Complex Geometry

Holomorphic functions of several variables

One reference (among many) for this material :
D. Huybrechts, *Complex Geometry, an introduction*, Springer Verlag (2005).

Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \to \mathbb{C}$ be a C^∞ function defined on U. We shall consider balls $\mathbb{B}(z, r)$ for the usual Hermitian norm on $\mathbb{C}^n$; the notation $\mathbb{D}(z,(r_1,\dots,r_n))$ will refer to the polydisc

$$\mathbb{D}(z,(r_1,\dots,r_n)) := \mathbb{D}(z_1,r_1)\times\cdots\times\mathbb{D}(z_n,r_n) \subset \mathbb{C}^n$$

centered at $z = (z_1,\dots,z_n)$ with polyradius $(r_1,\dots,r_n)$. We shall also use the following shorthand : if $z \in \mathbb{C}^n$ and $\alpha \in \mathbb{N}^n$ we set

$$z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}.$$

Definition. The function f is said to be holomorphic on U if one the following equivalent conditions holds :

(i) f is *analytic* : for any point $x \in U$ there exists $r > 0$ (with $\mathbb{B}(x,r) \subset U$) and $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ complex numbers such that

$$\forall z \in \mathbb{B}(x,r),\ f(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha (z-x)^\alpha$$

(the series having a convergence radius larger or equal to r) ;

(ii) the differential $df_z : \mathbb{C}^n \to C$ is $\mathbb{C}$-linear for any $z \in U$;

(iii) $\overline{\partial} f = 0$, *i.e.* :

$$\forall j = 1\dots n,\ \frac{\partial f}{\partial \overline{z_j}} = 0;$$

(iv) For any $x \in U$ and any $\underline{r} = (r_1,\dots,r_n)$ such that $\mathbb{D}(x,\underline{r}) \subset U$, we have the Cauchy formula :

$$\forall z \in \mathbb{D}(x,\underline{r}),\ f(z) = \frac{1}{(2i\pi)^n}\int_{\delta\mathbb{D}(x,\underline{r})} \frac{f(y)}{(y_1-z_1)\cdots(y_n-z_n)} dy_1\cdots dy_n$$

where

$$\delta\mathbb{D}(x,\underline{r}) := \partial\mathbb{D}(x_1,r_1)\times\cdots\times\partial\mathbb{D}(x_n,r_n)$$

is the distinguished boundary of $\mathbb{D}(x,\underline{r})$.

We shall denote $\mathcal{O}(U)$ the set of holomorphic functions defined on U.

Remark.

1. The distinguished boundary is not the topological boundary of the polydisc :
$$\delta\mathbb{D}(x, \underline{r}) \subsetneq \partial\mathbb{D}(x, \underline{r}).$$
2. The above definition amounts to saying that f is holomorphic separately in each variable.
3. In the expansion (i), we obviously have :
$$a_\alpha = \frac{f^{(\alpha)}(x)}{\alpha!} \text{ with } \alpha! = \prod_{j=1}^{n} \alpha_j! \text{ and } f^{(\alpha)}(x) := \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(x).$$

Most of properties are the same as in dimension 1

Holomorphic functions satisfy the following properties :

(a) if $f : U \to \mathbb{C}$ (with U connected) is holomorphic then it satisfies the **analytic continuation principle** : if f is vanishes on a non-empty open subset of U then f is identically zero on U.

(b) **Cauchy estimates** for holomorphic functions :
$$\forall\, r < R, \forall\, \alpha \in \mathbb{N}^n,\ \exists C_{\alpha,r,R} > 0, \quad \left\| f^{(\alpha)} \right\|_{\mathbb{B}(0,r)} \leq C_{\alpha,r,R} \left\| f \right\|_{\mathbb{B}(0,R)}$$
where f is any holomorphic function defined on a neighborhood of $\overline{\mathbb{B}(0, R)}$.

(c) **Maximum principle** : if $f : U \to \mathbb{C}$ (with U connected) is holomorphic and $|f|$ reaches a local maximum at a point $x \in U$ then f is constant on U.

(d) **Montel Theorem** : a collection of holomorphic functions that is locally uniformly bounded is relatively compact in $\mathcal{O}(U)$.

(e) **Riemann extension Theorem** : let $f : U \setminus H \to \mathbb{C}$ be a holomorphic function where $U \subset \mathbb{C}^n$ is an open subset and $H \subset \mathbb{C}^n$ is a complex hyperplane. If f is *locally bounded* near any point of $U \cap H$ then f extends holomorphically through H.

But there is a least one new phenomenon in several variables ! The following statement is known as the Hartogs phenomenon.

Theorem (Hartogs, 1906)**.** *Let $f : U \setminus H \to \mathbb{C}$ be a holomorphic function where $U \subset \mathbb{C}^n$ is an open subset and $H \subset \mathbb{C}^n$ is an affine subspace whose codimension is at least 2. Then f automatically extends to the whole of U.*

For instance, any holomorphic function $f : \mathbb{B}^2 \setminus \{0\} \to \mathbb{C}$ automatically extends to $\mathbb{B}^2$!