1 - (Weak mixing) Let $(X, B, \mu, T)$ be a measure preserving system. Show that the following are equivalent:

(i) $T$ is weakly mixing.
(ii) For every ergodic measure preserving system $(Y, C, \nu, S)$, the product system $(X \times Y, B \times C, \mu \times \nu, T \times S)$ is ergodic.

2 - (Weak mixing, again) Let $(X, B, \mu, T)$ be a probability measure preserving system. Show that the following properties are equivalent:

(i) $T$ is weakly mixing
(ii) For any $A, B, C \in B$ with positive measure, there exists $n \geq 1$ such that $T^{-n}A \cap B \neq \emptyset$ and $T^{-n}A \cap C \neq \emptyset$.

3 - (Strong mixing of torus automorphisms) For $d \geq 1$, let $X = \mathbb{R}^d/\mathbb{Z}^d$ be the $d$-dimensional torus, equipped with the Lebesgue measure $\lambda$. For $A \in GL_d(\mathbb{Z})$, let $T_A : X \rightarrow X$ the associated automorphism of $X$.

(i) Assume that $T_A$ is ergodic. Show that $T_A$ is strongly mixing. (Hint: Fourier analysis).
(ii) Let $x \in \mathbb{Q}^d/\mathbb{Z}^d$. Show that $x$ is $T_A$-periodic.
(iii) Show that $T_A$ is not uniquely ergodic.

4 - (Birkhoff sums and unique ergodicity) Let $X = \mathbb{S}^1 \times [0, 1]$. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $T : X \rightarrow X$ be defined by $T(x, t) = (x + \alpha, t)$.

(i) Show that, for every $f \in C(X)$, the Birkhoff sums

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n$$

converge uniformly on $X$.
(ii) Show that $T$ is not uniquely ergodic.
5 - (Tent map and logistic map) Let $S : [0, 1] \to [0, 1]$ be the “tent” map defined by

$$Sx = \begin{cases} 
2x & \text{if } x \in [0, 1/2] \\
2 - 2x & \text{if } x \in [1/2, 1]. 
\end{cases}$$

Let $\lambda$ be the Lebesgue measure on $[0, 1]$.

(i) Check that $\lambda$ is $S$-invariant.

(ii) Let $E_2 : [0, 1] \to [0, 1], x \mapsto \{2x\}$. Show that $([0, 1], E_2, \lambda)$ is strongly mixing.

(iii) Show that $S^{n+1} = S \circ E_2^n$ for every $n \geq 1$.

(iv) Show that the probability measure preserving system $([0, 1], S, \lambda)$ is strongly mixing.

Consider now the “logistic” map

$$T : [0, 1] \to [0, 1], \quad x \mapsto 4x(1 - x).$$

(v) Check that, for every $\theta \in \mathbb{R}$ and $n \in \mathbb{N}^*$, we have $T^n(\sin^2 \theta) = \sin^2(2^n \theta)$.

(vi) For $n \geq 1$, determine the points $x \in [0, 1]$ of period $n$ for $T$. Deduce that the $T$-periodic points are dense in $[0, 1]$.

(vii) Let $\varphi : [0, 1] \to [0, 1]$ be the bijection defined by $\varphi(x) = \sin^2(\pi x/2)$. Check that $T \circ \varphi = \varphi \circ S$.

Let $\mu$ be the probability measure defined on the Borel subsets of $[0, 1]$ by

$$\mu(A) = \frac{1}{\pi} \int_A \frac{dx}{\sqrt{x(1-x)}}$$

(viii) Show that $\mu$ is $T$-invariant.

(ix) Show that the probability measure preserving system $([0, 1], T, \mu)$ is ergodic.

(x) Let $\alpha \in [0, 1]$. Show that, for $\lambda$-almost every $x \in [0, 1]$, we have

$$\lim_{N \to +\infty} \frac{1}{N} \text{Card}\{n : 1 \leq n \leq N, T^n x \leq \alpha\} = \frac{2}{\pi} \arcsin(\sqrt{\alpha}).$$