Dynamics of geodesic and horocyclic flows

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1 Introduction

These notes were written for lectures at CIRM in spring 2014, where I presented in a unified way classical dynamical and ergodic properties of the horocyclic flow. Therefore, the writing is unformal. I will state several results, and sketch their proofs, because my aim is to show you how deeply the ergodic properties of the horocyclic flow and the geodesic flow are related.

Many good references exist. Among many others, let me recommand [EW11] (probably not for master students), and at a master level [Cou12] (in french), and [Dal11] (in french or english).

In this introduction, let me present you the main objects of interest, and state the main results that are discussed in the text.

Then I will come back with more details to the necessary geometric preliminaries, prove results on the topological dynamics of the horocycle flow, and then discuss invariant measures and ergodic properties.

The hyperbolic plane is defined as $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ and endowed with the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. The geodesics are the curves which minimize the distance. The hyperbolic geodesics are the vertical half-lines and the half-circles orthogonal to the boundary $\mathbb{R} \times \{0\}$. The isometries preserving orientation are the homographies $z \to \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with determinant 1. See exercise 1.1.

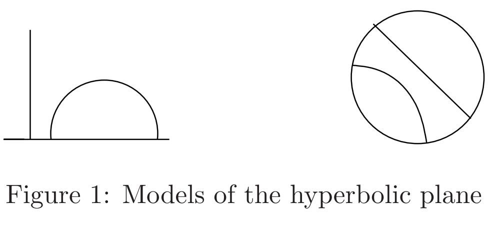

Figure 1: Models of the hyperbolic plane

The model of the disk is more natural geometrically. The hyperbolic disk is the open disk $D(0,1)$ in $\mathbb{C}$, endowed with the image metric from the hyperbolic metric of $\mathbb{H}$ through the map $z \mapsto \frac{z-i}{z+i}$. In the disk model, the geodesics are the diameters and the pieces of circles orthogonal to the boundary.

The geodesic flow $(g^t)_{t\in\mathbb{R}}$ is defined on the unit tangent bundle as follows. A vector $v \in T^1\mathbb{H}$ determines a unique geodesic $c : \mathbb{R} \to \mathbb{H}$ such that $c'(0) = v$. Now, follow the geodesic (at unit speed) until time t to get $g^t(v) := c'(t)$.

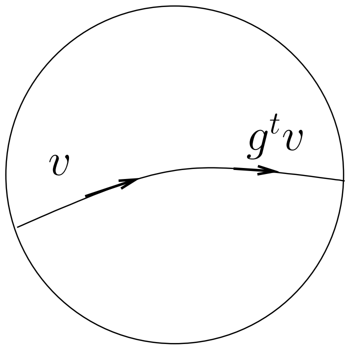

Figure 2: Geodesic flow

A horocycle is a (euclidean) circle tangent to the boundary, or a horizontal line, in the model of the upper half-plane. We can lift it to the unit tangent bundle $T^1\mathbb{H}$ in the following ways. A stable horocycle is the set of vectors orthogonal to such a circle, pointing inward. An unstable horocycle is the set of vectors that are orthogonal to a horocycle of $\mathbb{H}$ and pointing outwards. The stable horocycle flow $(h^s)_{s\in\mathbb{R}}$ acts on $T^1\mathbb{H}$ by moving a vector $v \in T^1\mathbb{H}$ along the stable horocycle that it defines. Thus, $h^s(v)$ is the vector based on the same horocycle, whose base point is at distance $|s|$ from the base point of v for the induced distance on the horocycle. One needs to choose a direction for s positive. See Exercise 1.2 for the matrix group description of these flows.

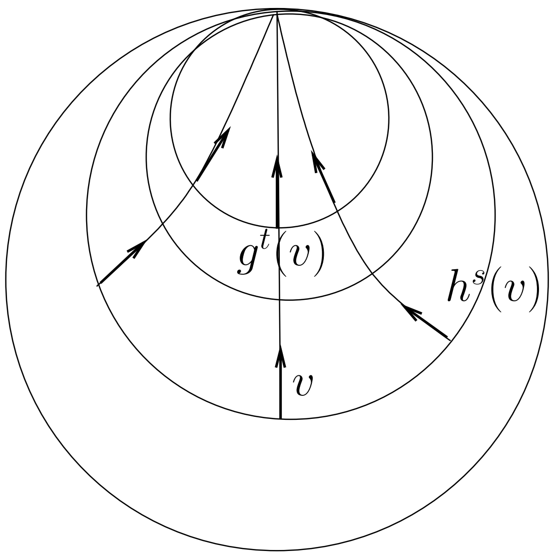

Figure 3: Horocyclic flow

The key point in everything concerning horocyclic flows is that

$$W^{ss}_{(g^t)}(v) = \{h^s(v),\ s \in \mathbb{R}\}\,.$$

In other words, the orbit of v for the horocyclic flow is exactly the strong stable manifold of the geodesic flow: A vector w belongs to $W^{ss}(v)$ iff $d(g^t v, g^t w) \to 0$ when $t \to +\infty$, iff there exists $s \in \mathbb{R}$ such that $w = h^s v$.

The aim of these lectures is to understand the behaviour of these two flows, and particularly of (h^s). On the unit tangent bundle of $T^1\mathbb{H}$, there is nothing interesting to say. All orbits of both flows go to infinity. The interesting study is the behaviour of (g^t) and (h^s) on the unit tangent bundle $T^1S = \Gamma\backslash PSL(2,\mathbb{R})$ of a hyperbolic surface $S = \Gamma\backslash\mathbb{H}$. The actions of (g^t) and (h^s) on $T^1\mathbb{H}$ commute with the action of $PSL(2,\mathbb{R})$ by isometries.

Here, Γ is a discrete group of isometries of $PSL(2,\mathbb{R})$. We will always assume that Γ be **nonelementary**. It means that Γ does not contain $\mathbb{Z}$ as a finite index subgroup. Or that any orbit $\Gamma.x$ of Γ has infinitely many accumulation points in $\overline{\mathbb{H}} = \mathbb{H} \cup S^1$. Any discrete group Γ which contains two hyperbolic isometries with different fixed points will be nonelementary. For example,

• the group $PSL(2,\mathbb{Z})$,

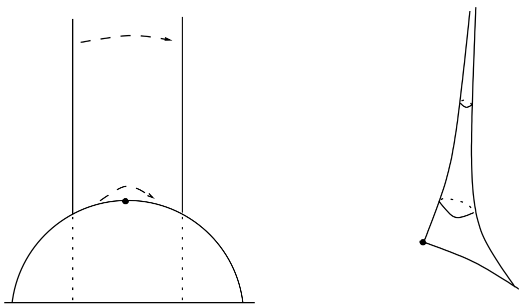

Figure 4: $PSL(2,\mathbb{Z})$ and the Modular surface

• a Schottky group

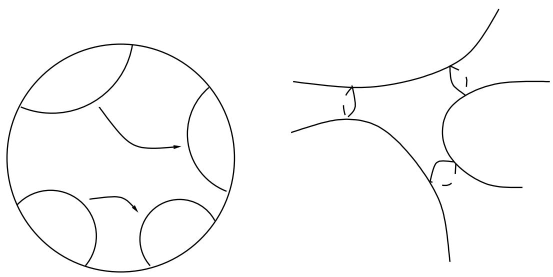

Figure 5: Schottky group and Schottky surface

• the fundamental group of a compact surface of genus g

The typical behaviour of these flows is the following. The geodesic flow is *hyperbolic*. The past and the future of a trajectory are independant. This can be formalized by the so-called *local product structure*: given a past trajectory and a future trajectory, there exists a geodesic line which has these prescribed past and future trajectories, asymptotically.

As a consequence, most trajectories of (h^s) are dense. Indeed, given a vector $u \in T^1S$, any vector v has a neighbour vector w with the same past than v and the same future as u. It says exactly that the horocyclic orbits are dense.

Another consequence of the hyperbolicity of the geodesic flow is that all behaviours that you could imagine will indeed be realized as geodesic trajectories.

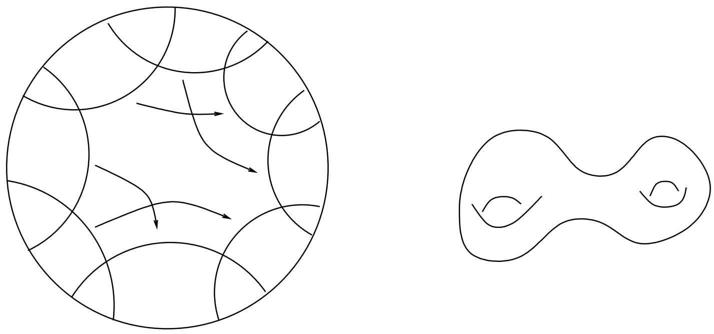

Figure 6: A hyperbolic octogon and associated genus two surface

The geodesic flow will have infinitely many periodic orbits, infinitely many ergodic invariant probability measures.

By contrast, the horocyclic flow will have very few invariant ergodic measures, and, heuristically, these ergodic components of the horocyclic flow reflect in which different ways and with which different speeds a geodesic trajectory can escape (or not) to infinity.

Let me now state (maybe formally with mistakes) the main results that I will discuss in these lectures.

A vector $v \in T^1S$ is said to be quasi-minimizing for the geodesic flow if there exists $C \geq 0$, such that for all $t \geq 0$, $d(g^t v, v) \geq t - C$. In other words, the geodesic $(g^t v)_{t \geq 0}$ goes to infinity at maximal speed.

Proposition 1.1 (Eberlein [Ebe72]). *Let S be a hyperbolic surface. The horocycle $(h^s v)_{s\in\mathbb{R}}$ is dense (in the nonwandering set $\mathcal{E} \subset T^1\mathbb{H}$) iff the geodesic $(g^t v)$ is* **not** *quasi-minimizing.*

As a corollary, we see that if S is a compact hyperbolic surface, the horocyclic flow is *topologically transitive*: all horocycles are dense.

One can even prove a refinement of this result and understand at which condition the positive orbit $(h^s v)_{s \geq 0}$ is dense or not [Sch11].

The *limit set* Λ_Γ is the set of accumulation points of any orbit $\Gamma.x$. See exercise 1.3. The *radial (conical) limit set* $\Lambda_{rad}(\Gamma)$ is the set of points $\xi \in \Lambda_\Gamma$ such that for any geodesic ray ending at ξ, there exists infinitely many points of $\Gamma.x$ at bounded distance of this ray. Points in the radial limit set correspond to asymptotic endpoints of geodesic rays that return infinitely often in a compact set.

The *horospherical limit set* $\Lambda_{hor}(\Gamma)$ is the set of points $\xi \in \Lambda_\Gamma$, such that for any horodisk centered at ξ, there exists infinitely many points of $\Gamma.x$ inside the disk. It contains the radial limit set.

We will see in proposition 2.2 that a geodesic $(g^t v)_{t\geq 0}$ is not quasi-minimizing iff any lift $\tilde{v}$ has a positive endpoint v^+ in the horospherical limit set.

In terms of the limit set, the above proposition becomes:

Proposition 1.2. *Let S be a hyperbolic surface. A horocycle $(h^s v)_{s\in\mathbb{R}}$ is dense (in the nonwandering set $\mathcal{E} \subset T^1S$ of the horocyclic flow) iff any lift $\tilde{v}$ of v to $T^1\mathbb{H}$ has a limit point v^+ in the horospherical limit set.*

We refer to Dal'bo [Dal99] and [Dal00] for precise references on the following result.

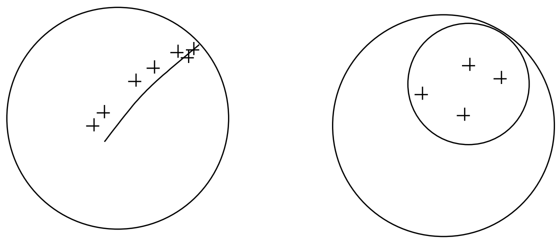

Figure 7: A radial (on the left) and a horospherical (on the right) limit point

The length spectrum $(l(\gamma))$ of a surface is the collection of lengths of closed geodesic of the surface that are associated to each conjugacy class of hyperbolic elements $\gamma \in \Gamma$.

Theorem 1.3 (Hedlund, Eberlein, Dal'bo). *The length spectrum $(l(\gamma))_{\gamma\in\Gamma}$ is non arithmetic iff the geodesic flow is topologically mixing iff there exists $v \in T^1S$ with $(h^s v)_{s\in\mathbb{R}}$ dense and such that $(g^t v)_{t\geq 0}$ admits an accumulation point.*

Note that, as we deal here only with hyperbolic surfaces, all statements of Theorem 1.3 are not only equivalent, but true because the length spectrum of a hyperbolic surface is known to be nonarithmetic. But this equivalence is relevant for more general nonpositively curved manifolds.

As a consequence of the above proposition 1.1 and our knowledge of the ends of a finite volume hyperbolic surface, it is easy to get the following result.

Theorem 1.4 (Hedlund). *If S is a surface with finite volume, then all horocycles $(h^s v)_{s\in\mathbb{R}}$ of T^1S are either periodic or dense.*

Let us describe now results about the ergodic theory of the horocyclic flow, that is results concerning invariant measures μ under the horocyclic flow on T^1S.

The Haar measure on $PSL(2,\mathbb{R})$ identified with $T^1\mathbb{H}$ can be written in coordinates as $d\mathcal{L}(v) = \frac{dxdy}{y^2}\frac{d\theta}{2\pi}$ where $x+iy$ is the base point of v in $\mathbb{H}$ and θ the angle of $v \in T^1_{x+iy}\mathbb{H} \simeq S^1$. This measure is invariant by left and right multiplication by elements of $PSL(2,\mathbb{R})$, so that it induces on $\Gamma\backslash PSL(2,\mathbb{R})$ a measure $\mathcal{L}$ which is invariant under both geodesic and horocyclic flows.

It is the most natural invariant measure for these flows.

Theorem 1.5 (Hopf, [Hop71]). *If S is a finite volume surface, the measure $\mathcal{L}$ on T^1S is ergodic and even mixing for the geodesic flow. It is the measure of maximal entropy of the geodesic flow.*

The ergodicity of $\mathcal{L}$ is due to Hopf through the now famous *Hopf argument*. The mixing property can be proven through a refinement of this Hopf argument (see for example Babillot [Bab02], Coudène [Cou07] and more recently [Cou13]. It maximizes entropy, but we will not develop this aspect here.

The main results that we shall discuss for the horocyclic flow are the following.

Theorem 1.6 (Furstenberg, 73). *Let S be a compact hyperbolic surface. Then the measure $\mathcal{L}$ is the unique invariant ergodic measure for the horocyclic flow.*

His proof used harmonic analysis on $\mathbb{R}^2$ but we will give purely dynamical arguments.

As an immediate corollary, we get the equidistribution of horocycles.

Corollary 1.7 (Furstenberg)**.** *Let S be a compact hyperbolic surface. For all $v \in T^1S$, and all continuous maps $\varphi : T^1S \to \mathbb{R}$, we have*

$$\frac{1}{t}\int_0^t \varphi(h^s v)\, ds \to \frac{1}{\mathcal{L}(T^1S)}\int_{T^1S} \varphi\, d\mathcal{L}\,.$$

Moreover, the convergence is uniform in $v \in T^1S$.

Theorem 1.8 (Dani [Dan78])**.** *Let S be a finite volume hyperbolic surface. Then the normalized Liouville measure $\mathcal{L}$ and the (normalized) measures supported by the periodic orbits are the only invariant ergodic probability measures.*

Moreover, for all $\varphi : T^1S \to \mathbb{R}$ continuous with compact support, and all $v \in T^1S$ whose horocyclic orbit is nonperiodic, we have

$$\frac{1}{t}\int_0^t \varphi(h^s v)\, ds \to \frac{1}{\mathcal{L}(T^1S)}\int_{T^1S} \varphi\, d\mathcal{L}\,.$$

Note that, contrarily to the compact case, the equidistributon of the horocycles is not a trivial consequence of the classification of invariant measures.

When Γ is finitely generated, Λ_Γ is infinite, and S has infinite volume, one says that S is geometrically finite.

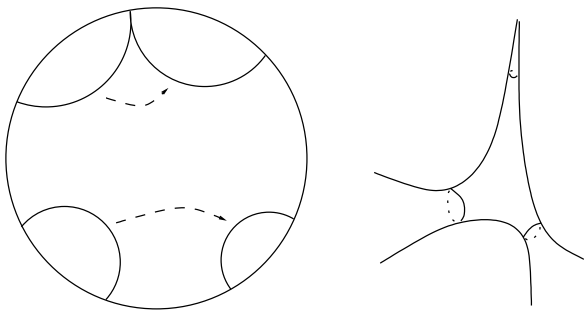

Figure 8: Geometrically finite surface

In this situation, the limit set Λ_Γ is strictly included in the circle at infinity, the interesting dynamics happens on smaller subsets of T^1S. One studies the dynamics of the geodesic flow on its nonwandering set Ω_Γ, which is the set of vectors $v \in T^1S$ which admit a lift $\tilde{v} \in T^1\mathbb{H}$ whose endpoints v^+ and v^- are in Λ_Γ. (Result due to Eberlein). The nonwandering set of the horocyclic flow $\mathcal{E}_\Gamma \supset \Omega_\Gamma$ is the set of vectors whose horocycle intersects Ω_Γ. Equivalently, it is the set of vectors $v \in T^1S$ which admit a lift $\tilde{v} \in T^1\mathbb{H}$ whose endpoint v^+ is in Λ_Γ (Dal'bo).

Theorem 1.9 (Burger (1990), Roblin (2003))**.** *Let S be a geometrically finite hyperbolic surface. There exists a unique invariant measure supported on the full nonwandering set $\mathcal{E}$ of the horocyclic flow, up to a multiplicative constant. It is an infinite Radon measure.*

This measure is now called the Burger-Roblin measure and denoted by m_{BR}.

Theorem 1.10 (Maucourant-Schapira, 2013)**.** *Let S be a geometrically finite hyperbolic surface (of infinite volume). For all $\varphi : T^1S \to \mathbb{R}$ continuous with compact support and all $v \in \mathcal{E}_\Gamma$ whose horocycle is nonperiodic, we have*

$$\int_{-t}^t \varphi(h^s v)\, ds \sim t_\Gamma^\delta \tau(v,t)\int_{T^1S} \varphi\, dm_{BR}$$

where $0 < \delta_\Gamma < 1$ is the critical exponent of the group and τ is a positive continuous function on $\mathcal{E}_\Gamma \times \mathbb{R}$.

1.1 First exercises

Exercise 1.1 (Hyperbolic metric and geodesics)**.** The hyperbolic length of a C^1-path $c : [a, b] \to \mathbb{H}$ is measured as follows:

$$l(c) = \int_a^b \|c'(t)\|_{c(t)}\, dt\,,$$

where $\|v\|_z = \frac{|v|}{y}$ if $z = x + iy \in \mathbb{H}$, $v \in T_z\mathbb{H} \simeq \mathbb{R}^2$, and $|v|$ is its euclidean norm. The hyperbolic distance between two points z and z' is the infimum of the lengths of all paths from z to z'. s The geodesics are the curves which minimize the distance. Prove the following claims.

• The vertical half-lines are geodesics.
• The homographies are hyperbolic isometries
• The half-circles orthogonal to the boundary are geodesics
• The group of direct isometries is isomorphic to $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm$.

Exercise 1.2 (Geodesic and horocyclic flows as one-parameter matrix flows)**.** • Prove that $PSL(2, \mathbb{R})$ acts transitively on $\mathbb{H}$ and simply transitively on $T^1\mathbb{H}$: given two vectors $v, w \in T^1\mathbb{H}$ there exists a unique $\gamma \in PSL(2, \mathbb{R})$ sending v to w.
One identifies $PSL(2, \mathbb{R})$ with $T^1\mathbb{H}$ through the map $\gamma \mapsto \gamma.v_0$ where v_0 is the vector $(0, 1)$ tangent to $\mathbb{H}$ at the point i.

• Show that in this identification, if $v = \gamma.v_0$, then $g^t(v) = \gamma.\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. In other words, the geodesic flow acts by multiplication to the right by the diagonal group $\left\{\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, t \in \mathbb{R}\right\}$.

• Show that the action of the stable horocyclic flow corresponds to the multiplication to the right by the unipotent group $\left\{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, s \in \mathbb{R}\right\}$.

• Show that the action of $PSL(2, \mathbb{R})$ by (differentials of) isometries on $T^1\mathbb{H}$ corresponds to the multiplication to the left by $PSL(2, \mathbb{R})$.

Exercise 1.3 (The limit set)**.** • *Show that the limit set Λ_Γ does not depend on the point $x \in \mathbb{H}$ used to define it.*
• *Show that if $\Gamma \triangleleft \Gamma_0$ is a normal subgroup of Γ, then $\Lambda_\Gamma = \Lambda_{\Gamma_0}$.*
• *Show that Λ_Γ is the smallest closed Γ-invariant set in S^1.*
• *Show that the action of Γ on Λ_Γ is minimal, i.e. for all $\xi \in \Lambda_\Gamma$, $\Gamma.\xi$ is dense in Λ_Γ.*
• *Show that the set of points γ^+ for $\gamma \in \Gamma$ a hyperbolic isometry, is dense in Λ_Γ.*
• *Show that the set of pairs (γ^-, γ^+), for $\gamma \in \Gamma$ a hyperbolic isometry, is dense in $\Lambda_\Gamma \times \Lambda_\Gamma$*

Exercise 1.4 (The Liouville measure)**.** *Show that the measure $\mathcal{L}$ is invariant under left and right multiplication.*

Exercise 1.5 (Unique ergodicity and equidistribution)**.** *Let $T : X \to X$ be a homeomorphism of a compact topological space. Show that T is uniquely ergodic (i.e. admits a unique invariant probability measure) iff for all continuous maps $\varphi : X \to \mathbb{R}$, the ergodic averages $\displaystyle\frac{1}{N}\sum_{k=0}^{N-1} \varphi \circ T^k(x)$ converge uniformly in x to a constant.*

2 Topological dynamics of the horocyclic flow

2.1 Nonarithmeticity, mixing of the geodesic flow, density of horocycles

The length spectrum $(l(\gamma))$ is the set of lengths of translation of the hyperbolic elements $\gamma \in \Gamma$. It is said to be *nonarithmetic* if it generates a dense subgroup of $\mathbb{R}$.

We will admit the following result.

Theorem 2.1 (Hedlund). *The length spectrum of a hyperbolic surface is nonarithmetic.*

It is more generally the case in constant curvature, or for any surface, or in presence of cusps, or when the limit set is connected. See [Dal99] proposition 2.1 for details and references on nonarithmeticity.

The geodesic flow is said to be *topologically mixing* (in restriction to the nonwandering set Ω_Γ) if for all open sets A, B of Ω_Γ there exists $T > 0$ such that for all $t \geq T$, $g^t A \cap B \neq \emptyset$. Theorem 1.3 combined with the above result ensures that the geodesic flow on the nonwandering set of a nonelementary hyperbolic surface is topologically mixing.

In general, on the unit tangent bundle of negatively curved manifolds, it is always at least *topologically transitive*, that is for all open sets A, B of Ω_Γ and all $T > 0$ there exists $t \geq T$ such that $g^t A \cap B \neq \emptyset$.

The geodesic flow admits a *local product structure.* If S is a hyperbolic surface, for all vectors v and w in T^1S close enough, there exists a vector $[v, w]$ close to v and w with the same past than v and the same future than w.

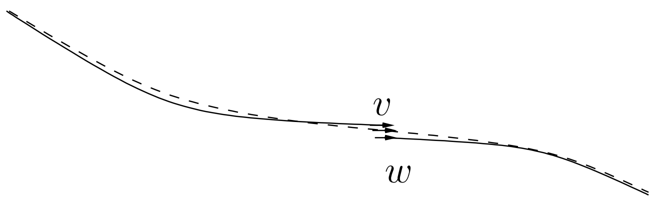

Figure 9: Local product structure

The geodesic flow of a nonelementary hyperbolic surface satisfies the Anosov *closing lemma.* For all T large enough and nonwandering vectors $v \in \Omega_\Gamma$, if $d(g^T v, v)$ is small enough, there exists a periodic vector of period almost T in a small neighbourhood of v.

Let us prove Theorem 1.3.

Proof. • Assume that the geodesic flow is topologically mixing. Let U be an open set of Ω_Γ. For all $t > 0$ large enough, $g^t U \cap U \neq \emptyset$. Choosing the diameter of U small enough, it implies that for all $t > 0$ large enough and $\epsilon > 0$, we can find a vector $v_{t,\varepsilon}$ such that $d(g^t v_{t,\varepsilon}, v_{t,\varepsilon}) \leq \varepsilon$. The closing lemma furnishes a periodic vector $p_{t,\varepsilon}$ with period in $[t - \varepsilon, t + \varepsilon]$ in the ε-neighbourhood of $v_{t,\varepsilon}$. It proves the nonarithmeticity of the length spectrum.

• Assume that the length spectrum is nonarithmetic. Thanks to the transitivity of the geodesic flow and the local product structure, it is enough to show that given any open set U, and all t large enough, $g^t U \cap U$ is nonempty. By continuity of the

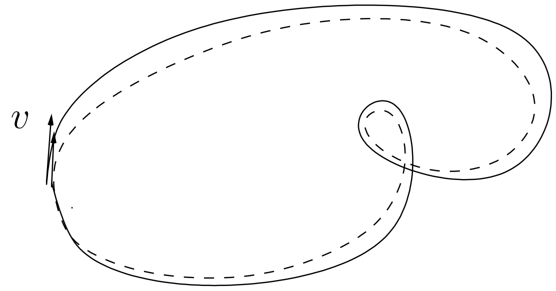

Figure 10: Anosov closing lemma

geodesic flow, for ε small enough, it is enough to prove that the set of $t \geq 0$ such that $g^t U \cap U \neq \emptyset$ is ε-dense in an interval $[T, +\infty[$.

We can use the density of periodic orbits to find a periodic vector $p_0 \in U$, and the nonarithmeticity of the length spectrum to find another periodic vector $p \in T^1 S$ such that there exist $n, m \in \mathbb{Z}$ with $|nl(p_0) - ml(p)| < \varepsilon$.

Now, using the local product structure, the transitivity and the closing lemma, one can construct an orbit which is negatively asymptotic to the negative orbit of p_0, then turns $k.m$ times around the orbit of p, then is positively asymptotic to the orbit of p_0. On the resulting orbits, one can find vectors v_k that belong to U and such that for some large T_k, $g^{T_k} v_k$ also belongs to U, and such that the set of numbers T_k, $k \in \mathbb{N}$, is ε-dense in $[T_0, +\infty[$.

• Assume that the geodesic flow is topologically mixing. Let $p \in T^1 S$ be a periodic vector for the geodesic flow. Let V be a small neighbourhood of p and U any open set. By topological mixing, there exists $T > 0$ such that for all $t \geq T$ there exists $u_t \in U$ with $g^t u_t \in V$. By local product one can glue the negative orbit of u_t with the positive orbit of p and obtain a vector in the weak stable manifold of v and in U. Choosing correctly t allows to ensure that this vector belongs to $W^{ss}(v)$, so that $W^{ss}(v)$ is dense in $T^1 S$.

• Assume that there exists $v \in T^1 S$ such that $(h^s v)_{s \in \mathbb{R}}$ is dense and $g^{t_n} v \to z \in T^1 S$. We refer to [Dal00] page 987 or [Cou08] chapter 2.1 for the proof of the fact that then, the length spectrum is nonarithmetic. We reproduce here the argument of Coudène.

Observe first that the density of $(h^s v)_{s \in \mathbb{R}}$ implies the density of $(h^s g^l v)_{s \in \mathbb{R}}$ for all $l \in \mathbb{R}$. Fix a small neighbourhood V of z, and assume that V is small enough to satisfy the closing lemma. Choose a vector $w \neq v$ in the intersection of the orbit $(h^s g^l v)_{s \in \mathbb{R}}$ and of V. Note that $d(g^t v, g^{t-l} w) \to 0$ when $t \to +\infty$. Observe that we can find a sequence $t_n \to \infty$ s.t. for all $n \in \mathbb{N}$ large enough, $g^{t_n} v \in V$ and $g^{t_n - l} w \in V$. We can apply the closing lemma to the geodesic orbits $(g^t v)_{t_0 \leq t \leq t_n}$ and $(g^t w)_{0 \leq t \leq t_n - l}$. Up to ε, we get periodic orbits of lengths $t_n - t_0$ and $t_n - l$. This is true for all $\varepsilon > 0$ and $l \in \mathbb{R}$. Therefore the length spectrum is nonarithmetic.

□

2.2 The horocycle $(h^s v)$ is dense iff the geodesic $(g^t v)$ is not quasiminimizing

Let us prove the following result.

Proposition 2.2 (Eberlein). *A vector $v \in T^1 S$ is not quasi-minimizing iff its endpoint v^+ is horospherical.*

Theorem 1.4 follows easily from the above proposition because quasi-minimizing vectors on a finite volume surface are well understood. They go straight away in some cusp and their horocycle is periodic.

Proof. Let $v \in T^1S$ be a nonwandering vector, $\tilde{v}$ any lift of v and v^+ the endpoint of $\tilde{v}$ in Λ_Γ. For simplicity, assume that the origin o of the disk is the basepoint of $\tilde{v}$. The endpoint v^+ is not horospherical iff there exists $t_0 \in \mathbb{R}$ such that $\Gamma.o$ does not intersect the horoball centered in v^+ and containing $\pi(g^{t_0}v)$ in its boundary. It is equivalent to say that for all $t \geq 0$, the ball centered in $\pi(g^{t_0+t})$ of radius t does not intersect $\Gamma.o$. It is also equivalent to say that for all $\gamma \in \Gamma$ and $t \geq 0$, the distance $d(\gamma.o, \pi(g^{t_0+t}\tilde{v})$ is at least t. On the quotient surface S it is equivalent to say that for all $t \geq 0$, the distance from $\pi(v)$ to $\pi(g^{t_0+t}v)$ is at least t, or in other words $d(\pi(v), \pi(g^t v)) \geq t - t_0$. □

Let us assume that the length spectrum is nonarithmetic, or equivalently that the geodesic flow is topologically mixing. Let us now prove that $(h^s v)_{s\in\mathbb{R}}$ is dense in $\mathcal{E}_\Gamma$ iff v^+ is horospherical, following arguments of Coudène [Cou08].

Proof. • If v^+ is horospherical, there exists sequences $t_i \to +\infty$ and $s_i \in \mathbb{R}$ such that $g^{t_i}h^{s_i}v$ converges to some vector w. Consider some very small open neighbourhood W of w on which the local product structure holds, and any open set U of Ω_Γ. For i large enough, $g^{t_i}h^{s_i}v$ belongs to W. The mixing property of the geodesic flow implies that for all t large enough there exists $u_t \in U$ such that $g^t u_t \in W$.

Now, use the local product structure to glue the past of $g^{t_i}u_{t_i}$ with the future of $g^{t_i}h^{s_i}v$. We obtain a vector $w_{t_i} \in W^{ss}(g^{t_i}v)$ with $g^{-t_i}v \in U$. But $g^{-t_i}v$ also belongs to $W^{ss}(v)$. It proves that $W^{ss}(v) = \{h^s v, s \in \mathbb{R}\}$ is dense in Ω_Γ.

• Conversely, assume that $(h^s v)_{s\in\mathbb{R}}$ is dense in Ω_Γ. Let $\tilde{v}$ be a lift of v to $T^1\mathbb{H}$ and v^+ its endpoint. Choose a periodic vector $p \in T^1S$, $\tilde{p}$ its lift to $T^1\mathbb{H}$ and $o \in \mathbb{H}$ his basepoint. By density, there exists a sequence $s_i \to +\infty$ such that $h^{s_i}v$ converges to p. In particular, for any $T_0 > 0$ there exists s_i large enough so that for all $0 \leq t \leq T$, $g^t h^{s_i}v$ and $g^t p$ stay very close one another. It implies easily that the horoball which contains $\pi(g^{T_0}v)$ in its boundary contains a point of the orbit $\Gamma.o$ (up to slightly cheeting with the constants). □

2.3 Geometrically finite surfaces

A hyperbolic surface S is geometrically finite iff $\pi_1(S)$ is finitely generated iff $\Gamma = \pi_1(S)$ admits a fundamental domain on $\mathbb{H}$ which is a domain bounded by finitely many geodesics, possibly with endpoints in the boundary. It is equivalent to say that S has a compact part, and finitely many ends, isometric either to a cusp $\{z \in \mathbb{H}, \Im(z) \geq 1\}/\{z \mapsto z+1\}$, or to a funnel $\{z \in \mathbb{H}, \Re(z) > 0\}/\{z \mapsto \lambda z\}$ for some $\lambda > 1$.

Proposition 2.3 (Hedlund, Eberlein, Dal'bo)**.** *Let S be a geometrically finite hyperbolic surface. If $v \in T^1S$, its horocycle $(h^s v)$ is either periodic, or dense in $\mathcal{E}_\Gamma$, or embedded nonperiodic. The last case corresponds to vectors that do not belong to $\mathcal{E}_\Gamma$.*

This proposition also follows easily from Proposition 2.2, as quasi-minimizing vectors go straight away in a cusp or in a funnel.

Remark 2.4. There exist examples, due to Coudène-Maucourant [CM10], of infinite volume surfaces where all horocycles are recurrent, i.e. return infinitely often in a compact set, but not all dense.

2.4 Some more exercises

Exercise 2.1. Let S be a hyperbolic surface whose limit set is infinite. Prove that there exists $v \in T^1S$ such that $(g^t v)_{t\geq 0}$ is dense in the nonwandering set Ω_Γ of the geodesic flow.

Prove that it is equivalent to the topological transitivity of the geodesic flow restricted to its nonwandering set.

3 Invariant measures for the horocyclic flow

3.1 The Hopf coordinates

A key tool is the following fact. The unit tangent bundle $T^1\mathbb{H}$ is homeomorphic to the product space $S^1 \times S^1 \setminus \text{Diagonal} \times \mathbb{R}$ through the map

$$v \in T^1\mathbb{H} \mapsto (v^-, v^+, \beta_{v^+}(o, \pi(v))\,,$$

where $\pi : T^1\mathbb{H} \to \mathbb{H}$ is the canonical projection, and β is the *Busemann cocycle*, defined for $x, y \in \mathbb{H}$ and $\xi \in S^1$ by

$$\beta_\xi(x, y) = \lim_{t\to+\infty} d(x, \xi_x(t)) - d(y, \xi_x(t)) = \text{"}d(x, \xi) - d(y, \xi)\text{"}.$$

In these coordinates, the geodesic flow acts by translation on the last factor: if $v = (v^-, v^+, \tau)$, then $g^t v = (v^-, v^+, \tau + t)$. The group $\Gamma = \pi_1(S)$ acts as follows.

$$\gamma.(v^-, v^+, \tau) = (\gamma.v^-, \gamma.v^+, \tau + \beta_{v^+}(\gamma^{-1}o, o))\,.$$

In these coordinates, it is easy to understand the orbits of the (stable) horocyclic flow. Indeed, if $v = (v^-, v^+, \tau(v))$, we have

$$W^{ss}(v) = \{h^s v, s \in \mathbb{R}\} = \{w = (w^-, v^+, \tau(v)\}\,.$$

In other terms the horocycle $(h^s v)_{s\in\mathbb{R}}$ of $T^1\mathbb{H}$ can be identified with $S^1 \setminus \{v^+\}$, and the space of horocycles can be identified with $S^1 \times \mathbb{R}$.

Proposition 3.1. *Let S be a hyperbolic surface. The set of invariant (ergodic) Radon measures for the geodesic flow on T^1S (resp. on Ω_Γ) is in $1-1$ correspondance with the set of invariant (ergodic) Radon measures for the action of Γ on $S^1 \times S^1 \setminus \{Diagonal\}$ (resp. on $\Lambda_\Gamma \times \Lambda_\Gamma \setminus \{Diagonal\}$).*

Similarly, the set of invariant (ergodic) Radon measures for the horocyclic flow on T^1S (resp. on $\mathcal{E}_\Gamma$) is in $1-1$ correspondance with the set of invariant (ergodic) Radon measures for the action of Γ on $S^1 \times \mathbb{R}$ (resp. on $\Lambda_\Gamma \times \mathbb{R}$).

Proof. Exercise $\square$

The above coordinates are extremely useful, because in terms of these coordinates, the Liouville measure has a product structure: if $\widetilde{\mathcal{L}}$ is the lift of the Liouville measure to $T^1\mathbb{H}$, this measure can be written as

$$d\widetilde{\mathcal{L}}(v) = \psi(v^-, v^+)d\lambda(v^-)d\lambda(v^+)dt,$$

where λ is the angular measure on the circle viewed from the origin and ψ is a continuous positive map on $S^1 \times S^1 \setminus \text{Diagonal}$.

It is a corollary of the property of *absolute continuity of the stable foliation*, due to Hopf for hyperbolic surfaces, and Anosov and Sinai much later in higher dimensions [Ano67], [Sin72].

In some proofs, we will ignore the map ψ above, as it is useful only to guarantee the Γ-invariance of $\widetilde{\mathcal{L}}$, but not to understand the arguments.

3.2 The Hopf argument, ergodicity and mixing of the Liouville measure

As soon as one knows that, in the Hopf coordinates, a measure has a product structure, the now classical *Hopf argument* implies ergodicity of the Liouville measure w.r.t. the geodesic flow. From this ergodicity, Hedlund deduced the ergodicity of the same measure w.r.t. the horocyclic flow. Then, he could deduce the mixing of the Liouville measure w.r.t. the geodesic flow from the ergodicity of the horocyclic flow.

Let us present the argument for the Liouville measure. We let the reader check that we only use the product structure.

Denote by $\widetilde{\mathcal{L}}$ the Γ-invariant lift of the Liouville measure. As said above, its product structure means that it can be written locally as $d\widetilde{\mathcal{L}}(v) = \psi(v^-, v^+)d\lambda(v^-)d\lambda(v^+)dt$, where λ is the angular measure on the circle viewed from the origin.

Let us prove that it is ergodic.

Proof. Let f be a continuous map with compact support, and consider the almost sure limits

$$f^+(v) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T f\circ g^t v\, dt \quad \text{and} \quad f^-(v) = \lim_{T\to-\infty} \frac{1}{T}\int_0^T f\circ g^t v\, dt\,.$$

These maps are defined $\mathcal{L}$-almost surely, and coincide $\mathcal{L}$-almost surely (the limit is the conditional expectation of f w.r.t the σ-algebra of invariant sets).

Let us call Ω the set of full measure of vectors v such that $f^+(v)$ and $f^-(v)$ are well defined and equal. Denote therefore by f_∞ a measurable map which coincides with both functions on Ω. Observe now that f^+ is invariant under the stable horocyclic flow and the geodesic flow, and f^- is invariant under the unstable horocyclic flow and the geodesic flow.

Therefore, almost surely, they coincide in the stable direction, the unstable direction and the direction of the flow, so that they have to be constant.

Remark 3.2. It seems maybe obvious to the reader. Because our intuition is based on product measures: when we have several coordinates on $\mathbb{R}^n$, we usually work with the Lebesgue measure which is a product measure. Imagine that we replace the Liouville measure by a (nonergodic) measure which is the average of two periodic measures on two periodic orbits. You can check that f_∞ coincides with f^+ and f^- almost surely but neither of these three maps is (even almost surely) constant.

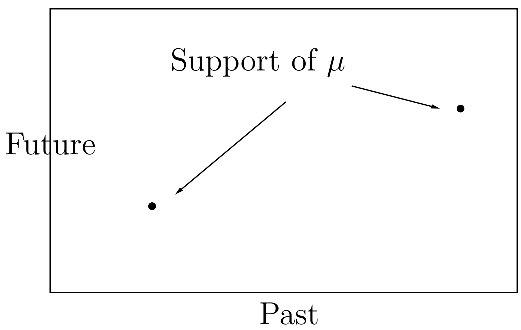

Figure 11: A non product measure

Let us come back to the requested proof, that you feel maybe more useful after the above example. To make rigorous the "almost surely", let us do the following computation. The local product structure of $\widetilde{\mathcal{L}}$ says that $\int \varphi d\mathcal{L} = \int_{S^1}\int_{S^1}\int_{\mathbb{R}} \tilde{\varphi} d\lambda^-(v^-)d\lambda^+(v^+)dt$ where $\tilde{\varphi}$ is a lift of φ to a fundamental domain of $T^1\mathbb{H}$ for the action of Γ.

Call ψ^+ (resp. ψ^-) a measurable map on S^1 such that $f^+(v) = \psi^+(v^+)$ $\mathcal{L}$-almost surely and $f^-(v) = \psi^-(v^-)$ $\mathcal{L}$-almost surely.

Forget the density ψ of $\mathcal{L}$ w.r.t. the product and do the following computations.

$$\begin{aligned}
\int_{S^1} (\psi^+)^2 d\lambda^+(v^+) &= \int_{S^1}\int_{S^1}\int_0^1 (\psi^+)^2 d\lambda^+(v^+)d\lambda^-(v^-)dt \\
&= \int_{T^1\mathbb{H}} (f^+(v))^2 d\mathcal{L}(v) \\
&= \int_{T^1\mathbb{H}} f^+(v)f^-(v)d\mathcal{L}(v) \\
&= \int_{S^1}\int_{S^1}\int_0^1 \psi^+(v^+)\psi^-(v^-)d\lambda^+(v^+)d\lambda^-(v^-)dt \\
&= \left(\int_{S^1} \psi^+(v^+)d\lambda^+(v^+)\right) . \left(\int_{S^1} \psi^-(v^-)d\lambda^-(v^-)\right) \\
&= \left(\int_{S^1} \psi^+(v^+)d\lambda^+(v^+)\right) . \left(\int_{S^1}\int_{S^1}\int_0^1 \psi^-(v^-)d\lambda^-(v^-)d\lambda^+(v^+)dt\right) \\
&= \left(\int_{S^1} \psi^+(v^+)d\lambda^+(v^+)\right) . \left(\int_{S^1}\int_{S^1}\int_0^1 f^-(v)d\mathcal{L}(v)\right) \\
&= \left(\int_{S^1} \psi^+(v^+)d\lambda^+(v^+)\right) . \left(\int_{S^1}\int_{S^1}\int_0^1 f^+(v)d\mathcal{L}(v)\right) \\
&= \left(\int_{S^1} \psi^+(v^+)d\lambda^+(v^+)\right) . \left(\int_{S^1} \psi^+(v^+)d\lambda^+(v^+)\right) \\
&= \left(\int_{S^1} \psi^+(v^+)d\lambda^+(v^+)\right)^2
\end{aligned}$$

The equality case in Cauchy-Schwarz inequality implies that ψ^+ is constant λ^+ almost surely, equal say to a constant c^+. (Similarly, ψ^- is constant λ^--almost surely.) Therefore, as $\mathcal{L}$ is equivalent to $\lambda^- \times \lambda^+ \times dt$, the set of vectors such that $f_\infty = c^+$ is of full $\mathcal{L}$-measure. It proves that the geodesic flow is ergodic. $\square$

Theorem 3.3 (Hedlund, 1936 [Hed39])**.** *Let S be a finite volume surface. The Liouville measure is ergodic w.r.t. the horocyclic flow.*

Proof. We follow verbatim [Cou13], whose argument is inspired from [Tho95]. Let $F \in L^2(T^1S, \mathcal{L})$ be a (h^s)-invariant map. The goal of the proof is to prove that F is invariant under the geodesic flow. The ergodicity of the geodesic flow w.r.t. the Liouville measure concludes the argument.

We want to see that F is almost surely invariant under the geodesic flow. The key point is that by doing alternatively long travels along stable horocycles and short travels along unstable horocycles, starting from a vector v, one can come back to some $g^t v$. Moreover, given a fixed t, we have enough flexibility to make the travels along unstable horocycles as short as we want, and get this almost sure invariance.

More precisely, for $s > 0$ and $\varepsilon > 0$, check that the following relation holds.

$$\begin{pmatrix} 1 & 0 \\ \frac{-\varepsilon}{S} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{S-1}{\varepsilon} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1/S-1}{\varepsilon} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 1/S \end{pmatrix}$$

Denote by (h^s) the stable horocyclic flow, and $(h^s_u)_{s\in\mathbb{R}}$ the unstable horocyclic flow. Now use the invariance of F under (h^s) and the invariance of $\mathcal{L}$ under both

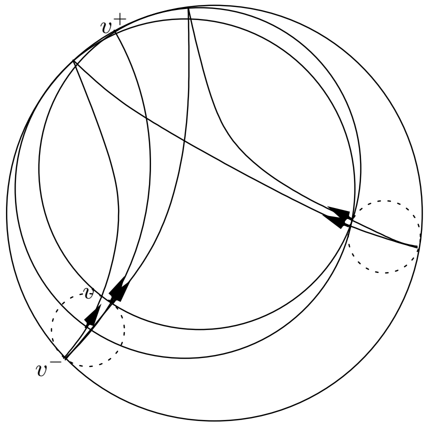

Figure 12: Ergodicity of the horocyclic flow

flow, and observe that

$$\begin{aligned} \|F\circ g^{2\ln S}-F\|_{L^2}^2 &= \|F\circ h^{\frac{S^{-1}-1}{\varepsilon}}\circ h_u^\varepsilon\circ h^{\frac{S-1}{\varepsilon}}\circ u^{-\frac{\varepsilon}{S}}-F\|_2^2 \\ &\le \|F\circ h_u^\varepsilon\circ h^{\frac{S-1}{\varepsilon}}\circ u^{-\frac{\varepsilon}{S}}-F\circ h^{\frac{S-1}{\varepsilon}}\circ u^{-\frac{\varepsilon}{S}}\|_2^2+\|F\circ h^{\frac{S-1}{\varepsilon}}\circ u^{-\frac{\varepsilon}{S}}-F\|_2^2 \\ &= \|F\circ h_u^\varepsilon-F\|_2^2+\|F\circ u^{-\frac{\varepsilon}{S}}-F\|_2^2 \end{aligned}$$

Let ε go to 0 to get that $F\circ g^{2\ln S}=F$ $\mathcal{L}$-almost surely. □

A measure μ invariant under the geodesic flow is *mixing* if for all borel sets A and B, one has

$$\mu(A\cap g^tB)\to\mu(A)\mu(B)\,.$$

Equivalently, it is mixing iff for all $f\in L^2(T^1S,\mathcal{L})$, the sequence $(f\circ g^t)_{t\geq 0}$ converges weakly towards a constant.

Theorem 3.4 (Hedlund, 1936 [Hed39])**.** *Let S be a finite volume surface. The Liouville measure is mixing w.r.t. the geodesic flow.*

Proof. We follow the arguments of [Cou07], see also [Cou08], who proves that all weak accumulation values of $(f\circ g^t)_{t\geq 0}$ are invariant under the stable and unstable horocyclic flow. Therefore, they are constant by ergodicity of this flow.

Let us show this invariance property. Assume that f is a bounded lipschitz function. Consider a sequence T_k such that $f\circ g^{T_k}$ weakly converges to a certain map $f_\infty\in L^2$. There exists a subsequence T_{k_i} such that $\frac{1}{N}\sum_{i=1}^N f\circ g^{T_{k_i}}$ converges in L^2 towards f_∞. And up to taking a subsequence N_j, $\frac{1}{N_j}\sum_{i=1}^{N_j} f\circ g^{T_{k_i}}$ converges almost surely towards f_∞.

For all $v\in T^1S$ and $w\in(h^sv)_{s\in\mathbb{R}}$ such that this convergence holds, we have

$$\left|\frac{1}{N_j}\sum_{i=1}^{N_j} f\circ g^{T_{k_i}}v-\sum_{i=1}^{N_j} f\circ g^{T_{k_i}}w\right|\leq\frac{C}{N_j}\sum_{i=0}^{N_j} d(g^{T_{k_i}}v,g^{T_{k_i}}w)\to 0\quad\text{when}N_j\to+\infty\,.$$

This proves that f_∞ is invariant under the horocyclic flow.

The ergodicity of the horocyclic flow concludes the proof for lipschitz functions, and a density argument of lipschitz functions in $L^2(T^1S,\mathcal{L})$ concludes the proof. □

Refinements of these arguments lead to the following theorem.

Theorem 3.5 (Babillot, [Bab02])**.** *Let M be a negatively curved manifold. The length spectrum is nonarithmetic iff all invariant measures under the geodesic flow that have a product structure are mixing.*

3.3 Unique ergodicity of the horocyclic flow

A corollary of the mixing is the following classical result.

Corollary 3.6. *Let S be a finite volume hyperbolic surface. Let $\varphi : T^1S \to \mathbb{R}$ be a continuous map with compact support. Then for all $v \in T^1S$, one has*

$$\frac{1}{2}\int_{-1}^{1} \varphi \circ g^{-t}(h^s v)\, ds \to \frac{1}{\mathcal{L}(T^1S)} \int_{T^1S} \varphi d\mathcal{L}\,.$$

Proof. The idea is simply to thicken slightly the set $\{h^s v, |s| \leq 1\}$ in the weak unstable direction into a relatively compact neighbourhood U_v of v. Pushing by the geodesic flow in negative time does not expand the weak unstable direction, so that the error due to thickening will not increase when t goes to infinity, and $\int_{-1}^{1} \varphi \circ g^{-t}(h^s v)\, ds$ stays close from $\int \varphi \circ g^{-t}(w).\mathbf{1}_{U_v}(w)\, d\mathcal{L}(w)$. Now, the mixing property ensures that this last quantity converges to $\frac{1}{\mathcal{L}(T^1S)}\left(\int_{T^1S} \varphi d\mathcal{L}\right) \times \mathcal{L}(U_v)$. It remains to let the size of the thickening go to 0 to get the desired result. $\square$

Remark 3.7. It is important to notice that once again, we did not use many assumptions, and in particular neither the fact that the Liouville measure is the Liouville measure nor the dimension 2 or the finiteness of volume or constant curvature assumption. We only used that we have a finite invariant mixing measure with a product structure (with continuous density).

Remark 3.8. Using similar arguments, it is possible to show that the family of maps $v \mapsto \int_{-1}^{1} \varphi \circ g^{-t}(h^s v)\, ds$ is equicontinuous in $t \geq 0$, so that the above convergence is uniform on compact sets.

Let us deduce from the above result the unique ergodicity of the horocyclic flow in the compact case (theorem 1.6).

Proof. Observe the following fundamental relation. For all $t, s \in \mathbb{R}$, we have

$$g^t \circ h^s = h^{se^{-t}} \circ g^t \tag{1}$$

This relation, easy to check on matrices of $PSL(2, \mathbb{R})$, means geometrically that the geodesic flow contracts exponentially the stable manifolds.

We deduce that $(g^{-t}h^s v)_{|s|\leq 1} = (h^s g^{-t} v)_{|s|\leq e^t}$.

As a corollary, one can check that

$$\frac{1}{2}\int_{-1}^{1} \varphi \circ g^{-t} h^s v\, ds = \frac{1}{2e^t}\int_{-e^t}^{e^t} \varphi \circ h^s(g^{-t}v)\, ds\,.$$

By the above corollary, the left term converges to $\frac{1}{\mathcal{L}(T^1S)}\int_{T^1S} \varphi d\mathcal{L}$, and by remark 3.8, this convergence is uniform on T^1S.

It implies that the averages $\int_{-e^t}^{e^t} \varphi \circ h^s(w)\, ds$ converge towards the same limit when $t \to +\infty$, uniformly in $t \geq 0$. It is the desired result, by exercise 1.5 . $\square$

3.4 The finite volume case

In the finite volume case, there is no uniform convergence anymore in Corollary 3.6. Therefore, one cannot deduce so easily the equidistribution of all orbits of the horocyclic flow.

However, it is possible to refine the argument to obtain the uniqueness of an invariant measure supported by the nonperiodic horocyclic orbits, and therefore the fact that the Liouville measure and the periodic measures are the only invariant ergodic measures.

3.4.1 About the proof of unique ergodicity

We follow the argument of Coudène [Cou09]. Roblin [Rob03] proved the same result in a more general context.

Observe that corollary 3.6 and remark 3.8 still hold on finite volume surfaces, so that the averages $\frac{1}{2}\int_{-1}^{1}\varphi\circ g^{-t}(h^s v)\,ds$ converge uniformly on compact sets towards $\frac{1}{\mathcal{L}(T^1S)}\int_{T^1S}\varphi d\mathcal{L}$.

Let μ be an (h^s)-invariant ergodic probability measure, which gives zero measure to the set of (h^s)-periodic vectors. It is equivalent to say that for μ-almost all $v\in T^1S$, there is a sequence $t_k\to+\infty$ such that $g^{t_k}v$ converges to some $w\in T^1S$. Choose such a vector v which is moreover generic for μ, that is satisfies the conclusion of Birkhoff theorem. For all $\varphi : T^1S\to\mathbb{R}$ continuous with compact support, we have $\frac{1}{2t}\int_{-t}^{t}\varphi(h^s v)\,ds\to\int\varphi\,d\mu$.

Let $K\subset T^1S$ be a compact set containing the sequence $g^{t_k}v$ and its limit w. The averages $\frac{1}{2}\int_{-1}^{1}\varphi\circ g^{-t}(h^s w)\,ds$ converge uniformly in $w\in K$ towards $\frac{1}{\mathcal{L}(T^1S)}\int_{T^1S}\varphi d\mathcal{L}$. In particular,

$$\frac{1}{2t_k}\int_{-t_k}^{t_k}\varphi(h^s v)\,ds=\frac{1}{2}\int_{-1}^{1}\varphi\circ g^{-t_k}(h^s g^{t_k}v)\,ds$$

converges uniformly to $\frac{1}{\mathcal{L}(T^1S)}\int_{T^1S}\varphi d\mathcal{L}$. As $\frac{1}{2t}\int_{-t}^{t}\varphi(h^s v)\,ds\to\int\varphi\,d\mu$, we deduce immediately that μ is the normalized Liouville measure.

To deduce the equidistribution of nonperiodic horocycles towards the Liouville measure, one needs additional ingredients.

3.4.2 Nondivergence of horocycles

Theorem 3.9 (Dani [Dan84])**.** *Let S be a finite volume hyperbolic surface. For all $\varepsilon>0$ there exists a compact $K\subset T^1S$ such that for all $v\in T^1S$ nonperiodic vectors (w.r.t. the horocyclic flow), there exists $T_{v,\varepsilon}$ such that for all $t\geq T_{v,\varepsilon}$, the proportion of time spent by $(h^s v)_{0\leq s\leq t}$ in K is at least $1-\varepsilon$:*

$$\frac{1}{t}\int_0^t \mathbf{1}_K(h^s v)\,ds\geq 1-\varepsilon\,.$$

This theorem ensures that all accumulation points of the family of measures $\frac{1}{t}\int_0^t\delta_{h^s v}\,ds$ are probability measures.

Let us prove Theorem 3.9. For the sake of simplicity assume that there exists only one cusp on S. On $\mathbb{H}$ this cusp cutted at an arbitrary initial height lifts to an infinite family of disjoint horoballs, centered at parabolic points.

Denote by K_0 the set of vectors based outside the cusp C_0. Denote by C_N the cusp cutted at a height N compared to C_0. Denote by K_N the set of vectors based outside C_N. let $\gamma_i\widetilde{C}_0$ and $\gamma_i\widetilde{C}_N$ be the collection of lifts of C_0 and C_N.

Consider $v\in K_0$ a nonperiodic vector (for the horocyclic flow). Observe that the proportion of time spent by $(h^s v)_{0\leq s\leq t}$ in K_N is greater than the time spent by $(h^s v)$ in $K_N\setminus K_0$ divided by the time spent by $(h^s v)$ outside K_0

To obtain a small upper bound of the proportion of time spent by the orbit outside K_N, it is enough to consider each horodisk of $\mathbb{H}$ separately. And to observe that the amount of time spent by a horocycle in some $\gamma_i\widetilde{C}_N$ compared to the amount of time spent in $\gamma_i\widetilde{C}_0$ is exponentially small, uniformly in the horocycle, and therefore in v. It concludes the proof of theorem 3.9.

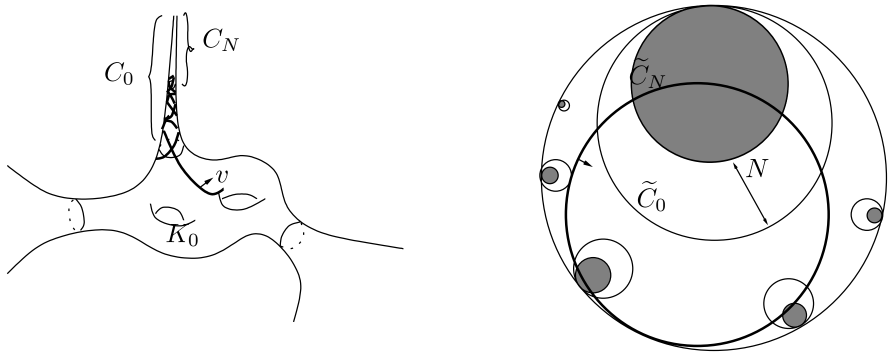

Figure 13: A cusp and corresponding horoball on the universal cover

3.4.3 Conclusion of the proof

A final argument shows that these accumulation measures give zero mass to periodic orbits of the horocyclic flow.

Consider now a fixed continuous map $\varphi : T^1S \to \mathbb{R}$ with compact support, and the ergodic averages $\frac{1}{t}\int_0^t \varphi(h^s v)\, ds$, for some vector $v \in T^1S$ which is nonperiodic. We aim to show that it converges to $\frac{1}{\mathcal{L}(T^1S)}\int_{T^1S} \varphi\, d\mathcal{L}$.

By theorem 3.9, all accumulation values of the sequence $\frac{1}{t}\int_0^t \delta_{h^s v}\, ds$ are probability measures. (See exercise 3.2.)

To deduce the equidistribution of horocycles from the classification of invariant measures and the nondivergence theorem 3.9, it remains to show that any limit measure μ of the family $\frac{1}{t}\int_0^t \delta_{h^s v}\, ds$ gives zero measure to the set of periodic vectors for the horocyclic flow. Let $\mu = \lim_{t_k\to+\infty} \frac{1}{t_k}\int_0^{t_k} \delta_{h^s v}\, ds$

Remember that a vector $v \in T^1S$ is periodic for the horocyclic flow iff $g^t v$ goes to infinity when $t \to +\infty$.

Assume by contradiction that there exists a compact subset of this set of periodic horocyclic vectors with $\mu(Q) \geq \alpha > 0$. For $\varepsilon = \alpha/4$, let K_ε be the compact given by theorem 3.9. By the above, one can find some $T_0 > 0$ such that for all $t \geq T_0$, $g^t Q$ does not intersect K_ε. For some $T_1 \geq T_0$ and all $t_k \geq T_1$, we have

$$\frac{1}{t_k}\int_0^{t_k} \mathbf{1}_Q(h^s v)\, ds \geq \frac{\alpha}{2}\,.$$

But we also have

$$\frac{1}{t_k}\int_0^{t_k} \mathbf{1}_Q(h^s v)\, ds = \frac{1}{t_k e^{-T_0}}\int_0^{t_k e^{-T_0}} \mathbf{1}_{g^{T_0}Q}(h^s g^{T_0} v)\, ds$$

The left term is at least $\alpha/2$ whereas the right term, for t_k large enough, is less than ε. It is a contradiction, which allows to conclude the proof of theorem 1.8.

3.5 Geometrically finite case

A geometrically finite hyperbolic surface has finitely many ends, that are cusps or funnels. The interesting geodesics are those which come back infinitely often in a compact set. They are exactly those which never enter a funnel. Heuristically the dynamics is the same as on the unit tangent bundle of finite volume surfaces. However, it is supported on a Cantor set of Liouville measure zero, and the Liouville measure is no more ergodic. Moreover, the nonwandering set of the horocyclic flow contains properly the nonwandering set of the geodesic flow.

These technical difficulties do not change the flavour of the results, just the length of the proofs.

3.5.1 The Patterson-Sullivan construction

Choose a point $o \in \mathbb{H}$. The Poincaré series $\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma.o)}$ converges for $s > \delta_\Gamma$ and diverges for $s < \delta_\Gamma$, where the *critical exponent* δ_Γ satisfies (exercise)

$$\delta_\Gamma = \lim_{T\to+\infty} \#\{\gamma \in \Gamma, d(o,\gamma.o) \in [T, T+1[\}$$

Consider the family of finite measures on $\Gamma.o$ defined by

$$\nu_{x,s} = \frac{1}{P(o,s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma.o)} \delta_{\gamma.o}$$

For all $x \in \mathbb{H}$, when s decreases to δ_Γ, up to considering subsequences, we can assume that it converges. Even more, thanks to a trick due to Patterson, one can assume that the limit ν_x is supported in the boundary $S^1 = \partial\mathbb{H}$, and therefore in Λ_Γ.

Theorem 3.10 (Patterson, Sullivan)**.** *When S is a geometrically finite hyperbolic surface, the above sequences converge to a family of measures $(\nu_x)_{x\in\mathbb{H}}$ of measures supported in $\Lambda_{rad}(\Gamma)$ and ergodic for the action of Γ on S^1.*

Moreover, this family (ν_x) satisfies for all $\gamma \in \Gamma$ and $x, y \in \mathbb{H}$, $\gamma_*\nu_x = \nu_{\gamma x}$ and for ν_x-almost all $\xi \in S^1$, $\frac{d\nu_x}{d\nu_y}(\xi) = \exp(\delta_\Gamma \beta_\xi(x,y))$.

The measure ν_o is the Hausdorff measure of the radial limit set for the visual distance on S^1 viewed from the point o.

Therefore, the measures (ν_x) are the good measures to play the role on Λ_Γ of the Lebesgue measures (λ_x) viewed from x on the circle S^1.

Now, define the measure

$$d\tilde{m}(v) = \exp(\delta_\Gamma \beta_{v^+}(o,\pi(v)) + \delta_\Gamma \beta_{v^-}(o,\pi(v))) d\nu_o(v^-) d\nu_o(v^+) dt\,.$$

This measure does not depend on the point o (exercise: check it), is invariant under the action of Γ (check it) and under the geodesic flow (by construction). Therefore it induces on the quotient on T^1S an invariant Radon measure for the horocyclic flow. In constant curvature, this measure is finite (Sullivan). The Hopf argument shows that it is ergodic, and even mixing ([Rud82], [Bab02], [Rob03]) One can show that it is the measure of maximal entropy of the geodesic flow (δ_Γ).

3.5.2 The Burger-Roblin measure

This measure is *not* invariant under the horocyclic flow. Indeed, call $\mu_{H^s(v)}$ the conditional measure of the Bowen-Margulis measure on the strong stable horocycles. This measure can be written as

$$d\mu_{H^s(v)}(w) = \exp(\delta_\Gamma \beta_{w^-}(o,\pi(w))) d\nu_o(w^-)$$

It is supported by a Cantor subset of the horocycle $(h^s v)_{s\in\mathbb{R}}$ and more precisely by the set of vectors $w = h^s v$ such that $w^- \in \Lambda_\Gamma$. Therefore, it can not be invariant under the horocyclic flow.

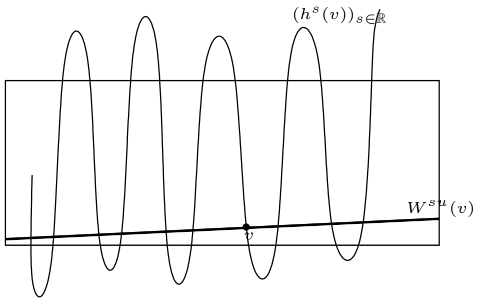

Figure 14: A chart of the horocyclic foliation

To get an invariant measure under the horocyclic flow, it is enough to replace these conditional measures by the Lebesgue measure induced by the parametrization of horocyclic flow.

The measure that we obtain is now called the *Burger-Roblin measure*, and can be written as

$$dm_{BR}(v) = \exp(\delta_\Gamma \beta_{v^+}(o,\pi(v)) + \beta_{v^-}(o,\pi(v)) d\lambda_o(v^-) d\nu_o(v^+) dt\,.$$

It is well defined and Γ-invariant on $T^1\mathbb{H}$ so that it induces a measure on T^1S, which is invariant under (h^s) (exercise), quasi-invariant under (g^t). But it is *infinite*.

However, one can show the following results.

Theorem 3.11 (Burger, Roblin)**.** *The measure m_{BR} is the unique invariant Radon measure for (h^s) giving zero measure to the periodic orbits of (h^s).*

Let us refer to [Sch15] for a proof of this Theorem using Coudene's argument, extended to this infinite volume situation.

Theorem 3.12 (Maucourant-Schapira [MS14])**.** *Let S be a geometrically finite hyperbolic surface. Let $v \in \mathcal{E}_\Gamma$ be a nonwandering nonperiodic vector for (h^s). For all continuous maps $\varphi : T^1S \to \mathbb{R}$ with compact support, we have*

$$\int_{-t}^{t} \varphi(h^s v)\, ds \sim_{t\to+\infty} \mu_{H^s(v)}((h^s v)_{|s|\le t}).\int_{T^1S} \varphi\, dm_{BR}\,.$$

Moreover, the quantity $\mu_{H^s(v)}((h^s v)_{|s|\le t})$ is (uniformly) comparable to t_Γ^δ when $g^{\ln t}v$ belongs to the compact part K of the manifold, and is comparable to $t^{\delta_\Gamma}.e^{(1-\delta_\Gamma)d(g^{\ln t}v,K)}$ when $g^{\ln t}v$ does not belong to K.

One has to deal with the two measures m_{BM} and m_{BR}, one of them being invariant under (g^t) and the other under (h^s). However, the strategy remains the same.

3.5.3 Equidistribution of horocycles towards the Bowen-Margulis measure

The mixing of the geodesic flow allows to prove the following theorem of equidistribution pushed by the geodesic flow

Theorem 3.13 (Babillot, Roblin)**.** *Let S be a geometrically finite surface, $v \in \mathcal{E}_\Gamma$ a nonwandering vector wrt the horocyclic flow and φ a continuous map with compact* *support. Then we have*

$$\int_0^1 \varphi \circ g^{-t}(h^s v)\, ds \to \int \varphi\, dm_{BM}$$

when $t \to +\infty$.

From this equidistribution pushed by the geodesic flow, and from theorem 3.11, one can deduce the equidistribution of horocycles parametrized by $(\mu_{H^s(v)})$, thanks to a nondivergence result.

Theorem 3.14 (S. 2004 [Sch04])**.** *Let S be a geometrically finite surface. For all $\varepsilon > 0$ there exists a compact K_ε such that for all $v \in \mathcal{E}_\Gamma$, there exists $T_{v,\varepsilon}$ such that for $t \geq T_{v,\varepsilon}$,*

$$\frac{1}{\mu_{H^s(v)}(\{h^s v, |s| \leq t\})} \int_{\{h^s v, |s| \leq t\}} \mathbf{1}_{K_\varepsilon}\, d\mu_{H^s(v)} \geq 1 - \varepsilon\,.$$

This theorem can be proven thanks to a refinement of the so-called Sullivan Shadow lemma, also due to the author. Using theorem 3.11 one deduces as in the finite volume case the following result.

Theorem 3.15 (S. 2005[Sch05])**.** *Let S be a geometrically finite surface. For all $v \in \mathcal{E}_\Gamma$, and $\varphi : T^1 S \to \mathbb{R}$ continuous with compact support, the averages of φ along $(h^s v)$ wrt $\mu_{H^s(v)}$ become equidistributed towards the Bowen-Margulis measure. More precisely,*

$$\frac{1}{\mu_{H^s(v)}(\{h^s v, |s| \leq t\})} \int_{\{h^s v, |s| \leq t\}} \varphi\, d\mu_{H^s(v)} \to \int \varphi\, dm_{BM} \quad \textit{when} \quad t \to +\infty$$

Now, the proof of theorem 3.12 follows by changing the parametrization of the horocycles, observing that the convergence in the above theorem implies a transversal equidistribution of the intersections of $(h^s v)$ with any transversal manifold, and that this transverse equidistribution implies theorem 3.12.

3.6 Geometrically infinite surfaces

We restricted our study to surfaces whose ends are finitely many, and easy to understand.

On reasonable geometrically infinite surfaces (when they admit a decomposition in pairs of pants with bounded diameters), Sarig [Sar04] proved that all nontrivial (i.e. nonperiodic) ergodic invariant measures under the horocyclic flow are quasi-invariant under the geodesic flow, and in $1-1$ correspondance with positive eigenfunctions of the Laplace-Beltrami operator on S. This result largely generalizes a result due to Babillot in the case of a nilpotent cover of a compact hyperbolic surface.

On abelian covers of compact hyperbolic surfaces, the set of ergodic invariant measures for the horocyclic flow is completely understood, and in $1-1$ correspondance with the set of *asymptotic cycles*, i.e. asymptotic speed of escape of a geodesic $(g^t v)$ in the cover group.

These results "show" that, heuristically, the ergodic components of the horocyclic flow are in $1-1$ correspondance with the different ways for a geodesic to escape to infinity.

3.7 Exercises

Exercise 3.1. Consider the horoballs $H_0 = \{z \in \mathbb{H}, \Im(z) \geq 1\}$ and $H_N = \{z \in \mathbb{H}, \Im(z) \geq e^N\}$. Consider a horocycle centered on the real line and prove that the time spent by this horocycle in H_N is exponentially small (in N) compared to the time spent in H_0.

Exercise 3.2. Let $T : X \to X$ be a homeomorphism of a noncompact topological space. Assume that the sequence $\mu_{x,N} = \frac{1}{N}\sum_{k=0}^{N-1}\delta_{T^k x}$ is tight: for all $\varepsilon > 0$, there exists a compact $K_{x,\varepsilon}$ such that for all $N \geq 0$, $\mu_{x,N}(K_{x,\varepsilon}) \geq 1 - \varepsilon$.

Show that all limit measures of the sequence $(\mu_{x,N})$ are probability measures.

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