1 - (Shift and $\times 2$-map) Let $X = S^1$ be equipped with the Lebesgue measure $\lambda$ and $T : X \to X$ defined by $Tx = \{2x\}$. Let $(\Sigma^+, \mu, \sigma)$ be the one-sided Bernoulli shift, where $\Sigma^+ = \{0, 1\}^\mathbb{N}$ is equipped with the measure $\mu = \nu^\otimes \mathbb{N}$ defined on the $\sigma$-algebra generated by the cylinders, $\nu$ is the probability $(1/2, 1/2)$ on $\{0, 1\}$ and $\sigma$ is the shift $\sigma((a_i)_{i \in \mathbb{N}}) = (a_{i+1})_{i \in \mathbb{N}}$.

(i) Show that the systems $(X, \lambda, T)$ and $(\Sigma^+, \mu, \sigma)$ are isomorphic

(ii) Determine the periodic points of $\sigma$ and $T$.

2 - (Shift and Baker transformation) Let $X = S^1 \times S^1$, equipped with the Lebesgue measure $\lambda$. Let $T : X \to X$ be the baker transformation. Let $\Sigma = \{0, 1\}^\mathbb{Z}$ and $\varphi : \Sigma \to X$ the map defined by $\varphi(a) = (x, y)$ with

$$x = \sum_{i=0}^{\infty} \frac{a_{-i}}{2^{i+1}}, \quad y = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$$

for $a = (a_i)_{i \in \mathbb{Z}} \in \Sigma$.

Show that $\varphi$ is an isomorphism between the two-sided Bernoulli shift $(\Sigma, \mu, \sigma)$ and $(X, \lambda, T)$, where $\Sigma$ is equipped with the measure $\mu = \nu^\otimes \mathbb{Z}$ defined on the $\sigma$-algebra generated by the cylinders with $\nu$ the probability $(1/2, 1/2)$ on $\{0, 1\}$ and $\sigma$ is the shift $\sigma((a_i)_{i \in \mathbb{Z}}) = (a_{i+1})_{i \in \mathbb{Z}}$.

3 - (Gauß map) Let $T : [0, 1] \to [0, 1]$ be the Gauß map defined by

$$T(x) = \begin{cases} \{\frac{1}{x}\} = \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

($\{y\}$ and $[y]$ denote the fractional and integer part of $y \in \mathbb{R}$, respectively).

Denote by $\lambda$ the Lebesgue measure $[0, 1]$ and let $\mu$ be the probability measure on $[0, 1]$ with density $\frac{1}{\log 2(1+x)}$ with respect to $\lambda$.

(i) Show that $T^{-1}(]a, b[) = \bigcup_{n=1}^{\infty} \frac{1}{n+b} - \frac{1}{n+a} [ \text{ for every } 0 \leq a < b \leq 1.$
(ii) Show that the Lebesgue measure \( \lambda \) is not \( T \)-invariant (Hint: you may consider \( T^{-1}[0,1/2[) \)).

(iii) Show that \( \mu(T^{-1}([a,b[)) = \mu([a,b[) \) for every \( 0 \leq a < b \leq 1 \) and deduce that \( \mu \) is \( T \)-invariant.

4 - (Skew products) Let \( (X,\mathcal{B},\mu,T) \) be a probability measure preserving dynamical system. Let \( \rho : X \to S^1 \) be a measurable map. Let \( \tilde{X} = X \times S^1 \), \( \tilde{\mu} = \mu \otimes \lambda \) and \( \tilde{T} : \tilde{X} \to \tilde{X} \) given by \( \tilde{T}(x,y) = (Tx, y + \rho(x)) \).

(i) Show that \( \tilde{T} \) preserves \( \tilde{\mu} \).

(ii) Show that \( (\tilde{X},\tilde{T}) \) is an extension of \( (X,T) \) (that is, \( (X,T) \) is a factor of \( (\tilde{X},\tilde{T}) \)).

(iii) Let \( X = S^1 \), \( T = R_{\alpha} \) for \( \alpha \) irrational and \( \rho(x) = x \). Show that \( (\tilde{X},\tilde{T}) \) is ergodic.

5 - (Ergodic theorem for finite systems) Let \( X = \{x_1,\ldots,x_r\} \) be a finite set and \( \sigma : X \to X \) a permutation of \( X \). Let \( f : X \to \mathbb{C} \) and \( x \in X \).

(i) Assume that \( \sigma \) is a cycle. Show that \( \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n(x)) = \frac{1}{r} (f(x_1) + \cdots + f(x_r)) \).

(ii) Let \( \sigma \) be arbitrary. Determine the limit \( \lim_{N \to +\infty} \sum_{n=0}^{N-1} f(\sigma^n(x)) \).

6 - (L\(^p\) ergodic theorem) Let \( (X,\mathcal{B},\mu,T) \) be a probability measure preserving dynamical system. Let \( p \in [1,\infty[ \). For \( f \in L^p(X,\mu) \) and \( N \in \mathbb{N}^* \), let \( f_N^+ \in L^p(X,\mu) \) be given by \( f_N^+(x) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^nx) \).

(i) Show that, for every \( f \in L^p(X,\mu) \), there exists a \( T \)-invariant function \( f^* \in L^p(X,\mu) \) such that \( \lim_{N \to +\infty} \|f_N^+ - f^*\|_p = 0 \).

[Hint: For \( p < 2 \), use the density of \( L^2(X,\mu) \) in \( L^p(X,\mu) \) and for \( p > 2 \) the density of \( L^\infty(X,\mu) \) in \( L^p(X,\mu) \).]

(ii) Does (i) remains true in the case \( p = +\infty \)?

7 - (Torus translation) For \( d \geq 1 \), let \( X = S^1 \times \cdots \times S^1 \) be the torus of dimension \( d \), equipped with the Lebesgue measure \( \lambda \). Let \( \alpha_1,\ldots,\alpha_d \in \mathbb{R} \). Define \( T : X \to X \) by

\[
T(x_1,\ldots,x_d) = (x_1 + \alpha_1,\ldots,x_d + \alpha_d) \mod 1.
\]

Show that \( T \) is ergodic if and only if \( 1,\alpha_1,\ldots,\alpha_d \) are linearly independent over \( \mathbb{Q} \).