Université de Rennes 1
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Master 2

Théorie Ergodique et Systèmes Dynamiques
Exercise sheet 4

1 - (Weak mixing) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Show that the following are equivalent:
(i) T is weakly mixing.
(ii) For every ergodic measure preserving system $(Y, \mathcal{C}, \nu, S)$, the product system $(X \times Y, \mathcal{B} \times \mathcal{C}, \mu \times \nu, T \times S)$ is ergodic.

2 - (Weak mixing, again) Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving system. Show that the following properties are equivalent:
(i) T is weakly mixing
(ii) For any $A, B, C \in \mathcal{B}$ with positive measure, there exists $n \geq 1$ such that $T^{-n}A \cap B \neq \emptyset$ and $T^{-n}A \cap C \neq \emptyset$.

3 - (Strong mixing of torus automorphisms) For $d \geq 1$, let $X = \mathbf{R}^d/\mathbf{Z}^d$ be the d-dimensional torus, equipped with the Lebesgue measure λ. For $A \in GL_d(\mathbf{Z})$, let $T_A : X \to X$ the associated automorphism of X.
(i) Assume that T_A is ergodic. Show that T_A is strongly mixing. (*Hint:* Fourier analysis).
(ii) Let $x \in \mathbf{Q}^d/\mathbf{Z}^d$. Show that x is T_A-periodic.
(iii) Show that T_A is not uniquely ergodic.

4 – (Birkhoff sums and unique ergodicity) Let $X = \mathbf{S}^1 \times [0, 1]$. For $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, let $T : X \to X$ be defined by $T(x, t) = (x + \alpha, t)$.
(i) Show that, for every $f \in C(X)$, the Birkhoff sums

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n$$

converge uniformly on X.
(ii) Show that T is not uniquely ergodic.

5 -(Tent map and logistic map) Let $S : [0,1] \to [0,1]$ be the "tent" map defined by

$$Sx = \begin{cases} 2x & \text{if } x \in [0,1/2] \\ 2-2x & \text{if } x \in [1/2,1]. \end{cases}$$

Let λ be the Lebesgue measure on $[0,1]$.
(i) Check that λ is S-invariant.
(ii) Let $E_2 : [0,1] \to [0,1], x \mapsto \{2x\}$. Show that $([0,1], E_2, \lambda)$ is strongly mixing.
(iii) Show that $S^{n+1} = S \circ E_2^n$ for every $n \geq 1$.
(iv) Show that the probability measure preserving system $([0,1], S, \lambda)$ is strongly mixing.

Cnsider now the "logistic" map

$$T : [0,1] \to [0,1], \qquad x \mapsto 4x(1-x).$$

(v) Check that, for every $\theta \in \mathbf{R}$ and $n \in \mathbf{N}^*$, we have $T^n(\sin^2\theta) = \sin^2(2^n\theta)$.
(vi) For $n \geq 1$, determine the points $x \in [0,1]$ of period n for T. Deduce that the T-periodic points are dense in $[0,1]$.
(vii) Let $\varphi : [0,1] \to [0,1]$ be the bijection defined by $\varphi(x) = \sin^2(\pi x/2)$. Check that $T \circ \varphi = \varphi \circ S$.

Let μ be the probability measure defined on the Borel subsets of $[0,1]$ by

$$\mu(A) = \frac{1}{\pi} \int_A \frac{dx}{\sqrt{x(1-x)}}$$

(viii) Show that μ is T-invariant.
(ix) Show that the probability measure preserving system $([0,1], T, \mu)$ is ergodic.
(x) Let $\alpha \in [0,1]$. Show that, for λ-almost every $x \in [0,1]$, we have

$$\lim_{N\to+\infty} \frac{1}{N} \mathrm{Card}\{n : \ 1 \leq n \leq N, T^n x \leq \alpha\} = \frac{2}{\pi} \mathrm{Arcsin}(\sqrt{\alpha}).$$