Complex Geometry

Class exam - 2021/12/13

Tentative rating:

Pb1: 7 points (1/3/3).

Pb2: 8 points (2/1/3/2).

Pb3: 5 points (3/2)

Problem 1 [Hodge numbers of complex tori]

Let $X = \mathbb{C}^n/\Lambda$ be a complex torus of dimension n.

1. Explain why X is a Kähler manifold.

2. Compute the Hodge numbers of X using Hodge theorem and a metric coming from $\mathbb{C}^n$.

3. Compute the Hodge numbers showing first that Ω^1_X is trivial (there exists a global basis of holomorphic 1-forms) and then using the various symmetries existing on Hodge numbers (Serre and Hodge).

Problem 2 [Surjective implies injective]

Let $f : X \to Y$ be a surjective holomorphic map between compact complex manifolds and let us assume that (X, ω_X) is Kähler. We will denote their dimensions by $m := \dim_{\mathbb{C}}(X)$ and $n := \dim_{\mathbb{C}}(Y)$.

1. Remind that the top de Rham cohomology group $H^{2n}(Y, \mathbb{R})$ is generated by the class $[\omega_Y^n]$ where ω_Y is any hermitian metric on Y. Show that the form

$$f^*(\omega_Y^n) \wedge \omega_X^{m-n}$$

is a non negative (m, m)-form that is positive on a non empty open subset of X.

2. Deduce that f^* is injective on $H^{2n}(Y, \mathbb{R})$.

3. Let α be a non zero class in $H^k(Y, \mathbb{R})$. Use Poincaré duality and the case $k = 2n$ to show that $f^*(\alpha)$ is non zero in $H^k(X, \mathbb{R})$. Conclude that f^* is injective on $H^k(Y, \mathbb{R})$ for any $k = 0 \dots 2n$.

4. Let S be the Hopf surface and let us consider its fibration $\pi : S \to \mathbb{P}^1$. Show that π^* is **not** injective for $k = 2$ (this gives a counterexample in the non Kähler setting).

Problem 3 [Riemann surfaces as algebraic curves]

To treat this part, you shall need the following statement, known as the $\partial\bar{\partial}$–lemma:

Lemma. *On a compact Kähler manifold* X*, let* α *be a closed real* $(1,1)$*-form such that* $[\alpha] = 0$ *in* $H^2(X, \mathbb{R})$*. Then there exists* $f : X \to \mathbb{R}$ *a* C^∞ *function such that* $\alpha = i\partial\bar{\partial}f$*.*

Let X be a compact curve and remind that $H^2(X, \mathbb{R}) \simeq \mathbb{R}$. Let L be a holomorphic line bundle on X and let us denote by $\lambda := c_1(L)$ its 1^{st} Chern class seen as a real number using the above mentioned isomorphism. Let also ω_X be a (Kähler) metric on X normalized so that $\int_X \omega_X = 1$.

1. Show that there exist a metric h on L such that $i\Theta_h(L) = \lambda\omega_X$ (*Hint:* fix h_0 any smooth metric on L and apply the $\partial\bar{\partial}$–lemma to $i\Theta_{h_0}(L)$ and $\lambda\omega_X$).

2. Deduce that L is ample if and only if $\lambda > 0$.

3. Bonus! Show that any curve carries a holomorphic line bundle L such that $c_1(L) > 0$ and infer that any curve can be embedded in a projective space (*Hint:* you can use the canonical bundle K_X or the line bundle $\mathcal{O}_X(\{x\})$ for some $x \in X$).