Université de Rennes 1
Année 2020/2021

Master 2

Théorie Ergodique et Systèmes Dynamiques
Exercise sheet 2

1 - (Shift and $\times 2$-map) Let $X = S^1$ be equipped with the Lebesgue measure λ and $T : X \to X$ defined by $Tx = \{2x\}$. Let (Σ^+, μ, σ) be the one-sided Bernoulli shift, where $\Sigma^+ = \{0,1\}^{\mathbf{N}}$ is equipped with the measure $\mu = \nu^{\otimes \mathbf{N}}$ defined on the σ-algebra generated by the cylinders, ν is the probability $(1/2, 1/2)$ on $\{0,1\}$ and σ is the shift $\sigma((a_i)_{i\in\mathbf{N}}) = (a_{i+1})_{i\in\mathbf{N}}$.
(i) Show that the systems (X, λ, T) and (Σ^+, μ, σ) are isomorphic
(ii) Determine the periodic points of σ and T.

2 - (Shift and Baker transformation) Let $X = S^1 \times S^1$, equipped with the Lebesgue measure λ. Let $T : X \to X$ be the baker transformation. Let $\Sigma = \{0,1\}^{\mathbf{Z}}$ and $\varphi : \Sigma \to X$ the map defined by $\varphi(a) = (x, y)$ with

$$x = \sum_{i=0}^{\infty} \frac{a_{-i}}{2^{i+1}}, \qquad y = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$$

for $a = (a_i)_{i\in\mathbf{Z}} \in \Sigma$.

Show that φ is an isomorphism between the two-sided Bernoulli shift (Σ, μ, S) and (X, λ, T), where Σ is equipped with the measure $\mu = \nu^{\otimes \mathbf{Z}}$ defined on the σ-algebra generated by the cylinders with ν the probability $(1/2, 1/2)$ on $\{0,1\}$ and σ is the shift $\sigma((a_i)_{i\in\mathbf{Z}}) = (a_{i+1})_{i\in\mathbf{Z}}$.

3 - (Gauß map) Let $T : [0,1] \to [0,1]$ be the Gauß map defined by

$$T(x) = \begin{cases} \left\{\dfrac{1}{x}\right\} = \dfrac{1}{x} - \left[\dfrac{1}{x}\right] & \text{if } x \neq 0 \\ 0 \quad \text{if } x = 0. \end{cases}$$

($\{y\}$ and $[y]$ denote the fractional and integer part of $y \in \mathbf{R}$, respectively).

Denote by λ the Lebesgue measure $[0,1]$ and let μ be the probability measure on $[0,1]$ with density $\dfrac{1}{\log 2}\dfrac{1}{1+x}$ with respect to λ.

(i) Show that $T^{-1}(]a,b[) = \bigcup_{n=1}^{\infty}]\frac{1}{n+b}, \frac{1}{n+a}[$ for every $0 \le a < b \le 1$.

(ii) Show that the Lebesgue measure λ is not T-invariant (Hint: you may consider $T^{-1}]0,1/2[$).
(iii) Show that $\mu(T^{-1}(]a,b[)) = \mu(]a,b[)$ for every $0 \le a < b \le 1$ and deduce that μ is T-invariant.

4 - (Skew products) Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving dynamical system. Let $\rho : X \to S^1$ be a measurable map. Let $\widetilde{X} = X \times S^1$, $\widetilde{\mu} = \mu \otimes \lambda$ and $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ given by $\widetilde{T}(x,y) = (Tx, y + \rho(x))$.
(i) Show that $\widetilde{T}$ preserves $\widetilde{\mu}$.
(ii) Show that $(\widetilde{X}, \widetilde{T})$ is an extension of (X,T) (that is, (X,T) is a factor of $(\widetilde{X}, \widetilde{T})$).
(iii) Let $X = S^1$, $T = R_\alpha$ for α irrational and $\rho(x) = x$. Show that $(\widetilde{X}, \widetilde{T})$ is ergodic.

5 - (Ergodic theorem for finite systems) Let $X = \{x_1, \ldots, x_r\}$ be a finite set and $\sigma : X \to X$ a permutation of X. Let $f : X \to \mathbf{C}$ and $x \in X$.
(i) Assume that σ is a cycle. Show that

$$\lim_N \sum_{n=0}^{N-1} f(\sigma^n(x)) = \frac{1}{r}(f(x_1) + \cdots + f(x_r)).$$

(ii) Let σ be arbitrary. Determine the limit $\lim_N \sum_{n=0}^{N-1} f(\sigma^n(x))$.

6 - (L^p ergodic theorem) Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving dynamical system. Let $p \in [1, \infty[$. For $f \in L^p(X,\mu)$ and $N \in \mathbf{N}^*$, let $f_N^+ \in L^p(X,\mu)$ be given by $f_N^+(x) = \dfrac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$.
(i) Show that, for every $f \in L^p(X,\mu)$, there exists a T-invariant function $f^* \in L^p(X,\mu)$ such that $\lim_{N \to +\infty} \|f_N^+ - f^*\|_p = 0$.
[*Hint:* For $p < 2$, use the density of $L^2(X,\mu)$ in $L^p(X,\mu)$ and for $p > 2$ the density of $L^\infty(X,\mu)$ in $L^p(X,\mu)$.]
(ii) Does (i) remains true in the case $p = +\infty$?

7 - (Torus translation) For $d \ge 1$, let $X = S^1 \times \cdots \times S^1$ be the torus of dimension d, equipped with the Lebesgue measure λ. Let $\alpha_1, \ldots, \alpha_d \in \mathbf{R}$. Define $T : X \to X$ by

$$T(x_1, \ldots, x_d) = (x_1 + \alpha_1, \ldots, x_d + \alpha_d) \mod 1.$$

Show that T is ergodic if and only if $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbf{Q}$.