Complex Geometry

Digest on differential forms

A standard reference for this material is Chapter I, §1-5 in :
R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer Verlag, 1982.

Let M be a differentiable manifold of dimension $m = \dim(M)$. A differential form α of degree k is a smooth section of the vector bundle $\bigwedge^k T_M^*$: for any point $x \in M$, $\alpha(x)$ is a k-multilinear skew-symmetric form on the vector space $T_{M,x}$.

In the sequel we will denote by $\mathcal{A}^k(U)$ the vector space of differential forms of degree k defined on an open set $U \subset M$. It is a $\mathcal{C}^\infty(U)$-module (a differential form can be multiplied by a smooth function).

Local description. If $(x_1, \dots, x_m)$ are local coordinates defined on an open set $U \subset M$, then α can be written

$$\alpha(x) = \sum_{|I|=k} \alpha_I(x) dx_I$$

where :

- ⋆ $I := \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ is an ordered set of indices (a multi-index) ;
- ⋆ $\alpha_I : U \longrightarrow \mathbb{R}$ is a smooth function for any I ;
- ⋆ $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is the I^{th}-term of canonical basis of $\bigwedge^k \mathbb{R}^m$.

In particular, the vector space of differential forms of degree k defined on a coordinate patch $U \subset M$ is a free module of rank $\binom{m}{k}$ over the ring $\mathcal{C}^\infty(U)$.

Wedge product. There is a natural product

$$\bigwedge^k V \otimes \bigwedge^p V \longrightarrow \bigwedge^{k+p} V$$

defined for any vector space V. In particular, it gives rise to a product for differential forms : if α is a k-form and β a p-form, then $\alpha \wedge \beta$ is a $(k+p)$-form. This *wedge product* has the following properties :

- ⋆ it is $\mathcal{C}^\infty$-linear : $(f\alpha) \wedge (g\beta) = fg(\alpha \wedge \beta)$ for any smooth functions f, g and for any differential forms α and β ;
- ⋆ it is anti-commutative : $\alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha$ (with k and p their degrees).

In particular, if α is a 1-form, then we have : $\alpha \wedge \alpha = -\alpha \wedge \alpha$ and $\alpha \wedge \alpha = 0$.

The product can be computed using the local description above. Let us consider $\alpha(x) = \sum_{|I|=k} \alpha_I(x) dx_I$ and $\beta(x) = \sum_{|J|=p} \beta_J(x) dx_J$. By linearity, we have :

$$\alpha \wedge \beta(x) = \sum_{I,J} \alpha_I(x)\beta_J(x) dx_I \wedge dx_J.$$

Using the fact that $dx_i \wedge dx_i = 0$, we see that $dx_I \wedge dx_J = 0$ as soon as $I \cap J \neq \emptyset$. On the contrary, if $I \cap J = \emptyset$, then $I \cup J$ is a $(k+p)$-tuples of indices and it can be reordered. Finally we have :

$$\alpha \wedge \beta(x) = \sum_{I \cap J = \emptyset} \alpha_I(x)\beta_J(x)\epsilon_{I,J} dx_{I \cup J}$$

where $\epsilon_{I,J} = \pm 1$ according to the number of permutations needed to reorder $I \cup J$.

Exterior derivative. Differential forms can be differentiate as smooth functions. There is a well-defined operator d acting on forms and increasing the degree by one :

$$d : \mathcal{A}^k(U) \longrightarrow \mathcal{A}^{k+1}(U)$$

and having the following properties :

1. if $\alpha = f$ is a function (*i.e.* a 0-form), then df coincides with the tangent map of f :

$$df_x : T_{M,x} \longrightarrow T_{\mathbb{R}, f(x)} = \mathbb{R}.$$

2. the operator d satisfies the Leibniz rule :

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

for any forms α, β with $\deg(\alpha) = k$.

3. the square of d vanishes : $d(d\alpha) = 0$ for any form α.

In local coordinates, the operator d has the following expression. First, observe that if $\alpha = f$ is a function, then

$$df(x) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(x) dx_i$$

according to Point 1 above. Points 2 and 3 imply that

$$d\alpha(x) = \sum_{i=1}^{n} \sum_{|I|=k} \frac{\partial \alpha_I}{\partial x_i}(x) dx_i \wedge dx_I = \sum_{i \notin I} \frac{\partial \alpha_I}{\partial x_i}(x)\epsilon_{i,I} dx_{i \cup I}$$

where $\epsilon_{i,I} = \pm 1$ is the signature of the reordering of $\{i\} \cup I$.

Behavior with respect to smooth mappings. Any $\mathcal{C}^\infty$ map $f : M \longrightarrow N$ between smooth manifolds M and N of respective dimensions $m = \dim(M)$ and $n = \dim(N)$ induces a morphism (called *pull-back*) :

$$f^* : \mathcal{A}^k(N) \longrightarrow \mathcal{A}^k(M).$$

Intrinsically, it is defined as follows :

$$f^*(\alpha)_x(v_1, \dots, v_k) = \alpha_{f(x)}(df_x(v_1), \dots, df_x(v_k))$$

where α is a k-form on N, $x \in M$ and $v_1, \dots, v_k$ are tangent vectors to M at the point x.

It is not difficult to check that f^* is compatible with d and with the wedge product in the following sense :

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta, \quad \forall\, \alpha, \beta \in \mathcal{A}^\bullet(N)$$
$$d(f^*\alpha) = f^*(d\alpha), \quad \forall\, \alpha \in \mathcal{A}^\bullet(N).$$

If $(x_1, \dots, x_m)$ are local coordinates near a point $x \in M$ and $(y_1, \dots, y_n)$ are local coordinates near $y = f(x)$ then the map f can be written $f = (f_1, \dots, f_n)$. The pull-back of a form $\alpha(y) = \sum_{|I|=k} \alpha_I(y) dx_I$ is given by :

$$f^*(\alpha)(x) = \sum_{I=\{i_1<\dots<i_k\}} \alpha_I(f(x)) df_{i_1}(x) \wedge \dots \wedge df_{i_k}(x).$$

In the special case $n = m = k$, the form α is just $\alpha(y) = g(y) dy_1 \wedge \dots \wedge dy_n$ and the pull-back formula can be read :

$$f^*\alpha(x) = g(f(x)) \mathrm{Jac}(df_x) dx_1 \wedge \dots \wedge dx_n.$$

De Rham cohomology. The property $d^2 = 0$ implies that

$$\mathrm{Im}(d : \mathcal{A}^{k-1}(M) \to \mathcal{A}^k(M)) \subset \mathrm{Ker}(d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M))$$

and it legitimates the following definition.

Definition (de Rham cohomology groups)**.** The k^{th} de Rham cohomology group of a smooth manifold M is denoted by :

$$H^k_{dR}(M, \mathbb{R}) := \frac{\mathrm{Ker}(d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M))}{\mathrm{Im}(d : \mathcal{A}^{k-1}(M) \to \mathcal{A}^k(M))}.$$

It is naturally a $\mathbb{R}$-vector space.

A typical element in this cohomology group is a *closed* form (*i.e.* a form α with $d\alpha = 0$) up to the addition of an *exact* form (*i.e.* $\alpha = d\beta$). Leibniz rule implies that the product of cohomology classes is well defined (check it !) and the compatibility of f^* and d can be rephrase by saying that $f : M \to N$ induces a map between cohomology groups :

$$f^* : H^k_{dR}(N, \mathbb{R}) \longrightarrow H^k_{dR}(M, \mathbb{R})$$

which is a morphism of algebras (with the wedge product).

Easy computations.

1. If M is connected, then $H^0_{dR}(M, \mathbb{R}) = \mathbb{R}$.
2. If M is compact and oriented then $H^m_{dR}(M, \mathbb{R}) \simeq \mathbb{R}$ (with $m = \dim(M)$) and the isomorphism is given by the integration $[\alpha] \mapsto \int_M \alpha$ (*cf.* [Bott-Tu, Corollary 5.8]). The fact that the latter map is well-defined boils down to the so-called Stokes' formula.
3. The Poincaré lemma (*cf.* [Bott-Tu, Proposition 4.1]) states that :

$$\forall k \geq 1, \quad H^k(\mathbb{R}^n, \mathbb{R}) = 0$$

(any closed k-form is exact on $\mathbb{R}^n$ or more generally on a contractible manifold).

The Künneth formula This gives the cohomology of a product. It says the following :

Proposition. *Let M and N be compact manifolds. The cohomology algebra of $M \times N$ is given by :*

$$H^*(M \times N, \mathbb{R}) \simeq H^*(M, \mathbb{R}) \otimes H^*(N, \mathbb{R}).$$

The tensor product of graded algebra has the following meaning :

$$\forall k \geq 0, \; H^k(M \times N, \mathbb{R}) \simeq \bigoplus_{i+j=k} H^i(M, \mathbb{R}) \otimes H^j(N, \mathbb{R}).$$

It has to be noticed that both projections

$$M \xleftarrow{p} M \times N \xrightarrow{q} N$$

can be used to define a map on forms :

$$\left\{ \begin{array}{rcl} \mathcal{A}^i(M) \otimes \mathcal{A}^j(N) & \longrightarrow & \mathcal{A}^{i+j}(M \times N) \\ \alpha \otimes \beta & \mapsto & p^*(\alpha) \wedge q^*(\beta) \end{array} \right.$$

that induces the isomorphism in the Künneth formula above.