UNIVERSITÉ DE RENNES 1

Complex Geometry

Homework

Problem 1 [Pre-Hodge theory]

Let X be a compact complex manifold and $E \longrightarrow X$ be a rank r holomorphic vector bundle. Show that $H^0(X, E)$ is finite dimensional without using Hodge theory.

To do so, let us fix h a Hermitian metric on E and ω a metric on X. We shall consider the following norm for sections $s \in H^0(X, E)$:

$$\|s\|^2 := \int_X |s(z)|_h^2 \, dV_\omega.$$

1. Prove that $\left(H^0(X, E), \|\cdot\|\right)$ is a Hilbert space (*Hint :* choose a neighborhood U of a given point $x \in X$ where E is trivial and remark that, up to shrinking a bit U, the norm is comparable to a L^2-norm ; using Cauchy estimates, show that norm convergence implies uniform convergence).

2. Using Montel's theorem, show that the unit ball of $H^0(X, E)$ is compact.

3. Conclude.

Problem 2 [Normal and canonical bundles]

1. Let us consider an exact sequence of vector bundles

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0 \tag{1}$$

defined over a complex manifold X. Show that it induces an injective morphism of vector bundles

$$\det(F) \otimes \bigwedge^k Q \hookrightarrow \bigwedge^{k+r} E$$

for any $k \geq 0$ with $r := \operatorname{rk}(F)$. Is this morphism always surjective ?

2. Deduce the existence of an isomorphism

$$\det(E) \simeq \det(F) \otimes \det(Q).$$

3. In the case $r = 1$, show that we have an exact sequence

$$0 \longrightarrow F \otimes \bigwedge^k Q \longrightarrow \bigwedge^{k+1} E \longrightarrow \bigwedge^{k+1} Q \longrightarrow 0 \tag{2}$$

for any $0 \leq k < \operatorname{rk}(E)$.

Let $Y \subset X$ be a smooth submanifold of X. The **normal bundle** of Y in X is defined to be the quotient

$$N_{Y|X} := \frac{(T_X)_Y}{T_Y}.$$

It sits in the exact sequence

$$0 \longrightarrow T_Y \longrightarrow (T_X)_Y \longrightarrow N_{Y|X} \longrightarrow 0. \qquad (3)$$

4. Infer that $K_Y \simeq (K_X)_{|Y} \otimes \det(N_{Y|X})$.

5. If Y is smooth hypersurface, show that $N_{Y|X} \simeq \mathcal{O}_X(Y)_{|Y}$ and conclude that

$$K_Y \simeq (K_X \otimes \mathcal{O}_X(Y))_{|Y}.$$

Hint : let $(U_i)_{i\in I}$ be an open cover of X such that $Y_{|U_i} = (f_i = 0)$ for some holomorphic functions $f_i : U_i \to \mathbb{C}$. Show that the collection $(df_{i|Y})_{i\in I}$ can be interpreted as a global map $(T_X)_{|Y} \to \mathcal{O}_X(Y)_{|Y}$ (recall that the defining cocycle of $\mathcal{O}_X(Y)$ is $g_{ij} := \frac{f_i}{f_j}$) and conclude.

Problem 3 [Hypersurfaces of the projective spaces]

1. Let $n \geq 1$ be an integer and let us consider the canonical projection $\pi : \mathbb{C}^{n+1} \to \mathbb{P}^n$. Show that the map

$$\begin{cases} \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{n+1} & \longrightarrow & T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \\ (x, v) & \mapsto & d\pi_x(v) \otimes x \end{cases}$$

can be used to get a map of vector bundles on $\mathbb{P}^n$:

$$\underline{\mathbb{C}}^{n+1} \longrightarrow T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$$

where the left hand side is the trivial bundle of rank $n+1$ on $\mathbb{P}^n$.

2. Show that on $\mathbb{P}^n$ there exists an exact sequence called the *Euler exact sequence* :

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0. \qquad (4)$$

3. Infer that the canonical bundle of $\mathbb{P}^n$ is given by

$$K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

4. Conclude that if $Y_d = (P = 0)$ is a smooth hypersurface defined by a homogeneous polynomial P of degree d (well chosen such that Y is smooth), then its canonical bundle is given by :

$$K_Y \simeq \mathcal{O}_{\mathbb{P}^n}(d-n-1)_{|Y}.$$

Problem 4 [Lefschetz theorem on hyperplane sections]

1. Let L be an positive line bundle on a compact complex manifold Z. Show that Akizuki–Kodaira–Nakano vanishing theorem can be stated as :

$$\forall\, p+q<\dim(Z), \quad H^q(Z,\Omega_Z^p\otimes L^*)=0$$

From now on, we shall consider a compact complex manifold X and a hypersurface $Y\subset X$ such that the line bundle $\mathcal{O}_X(Y)$ is positive.

2. Show that we have an exact sequence

$$0\longrightarrow \Omega_X^p\otimes\mathcal{O}_X(-Y)\longrightarrow\Omega_X^p\longrightarrow\Omega_{X|Y}^p\longrightarrow 0 \tag{5}$$

by considering restriction to Y. Using the long exact sequence associated with (5), show that the natural morphisms

$$H^q(X,\Omega_X^p)\longrightarrow H^q(Y,\Omega_{X|Y}^p)$$

are ismorphisms for $p+q<\dim(Y)$ and injective for $p+q=\dim(Y)$.

3. Show that for any $p\geq 1$ we have a short exact sequence (on Y)

$$0\longrightarrow\Omega_Y^{p-1}\otimes\mathcal{O}_Y(-Y)\longrightarrow\Omega_X^p\longrightarrow\Omega_Y^p\longrightarrow 0. \tag{6}$$

where $\mathcal{O}_Y(-Y)$ is nothing but the restriction $\mathcal{O}_X(-Y)_{|Y}=\mathcal{O}_X(Y)^*_{|Y}$ (*Hint :* for $p=1$ it reduces to dualizing the sequence (3) ; then use the sequence (2)).

4. Show that the morphisms

$$H^q(Y,\Omega_{X|Y}^p)\longrightarrow H^q(Y,\Omega_Y^p)$$

are ismorphisms for $p+q<\dim(Y)$ and injective for $p+q=\dim(Y)$ (unwrap (6) !).

5. Conclude that the morphims

$$j^*:H^k(X,\mathbb{C})\longrightarrow H^k(Y,\mathbb{C})$$

induced by the inclusion $j:Y\hookrightarrow X$ are isomorphisms for $k<\dim(X)-1$ and injective for $k=\dim(X)-1$ (think of Hodge decomposition).

6. Let X be a smooth quartic surface [1] in $\mathbb{P}^3$ (*i.e.* $X=(f=0)$ where f is a general homogeneous polynomial of degree 4). Compute the canonical bundle K_X of X and deduce that the Betti numbers of X satisfy :

$$b_0=b_4=1,\quad b_1=b_3=0,\quad \text{and}\quad b_2\geq 3.$$

1. Such a surface is an example of K3 surface, a very important class of Kähler surfaces. It can be shown that $b_2=22$ in that case. For example, computations can be carried out using the whole long exact sequences associated with (6) and (4) restricted to X... but it is a bit lengthy !