# Density of half-horocycles on geometrically infinite hyperbolic surfaces

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[1]

**Abstract**

On the unit tangent bundle of a hyperbolic surface, we study the density of positive orbits $(h^s v)_{s\geq 0}$ under the horocyclic flow. More precisely, given a full orbit $(h^s v)_{s\in\mathbb{R}}$, we prove that under a weak assumption on the vector $v$, both half-orbits $(h^s v)_{s\geq 0}$ and $(h^s v)_{s\leq 0}$ are simultaneously dense or not in the nonwandering set $\mathcal{E}$ of the horocyclic flow. We give also a counter-example to this result when this assumption is not satisfied.

## 1 Introduction

On the unit tangent bundle of a finite volume hyperbolic surface, Hedlund [**H**] proved that all positive non periodic orbits $(h^s v)_{s\geq 0}$ of the horocyclic flow are dense.

On the other hand, there exists today a complete geometric criterion (see [**Da**] for a general result and full references) to know whether, on any hyperbolic surface, a full horocycle $(h^s v)_{s\in\mathbb{R}}$ is dense or not in the nonwandering set $\mathcal{E}$ of the horocyclic flow. It is dense if and only if $v$ is *horospherical* (see section 3).

When such a full horocycle is dense, a natural question from a dynamical point of view is to know whether both half-horocycles $(h^s v)_{s\geq 0}$ and $(h^s v)_{s\leq 0}$ are dense or not.

Hedlund had a partial positive answer, on the so-called "first kind surfaces" for *radial vectors*, that is vectors $v$ whose backward geodesic orbit returns infinitely often in a compact set. (We consider backward geodesics when we study the unstable horocyclic flow.)

In [**Scha**], we solved the problem on hyperbolic surfaces whose fundamental group is finitely generated, and we proved that the answer is always positive (both positive and negative horocyclic orbits are simultaneously dense or not), except in the case (trivial obstruction, see figure 1) of vectors $v$ such that one half-horocycle is dense in $\mathcal{E}$, and the other eventually leaves all compact sets.

In this work, we obtain an almost complete answer to this question, on any (infinitely generated) oriented hyperbolic surface.

Let us introduce some definitions and notations. Let $S$ be a connected oriented hyperbolic surface. It can be written $S = \Gamma \setminus \mathbb{D}$, where $\Gamma$ is a discrete group of isometries of the hyperbolic disc $\mathbb{D}$, and its unit tangent bundle $T^1 S$ identifies with the quotient $\Gamma \setminus T^1\mathbb{D}$ of the unit tangent bundle of the hyperbolic disc by $\Gamma$. Let

[1] 37A40,37A17, 37B99, 37D40

$\pi : T^1S \to S$ be the canonical projection. If $u \in T^1S$, and $\widetilde{u}$ is a lift of $u$ to $T^1\mathbb{D}$, we denote by $u^- \in S^1$ (resp. $u^+$) the negative (resp. positive) endpoint of the geodesic $(g^t\widetilde{u})_{t\in\mathbb{R}}$ defined by $\widetilde{u}$ in the boundary at infinity $S^1 = \partial\mathbb{D}$. The limit set $\Lambda_\Gamma \subset S^1$ is the set $\overline{\Gamma.o} \setminus \Gamma.o$ of limit points of an orbit. In this work, we assume $\Gamma$ to be nonelementary, that is not virtually abelian. It is equivalent to say that the limit set $\Lambda_\Gamma$ is infinite.

We endow the boundary $S^1$ with its counterclockwise orientation.

A horocycle is a euclidean circle of $\mathbb{D}$, tangent to $\partial\mathbb{D}$.

An unstable horocycle is the set of unit vectors orthogonal to a given horocycle, and pointing outwards. We study here the *unstable horocyclic flow.* This flow $(h^s)_{s\in\mathbb{R}}$ acts on $T^1\mathbb{D}$; its orbits are unstable horocycles, and the map $h^s$ turns vectors of a distance $|s|$ (for the induced Riemannian distance) on the unstable horocycle, in the clockwise direction. This flow induces on $T^1S$ the unstable horocycle flow of $T^1S$.

A vector $v \in T^1S$ is *radial* if its geodesic orbit $(g^{-t}v)_{t\geq 0}$ returns infinitely often in a compact set (here, $(g^t)_{t\in\mathbb{R}}$ denotes the geodesic flow, which moves vectors of a distance $t$ along the geodesic line that they define). It is well known that if $v$ is radial, its horocycle $(h^sv)_{s\in\mathbb{R}}$ is dense in the nonwandering set $\mathcal{E}$ of the horocyclic flow, that is the set $\mathcal{E}$ of vectors of $T^1S$ such that all neighbourhoods $V$ satisfy $h^{s_n}V \cap V \neq \emptyset$ for a sequence $s_n \to +\infty$. (It is the set which contains the interesting dynamics.)

Hedlund's result extends to :

**Theorem 1.1** *Let $S$ be a nonelementary oriented hyperbolic surface. Let $v \in T^1S$ be a radial vector. Then its positive half-horocycle $(h^sv)_{s\geq 0}$ is dense in the nonwandering set $\mathcal{E}$ of the horocyclic flow if and only if $v^-$ is not the first endpoint (in the counterclockwise direction) of an interval of $S^1 \setminus \Lambda_\Gamma$.*

The above result is stated and proved because its proof is simple and short, and uses simplified versions of Hedlund's ideas. But I prove below a much more general result.

**Theorem 1.2** *Let $S$ be a nonelementary oriented hyperbolic surface. Let $v$ be a vector whose full horocycle $(h^sv)_{s\in\mathbb{R}}$ is dense in $\mathcal{E}$, and such that there exist two constants $\Lambda > 0$ and $0 < \alpha_0 \leq \pi/2$, such that the geodesic ray $(\pi(g^{-t}v))$, $t \geq 0$, intersects infinitely many closed geodesics of length at most $\Lambda$, with an angle of intersection at least $\alpha_0$. Then both half-orbits $(h^sv)_{s\geq 0}$ and $(h^sv)_{s\leq 0}$ are dense in $\mathcal{E}$.*

In the assumptions of this theorem, the geodesic ray $(\pi(g^{-t}v)_{t\geq 0}$ can intersect (with an angle of intersection bouded from below) infinitely many times finitely many closed geodesics, or infinitely many distinct closed geodesics.

A radial vector satisfies this assumption, because closed geodesics are dense in $\mathcal{E}$, and if $v$ is radial, $(g^{-t}v)$ returns infinitely often in a closed ball of $\mathcal{E}$. One can show that the assumption of theorem 1.2 is therefore satisfied, by using for example a pants decomposition of the surface.

It is worth noticing that a very close assumption is used in a recent work of Omri Sarig [**Sa**] on the horocyclic flow.

We can ask whether this result is optimal or not. We build a counterexample to a completely general result.

**Theorem 1.3** *There exist nonelementary oriented hyperbolic surfaces containing vectors $v$ such that $(h^sv)_{s\geq 0}$ is dense in $\mathcal{E}$, $(h^sv)_{s\leq 0}$ is not dense in $\mathcal{E}$, $v^-$ is not the endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, and $(g^{-t}v)_{t\geq 0}$ intersects infinitely many closed geodesics of length going to infinity.*

*Depending on the examples, the intersection angle of $(g^{-t}v)_{t\geq 0}$ with these geodesics can be uniformly bounded away from zero, or go to zero.*

Section 2 is devoted to preliminaries, and the three other sections to proofs of the three above results.

I thank Yves Coudene, Antonin Guilloux and Omri Sarig for discussions on this work.

# 2 Preliminaries

## Hyperbolic geometry

The hyperbolic disc $\mathbb{D} = D(0,1)$ is endowed with the hyperbolic metric $\frac{4dx^2}{(1-|x|^2)^2}$. Let $o$ be the origin of the disc, and $\pi : T^1\mathbb{D} \to \mathbb{D}$ the canonical projection. The boundary at infinity of $\mathbb{D}$ is $S^1 = \partial\mathbb{D}$. We denote by $d$ both riemannian distances on $\mathbb{D}$ and $T^1\mathbb{D}$.

The Busemann cocycle is the continuous map defined on $S^1 \times \mathbb{D}^2$ by

$$\beta_\xi(x,y) = \lim_{z\to\xi} \left(d(x,z) - d(y,z)\right) .$$

The map

$$v \in T^1\mathbb{D} \mapsto (v^-, v^+, \beta_{v^-}(\pi(v), o)) \in (S^1 \times S^1) \setminus \Delta \times \mathbb{R} \,,$$

where $v^\pm$ are the endpoints of the geodesic $(g^t v)_{t\in\mathbb{R}}$ in $S^1$, and $\Delta$ is the diagonal of $S^1 \times S^1$, is an homeomorphism. In the sequel, we will identify $T^1\mathbb{D}$ with the set of *Hopf coordinates* $(S^1 \times S^1) \setminus \Delta \times \mathbb{R}$.

The classical identification of $\mathbb{D}$ with $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ through the homography $z \mapsto i\frac{1+z}{1-z}$ allows to identify the group of isometries preserving orientation of $\mathbb{D}$ with $PSL(2,\mathbb{R})$ acting by homographies on $\mathbb{H}$. This action extends to a simply transitive action on $T^1\mathbb{D}$ (or $T^1\mathbb{H}$). We identify therefore $T^1\mathbb{D}$ with $PSL(2,\mathbb{R})$. In the Hopf coordinates, an element $\gamma \in PSL(2,\mathbb{R})$ acts as follows:

$$\gamma.(v^-, v^+, t) = (\gamma.v^-, \gamma.v^+, t + \beta_{v^-}(o, \gamma^{-1}.o)) \,.$$

If $\Gamma \subset PSL(2,\mathbb{R})$ is a discrete group, we identify the unit tangent bundle $T^1S = \Gamma\backslash T^1\mathbb{D}$ of $\Gamma\backslash\mathbb{D}$ with the quotient $\Gamma\backslash\left((S^1 \times S^1) \setminus \Delta \times \mathbb{R}\right)$.

The *limit set* $\Lambda_\Gamma$ of the group is defined as $\Lambda_\Gamma = \overline{\Gamma.o} \setminus \Gamma.o \subset S^1$. It is also the smallest non empty closed $\Gamma$-invariant subset of $S^1$. We will often use the minimality of the action of $\Gamma$ on $\Lambda_\Gamma$: for all $\xi \in \Lambda_\Gamma$, $\Gamma.\xi$ is dense in $\Lambda_\Gamma$.

A point $\xi \in \Lambda_\Gamma$ is *radial* if it is the limit of a sequence $(\gamma_n.o)$ of points of $\Gamma.o$ that stay at bounded hyperbolic distance of the geodesic ray $[o\xi)$ from $o$ to $\xi$. We denote by $\Lambda_{\mathrm{rad}}$ the *radial limit set.* The set of points of $\Lambda_\Gamma$ that are fixed by an hyperbolic isometry (see definition below) of $\Gamma$ is included in $\Lambda_{\mathrm{rad}}$.

An *horocycle* of $\mathbb{D}$ is a euclidean circle tangent to $S^1$; it is also a level set of a Busemman function. An *horoball* is a euclidean disk bounded by an horocycle. A point $\xi \in \Lambda_\Gamma$ is *horospherical* if all horoballs centered at $\xi$ contains infinitely many points of the orbit $\Gamma.o$. In particular, $\Lambda_{\mathrm{rad}}$ is included in the set of horospherical limit points, denoted by $\Lambda_{\mathrm{hor}}$.

An isometry of $PSL(2,\mathbb{R})$ is said *hyperbolic* if it has exactly two fixed points on $S^1$, *parabolic* if it fixes exactly one point of $S^1$, and *elliptic* in other cases. We denote by $\Lambda_{\mathrm{p}} \subset \Lambda_\Gamma$ the set of *parabolic* limit points, that is the fixed points of a parabolic isometry of $\Gamma$.

Any oriented hyperbolic surface can be written as the quotient $S = \Gamma\backslash\mathbb{D}$ of $\mathbb{D}$ by a discrete subgroup $\Gamma$ of $PSL(2,\mathbb{R})$ without elliptic element, and its unit tangent bundle $T^1S = \Gamma\backslash T^1\mathbb{D}$ identifies with $\Gamma\backslash PSL(2,\mathbb{R})$.

In this note, we will always assume $\Gamma$ to be *nonelementary*, i.e. $\#\Lambda_\Gamma = +\infty$.

When $S$ is compact, then $\Lambda_\Gamma = \Lambda_{\rm rad} = S^1$. The surface is *convex-cocompact* when $\Gamma$ is finitely generated and contains only hyperbolic isometries. In this case, $\Lambda_\Gamma = \Lambda_{\rm rad}$ is strictly included in $S^1$, and $\Gamma$ acts cocompactly on the set $(\Lambda_\Gamma \times \Lambda_\Gamma)\setminus$ Diagonale $\times \mathbb{R} \subset T^1\mathbb{D}$. When $S$ has finite volume, all ends are cusps isometric to $\{z \in \mathbb{H}, |z| > 1\}/ < z \mapsto z+1 >$, and $\Lambda_\Gamma = \Lambda_{\rm rad} \sqcup \Lambda_{\rm p} = S^1$.

## Geodesic and horocyclic flows

A hyperbolic geodesic of $\mathbb{D}$ is a diameter or a half-circle orthogonal to $S^1$. A vector $v \in T^1\mathbb{D}$ is tangent to a unique geodesic, and orthogonal to exactly two horocycles containing its basepoint, tangent to $S^1$ respectively at $v^+$ and $v^-$. The set of vectors $w \in T^1\mathbb{D}$ such that $w^- = v^-$ and whose basepoint belongs to the horocycle tangent to $S^1$ at $v^-$ and containing $\pi(v)$ is the *strong unstable horocycle*, or strong unstable manifold of $v$. We denote it by $W^{su}(v) = \{h^s v, s \in \mathbb{R}\}$. The *strong stable horocycle* is defined similarly.

The *geodesic flow* $(g^t)_{t\in\mathbb{R}}$ acts on $T^1\mathbb{D}$ by moving a vector $v$ of a distance $t$ along the geodesic that it defines. In the identification of $T^1\mathbb{D}$ with $PSL(2,\mathbb{R})$, this flow corresponds to the right action by multiplication of the one-parameter subgroup

$$\left\{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \, t \in \mathbb{R} \right\}.$$

The *unstable horocyclic flow* $(h^s)_{s\in\mathbb{R}}$ acts on $T^1\mathbb{D}$ by moving a vector $v$ of a distance $|s|$ along its strong unstable horocycle. There are two possible orientations for such a flow, and we choose the orientation which corresponds to the right action by multiplication by the one-parameter subgroup

$$\left\{ n_s := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \, s \in \mathbb{R} \right\}$$

on $PSl(2,\mathbb{R})$. This flow makes the vectors turn around their strong unstable horocycle so that the orbit $\{h^s v, \, s \in \mathbb{R}\}$ is equal to the full strong unstable horocycle.

Moreover, for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$, these geodesic and horocyclic flows satisfy the following fundamental relation:

$$g^t \circ h^s = h^{se^t} \circ g^t \,. \tag{2.1}$$

**Remark 2.1** With our choice of orientation of $S^1$, when $s \to +\infty$, if $u \in T^1\mathbb{D}$ and $u_s^+ \in S^1$ is the positive endpoint of the geodesic determined by $h^s u$, then $u_s^+$ converges to $u^-$, with $u_s^+ \geq u^-$ in the counterclockwise orientation of $S^1$.

These two right actions commute with the left action by multiplication of $PSL(2,\mathbb{R})$ on itself, so that they are well defined on the quotient on $T^1S \simeq \Gamma\backslash PSL(2,\mathbb{R})$.

**Definition 2.2** *Let $(\phi^t)_{t\in\mathbb{R}}$ be a flow acting by homeomorphisms on a topological space $X$. The* nonwandering set *of $\phi$ is the set of points $x \in X$ such that for all neighbourhoods $V$ of $x$, there exists a sequence $t_n \to +\infty$ such that $\phi^{t_n} V \cap V \neq \emptyset$.*

The first part of the following lemma is due to Eberlein [**E1**], and the second part is proved in [**Scha**].

**Lemma 2.3** *The nonwandering set of the geodesic flow acting on $T^1S$ is*

$$\Omega := \Gamma\backslash \left( (\Lambda_\Gamma \times \Lambda_\Gamma) \setminus \Delta \times \mathbb{R} \right).$$

*The nonwandering set of the horocyclic flow acting on $T^1S$ is*

$$\mathcal{E} := \Gamma\backslash \left( (\Lambda_\Gamma \times S^1) \setminus \Delta \times \mathbb{R} \right).$$

*Moreover, we have*

$$\mathcal{E} = \cup_{s\in\mathbb{R}} h^s\Omega \quad \textit{and} \quad \mathcal{E} = \overline{\cup_{s\geq 0} h^s\Omega}$$

Recall that $W^{su}(v) = \{h^s v,\, s \in \mathbb{R}\}$ is compact if and only if $v^- \in \Lambda_{\mathrm{p}}$, and dense in $\mathcal{E}$ if and only if $v^- \in \Lambda_{\mathrm{hor}}$. Denote by $W^{su}_+(v) = \{h^s v,\, s \geq 0\}$ the positive half-horocycle of $v$.

We assume in the sequel that $S^1$ is oriented in the counterclockwise direction.

## Funnels

Recall (see for example [**Scha**]) the standard following facts.

**Remark 2.4** If the surface $S$ viewed as $\Gamma\backslash\mathbb{H}$ has a *funnel* isometric to $\{z \in \mathbb{H}, \mathrm{Re}(z) \geq 0\}/ < z \mapsto az >$, for some fixed $a > 1$, the geodesic $Re(z) = 0$ induces on the quotient a closed geodesic bounding the funnel. A geodesic intersecting this closed geodesic and entering in the funnel will never leave the funnel. In particular, the limit set $\Lambda_\Gamma$ viewed in $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ does not intersect $\mathbb{R}^*_+$.

A horocycle centered in $\mathbb{R}^*_+$, viewed on the quotient on $S$, will stay in the funnel except at most during a finite interval of time. A horocycle centered at 0, viewed on $S$, will have a side which will never enter the funnel, and the other side which will stay in the funnel and never go outside.

**Fact 2.5** In the case of a geometrically finite hyperbolic surface, the endpoints of an interval of $S^1 \setminus \Lambda_\Gamma$ are hyperbolic; they are the endpoints of the axis of a lift of the closed geodesic which bounds the funnel. It is not necessary the case on a general hyperbolic surface ([2]) .

**Fact 2.6** If $v^- \in \Lambda_\Gamma$ is the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, then $W^{su}_+(v) = \{h^s v, s \geq 0\}$ is not dense in $\mathcal{E}$.

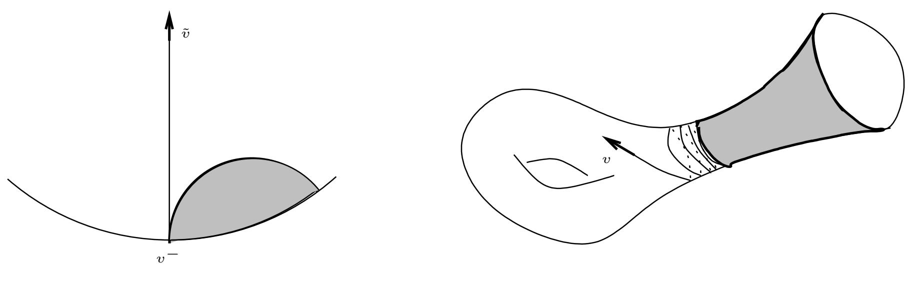


Figure 1: A vector $v$ whose positive horocycle $(h^s v)_{s\geq 0}$ is not dense in $\mathcal{E}$

[2]We can consider (example given by M. Peigné) the group $\Gamma =< \alpha^n h \alpha^{-n}, n \in \mathbb{Z} >$, where $h$ and $\alpha$ are two hyperbolic isometries generating a Schottky group, and $\alpha \notin \Gamma$, so that its fixed points $\alpha^-$ are $\alpha^+$ the endpoints of an interval of $S^1 \setminus \Lambda_\Gamma$, but it does not correspond to a funnel on the quotient surface.

# 3 Proof of theorem 1.1

## Right horocyclic points and vectors

If $v \in T^1\mathbb{D}$, let $v^{\pm}$ be its endpoints in $\partial\mathbb{D}$, $Hor(v) \subset \mathbb{D}$ the horoball centered at $v^-$ and containing its base point $\pi(v)$. The *right horoball* $Hor^+(v) \subset Hor(v)$ is the set of basepoints of vectors of $\cup_{t\geq 0} \cup_{s\geq 0} h^s g^{-t} v = \cup_{t\geq 0} \cup_{s\geq 0} g^{-t} h^s v$ (see relation (2.1)). Similarly, the *left horoball* $Hor^-(v)$ is defined as the other side of $Hor(v)$, that is $Hor^-(v) = \overline{Hor(v) \setminus Hor^+(v)}$ is the set of basepoints of vectors of $\cup_{t\geq 0} \cup_{s\leq 0} h^s g^{-t} v = \cup_{t\geq 0} \cup_{s\leq 0} g^{-t} h^s v$.

**Definition 3.1** *If $v \in T^1\mathbb{D}$, and $\alpha > 0$, the* cone *of width $\alpha$ around $v$ is the set $\mathcal{C}(v,\alpha)$ of points $x \in Hor(v)$ at (hyperbolic) distance at most $\alpha$ of the geodesic ray $(g^{-t}v)_{t\geq 0}$. It is the intersection of $Hor(v)$ with a euclidean cone.*

**Definition 3.2** *A vector $v \in T^1S$ is* right horocyclic *if it admits a lift $\tilde{v} \in T^1\mathbb{D}$, such that for all $\alpha > 0$ and $D > 0$, the orbit $\Gamma.o$ intersects the right part of the horoball $Hor^+(g^{-D}\tilde{v})$ minus the cone $\mathcal{C}(g^{-D}\tilde{v},\alpha)$.*

*A point $\xi \in \Lambda_\Gamma$ is* right horocyclic *if there exists a right horocyclic vector $v \in T^1S$ such that $\xi = v^-$ (see figure 3).*

Recall that a vector $v$ is horospherical if all horoballs $Hor^+(g^{-D}v)$ contain infinitely many points of $\Gamma.o$.

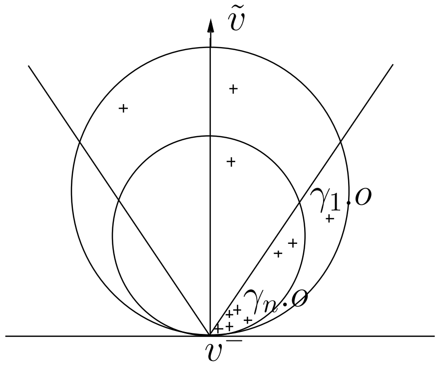


Figure 2: Lift of a right horocyclic vector

Recall the proposition

**Proposition 3.3 ([Scha])** *Let $S$ be a hyperbolic surface. A vector $v \in T^1S$ is right horocyclic if and only if $(h^s v)_{s\geq 0}$ is dense in $\mathcal{E}$.*

Recall also the following result.

**Proposition 3.4 ([Scha])** *Let $S$ be an oriented hyperbolic surface. If $p \in \Omega$ is a periodic vector for the geodesic flow, then its positive half-horocycle $(h^s(p))_{s\geq 0}$ is dense in the nonwandering set $\mathcal{E}$ of the horocyclic flow if and only if $p^-$ is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$.*

**Theorem 3.5 (Hedlund, [H], thm 4.2 )** *Let $S = \Gamma\backslash\mathbb{D}$ be an oriented hyperbolic surface of the first kind, that is such that $\Lambda_\Gamma = S^1$. Let $v \in T^1S$ be such that $(g^{-t}v)_{t\geq 0}$ returns infinitely often in a compact set, that is such that $v^- \in \Lambda_{\rm rad}$. Then its positive half-horocycle $(h^s v)_{s\geq 0}$ is dense in $T^1S$.*

## 3.1 Proof of theorem 1.1

If $v$ is a radial vector, and $v^-$ is the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, using lemma 2.3, we see that for $s > 0$ large enough, $h^s v$ will definitively leave the non-wandering set $\mathcal{E}$ of the horocyclic flow. Thus it cannot be dense in $\mathcal{E}$.

We will prove that a radial vector $v \in T^1 S$ such that $v^-$ is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$ is right horocyclic. To simplify notations, without ambiguity, we shall denote $v$ for a lift of $v$ to $T^1\mathbb{D}$. Let $R > 0$ and $D > 0$ be two constants, large enough. Let us prove that $\#\Gamma.o \cap Hor^+(g^{-D}v) \setminus \mathcal{C}(v, R) = +\infty$.

• As $v$ is radial, there exists $R_0$, and a sequence $\gamma_n.o \to v^-$, such that $d(\gamma_n.o, (v^-, v]) \leq R_0$. Let $g^{-t_n}v$, $t_n \to +\infty$, be a sequence of vectors at distance at most $R_0$ of $\gamma_n.o$. Letting $\gamma_n^{-1}$ act, we get a sequence of vectors $\gamma_n^{-1} g^{-t_n} v$ on the unit tangent bundle $T^1 B(o, R_0)$ of the ball $B(o, R_0)$. Up to a subsequence, we can assume that this sequence converges to a vector $w_\infty \in T^1 B(o, R_0)$, with endpoints $w_\infty^\pm \in \partial\mathbb{D}$.

Note that (see figure 3) $\gamma_n^{-1}.v^- \to w_\infty^-$, and that the half-horocycle of $\mathbb{D}$ $\pi(\gamma_n^{-1}.\cup_{s\geq 0} h^s v)$ converges (in the Hausdorff topology of closed subsets of $\overline{\mathbb{D}}$) to the half-circle $[w_\infty^-, w_\infty^+]$ of the boundary, oriented in the counterclockwise direction.

• As $v^-$ is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, and the $\gamma_n$ preserve orientation, $w_\infty^-$ is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$.

• Choose a hyperbolic isometry $h \in \Gamma$, with fixed points $h^\pm \in \Lambda_\Gamma$, both in the interval $[w_\infty^-, w_\infty^+]$, on the right side of $w_\infty^-$, close to $w_\infty^-$. The axis of $h$ projects on $S$ to a closed geodesic. Let $D_0 > 0$ be the distance from this geodesic to the projection of the origin $o$ of the disk $\mathbb{D}$.

Up to conjugate $h$ by another hyperbolic isometry wich has its fixed points on the right side of $w_\infty^-$, we can assume that the axis $(h^-, h^+)$ of the isometry $h$ is at distance at least $2R + R_0 + D_0$ of the geodesic $(w_\infty^-, w_\infty^+)$.

• As the half-horocycle $\pi(\cup_{s\geq 0} h^s \gamma_n^{-1} v)$ converges to $[w_\infty^-, w_\infty^+]$ (in the Hausdorff topology on closed subsets of $\overline{\mathbb{D}}$) for $n$ large enough, this half-horocycle intersects the geodesic $(h^- h^+)$ at two points $x_n^-, x_n^+$, with $d(x_n^-, x_n^+) \to \infty$.

We see also that if $y_n$ is the point of the geodesic segment $[x_n^-, x_n^+]$ at the largest distance of the horocycle $\pi(\cup_{s\geq 0} h^s \gamma_n^{-1} v)$, then this distance from $y_n$ to this horocycle goes to infinity when $n \to +\infty$.

But this point $y_n$ is at distance at most $D_0$ of a point, say $g_n.o$, of the orbit $\Gamma.o$. In other words, for $n$ large enough, $g_n.o$ belongs to the horoball $Hor^+(g^{-D}\gamma_n^{-1} v)$.

The point $y_n$ lies on $(h^- h^+)$, and is therefore at distance at least $2R + R_0 + D_0$ of the geodesic $(w_\infty^-, w_\infty^+)$. We deduce that $g_n.o$ is at distance at least $2R$ of this geodesic. For $n$ large enough, $g_n.o$ is at distance at least $R$ of the geodesic $(\gamma_n^{-1}.v^-, \gamma_n^{-1}.v^+)$.

• Let us come back with $\gamma_n$: denote $h_n^\pm = \gamma_n.h^\pm$, where $h_n = \gamma_n \circ h \circ \gamma_n^{-1}$ is the corresponding hyperbolic isometry. The points $\gamma_n.g_n.o$, for $n$ large enough, are all in the horoball $Hor^+(g^{-D}v)$ but not in the cone $\mathcal{C}(v, R)$. It is the desired result.

# 4 And more generally

Before the proof of theorem 1.2, recall some classical lemmas of hyperbolic geometry. We refer to [**G-Ha**] or [**C-D-P**] for example.

**Lemma 4.1** *Let $(a, b, c)$ be a hyperbolic triangle (possibly infinite). If the angle at $a$ is greater than $\alpha_0 > 0$, there exists a constant $C(\alpha_0) > 0$, such that*

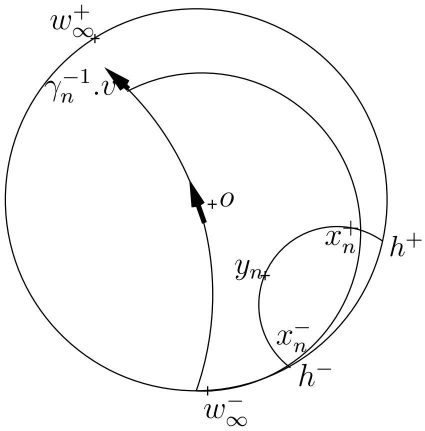


Figure 3: Proof of theorem 1.1

1. *we have* $d(a,b) + d(b,c) - C(\alpha_0) \leq d(b,c) \leq d(a,b) + d(a,c)$*,*
2. *the distance from* $a$ *to* $[b,c]$ *is smaller than* $C(\alpha_0)$*.*

The converse to this lemma is also true.

**Lemma 4.2** *Let* $k > 0$*. There exist constants* $\alpha(k) > 0$*,* $d(k) > 0$ *and* $C(k) = C(\alpha(k))$ *such that if* $(a,b,c)$ *is a hyperbolic triangle (possibly infinite) such that* $d(a,[bc]) \leq k$ *and* $d(b,c) \geq d(k)$*, then the angle at the vertex* $a$ *of the triangle* $(a,b,c)$ *is greater than or equal to* $\alpha(k)$*, and we have*

$$d(a,b) + d(b,c) - C(k) \leq d(b,c) \leq d(a,b) + d(a,c)\,.$$

Let us state another lemma which will be useful in the sequel.

**Lemma 4.3** *There exists a constant* $\delta > 0$*, such that for all* $\xi \in \partial\mathbb{D}$ *and* $p, q \in \mathbb{D}$ *such that* $\beta_\xi(p,q) = 0$*, there exists an "interior triangle"* $\alpha, \beta, \gamma$ *satisfying* $\alpha \in (\xi, q]$*,* $\beta \in (\xi, p]$*,* $\gamma \in [p,q]$*,* $d(\alpha,\beta) \leq \delta$*,* $d(\alpha,\gamma) \leq \delta$*,* $d(\beta,\gamma) \leq \delta$*, and moreover* $\beta_\xi(\alpha,\beta) = 0$*,* $d(\alpha,q) = d(\gamma,q)$*,* $d(\beta,p) = d(\gamma,p)$*.*

*In this situation, we also have* $d(p,q) = 2d(p,\gamma) = 2d(p,\beta) = 2d(q,\gamma) = 2d(q,\alpha)$*.*

Let us prove now theorem 1.2.

*Proof:* Let us prove first that one of the two sides $(h^s v)_{s\geq 0}$ or $(h^s v)_{s\leq 0}$ is dense in $\mathcal{E}$, or equivalently that $v^-$ is left or right horocyclic. As $(h^s v)_{s\in\mathbb{R}}$ is dense in $\mathcal{E}$, the point $v^- \in \Lambda_\Gamma$ is horospherical. If if $v^-$ were horocyclic but neither left nor right horocyclic, by definition, the horoball $Hor(\widetilde{v})$ minus a certain cone $\mathcal{C}(\widetilde{v}, R)$ would contain no element of $\Gamma.o$. It implies that $\Gamma.o \cap \mathcal{C}(\widetilde{v}, R)$ is infinite, so that $v^-$ is radial. As $v^-$ is in the limit set, it cannot be the endpoint of two distinct connected components of $S^1 \setminus \Lambda_\Gamma$. But Theorem 1.1 implies in this case that one of the two sides $(h^s v)_{s\geq 0}$ or $(h^s v)_{s\leq 0}$ is dense in $\mathcal{E}$.

Assume therefore that $(h^s v)_{s\geq 0}$ (for example) is dense, or equivalently that $v^-$ is right-horocyclic, and let us prove that it is also left-horocyclic, or equivalently that $(h^s v)_{s\leq 0}$ is dense. The idea is as follows.

Let $\gamma$ be a hyperbolic isometry coresponding to one of the closed geodesics, of length at most $\Lambda$, that are intersected by $(g^{-t} v)_{t\geq 0}$. This isometry $\gamma$ let globally the orbit $\Gamma.o$ invariant. Given two constants $D > 0$ and $R > 0$, this isometry, iterated in a convenient way, should send a point of $\Gamma.o \cap Hor^+(g^{-D}v) \setminus \mathcal{C}(v, R)$ in $\Gamma.o \cap Hor^{\mp}(g^{-D+const}v) \setminus \mathcal{C}(v, R - const')$, for some constants $const$ and $const'$ depending on $D, R$, but not on $\gamma$.

It will allow us to prove that there exist infinitely many points of $\Gamma.o$ inside $Hor^-(g^{-D'}v) \setminus \mathcal{C}(v, R')$, for all $D', R' > 0$.

Let us introduce some notations. We lift $v$ to $T^1\mathbb{D}$, and still denote it by $v$. Assume that $\gamma^- \geq v^- \geq \gamma^+$, on the circle oriented in the counterclockwise direction, so that $\gamma$ will roughly move points from $Hor^+$ to $Hor^-$, and not in the other direction. Assume therefore that $v^-$ is right horocyclic, and let us show that it is left horocyclic.

Fix first $D; R$, big compared to all constants appearing in the statement and in all lemmas used in the proof: the bound $\Lambda$ on the lengths of closed geodesics that are intersected, the angle $\alpha_0$ which is a lower bound for the angle of intersection between $(g^{-t}v)$ and these closed geodesics, and the constants $C(\alpha_0)$ of lemma 4.1 and $C(k), d(k)$ of lemma 4.2 for $k = C(\pi/2)$ above, and the constant $C_2(\alpha_0)$ which appears below.

Choose now a point $y_0 \in \Gamma.o$ in $Hor^{\pm}(g^{-D}v) \setminus \mathcal{C}(v, R)$, and $x_0$ its projection on the axis $(\gamma^-, \gamma^+)$. Let $x_n = \gamma^n(x_0)$ and $y_n = \gamma^n(y_0)$.

Consider at last an isometry $\gamma$ such that the axis $(\gamma^-\gamma^+)$ of $\gamma$ induces on $S$ a closed geodesic intersecting $(g^{-t}v)_{t\geq 0}$, such that the angle on $\mathbb{D}$ between the axis of $\gamma$ and $(g^{-t}v)$ is at least $\alpha_0$, and such that $y_0$ belongs to the bounded connected component of $Hor(v)\setminus(\gamma^-\gamma^+)$. As there exist infinitely many such geodesics $(\gamma^-\gamma^+)$ arbitrarily far from $y_0$, we shall assume that the distance from $y_0$ to $(\gamma^-\gamma^+)$ is greater than the constant $d(k)$ of lemma 4.2, for $k = C(\pi/2)$, the constant given by lemma 4.1.

Denote by $w = h^s v$ the vector of the horocycle of $v$ such that $(g^{-t}w)$ intersects $(\gamma^-\gamma^+)$ orthogonally. Denote respectively by $I_v$ and $I_w$ the intersection points of $(g^{-t}v)$ (resp. $(g^{-t}w)$) with $(\gamma^-\gamma^+)$.

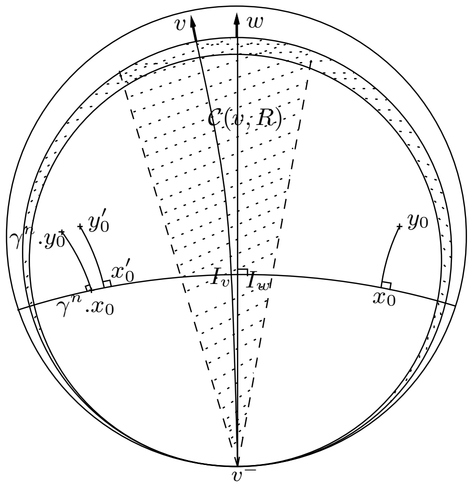


Figure 4: Proof of theorem 1.2

**Lemma 4.4** *If the angle at $I_v$ between $(\gamma^-\gamma^+)$ and $(g^{-t}v)_{t\geq 0}$ is bounded from below by $\alpha_0$, the distance between $I_v$ and $I_w$ is bounded (from above) by $C(\alpha_0)$.*

*Proof:* Consider the infinite triangle $v^-, I_v, I_w$. By lemma 4.1, we get $d(I_v, (v^- I_w)) \leq C(\alpha_0)$. As the angle at $I_w$ between $(I_v I_w)$ and $(v^- I_w)$ is equal to $\pi/2$, this distance equals $d(I_v, I_w)$. The lemma is proven. □

**Lemma 4.5** *Let $x_0$ be the projection of $y_0$ on $(\gamma^-\gamma^+)$, and $x_n = \gamma^n x_0$, $y_n = \gamma^n y_0$. There exist constants $C_2(\alpha_0)$ and $R(\alpha_0)$, depending only on $\alpha_0$, such that if $d(x_n, I_w) \geq R(\alpha_0)$, and $d(x_n, I_v) \geq R(\alpha_0/4)$, then*

$$\begin{array}{rcl} d(y_0,x_0)+d(x_n,I_w)-C_2(\alpha_0) & \leq & d(y_n,(g^{-t}w)_{t\geq 0}) \leq d(y_0,x_0)+d(x_n,I_w) \\ d(y_0,x_0)+d(x_n,I_v)-C_2(\alpha_0) & \leq & d(y_n,(g^{-t}v)_{t\geq 0}) \leq d(y_0,x_0)+d(x_n,I_v) \end{array}$$

*As a consequence, we have*

$$\left|d(y_n,(g^{-t}w)_{t\in\mathbb{R}})-d(y_n,(g^{-t}v)_{t\in\mathbb{R}})\right| \leq d(I_v,I_w)+C_2(\alpha_0) \leq C_1(\alpha_0)+C_2(\alpha_0)\,.$$

*Proof:* The third line follows directly from the first two lines and lemma 4.4. It is therefore enough to prove the first two lines of inequalities.

In both cases, the right inequality follows immediately from the standard triangular inequality and from the fact that $d(x_n, y_n) = d(x_0, y_0)$.

The triangle $y_n, x_n, I_w$ has a right angle at $x_n$, so that by lemma 4.1, $d(y_n, I_w) \geq d(y_n, x_n) + d(x_n, I_w) - C(\pi/2) = d(y_0, x_0) + d(x_n, I_w) - C(\pi/2)$. In the same way, $d(y_n, I_v) \geq d(y_0, x_0) + d(x_n, I_v) - C(\alpha_0)$.

We still have to prove that the distance from $y_n$ to $(g^{-t}w)_{t\geq 0}$ (resp. $(g^{-t}v)_{t\geq 0}$), is, up to uniform constants, realised by $d(y_n, I_w)$ (resp. $d(y_n, I_v)$).

Let $p$ be the projection of $y_n$ on $(g^{-t}w)_{t\in\mathbb{R}}$. Assume that $d(x_n, I_w) \geq C(\alpha_0/4)$. Lemma 4.1 in the triangle $y_n, I_w, x_n$ implies that the angle at $I_w$ between $[I_w, x_n]$ and $[I_w, y_n]$ is bounded from above by $\alpha_0/4$.

Then the angle at $I_w$ between $[I_w p]$ and $[I_w, y_n]$ is larger than $\pi/2 - \alpha_0/4 > 0$. Lemma 4.1 in the triangle $(p, I_w, y_n)$ gives therefore $d(y_n, p) \geq d(y_n, I_w) + d(I_w, p) - C(\pi/2 - \alpha_0/4)$. Thus, we proved that $d(y_n, (g^{-t}w)_{t\geq 0}) = d(y_n, p) \geq d(y_n, I_w) - C(\pi/2 - \alpha_0/4) \geq d(y_0, x_0) + d(x_n, I_w) - C(\pi/2) - C(\pi/2 - \alpha_0)$.

The same reasoning, replacing $I_w$ by $I_v$, gives $d(y_n, (g^{-t}v)_{t\geq 0}) \geq d(x_0, y_0) + d(x_n, I_v) - C(\alpha_0) - C(3\alpha_0/4)$. It concludes the proof of the lemma. □

Let us now conclude the proof of the theorem. Recall that $y_0 \in \Gamma.o$ belongs to $Hor^+(g^{-D}v) \setminus \mathcal{C}(v, R)$, and that we want to show that for a suitable $n$, $\gamma^n.y_0$ belongs to $Hor^-(g^{-D\pm cste}v) \setminus \mathcal{C}(v, R \pm Cste)$. Recall also that $w$ is the vector on $(h^s v)_{s\in\mathbb{R}}$ such that $(v^- w^+)$ intersects orthogonally $(\gamma^-\gamma^+)$.

Let $y_0'$ and $x_0'$ be respectively the images of $y_0$ and $x_0$ under the symmetry of axis $(v^- w^+)$. As the iterates $\gamma^n.x_0$ are its translates from a distance $l(\gamma) \leq \Lambda$ on the axis $(\gamma^-, \gamma^+)$, there exists $n \geq 1$, such that $d(\gamma^n x_0, x_0') \leq \Lambda$, and $d(\gamma^n x_0, x_0) \geq d(x_0', x_0)$. By symmetry, we have $d(y_0', \partial Hor(v)) = d(y_0, \partial Hor(v))$ and $d(y_0', (g^{-t}v)_{t\geq 0}) = d(y_0, (g^{-t}v)_{t\in\mathbb{R}})$.

Denote $\mathcal{H} = Hor(v)$ and compare first $d(y_0, \partial\mathcal{H}) \geq D$ with $d(y_n, \partial\mathcal{H})$. By symmetry, we have of course $d(y_0', \partial\mathcal{H}) \geq D$. As $x_n = \gamma^n.x_0$ and $x_0'$ are at distance at most $\Lambda$, we deduce that $|d(x_n, \partial\mathcal{H}) - d(x_0', \partial\mathcal{H})| \leq \Lambda$. We will try to bound by below $d(y_n, \partial\mathcal{H}) - d(y_0', \partial\mathcal{H})$ by a uniform constant. It will imply $d(y_n, \partial\mathcal{H}) \geq D - cste$ for $D$ large enough.

Denote respectively by $q_n$ and $q'$ the projections of $y_n = \gamma^n.x_0$ and $y_0'$ on $\partial\mathcal{H}$. If $d(y_n, \partial\mathcal{H}) \geq d(y_0', \partial\mathcal{H}) \geq D$, it is perfect. Assume now that $d(y_n, \partial\mathcal{H}) \leq d(y_0', \partial\mathcal{H})$, and let us prove that this distance cannot be too small compared to $D$.

The triangle $(y_n, x_n, v^-)$ has an angle greater than $\pi/2$ at $x_n$ (because $x_n$ is the projection of $y_n$ on $(\gamma^-, \gamma^+)$ which intersects $[y_n, v^-)$). By lemma 4.1, we have

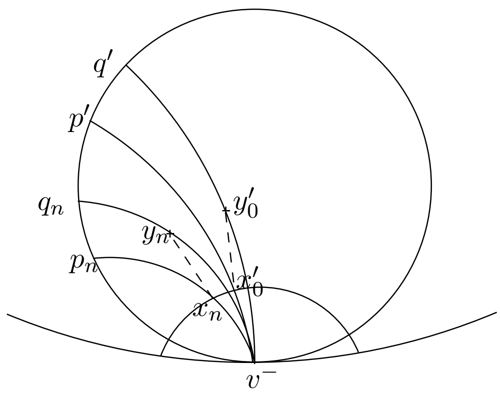


Figure 5: Proof of theorem 1.2

$d(x_n, [y_n, v^-)) \leq C(\pi/2)$. If we consider the triangle $(q_n, x_n, v^-)$, we deduce in particular that the distance from $y_n$ to the side $[q_n, x_n]$ satisfies $d(y_n, [q_n, x_n]) \leq C(\pi/2)$. We assumed that the distance $d(y_n, x_n) = d(y_n, (\gamma^-, \gamma^+))$ is large compared to $k(C(\pi/2))$. Lemma 4.2 gives therefore $d(x_n, y_n) + d(y_n, q_n) - C(k(C(\pi/2))) \leq d(x_n, q_n) \leq d(x_n, y_n) + d(y_n, q_n)$. The same reasoning applies if we replace $x_n, y_n, q_n$ by $x_0', y_0', q'$. We deduce therefore (using the fact that $d(x_n, y_n) = d(x_0', y_0')$) that

$$d(y_n, \partial\mathcal{H}) - d(y_0', \partial\mathcal{H}) = d(y_n, q_n) - d(y_0', q') \geq d(x_n, q_n) - d(x_0', q') - C(k(C(\pi/2)))\,.$$

On the other hand, we know that $|d(x_n, \partial\mathcal{H}) - d(x_0', \partial\mathcal{H})| \leq \Lambda$. We have therefore to compare $d(x_n, q_n)$ with $d(x_n, \partial\mathcal{H})$ on the one hand, and $d(x_0', q')$ with $d(x_0', \partial\mathcal{H})$ on the other hand. We clearly have $d(x_n, q_n) \geq d(x_n, \partial\mathcal{H})$. Let us show that $d(x_0', q)$ is close to $d(x_0', \partial\mathcal{H})$.

Apply lemma 4.3 in the triangle $(p', q', v^-)$. With the notations of this lemma, the point $x_0'$ lies on $(v^-, p']$. If $x_0' \in [\beta, v^-)$, we deduce easily that $|d(x_0', p') - d(x_0', q)| \leq \delta$. If $x_0'$ were on $]\beta, p']$, there vould exist a point of $(\gamma^-\gamma^+)$ inside the triangle $(\alpha, \beta, \gamma)$ closer from $y_0'$ than from $x_0'$, which would contradict the definition of $x_0'$.

All the above imply that

$$|d(x_0', \partial\mathcal{H}) - d(x_0', q')| = |d(x_0', p') - d(x_0', q')| \leq \delta\,.$$

Putting all these inequalities together, we proved that

$$\begin{aligned} d(y_n, \partial\mathcal{H}) - d(y_0', \partial\mathcal{H}) &\geq d(x_n, \partial\mathcal{H}) - d(x_0', \partial\mathcal{H}) - \delta - C(k(C(\pi/2))) \\ &\geq -\Lambda - \delta - C(k(C(\pi/2)))\,. \end{aligned}$$

In particular, we deduce the desired result, that is that for all $D > 0$ large enough, if $d(y_0, \partial\mathcal{H}) \geq D$, then $d(y_n, \partial\mathcal{H}) \geq D - \Lambda - \delta - C(k(C(\pi/2)))$.

We still have to understand the distance from $y_n = \gamma^n.y_0$ to the geodesic $(g^{-t}v)_{t\geq 0}$. We assumed that $d(y_0, (g^{-t}v)_{t\geq 0}) \geq R$. If $y_0'$ is larger than $y_0$ from this geodesic, we have $d(y_0', (g^{-t}v)_{t\in\mathbb{R}}) \geq R$. In the other case, it means that $y_0'$ is closer to $(g^{-t}v)_{t\geq 0}$ than to $(g^{-t}w)_{t\geq 0}$. By symmetry and by lemma 4.5, we have

$$\begin{aligned} d(y_0', (g^{-t}w)_{t\geq 0}) &= d(y_0, (g^{-t}w)_{t\geq 0}) \\ &\geq d(y_0, (g^{-t}v)_{t\geq 0}) - C(\alpha_0) - C_2(\alpha_0) \geq R - C(\alpha_0) - C_2(\alpha_0). \end{aligned}$$

By lemma 4.5, $d(y_n, (g^{-t}v)_{t\geq 0}) \geq d(y_n, (g^{-t}w)_{t\geq 0}) - C(\alpha_0) - C_2(\alpha_0)$.

By definition of $y'_0$ and $y_n$, $y_n$ is farther than $y'_0$ from $(g^{-t}w)_{t\in\mathbb{R}}$. We deduce from what precedes that $d(y_n, (g^{-t}v)_{t\geq 0}) \geq R - 2C(\alpha_0) - 2C_2(\alpha_0)$.

In other words, we associated to the point $y_0$ of $\Gamma.o \cap Hor^+(g^{-D}v) \setminus \mathcal{C}(v,R)$ another point $\gamma^n y_0$ which belongs to $Hor^-(g^{-D+\Lambda+\delta+C(k(C(\pi/2)))}v) \setminus \mathcal{C}(v, R-2C(\alpha_0)-2C_2(\alpha_0))$.

As $v$ was assumed to be right horocyclic, we can choose successively for $k = 1, 2, \ldots$ infinitely many such points $y_0^{(k)}$ in $\Gamma.o \cap Hor^+(g^{-D}v) \setminus \mathcal{C}(v,R)$ and construct for each of them another point in $\Gamma.o \cap Hor^-(g^{-D+const}v) \setminus \mathcal{C}(v, R - const')$. We can choose a sequence of points $(y_0^{(k)})_{k\in\mathbb{N}}$ of the right side in such a manner that their "images" on the left are pairwise distinct (for example by letting the distance from $y_0^{(k)}$ to the boundary $\partial Hor(v)$ go to infinity with $k$).

In particular, as $v$ is right horocyclic, it proves that it is also necessarily left horocyclic. In view of proposition 3.3, it implies that under the assumptions of theorem 1.2, $(h^s v)_{s\geq 0}$ and $(h^s v)_{s\leq 0}$ are simultaneously dense in $\mathcal{E}$, and concludes the proof of theorem 1.2. □

# 5 A counterexample

We prove the following result:

**Theorem 1.3** *There exists hyperbolic surfaces whose unit tangent bundle contains vectors $v$ such that $(h^s v)_{s\geq 0}$ is dense in $\mathcal{E}$, $(h^s v)_{s\leq 0}$ is not dense in $\mathcal{E}$, $v^-$ is not the endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, and $(g^{-t}v)_{t\geq 0}$ intersects infinitely many closed geodesics with length going to infinity, with an angle which can, depending on the surface, go to 0, or be uniformly bounded from below.*

The idea of the construction is as follows. Fix $v^- = \infty$, $v^+ = 0$, and study the orbit of $o = i$.

Choose on $\mathbb{R}_+$ half-circles $C_n^+$, so that $C_n^+$ is tangent to $C_{n\pm 1}^+$, their euclidean radius is bounded, let say equal to 1, and they are centered at $2n+1$, $n \geq 0$, $n \to +\infty$. Choose on $\mathbb{R}_-$ half-circles $C_n^-$, centered at $-x_n$, so that $C_n^+$ is tangent to $C_{n\pm 1}^+$, of radius $R_n \to +\infty$. By an immediate induction, we get $x_1 = R_1$, and $x_n = \sum_{k=0}^{n-1} 2R_k + R_n$. Choose hyperbolic isometries $\gamma_n$ of translation length going to $+\infty$, of fixed points $\gamma_n^- = 2n+1$ and $\gamma_n^+ = x_n$, which send $C_n^+$ onto $C_n^-$.

A classical ping pong argument gives the following lemma.

**Lemma 5.1** *The $(\gamma_n)_{n\in\mathbb{N}}$ generate a discrete, free (Schottky) group with infinitely many generators.*

**Remark 5.2** It is unclear whether the above group satisfies $\Lambda_\Gamma = S^1$. If it is not the case, it would be interesting to construct another counterexample satisfying $\Lambda_\Gamma = S^1$.

Now, it is easy to see that the vector $v$ based at $o = i$, with $v^- = \infty$, $v^+ = 0$, cannot be left horospherical ([3]) because the orbit of $o = i$ does not intersect $Hor^-(v)$ at all.

It is also clear that if $\xi \in \Lambda_\Gamma$, $\xi \neq v^\pm$, $\gamma_n \xi$ converges to $v^-$ on the right when $n \to +\infty$, and $\gamma_n^{-1}\xi$ converges to $v^-$ on the left. In other words, $v^-$ is not in the boundary of $S^1 \setminus \Lambda_\Gamma$.

[3]the terminology *left or right horocyclic* seems to be absurd on this example, but it is convenient for all points $v^-$ on the real axis on the boundary at infinity.

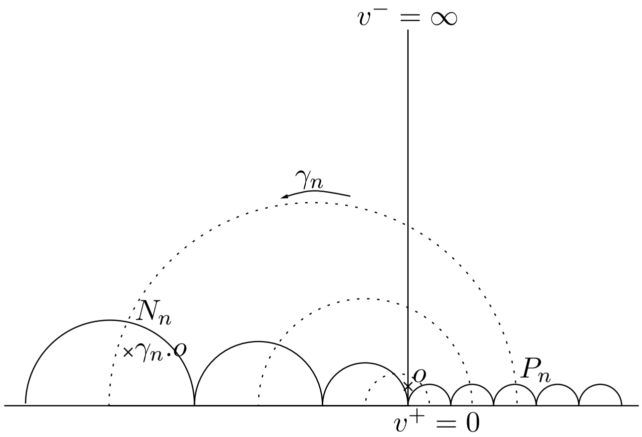


Figure 6: Proof of theorem 1.3

We just have to check that for a suitable choice of $x_n$ and $r_n$, $v^-$ is right horocyclic. To prove it, show that the vertical coordinate of $\gamma_n.o$ goes to $+\infty$ (the horizontal coordinate goes to $-\infty$ by construction.)

Denote by $z_n$ the point of coordinates $(2n+1, 1)$, and $P_n$ the intersection point of the axis of $\gamma_n$ and the circle $C_n^+$, that is the intersection of both half circles of equations $(x-(2n+1))^2 + y^2 = 1$ and $\left(x - \left(\frac{2n+1-x_n}{2}\right)\right)^2 + y^2 = \left(\frac{2n+1+x_n}{2}\right)^2$. Its coordinates are $\left(2n+1-\frac{1}{x_n+2n+1}, \sqrt{1-\frac{1}{(x_n+2n+1)^2}}\right)$, so that the hyperbolic distance from $P_n$ to $z_n$ satisfies $d(P_n, z_n) \to 0$ when $n \to \infty$, and for $n$ large enough $n$ (independently of the choice of $x_n$), $d(P_n, z_n) \leq 1$.

Besides, a classical computation gives $d(o, z_n) \sim 2\ln n$ when $n \to +\infty$. Therefore, we have $d(o, P_n) \leq 3\ln n$ when $n \to +\infty$.

The image of $P_n$ under $\gamma_n$ is the intersection point $N_n = \gamma_n.P_n$ of the half-circle $C_n^-$ and the axis of $\gamma_n$, whose equations are $(x+x_n)^2+y^2 = r_n^2$ and $(x-\frac{2n+1-x_n}{2})^2 + y^2 = \left(\frac{2n+1+x_n}{2}\right)^2$. The point $N_n$ has therefore its coordinates equal to

$$\left(-x_n + \frac{r_n^2}{2n+1+x_n}, r_n\sqrt{1-\frac{r_n^2}{(2n+1+x_n)^2}}\right).$$

Now, we know that the distance from $\gamma_n.o$ to $N_n$ is at most $3\ln n$ for $n$ large. We wish that $\gamma_n.o$ have an imaginary part as large as possible. But $\mathrm{Im}(\gamma_n.o) \geq r_n\sqrt{1-\frac{r_n^2}{(2n+1+x_n)^2}} - 3\ln n$.

Observe that $x_n + r_n = 2\sum_{k=1}^n r_k$. If we choose for all $k \geq 1$, $r_k = k$, we obtain $x_n + n = n(n+1)$, so that $x_n = n^2$, and $r_n\sqrt{1-\frac{r_n^2}{(2n+1+x_n)^2}} = n\sqrt{1-\frac{n^2}{(n+1)^4}} \sim n$ when $n \to \infty$. In other words, this choice of $(r_n)_{n\geq 1}$ is convenient.

If we choose for all $k \geq 1$, $r_k = \alpha^k$, $\alpha > 1$, we get $x_n + \alpha^n = 2\alpha(\alpha^n - 1)/(\alpha - 1)$, so that $x_n = \alpha^n\frac{\alpha+1}{\alpha-1} - \frac{2\alpha^{1-n}}{\alpha-1}$. Denote by $y_n$ the vertical coordinate of $N_n$. An immediate verification gives $y_n^2 \sim \alpha^{2n}\frac{4\alpha}{(\alpha+1)^2} >> (3\ln n)^2$. Therefore, $\mathrm{Im}(\gamma_n.o) \to +\infty$ when $n \to \infty$, and this choice of $(r_n)_{n\in\mathbb{N}}$ is also convenient.

**Remark 5.3** It is clear by construction that on the unit tangent bundle of the quotient surface $S$, the geodesic $(g^{-t}v)_{t\geq 0}$ intersects infinitely many closed geodesics, of length going to $+\infty$, which are the projections on the surface of the axis of $\gamma_n$.

A computation shows that the angle $\theta_n$ between the geodesic $(v^- v^+) = i\mathbb{R}$ and the axis of $\gamma_n$ satisfies $\cos\theta_n = \frac{x_n-(2n+1)}{x_n+2n+1}$. In both examples above, we have $2n+1 = o(x_n)$, whereas $\cos\theta_n \to 1$ and $\theta_n \to 0$.

In other words, the above example satisfies none of the two assumptions of theorem 1.2.

We can modify it so that it satisfies one of the two assumptions. The reader will easily check that if we keep the circles $C_n^-$ unchanged, centered at $-x_n$, of radius $R_n \to +\infty$, but the circles $C_n^+$ are now centered at $+x_n$, and still of radius 1 (they are not tangent anymore one to another), then the geodesic $(v^- v^+) = i\mathbb{R}$ intersects the axis $(-x_n, x_n)$ of $\gamma_n$ orthogonally. The distance from $o$ to $P_n$ is equivalent to $2\ln x_n$, so that $d(\gamma_n.o, N_n) \sim 2\ln x_n$, whereas the vertical coordinate of the point $N_n$ equals $y_n = r_n\sqrt{1-\frac{r_n^2}{4x_n^2}}$. If $R_n = n$ et $x_n = n^2$, the imaginary part of $\gamma_n.o$ goes to $+\infty$, so that the point $+\infty$ is still right horocyclic but not left horocyclic.

Besides, one can also add isometries that send circles of bounded height one to another, to “fill the gaps” between the half-circles $C_n^+$. It is still not sure that it would give a limit set $\Lambda_\Gamma = S^1$, because of the unbounded radius of the circles $C_n^-$.

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