_0^{-1}(U_i)$ times the identity. $\square$

[2]By Definition 2.4 these covers exist up to replacing B by a smaller open subset.

We follow the convention of [Ser06, Appendix B, p.287]: given a morphism $f : X \to S$ of normal varieties, we denote by $T_{X/S}$ the dual of the sheaf of Kähler differentials $\Omega_{X/S}$. In particular $T_{X/S}$ is always a reflexive sheaf.

3.30. Lemma. *Let $p^\eta : \mathbb{W}^\eta \to S$ be a minimal local Weierstraß fibration over a smooth base S. Then we have*

$$T_{\mathbb{W}^\eta/S} \simeq (p^\eta)^*\mathcal{L}. \tag{14}$$

Moreover, let $\mu : X \to \mathbb{W}^\eta$ be a functorial resolution of singularities and set $f := p^\eta \circ \mu$. Then there exists a natural injection $\mu^* T_{\mathbb{W}^\eta/S} \to T_{X/S}$ inducing a map

$$H^1(\mathbb{W}^\eta, T_{\mathbb{W}^\eta/S}) \to H^1(X, T_{X/S}). \tag{15}$$

Proof. All the fibres of p^η are reduced plane cubics, so there exists a codimension two subset $Z \subset \mathbb{W}^\eta$ such that $p^\eta|_{\mathbb{W}^\eta \setminus Z}$ is a smooth fibration. On this smooth locus we have by construction $T_{\mathbb{W}^\eta \setminus Z/S} \simeq (p^\eta|_{\mathbb{W}^\eta \setminus Z})^*\mathcal{L}$. Since $T_{\mathbb{W}^\eta/S}$ and $p^*\mathcal{L}$ are both reflexive and $\mathbb{W}^\eta$ is normal, the isomorphism extends to an isomorphism on $\mathbb{W}^\eta$. This shows (14), in particular $T_{\mathbb{W}^\eta/S}$ is locally free.

Since the resolution μ is functorial, the direct image sheaf $\mu_*(T_X) \subset T_{\mathbb{W}^\eta}$ is reflexive [GKK10, Corollary 4.7]. Thus for any open subset $U \subset \mathbb{W}^\eta$, the restriction map

$$\Gamma(\mu^{-1}(U), T_X) \to \Gamma(\mu^{-1}(U) \setminus \mathrm{Exc}(\mu), T_X)$$

is surjective. Using the exact sequence

$$0 \to T_{X/S} \to T_X \to f^*T_S$$

and the fact that f^*T_S is torsion-free, we obtain that

$$\Gamma(\mu^{-1}(U), T_{X/S}) \to \Gamma(\mu^{-1}(U) \setminus \mathrm{Exc}(\mu), T_{X/S})$$

is surjective. Thus $\mu_*(T_{X/S})$ is reflexive, and the natural map $\mu_*(T_{X/S}) \to T_{\mathbb{W}^\eta/S}$ is an isomorphism. By applying the projection formula to the inverse $T_{\mathbb{W}^\eta/S} \to \mu_*(T_{X/S})$ we obtain an injective morphism

$$\mu^* T_{\mathbb{W}^\eta/S} \to T_{X/S}.$$

Since $T_{\mathbb{W}^\eta/S}$ is locally free and $\mathbb{W}^\eta$ has rational singularities, we have an isomorphism

$$H^1(\mathbb{W}^\eta, T_{\mathbb{W}^\eta/S}) \simeq H^1(X, \mu^* T_{\mathbb{W}^\eta/S})$$

given by μ^*. The statement follows by composing this isomorphism with the map $H^1(X, \mu^* T_{\mathbb{W}^\eta/S}) \to H^1(X, T_{X/S})$. $\square$

Following [Ser06, Chapter 3.4.2] (cf. [FK87] for a presentation in the analytic setting) we consider the functor of locally trivial deformations of $f : X \to S$ with fixed target S. By [Ser06, Lemma 3.4.7.b) and Theorem 3.4.8] we have an injection

$$H^1(X, T_{X/S}) \hookrightarrow D_{X/S},$$

where $D_{X/S}$ is the tangent space of the semiuniversal deformation. Given a locally trivial deformation

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & S \times B \arrow[d, "pr_B"] \\ & B \end{tikzcd}$$

parametrised by a smooth base B, we have for every $b \in B$ a Kodaira-Spencer map

$$\kappa\varsigma_\Phi : T_{B,b} \to D_{X/S}$$

associating a tangent vector with the corresponding first-order deformation.

Proof of Theorem 3.25. Fix $b \in H^1(S, \mathcal{L})$. By Lemma 3.29 there exists (up to replacing the base $H^1(S, \mathcal{L})$ by a neighbourhood of the point b) a simultaneous functorial resolution of the tautological family:

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "\mu"] \arrow[dr, "\Phi"] \arrow[ddr, bend right, "\pi"'] & & \mathcal{W} \arrow[dl, "p"'] \arrow[ddl, bend left] \\
& S \times H^1(S, \mathcal{L}) \arrow[d] & \\
& H^1(S, \mathcal{L}) &
\end{tikzcd}$$

In order to simplify the notation, we replace $p_b : \mathbb{W}^{\exp(b)} \to S$ by $p_b : \mathbb{W}_b \to S$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_b \arrow[rr, "\mu_b"] \arrow[dr, "f_b"'] & & \mathbb{W}_b \arrow[dl, "p_b"] \\
& S &
\end{tikzcd}$$

where μ_b is a functorial resolution of singularities of $\mathbb{W}_b$.

Step 1. The Kodaira-Spencer map $\kappa\varsigma_{\Phi,b}$ is given by f_b^*. Using (14) one shows easily that the Kodaira-Spencer map $\kappa\varsigma_p$ for any point $b \in H^1(S, \mathcal{L})$ identifies to the pull-back

$$p_b^* : H^1(S, \mathcal{L}) \to H^1(\mathbb{W}_b, p_b^*\mathcal{L}) \simeq H^1(\mathbb{W}_b, T_{\mathbb{W}_b/S}).$$

By functoriality of the Kodaira-Spencer map we have a factorisation

$$\begin{tikzcd}
H^1(S, \mathcal{L}) \arrow[r, "\kappa\varsigma_{\Phi,b}"] \arrow[dr, "\kappa\varsigma_{p,b}"'] & H^1(\mathcal{X}_b, T_{\mathcal{X}_b/S}) \arrow[d] \\
& H^1(\mathbb{W}_b, T_{\mathbb{W}_b/S}).
\end{tikzcd}$$

By (15) the right column has an inverse $H^1(\mathbb{W}_b, T_{\mathbb{W}_b/S}) \to H^1(\mathcal{X}_b, T_{\mathcal{X}_b/S})$ defined by μ_b^*. Since $\kappa\varsigma_{p,b}$ identifies to the pull-back p_b^* we obtain that $\kappa\varsigma_{\Phi,b}$ identifies to

$$f_b^* : H^1(S, \mathcal{L}) \to H^1(\mathcal{X}_B, (f_b)^*\mathcal{L}).$$

Step 2. Applying the relative Buchdahl criterion. We want to apply Lemma 3.28 to the family $\mathcal{X} \to S \times H^1(S, \mathcal{L}) \to H^1(S, \mathcal{L})$. Fix a point $b \in H^1(S, \mathcal{L})$, and observe that

$$\mathbb{V}_b^{0,2} = H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})/H^2(S, \mathcal{O}_S).$$

We have to check that the composed map

$$H^1(S, \mathcal{L}) \xrightarrow{\kappa\varsigma_{\Phi,b}} H^1(\mathcal{X}_b, T_{\mathcal{X}_b}) \xrightarrow{\bullet\wedge[\omega]} H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) \longrightarrow H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})/H^2(S, \mathcal{O}_S) \tag{16}$$

is surjective.

Claim: the quotient $H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})/H^2(S, \mathcal{O}_S)$ is isomorphic to $H^1(S, \mathcal{L})$ and the projection $H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) \longrightarrow H^1(S, \mathcal{L})$ is induced by $(f_b)_*$.

We have $R^1(f_b)_*\mathcal{O}_{\mathcal{X}_b} = \mathcal{L}$ by definition and $R^2(f_b)_*\mathcal{O}_{\mathcal{X}_b} = 0$ by Theorem 3.9. Moreover, since $\mathcal{X}_b$ is compact Kähler, the natural map $H^2(S, \mathcal{O}_S) \to H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})$ is injective. Thus the Leray spectral sequence yields an exact sequence

$$0 \to H^2(S, \mathcal{O}_S) \to H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) \to H^1(S, R^1(f_b)_*\mathcal{O}_{\mathcal{X}_b}) \simeq H^1(S, \mathcal{L}) \to 0.$$

This shows the claim.

Combining Step 1 and the claim the composed map (16) is given by:

$$H^1(S, \mathcal{L}) \xrightarrow{f_b^*} H^1(\mathcal{X}_b, T_{\mathcal{X}_b}) \xrightarrow{\bullet \wedge [\omega]} H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) \xrightarrow{(f_b)_*} H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})/H^2(S, \mathcal{O}_S).$$

Using Dolbeault representatives we see that

$$(f_b)_*(\omega \wedge f_b^*(\alpha)) = c_\omega \alpha$$

where $c_\omega = \omega \cdot F$ with F a general f_b-fibre. Thus (16) is even an isomorphism. $\square$

4. Proofs of the main results

4.A. **Proof of Theorem 1.4 and Corollary 1.5.**

Proof of Theorem 1.4. Since $a(X) = \dim X - 1$, the algebraic reduction of X shows that up to replacing X with a bimeromorphic model, there exists a holomorphic map $X \to S$ onto a projective manifold of dimension $\dim X - 1$ whose general fiber is an elliptic curve [Ue75, Theorem 12.4]. Up to replacing X with another bimeromorphic model, we can assume that $X \to S$ is an elliptic fibration such that both X and S are smooth, X is Kähler, S is projective, and the discriminant locus $\Delta \subset S$ is an SNC divisor. By Proposition 3.11, there exists a finite Galois cover $r : \tilde{S} \to S$ such that the elliptic fibration $\tilde{X} := X \times_S \tilde{S} \to \tilde{S}$ has local meromorphic sections at every point of $\tilde{S}$; let $G := \mathrm{Gal}(\tilde{S}/S)$.

Let $\eta \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})_{mer})$ be the element associated to $\tilde{X} \to \tilde{S}$ where $\tilde{H} := r^*H$ and let $V := H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})$. The G-action on $\tilde{H}$ induces a G-action on V; let V^G be the G-invariant part.

4.1. Lemma. *There exists a family $\Pi : \widetilde{\mathcal{X}} \xrightarrow{q} \tilde{S} \times V^G \to V^G$ of elliptic fibrations over $\tilde{S}$ parameterized by V^G together with a G-action on $\widetilde{\mathcal{X}}$ such that q is G-equivariant and satisfies the following properties:*

i) The G-equivariant elliptic fibration $\widetilde{\mathcal{X}}_\phi \to \tilde{S}$ parameterized by $\phi \in V^G$ corresponds to

$$\exp_1(\phi) + \eta \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})_{mer})$$

where $\exp_1$ is the composition of $\exp : H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}}) \to H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}})$ with $H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}}) \to H^1(\tilde{S}, \mathcal{J}(\tilde{H})_{mer})$. In particular, the central fiber $\widetilde{\mathcal{X}}_0 \to \tilde{S}$ of Π is G-equivariantly bimeromorphic to $\tilde{X} \to \tilde{S}$. Furthermore, $\widetilde{\mathcal{X}}_0$ is a compact Kähler variety.

ii) There exist a G-invariant open cover $\{U_i\}$ of $\tilde{S}$ and elliptic fibrations $\mathcal{W}_i \to U_i$ such that $q^{-1}(U_i \times V^G) \simeq \mathcal{W}_i \times V^G$ over $U_i \times V^G$ (in particular Π is locally trivial in the sense of Definition 2.4).

iii) Moreover, there exists a G-action on $\{\mathcal{W}_i \to U_i\}$ compatible with the G-action on $\{U_i\}$ such that for each $g \in G$, the restriction of the G-action on $\widetilde{\mathcal{X}}$ to $q^{-1}(U_{gi} \times V^G)$ is isomorphic to $(\Theta_i^g \times \mathrm{Id}) : \mathcal{W}_{gi} \times V^G \xrightarrow{\sim} \mathcal{W}_i \times V^G$ where $\Theta_i^g : \mathcal{W}_{gi} \to \mathcal{W}_i$ is the map defining the G-action on $\{\mathcal{W}_i \to U_i\}$.

As an immediate consequence of ii) and iii), the quotient $\widetilde{\mathcal{X}}/G \to S \times V^G \to V^G$ of $\widetilde{\mathcal{X}} \to \tilde{S} \times V^G \to V^G$ is a locally trivial family of elliptic fibrations over S parameterized by V^G.

Proof. First we define 1-cocycle classes λ_G and θ_G which will be used to construct the family Π and the G-action. Let $\xi \in V^G \otimes H^0(V^G, \mathcal{O}_{V^G})$ be the elements corresponding to the identity maps Id_{V^G}. Let $\eta_G \in H^1_G(\tilde{S}, \mathcal{J}(\tilde{H})_{mer})$ be the element associated to the G-equivariant elliptic fibration $\tilde{X} \to \tilde{S}$ as we defined in 3.E. By Lemma 3.19 and Lemma 3.24, up to replacing m by a larger multiple, there exists $\eta'_G \in H^1_G(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}})$ such that $m\eta_G = \imath^W_G(\eta'_G)$ where we recall that

$$\imath^W_G : H^1_G(S, \mathcal{J}(H)^{\mathbb{W}}) \to H^1_G(S, \mathcal{J}(H)_{mer})$$

is the map induced by the inclusion $\mathcal{J}(H)^{\mathbb{W}} \subset \mathcal{J}(H)_{mer}$. Define

$$\begin{aligned} \lambda_G := (\eta_G \otimes 1) + (\exp^G_1 \otimes \mathrm{Id})(\xi) &\in H^1_G(\tilde{S}, \mathcal{J}(\tilde{H})_{mer}) \otimes H^0(V^G, \mathcal{O}_{V^G}) \\ &\simeq H^1_G\left(\tilde{S} \times V^G, \mathcal{J}(\mathrm{pr}_1^{-1}\tilde{H})_{mer}\right) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \theta_G := (\eta'_G \otimes 1) + (\exp^G \otimes \mathrm{Id})(m\xi) &\in H^1_G(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}}) \otimes H^0(V^G, \mathcal{O}_{V^G}) \\ &\simeq H^1_G\left(\tilde{S} \times V^G, \mathcal{J}(\mathrm{pr}_1^{-1}\tilde{H})^{\mathbb{W}}\right), \end{aligned} \tag{18}$$

where

$$\exp^G : H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})^G \to H^1_G(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}})$$

is induced by $\exp : \mathcal{L}_{\tilde{H}/\tilde{S}} \to \mathcal{J}(\tilde{H})^{\mathbb{W}}$ in (5) and $\exp^G_1$ is defined as the composition of $\exp^G$ with $H^1_G(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}}) \to H^1_G(\tilde{S}, \mathcal{J}(\tilde{H})_{mer})$. Then $m\lambda_G$ coincides with the image of θ_G in $H^1_G\left(\tilde{S} \times V^G, \mathcal{J}(\mathrm{pr}_1^{-1}\tilde{H})_{mer}\right)$.

Let $\{U_i\}_{i \in I}$ be a G-invariant good open cover of $\tilde{S}$ and $U_{ij} := U_i \cap U_j$. Let $p : \mathbb{W} \to \tilde{S}$ be the G-equivariant minimal Weierstraß fibration associated to $\tilde{H}$ and let $\mathbb{W}_i := p^{-1}(U_i)$. As V^G is contractible, $\{U_i \times V^G\}$ is a good cover of $\tilde{S} \times V^G$. Since $m\eta_G = \imath^W_G(\eta'_G)$, we can choose 1-cocycles $\{(\lambda_{ij}), (\lambda_i^g)\}$ and $\{(\theta_{ij}), (\theta_i^g)\}$ representing λ_G and θ_G in such a way that the diagrams

$$\begin{tikzcd} \mathbb{W}_{ij} \times V^G \arrow[r, dashed, "\mathrm{tr}(\lambda_{ij})"] \arrow[d, dashed, "\mu_{ij} \times \mathrm{Id}"'] & \mathbb{W}_{ij} \times V^G \arrow[d, dashed, "\mu_{ij} \times \mathrm{Id}"] \\ \mathbb{W}_{ij} \times V^G \arrow[r, "\sim", "\mathrm{tr}(\theta_{ij})"'] & \mathbb{W}_{ij} \times V^G \end{tikzcd} \qquad \begin{tikzcd} \mathbb{W}_{gi} \times V^G \arrow[r, dashed, "\psi^i_g"] \arrow[d, dashed, "\mu_{gi} \times \mathrm{Id}"'] & \mathbb{W}_i \times V^G \arrow[d, dashed, "\mu_i \times \mathrm{Id}"] \\ \mathbb{W}_{gi} \times V^G \arrow[r, "\sim", "\phi_g \times \mathrm{Id}"'] & \mathbb{W}_i \times V^G \end{tikzcd} \tag{19}$$

are commutative, where $\mu_i : \mathbb{W}_i \dashrightarrow \mathbb{W}_i$ is multiplication-by-m composed with some meromorphic translation, $\phi_g : \mathbb{W} \to \mathbb{W}$ is the action of $g \in G$ on $\mathbb{W}$ induced by the G-action on $\tilde{H}$, and $\psi^i_g := \mathrm{tr}(\lambda^g_i) \circ (\phi_g \times \mathrm{Id})$.

Let $\mu_i' : \mathcal{W}_i \to \mathbb{W}_i$ be the finite map in the Stein factorization of μ_i. Then there exist bimeromorphic maps $h_i : \mathcal{W}_i \dashrightarrow \mathbb{W}_i$ such that the diagrams

$$
\begin{tikzcd}
\mathcal{W}_{ij} \times V^G \arrow[r, dashed, "H_{ij}"] \arrow[d, "\mu_i' \times \mathrm{Id}"'] & \mathcal{W}_{ij} \times V^G \arrow[d, "\mu_j' \times \mathrm{Id}"] \\
\mathbb{W}_{ij} \times V^G \arrow[r, "\mathrm{tr}(\theta_{ij})"] & \mathbb{W}_{ij} \times V^G
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{W}_{gi} \times V^G \arrow[r, dashed, "\Theta_i^g"] \arrow[d, "\mu_{gi}' \times \mathrm{Id}"'] & \mathcal{W}_i \times V^G \arrow[d, "\mu_i' \times \mathrm{Id}"] \\
\mathbb{W}_{gi} \times V^G \arrow[r, "\phi_g \times \mathrm{Id}"] & \mathbb{W}_i \times V^G
\end{tikzcd}
\tag{20}
$$

are commutative, where $H_{ij} := (h_j \times \mathrm{Id})^{-1} \circ \mathrm{tr}(\lambda_{ij}) \circ (h_i \times \mathrm{Id})$ and $\Theta_i^g := (h_i \times \mathrm{Id})^{-1} \circ \psi_i^g \circ (h_{gi} \times \mathrm{Id})$. As both $\mathcal{W}_i$ and $\mathbb{W}_i$ are normal and μ_i' is finite, the bimeromorphic maps H_{ij} and Θ_i^g are biholomorphic. Thus we obtain an elliptic fibration $q : \widetilde{\mathcal{X}} \to \tilde{S} \times V^G$ by gluing the $\mathcal{W}_i \times V^G$ together using the 1-cocycle of biholomorphic maps $\{H_{ij}\}$. This is the construction of the family $\Pi : \widetilde{\mathcal{X}} \xrightarrow{q} \tilde{S} \times V^G \to V^G$ and *ii)* follows directly. Using $\{H_{ij}\}$ the maps Θ_i^g can be glued to a biholomorphic map $\Theta^g : \widetilde{\mathcal{X}}^G \to \widetilde{\mathcal{X}}^G$ and $g \mapsto \Theta^g$ defines a G-action on $\widetilde{\mathcal{X}}^G$ such that q is G-equivariant. Property *iii)* also follows easily from the construction.

By construction, each fiber of $\Pi : \widetilde{\mathcal{X}} \to V^G$ is G-stable and the G-equivariant elliptic fibration parameterized by $\phi \in V^G$ represents $\eta_G + \exp_1^G(\phi) \in H^1_G(\tilde{S}, \mathcal{J}(\tilde{H})_{mer})$. In particular if we forget the G-action, then the elliptic fibration parameterized by $\phi \in V^G$ represents $\eta + \exp_1(\phi) \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})_{mer})$. Let $\eta' \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}})$ be the image of η_G'. Since the multiplication-by-m induces a generically finite dominant map $\tilde{X} \dashrightarrow \mathbb{W}^{\eta'}$ and $\tilde{X}$ is bimeromorphic to a compact Kähler manifold, so is $\mathbb{W}^{\eta'}$. So $c(\eta')$ is torsion by Theorem 3.20 and up to replacing m with a larger multiple, we may assume that $\eta' = \exp(\phi_0)$ for some $\phi_0 \in V^G$ by the long exact sequence coming from (5). By Corollary 3.27, the pre-images $\Sigma_0 \subset H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})^G$ of torsion points of $H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}})$ under the restriction of the exponential map $V \to H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}})$ to V^G form a dense subset in V^G, the image of $H^1(\tilde{S}, j_*\tilde{H})^G$ in V^G contains a lattice Λ of V^G. Hence up to replacing m with a larger multiple, we may assume that η' is closed enough to Λ by Kronecker's theorem. As Λ parameterizes elliptic fibrations isomorphic to $\mathbb{W} \to \tilde{S}$ and $\mathbb{W}$ is Kähler, it follows that $\mathbb{W}^{\eta'}$, being a small deformation of $\mathbb{W}$, is also Kähler. Since there is a finite holomorphic map $\widetilde{\mathcal{X}}_0 \to \mathbb{W}^{\eta'}$ obtained by gluing the $\mu_i : \mathcal{W}_i \to \mathbb{W}_i$ using (20), $\widetilde{\mathcal{X}}_0$ is also Kähler by Remark 2.2. □

There is a dense subset Σ of V^G parameterizing fibers of $\widetilde{\mathcal{X}} \to V^G$ which are algebraic. Indeed, as we already saw in the proof of Lemma 4.1 that Σ_0 is dense in V^G,

$$\Sigma := \left\{\phi \in V^G \mid \exp(\phi_0 + m\phi) \text{ is torsion}\right\} = \frac{1}{m}(\Sigma_0 - \phi_0)$$

is also a dense subset of V^G, where we recall that $\phi_0 \in V^G$ is an element such that $\eta' = \exp(\phi_0)$ and $\eta' \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\mathbb{W}})$ is an element lifting $m\eta$. Since $\phi \in \Sigma$ implies that $\eta + \exp_1(\phi) \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})_{mer})$ is torsion, fibers of $\widetilde{\mathcal{X}} \to V^G$ parameterized by Σ are algebraic by [Nak02a, Proposition 5.5.4].

By Lemma 4.1, the quotient $\widetilde{\mathcal{X}}/G \to S \times V^G$ is a locally trivial family of elliptic fibrations over S parameterized by V^G. Let $\mathcal{X}$ be the functorial desingularization of $\widetilde{\mathcal{X}}/G$. The family $\mathcal{X} \to V^G$ is an algebraic approximation of the central fiber, which is smooth by Lemma 3.29, and is bimeromorphic to X, again by Lemma 4.1. □

Proof of Corollary 1.5. By Theorem 1.4, X is bimeromorphic to a compact Kähler manifold X' which has an algebraic approximation. In particular, $\pi_1(X')$ is projective. Since the fundamental group of a compact Kähler manifold is invariant under bimeromorphic transformations, we have $\pi_1(X) = \pi_1(X')$. □

4.B. **Proof of Theorem 1.2.** Let X be a compact Kähler manifold, and consider $g : X \to Y$ a fibration onto a compact Kähler manifold Y. If F is a general fibre, denote by $\pi_1(F)_X$ the image of the morphism $\pi_1(F) \to \pi_1(X)$. Up to blowing up X and Y we can suppose that the fibration g is neat in the sense of [Cam04, Definition 1.2]. By [Cam11, Corollary 11.9] we then have an exact sequence

$$1 \to \pi_1(F)_X \to \pi_1(X) \to \pi_1(Y, \Delta) \to 1. \tag{21}$$

where Δ is the orbifold divisor defined in (1).

Let us also recall that by [Cam94, Kol93] every compact Kähler manifold X admits a (unique up to bimeromorphic equivalence of fibrations) almost holomorphic fibration

$$g : X \to \Gamma(X)$$

with the following property: let Z be a subspace with normalisation $Z' \to Z$ passing through a very general point $x \in X$. Then Z is contained in the fibre through x if and only if the natural map $\pi_1(Z') \to \pi_1(X)$ has finite image. This fibration is called the Γ-reduction of X (Shafarevich map in the terminology of [Kol93]). Up to replacing γ by some neat holomorphic model we thus obtain a fibration such that $\pi_1(F)_X$ is finite and such that the dimension of the base is minimal among all fibrations with this property. We call $\gamma \dim(X) := \dim(\Gamma(X))$ the γ-dimension of X.

In geometric situations it is often necessary to replace X by some étale cover. It is easily seen that the situation can be made equivariant under the Galois group of the cover.

4.2. Lemma. *Let X be a compact Kähler manifold acted upon by a finite group G. Then there exists a proper modification $\mu : \tilde{X} \to X$ and a holomorphic map $g : \tilde{X} \to Y$ such that:*

(*i*) *$\tilde{X}$ and Y are compact Kähler manifolds.*
(*ii*) *G acts on $\tilde{X}$ and Y.*
(*iii*) *μ and g are G-equivariant for these actions.*
(*iv*) *g is a neat model* [Cam04, Definition 1.2] *of the Γ-reduction of X.*

Proof. Let us consider S the (normalisation of the) irreducible component of the cycle space $\mathcal{C}(X)$ which parametrizes the fibres of the Γ-reduction. By uniqueness of the latter, the group G acts on S and the natural meromorphic map $X \dashrightarrow S$ is G-equivariant. Now it is enough to perform G-equivariant resolution of singularities for S and G-equivariant blow-ups on X in order to make the latter map holomorphic, neat and G-equivariant. □

The following remark allows to control the extensions appearing in these covers.

4.3. Remark. Let $1 \to K \to H \to G \to 1$ be an exact sequence of groups. It is well known that this extension determines a morphism $\varphi_H : G \to \mathrm{Out}(K)$ to the group of outer automorphisms of K (induced by the conjugation in H). To recover the extension, it is needed to prescribe an additional information: the class $c_H \in H^2(G, Z(K))$ (see [Bro82, Chapter IV, §6] for details).

On the reverse direction, when G is a finite group acting by homeomorphisms on a topological space Z, we also have an induced morphism $\varphi_Z : G \to \mathrm{Out}(\pi_1(Z))$. In this case, there is an extension

$$1 \longrightarrow \pi_1(Z) \longrightarrow H \longrightarrow G \longrightarrow 1$$

induced by the action of G on Z. It can be explicitly constructed in the following way. By [Ser58] there exists a projective simply connected manifold P on which G acts freely and we can look at the natural projection $Z \times P \to (Z \times P)/G$. This is a finite étale cover of Galois group G and the homotopy exact sequence is:

$$1 \longrightarrow \pi_1(Z) \simeq \pi_1(Z \times P) \longrightarrow \pi_1((Z \times P)/G) \longrightarrow G \longrightarrow 1. \tag{22}$$

This is the sought exact sequence. It is stated in [Cla16, Lemma 3.9] that the extension (22) does not depend on P and that this extension is the usual one if G acts freely on Z (*i.e.* the one corresponding to $Z \to Z/G$).

Using the construction above we obtain the following technical result.

4.4. Lemma. *Let $H := \pi_1(X)$ be a Kähler group and $K \triangleleft H$ be finite index normal subgroup with quotient $G := H/K$. Let us denote by $\tilde{X}$ the corresponding étale cover of X. Assume now that there exists a continuous map $g : \tilde{X} \to Z$ to a projective manifold Z which is G-equivariant (so that G acts on Z) and which induces an isomorphism at the level of fundamental groups. Then H is a projective group.*

Proof. The G-equivariant map

$$\tilde{X} \times P \xrightarrow{g} Z \times P$$

induces an isomorphism on the fundamental group and, using Remark 4.3, we infer that

$$\pi_1(X) \simeq \pi_1((Z \times P)/G)$$

is a projective group. □

Let us recall that a group G is virtually torsion-free if there exist a subgroup $H \subset G$ of finite index that is torsion free.

4.5. Lemma. *Let X be a compact Kähler manifold admitting a fibration onto a curve $f : X \to C$ such that the fundamental group $\pi_1(F)$ of a general fibre F is abelian. Then the group $\pi_1(X)$ is virtually torsion-free.*

Proof. Applying [Cam98, Appendix C] we can take a finite étale cover such that $\pi_1(F)_X$ coincides with $K := \ker(f_* : \pi_1(X) \to \pi_1(C))$ and the latter is thus finitely generated. If $C \simeq \mathbb{P}^1$ we obtain that $\pi_1(X) \simeq \pi_1(F)_X$ is abelian, so virtually torsion-free. We can thus suppose that $g(C) \geq 1$. By [Ara11, Theorem 5.1] the cohomology class $e \in H^2(\pi_1(C), K)$ corresponding to the extension

$$1 \longrightarrow K \longrightarrow \pi_1(X) \longrightarrow \pi_1(C) \longrightarrow 1$$

is torsion. Then so is the class $e' \in H^2(\pi_1(C), K/K_{tor})$ corresponding to the extension

$$1 \longrightarrow K/K_{tor} \longrightarrow \pi_1(X)/K_{tor} \longrightarrow \pi_1(C) \longrightarrow 1.$$

Arguing as in [CCE14, §2.1] we can assume that the latter cohomology class e' vanishes (up to replacing $\pi_1(C)$ with a finite index subgroup). Using the following piece of long exact sequence of cohomology of $\pi_1(C)$-modules

$$\cdots \longrightarrow H^2(\pi_1(C), K_{tor}) \longrightarrow H^2(\pi_1(C), K) \longrightarrow H^2(\pi_1(C), K/K_{tor}) \longrightarrow \cdots$$

we see the cohomology class e comes from $H^2(\pi_1(C), K_{tor})$. It is then easily observed[3] that this class is annihilated when restricted to a finite index subgroup of $\pi_1(C)$. This means that the following exact sequence of groups

$$1 \longrightarrow K_{tor} \longrightarrow \pi_1(X) \longrightarrow \pi_1(X)/K_{tor} \longrightarrow 1$$

splits when restricted to a finite index subgroup and it proves that $\pi_1(X)$ is virtually torsion-free since $\pi_1(X)/K_{tor}$ is. □

Now we give some criteria to decide whether a Kähler group is projective.

4.6. Lemma. *Let X be a compact Kähler manifold having $\gamma\dim(X) \leq 1$. Its fundamental group is then projective.*

Proof. If $\gamma\dim(X) = 0$ its fundamental group is finite so [Ser58] applies. If $\gamma\dim(X) = 1$ we know from [Cla10, Théorème 1.2] (which is just a rephrase of [Siu87]) that there exists a finite étale Galois cover $\pi : \tilde{X} \to X$ with group G such that the Γ-reduction of $\tilde{X}$ is a fibration $\tilde{g} : \tilde{X} \to C$ onto a curve inducing an isomorphism $\pi_1(X) \simeq \pi_1(C)$. It is also G-equivariant according to Lemma 4.2. We conclude by Lemma 4.4. □

4.7. Lemma. *Let X be a compact Kähler manifold such that $\gamma\dim(X) = 2$. If $\pi_1(X)$ is virtually torsion-free it is a projective group.*

Proof. Let $K \lhd \pi_1(X)$ be a finite index subgroup which is normal and torsion-free, and set $G := \pi_1(X)/K$. Applying Lemma 4.2 to the finite étale cover corresponding to G, we know that we can find $\tilde{X} \to X$ which is a composition of a finite étale cover and a modification such that the γ-reduction $g : \tilde{X} \to Y$ is neat, Y is smooth Kähler surface and g is equivariant for the natural actions of G on $\tilde{X}$ and Y. Consider now the exact sequence (21): the group $\pi_1(\tilde{X})$ is torsion-free and $\pi_1(F)_X$ is finite, so we have $\pi_1(F)_X = 1$. Thus $\pi_1(\tilde{X}) \simeq \pi_1(Y, \Delta^*(g))$ is torsion-free, by Remark 2.8 this implies that

$$\pi_1(\tilde{X}) \simeq \pi_1(Y, \Delta^*(g)) \simeq \pi_1(Y).$$

We can now argue according to the algebraic dimension of Y.

1.) If $a(Y) = 0$ then by the classification of surfaces $\pi_1(Y)$ is abelian. Thus $\pi_1(X)$ is virtually abelian and [BR11, Theorem 1.4] applies.
2.) If $a(Y) = 1$ then the algebraic reduction $Y \to C$ is an elliptic fibration over a curve C. Since the algebraic reduction is unique, it is G-equivariant. By [Kod63, Theorems 14.1-3-5] we know that there exists a G-equivariant deformation of Y to an algebraic elliptic surface, so we can again conclude by Lemma 4.4.
3.) If $a(Y) = 2$ the surface Y is projective and Lemma 4.4 applies.

[3]If A is any finite $\pi_1(C)$-module then there is a finite index subgroup $\pi_1(C')$ of $\pi_1(C)$ such that the whole of the cohomology group $H^2(\pi_1(C), A)$ vanishes when restricted to $\pi_1(C')$. This is a consequence of the fact that a curve of positive genus admits finite étale covers of any given degree.

□

Remark. Although Lemma 4.6 and 4.7 are stated in a very similar manner, they are of different nature: the former is a group theoretic statement whereas the latter is not. Indeed as a consequence of [Siu87] it is known that the property $\gamma d(X) = 1$ is equivalent to having a fundamental group commensurable with the fundamental group of a curve; this property does thus depend only on the fundamental group. In general it is however possible to realize a given Kähler group as the fundamental group of several manifolds having different γ-dimensions.

Proof of Theorem 1.2. We argue according to the algebraic dimension of X, the case $a(X) = 3$ being trivial since a Kähler Moishezon manifold is projective.

1.) If $a(X) = 0$ then X is special in the sense of Campana. Thus the fundamental group is virtually abelian [CC14, Theorem 1.1] and thus projective [BR11, Theorem 1.4].
2.) If $a(X) = 1$, we replace X by some blowup such that the algebraic reduction is a holomorphic fibration $f : X \to C$ onto a curve. By [CP00] the general fibre of F is bimeromorphic to a K3 surface, torus or ruled surface over an elliptic curve, so its fundamental group is abelian. By Lemma 4.5 the group $\pi_1(X)$ is virtually torsion free. If $\gamma \dim(X) \leq 2$ we can thus apply Lemma 4.6 and Lemma 4.7. If $\gamma \dim(X) = 3$ it is shown in [CZ05, Theorem 1] that up to bimeromorphic transformations and étale cover f is a smooth morphism. Thus we can apply [Cla16, Corollary 1.2].
3.) If $a(X) = 2$ the algebraic reduction makes X into an elliptic fibre space over a projective surface and we can apply Corollary 1.5.

□

Appendix A. Elliptic surfaces

A.1. Proposition. *Let S be a non-algebraic compact Kähler surface that admits an elliptic fibration $f : S \to \mathbb{P}^1$. Then f has at least three singular fibres.*

Proof. We can suppose without loss of generality that f is relatively minimal, namely K_X is $\mathbb{Q}$-Cartier and there exists a line bundle L on S such that $mK_X \simeq f^*L$ for some $m \in \mathbb{N}^*$. Thus we know by the canonical bundle formula [Kod63] that

$$K_S \simeq f^*(K_{\mathbb{P}^1} + M + \sum_{c \in \mathbb{P}^1} m_c S_c)$$

where M is the modular part defined by the j-function and $\sum_{c \in \mathbb{P}^1} m_c S_c$ the discriminant divisor.

Let $C \subset S$ be an irreducible curve. If $f(C) = \mathbb{P}^1$ the surface S is connected by curves, hence algebraic by a theorem of Campana. By our assumption this shows that $f(C)$ is a point. Thus $K_S \cdot C \geq 0$ since f is relatively minimal. Recall that a non-algebraic Kähler surface has a pseudoeffective canonical bundle. Thus K_S is pseudoeffective and non-negative on all curves, hence it is nef.

Suppose first that f is isotrivial, i.e. we have $M \equiv 0$. Then [BHPVdV04, Chapter V, Table 6] shows that the singular fibres are either multiples of smooth elliptic curves or of type I_0^*. For a multiple fibre we have $m_c \leq \frac{1}{2}$ and for a fibre of type

I_0^* we have $m_c = \frac{1}{2}$. Since $K_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ we see that there are at least 4 singular fibres.

Suppose now that f is not isotrivial. Then we can use the argument from [Bea81, Proposition 1]: let $C^\star \subset \mathbb{P}^1$ be the maximal open set over which f is smooth. The j-function defines a non-constant holomorphic map $\tilde{C}^\star \to \mathbb{H}$ from the universal cover $\tilde{C}^\star \to C^\star$ to the upper half plane $\mathbb{H}$. In particular $\tilde{C}^\star$ is not $\mathbb{C}$ or $\mathbb{P}^1$, hence $\mathbb{P}^1 \setminus C^\star$ has at least three points. $\square$

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