THE PLANCHEREL FORMULA FOR COUNTABLE GROUPS

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Abstract. We discuss a Plancherel formula for countable groups, which provides a canonical decomposition of the regular representation of such a group Γ into a direct integral of factor representations. Our main result gives a precise description of this decomposition in terms of the Plancherel formula of the FC-center $\Gamma_{\rm fc}$ of Γ (that is, the normal sugbroup of Γ consisting of elements with a finite conjugacy class); this description involves the action of an appropriate totally disconnected compact group of automorphisms of $\Gamma_{\rm fc}$. As an application, we determine the Plancherel formula for linear groups. In an appendix, we use the Plancherel formula to provide a unified proof for Thoma's and Kaniuth's theorems which respectively characterize countable groups which are of type I and those whose regular representation is of type II.

1. Introduction

Given a second countable locally compact group G, a fundamental object to study is its **unitary dual space** $\widehat{G}$, that is, the set irreducible unitary representations of G up to unitary equivalence. The space $\widehat{G}$ carries a natural Borel structure, called the Mackey Borel structure (see [Mac57, §6] or [Dix77, §18.6]). A classification of $\widehat{G}$ is considered as being possible only if $\widehat{G}$ is a standard Borel space; according to Glimm's celebrated theorem ([Gli61]), this is the case if and only if G is of type I in the following sense.

Recall that a von Neumann algebra is a self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed for the weak operator topology of $\mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. A von Neumann algebra is a factor if its center only consists of the scalar operators.

Let π be a unitary representation of G in a Hilbert space $\mathcal{H}$ (as we will only consider representations which are unitary, we will often drop the adjective "unitary"). The von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$

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generated by $\pi(G)$ coincides with the bicommutant $\pi(G)''$ of $\pi(G)$ in $\mathcal{L}(\mathcal{H})$; we say that π is a factor representation if $\pi(G)''$ is a factor.

Definition 1. The group G is of **type I** if, for every factor representation π of G, the factor $\pi(G)''$ is of type I, that is, $\pi(G)''$ is isomorphic to the von Neumann algebra $\mathcal{L}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$; equivalently, the Hilbert space $\mathcal{H}$ of π can written as tensor product $\mathcal{K} \otimes \mathcal{K}'$ of Hilbert spaces in such a way that π is equivalent to $\sigma \otimes I_{\mathcal{K}'}$ for an irreducible representation σ of G on $\mathcal{K}$.

Important classes of groups are known to be of type I, such as semisimple or nilpotent Lie groups. A major problem in harmonic analysis is to decompose the left regular representation λ_G on $L^2(G, \mu_G)$ for a Haar measure μ_G as a direct integral of irreducible representations. When G is of type I and unimodular, this is the content of the classical Plancherel theorem: there exist a unique measure μ on $\widehat{G}$ and a unitary isomorphism between $L^2(G, \mu_G)$ and the direct integral of Hilbert spaces $\int_{\widehat{G}}^{\oplus} (\mathcal{H}_\pi \otimes \overline{\mathcal{H}_\pi}) d\mu(\pi)$ which transforms λ_G into $\int_{\widehat{G}}^{\oplus} (\pi \otimes I_{\overline{\mathcal{H}_\pi}}) d\mu(x)$, where $\overline{\mathcal{H}_\pi}$ is the conjugate of the Hilbert space $\mathcal{H}_\pi$ of π; in particular, we have a Plancherel formula

$$\|f\|_2^2 = \int_{\widehat{G}} \mathrm{Tr}(\pi(f)^* \pi(f)) d\mu(\pi) \qquad \text{for all} \quad f \in L^1(G, \mu_G) \cap L^2(G, \mu_G),$$

where $\|f\|_2$ is the L^2-norm of f, $\pi(f)$ is the value at f of the natural extension of π to a representation of $L^1(G, \mu_G)$, and Tr denotes the standard trace on $\mathcal{L}(\mathcal{H}_\pi)$; for all this, see [Dix77, 18.8.1].

When G is not type I, λ_G usually admits several integral decompositions into irreducible representations and it is not possible to single out a canonical one among them. However, when G is unimodular, λ_G does admit a canonical integral decomposition into factor representations representations; this the content of a Plancherel theorem which we will discuss in the case of a discrete group (see Theorem A).

Let Γ be a countable group. As discussed below (see Theorem E), Γ is usually not of type I. In order to state the Plancherel theorem for Γ, we need to replace the dual space $\widehat{\Gamma}$ by the consideration of Thoma's dual space $\mathrm{Ch}(\Gamma)$ which we now introduce.

Recall that a function $t : \Gamma \to \mathbf{C}$ is of positive type if the complex-valued matrix $(t(\gamma_j^{-1}\gamma_i))_{1 \leq i,j \leq n}$ is positive semi-definite for any $\gamma_1, \ldots, \gamma_n$ in Γ

A function of positive type t on Γ which is constant on conjugacy classes and normalized (that is, $t(e) = 1$) will be called a **trace** on G. The set of traces on Γ will be denoted by $\mathrm{Tr}(\Gamma)$.

Let $t \in \mathrm{Tr}(\Gamma)$ and $(\pi_t, \mathcal{H}_t, \xi_t)$ be the associated GNS triple (see [BHV08, C.4]). Then $\tau_t : \pi(\Gamma)'' \to \mathbf{C}$, defined by $\tau_t(T) = \langle T\xi_t \mid \xi_t \rangle$ is a trace on the von Neumann algebra $\pi_t(\Gamma)''$, that is, $\tau_t(T^*T) \geq 0$ and $\tau_t(TS) = \tau_t(ST)$ for all $T, S \in \pi_t(\Gamma)''$; moreover, τ_t is faithful in the sense that $\tau_t(T^*T) > 0$ for every $T \in \pi_t(\Gamma)'', T \neq 0$. Observe that $\tau_t(\pi(f)) = t(f)$ for $f \in \mathbf{C}[\Gamma]$, where t denotes the linear extension of t to the group algebra $\mathbf{C}[\Gamma]$.

The set $\mathrm{Tr}(\Gamma)$ is a convex subset of the unit ball of $\ell^\infty(\Gamma)$ which is compact in the topology of pointwise convergence. An extreme point of $\mathrm{Tr}(\Gamma)$ is called a **character** of Γ; we will refer to $\mathrm{Ch}(\Gamma)$ as **Thoma's dual space**.

Since Γ is countable, $\mathrm{Tr}(\Gamma)$ is a compact metrizable space and $\mathrm{Ch}(\Gamma)$ is easily seen to be a G_δ subset of $\mathrm{Tr}(\Gamma)$. So, in contrast to $\widehat{\Gamma}$, Thoma's dual space $\mathrm{Ch}(\Gamma)$ is always a standard Borel space.

An important fact is that $\mathrm{Tr}(\Gamma)$ is a simplex (see [Tho64, Satz 1] or [Sak71, 3.1.18]); by Choquet theory, this implies that every $\tau \in \mathrm{Tr}(\Gamma)$ can be represented as integral $\tau = \int_{\mathrm{Ch}(\Gamma)} t d\mu(t)$ for a *unique* probability measure μ on $\mathrm{Ch}(\Gamma)$.

As we now explain, the set of characters of Γ parametrizes the factor representations of finite type of Γ, up to quasi-equivalence; for more details, see [Dix77, §17.3] or [BH, §11.C].

Recall first that two representations π_1 and π_2 of Γ are quasi-equivalent if there exists an isomorphism $\Phi : \pi_1(\Gamma)'' \to \pi_2(\Gamma)''$ of von Neumann algebras such that $\Phi(\pi_1(\gamma)) = \pi_2(\gamma)$ for every $\gamma \in \Gamma$.

Let $t \in \mathrm{Ch}(\Gamma)$ and π_t the associated GNS representation. Then $\pi_t(\Gamma)''$ is a factor of finite type. Conversely, let π be a representation of Γ such that $\pi(\Gamma)''$ is a factor of finite type and let τ be the unique normalized trace on $\pi(\Gamma)''$. Then $t := \tau \circ \pi$ belongs to $\mathrm{Ch}(\Gamma)$ and only depends on the quasi-equivalence class $[\pi]$ of π.

The map $t \to [\pi_t]$ is a bijection between $\mathrm{Ch}(\Gamma)$ and the set of quasi-equivalence classes of factor representations of finite type of Γ.

The following result is a version for countable groups of a Plancherel theorem due to Mautner [Mau50] and Segal [Seg50] which holds more generally for any unimodular second countable locally compact group; its proof is easier in our setting and will be given in Section 3 for the convenience of the reader.

Theorem A. ***(Plancherel theorem for countable groups)*** *Let* Γ *be a countable group. There exists a probability measure* μ *on* $\mathrm{Ch}(\Gamma)$*, a measurable field of representations* $t \mapsto (\pi_t, \mathcal{H}_t)$ *of* Γ *on the standard Borel space* $\mathrm{Ch}(\Gamma)$*, and an isomorphism of Hilbert spaces between* $\ell^2(\Gamma)$

and $\int_{\mathrm{Ch}(\Gamma)}^{\oplus} \mathcal{H}_t d\mu(t)$ *which transforms* λ_Γ *into* $\int_{\mathrm{Ch}(\Gamma)}^{\oplus} \pi_t d\mu(t)$ *and has the following properties:*

(i) π_t *is quasi-equivalent to the GNS representation associated to* t*; in particular, the* π_t*'s are mutually disjoint factor representations of finite type, for* μ*-almost every* $t \in \mathrm{Ch}(\Gamma)$*;*

(ii) *the von Neumann algebra* $L(\Gamma) := \lambda_\Gamma(\Gamma)''$ *is mapped* onto *the direct integral* $\int_{\mathrm{Ch}(\Gamma)}^{\oplus} \pi_t(\Gamma)'' d\nu(t)$ *of factors;*

(iii) *for every* $f \in \mathbf{C}[\Gamma]$*, the following Plancherel formula holds:*

$$\|f\|_2^2 = \int_{\mathrm{Ch}(\Gamma)} \tau_t(\pi_t(f)^* \pi_t(f)) d\mu(t) = \int_{\mathrm{Ch}(\Gamma)} t(f^* * f) d\mu(t).$$

The measure μ *is the unique probability measure on* $\mathrm{Ch}(\Gamma)$ *such that the Plancherel formula above holds.*

The probability measure μ on $\mathrm{Ch}(\Gamma)$ from Theorem A is called the **Plancherel measure** of Γ.

Remark 2. The Plancherel measure gives rise to what seems to be an interesting dynamical system on $\mathrm{Ch}(\Gamma)$ involving the group $\mathrm{Aut}(\Gamma)$ of automorphisms of Γ. We will equip $\mathrm{Aut}(\Gamma)$ with the topology of pointwise convergence on Γ, for which it is a totally disconnected topological group. The natural action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Ch}(\Gamma)$, given by $t^g(\gamma) = t(g^{-1}(\gamma))$ for $g \in \mathrm{Aut}(\Gamma)$ and $t \in \mathrm{Ch}(\Gamma)$, is clearly continuous.

Since the induced action of $\mathrm{Aut}(\Gamma)$ on $\ell^2(\Gamma)$ is isometric, the following fact is an immediate consequence of the uniqueness of the Plancherel measure μ of Γ :

the action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Ch}(\Gamma)$ preserves μ.

For example, when $\Gamma = \mathbf{Z}^d$, Thoma's dual $\mathrm{Ch}(\Gamma)$ is the torus $\mathbf{T}^d$, the Plancherel measure μ is the normalized Lebesgue measure on $\mathbf{T}^d$ which is indeed preserved by the group $\mathrm{Aut}(\mathbf{Z}^d) = \mathrm{GL}_d(\mathbf{Z})$. Dynamical systems of the form $(\Lambda, \mathrm{Ch}(\Gamma), \mu)$ for a subgroup Λ of $\mathrm{Aut}(\Gamma)$ may be viewed as generalizations of this example.

We denote by Γ_{fc} the FC-centre of Γ, that is, the normal subgroup of elements in Γ with a finite conjugacy class. It turns out (see Remark 6) that $t = 0$ on $\Gamma \setminus \Gamma_{\mathrm{fc}}$ for μ-almost every $t \in \mathrm{Ch}(\Gamma)$. In particular, when Γ is ICC, that is, when $\Gamma_{\mathrm{fc}} = \{e\}$, the regular representation λ_Γ is factorial (see also Corollary 5) so that the Plancherel formula is vacuous in this case.

In fact, as we now see, the Plancherel measure of Γ is entirely determined by the Plancherel measure of Γ_{fc}. Roughly speaking, we will see that the Plancherel measure of Γ is the image of the Plancherel measure of $\Gamma_{\rm fc}$ under the quotient map $\mathrm{Ch}(\Gamma_{\rm fc}) \to \mathrm{Ch}(\Gamma_{\rm fc})/K_\Gamma$, for a compact group K_Γ which we now define.

Let K_Γ be the closure in $\mathrm{Aut}(\Gamma_{\rm fc})$ of the subgroup $\mathrm{Ad}(\Gamma)|_{\Gamma_{\rm fc}}$ given by conjugation with elements from Γ. Since every Γ-conjugation class in $\Gamma_{\rm fc}$ is finite, K_Γ is a compact group. By a general fact about actions of compact groups on Borel spaces (see Corollary 2.1.21 and Appendix in [Zim84]), the quotient space $\mathrm{Ch}(\Gamma_{\rm fc})/K_\Gamma$ is a standard Borel space.

Given a function $t : H \to \mathbf{C}$ of positive type on a subgroup H of a group Γ, we denote by $\widetilde{t}$ the extension of t to Γ given by $\widetilde{t} = 0$ outside H. Observe that $\widetilde{t}$ is of positive type on Γ (see for instance [BH, 1.F.10]).

Here is our main result.

Theorem B. ***(Plancherel measure: reduction to the FC-center)*** *Let Γ be a countable group. Let ν be the Plancherel measure of $\Gamma_{\rm fc}$ and $\lambda_{\Gamma_{\rm fc}} = \int_{\mathrm{Ch}(\Gamma_{\rm fc})}^{\oplus} \pi_t d\nu(t)$ the integral decomposition of the regular representation of $\Gamma_{\rm fc}$ as in Theorem A. Let $\dot{\nu}$ be the image of ν under the quotient map $\mathrm{Ch}(\Gamma_{\rm fc}) \to \mathrm{Ch}(\Gamma_{\rm fc})/K_\Gamma$.*

(i) *For every K_Γ-orbit $\mathcal{O}$ in $\mathrm{Ch}(\Gamma_{\rm fc})$, let $m_\mathcal{O}$ be the unique normalized K_Γ-invariant probability measure on $\mathcal{O}$ and let $\pi_\mathcal{O} := \int_\mathcal{O}^\oplus \pi_t m_\mathcal{O}(t)$. Then the induced representation $\widetilde{\pi_\mathcal{O}} := \mathrm{Ind}_{\Gamma_{\rm fc}}^{\Gamma} \pi_\mathcal{O}$ is factorial for $\dot{\nu}$-almost every $\mathcal{O}$ and we have a direct integral decomposition of the von Neumann algebra $L(\Gamma)$ into factors*

$$L(\Gamma) = \int_{\mathrm{Ch}(\Gamma_{\rm fc})/K_\Gamma}^{\oplus} \widetilde{\pi_\mathcal{O}}(\Gamma)'' d\dot{\nu}(\mathcal{O}).$$

(ii) *The Plancherel measure of Γ is the image $\Phi_*(\nu)$ of ν under the map*

$$\Phi : \mathrm{Ch}(\Gamma_{\rm fc}) \to \mathrm{Tr}(\Gamma), \qquad t \mapsto \int_{K_\Gamma} \widetilde{t^g} dm(g),$$

where m is the normalized Haar measure on K_Γ.

It is worth mentioning that the *support* of μ was determined in [Tho67]. For an expression of the map Φ as in Theorem B.ii without reference to the group K_Γ, see Remark 7.

As we next see, the Plancherel measure on $\Gamma_{\rm fc}$ can be explicitly described in the case of a linear group Γ. We first need to discuss the Plancherel formula for a so-called **central group**, that is, a central extension of a finite group.

Let Λ be a central group. Then Λ is of type I (see Theorem E). In fact, $\widehat{\Lambda}$ can be described as follows; let $r : \widehat{\Lambda} \to \widehat{Z(\Lambda)}$ be the restriction map, where $Z(\Lambda)$ is the center of Λ. Then, for $\chi \in \widehat{Z(\Lambda)}$, every

$\pi \in r^{-1}(\chi)$ is equivalent to a subrepresentation of the finite dimensional representation $\mathrm{Ind}_{Z(\Lambda)}^{\Lambda} \chi$, by a generalized Frobenius reciprocity theorem (see [Mac52, Theorem 8.2]); in particular, $r^{-1}(\chi)$ is finite. The Plancherel measure ν on $\mathrm{Ch}(\Lambda)$ is, in principle, easy to determine: we identify every $\pi \in \widehat{\Lambda}$ with its normalized character given by $x \mapsto \dfrac{1}{\dim \pi} \mathrm{Tr}\, \pi(x)$; for every Borel subset A of $\mathrm{Ch}(\Lambda)$, we have

$$\nu(A) = \int_{\widehat{Z(\Lambda)}} \frac{\#(A \cap r^{-1}(\chi))}{\sum_{\pi \in r^{-1}(\chi)} (\dim \pi)^2} d\chi,$$

where $d\chi$ is the normalized Haar measure on the abelian compact group $\widehat{Z(\Lambda)}$.

Corollary C. ***(The Plancherel measure for linear groups)*** *Let Γ be a countable linear group.*

(i) Γ_{fc} *is a central group;*
(ii) K_Γ *coincides with* $\mathrm{Ad}(\Gamma)|_{\Gamma_{\mathrm{fc}}}$ *and is a finite group;*
(iii) *the Plancherel measure of Γ is the image of the Plancherel measure of Γ_{fc} under the map $\Phi : \mathrm{Ch}(\Gamma_{\mathrm{fc}}) \to \mathrm{Tr}(\Gamma)$ given by*

$$\Phi(t) = \frac{1}{\#\mathrm{Ad}(\Gamma)|_{\Gamma_{\mathrm{fc}}}} \sum_{s \in \mathrm{Ad}(\Gamma)|_{\Gamma_{\mathrm{fc}}}} t^s.$$

When the Zariski closure of the linear group Γ is connected, the Plancherel measure of Γ has a particularly simple form.

Corollary D. ***(The Plancherel measure for linear groups-bis)*** *Let $\mathbf{G}$ be a* connected *linear algebraic group over a field $\mathbf{k}$ and let Γ be a countable Zariski dense subgroup of $\mathbf{G}$. The Plancherel measure of Γ is the image of the normalized Haar measure $d\chi$ on $\widehat{Z(\Gamma)}$ under the map*

$$\widehat{Z(\Gamma)} \to \mathrm{Tr}(\Gamma), \qquad \chi \mapsto \widetilde{\chi}$$

and the Plancherel formula is given for every $f \in \mathbf{C}[\Gamma]$ by

$$\|f\|_2^2 = \int_{\widehat{Z(\Gamma)}} \mathcal{F}((f^* * f)|_{Z(\Gamma)})(\chi) d\chi,$$

where $\mathcal{F}$ is the Fourier transform on the abelian group $Z(\Gamma)$.
The previous conclusion holds in the following two cases:

(i) $\mathbf{k}$ *is a countable field of characteristic* 0 *and $\Gamma = \mathbf{G}(\mathbf{k})$ is the group of $\mathbf{k}$-rational points in $\mathbf{G}$;*
(ii) $\mathbf{k}$ *is a local field (that is, a non discrete locally compact field), $\mathbf{G}$ has no proper $\mathbf{k}$-subgroup $\mathbf{H}$ such that $(\mathbf{G}/\mathbf{H})(\mathbf{k})$ is compact, and Γ is a lattice in $\mathbf{G}(\mathbf{k})$.*

Corollary D generalizes the Plancherel theorem obtained in [CPJ94, Theorem 4] for $\Gamma = \mathbf{G}(\mathbf{Q})$ and in [PJ95, Theorem 3.6] for $\Gamma = \mathbf{G}(\mathbf{Z})$, in the case where $\mathbf{G}$ is a unipotent linear algebraic group over $\mathbf{Q}$; indeed, $\mathbf{G}$ is connected (since the exponential map identifies $\mathbf{G}$ with its Lie algebra, as affine varieties) and these two results follow from (i) and (ii) respectively.

In an appendix to this article, we use Theorem A to give a unified proof of Thoma's and Kaniuth's results ([Tho64],[Tho68], [Kan69]) as stated in the following theorem. For a group Γ, we denote by $[\Gamma, \Gamma]$ its commutator subgroup. Recall that Γ is said to be virtually abelian if it contains an abelian subgroup of finite index.

The regular representation λ_Γ is of type I (or type II) if the von Neumann algebra $L(\Gamma)$ is of type I (or type II); equivalently (see Corollaire 2 in [Dix69, Chap. II, § 3, 5]), if $\pi_t(\Gamma)''$ is a finite dimensional factor (or a factor of type II) for μ-almost every $t \in \mathrm{Ch}(\Gamma)$ in the Plancherel decomposition $\lambda_\Gamma = \int_{\mathrm{Ch}(\Gamma)}^{\oplus} \pi_t d\mu(t)$ from Theorem A.

Theorem E (**Thoma**, **Kaniuth**). *Let Γ be a countable group. The following properties are equivalent:*

- (i) Γ *is type I;*
- (ii) Γ *is virtually abelian;*
- (iii) *the regular representation λ_Γ is of type I;*
- (iv) *every irreducible unitary representation of Γ is finite dimensional;*
- (iv') *there exists an integer $n \geq 1$ such that every irreducible unitary representation of Γ has dimension $\leq n$.*

Moreover, the following properties are equivalent:

- (v) λ_Γ *is of type II;*
- (vi) *either $[\Gamma : \Gamma_{\mathrm{fc}}] = \infty$ or $[\Gamma : \Gamma_{\mathrm{fc}}] < \infty$ and $[\Gamma, \Gamma]$ is infinite.*

Our proof of Theorem E is not completely new as it uses several crucial ideas from [Tho64] and especially from [Kan69] (compare with the remarks on p.336 after Lemma in [Kan69]); however, we felt it could be useful to have a short and common treatment of both results in the literature. Observe that the equivalence between (i) and (iii) above does not carry over to non discrete groups (see [Mac61]).

Remark 3. Theorem E holds also for non countable discrete groups. Write such a group Γ as $\Gamma = \cup_j H_j$ for a directed net of countable subgroups H_j. If $L(\Gamma)''$ is not of type II (or is of type I), then $L(H_j)$ is not of type II (or is of type I) for j large enough. This is the crucial tool for the extension of the proof of Theorem E to Γ; for more details, proofs of Satz 1 and Satz 2 in [Kan69].

This paper is organized as follows. In Section 2, we recall the well-known description of the center of the von Neumann algebra of a discrete group. Sections 3 and 4 contains the proofs of Theorems A and B. In Section 5, we prove Corollaries C and D. Section 6 is devoted to the explicit computation of the Plancherel formula for a few examples of countable groups. Appendix A contains the proof of Theorem E.
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2. On the center of the group von Neumann algebra

Let Γ be a countable group. We will often use the following well-known description of the center $\mathcal{Z} = \lambda(\Gamma)'' \cap \lambda(\Gamma)'$ of $L(\Gamma) = \lambda_\Gamma(\Gamma)''$.

Observe that $\lambda_\Gamma(H)''$ is a von Neumann subalgebra of $L(\Gamma)$, for every subgroup H of Γ. For $h \in \Gamma_{\mathrm{fc}}$, we set

$$T_{[h]} := \lambda_\Gamma(\mathbf{1}_{[h]}) = \sum_{x\in[h]} \lambda_\Gamma(x) \in \lambda_\Gamma(\Gamma_{\mathrm{fc}})'',$$

where $[h]$ denotes the Γ-conjugacy class of h.

Lemma 4. *The center $\mathcal{Z}$ of $L(\Gamma) = \lambda_\Gamma(\Gamma)''$ coincides with the closure of the linear span of $\{T_{[h]} \mid h \in \Gamma_{\mathrm{fc}}\}$, for the strong operator topology; in particular, $\mathcal{Z}$ is contained in $\lambda_\Gamma(\Gamma_{\mathrm{fc}})''$.*

Proof. It is clear that $T_{[h]} \in \mathcal{Z}$ for every $h \in \Gamma_{\mathrm{fc}}$. Observe also that the linear span of $\{T_{[h]} \mid h \in \Gamma_{\mathrm{fc}}\}$ is a unital selfadjoint algebra; indeed, $T_{[h^{-1}]} = T^*_{[h]}$ for every $h \in \Gamma_{\mathrm{fc}}$ and $\{\mathbf{1}_{[h]} \mid h \in \Gamma_{\mathrm{fc}}\}$ is a vector space basis of the algebra $\mathbf{C}[\Gamma_{\mathrm{fc}}]^\Gamma$ of Γ-invariant functions in $\mathbf{C}[\Gamma_{\mathrm{fc}}]$.

Let $T \in \mathcal{Z}$. We have to show that $T \in \{T_{[h]} \mid h \in \Gamma_{\mathrm{fc}}\}''$. For every $\gamma \in \Gamma$, we have

$$\lambda_\Gamma(\gamma)\rho_\Gamma(\gamma)(T\delta_e) \;=\; (\lambda_\Gamma(\gamma)\rho_\Gamma(\gamma)T)\delta_e \;=\; (T\lambda_\Gamma(\gamma)\rho_\Gamma(\gamma))\delta_e \;=\; T\delta_e.$$

and this shows that $f := T\delta_e$, which is a function in $\ell^2(\Gamma)$, is invariant under conjugation by γ. The support of f is therefore contained in Γ_{fc}.

Write $f = \sum_{[h]\in\mathcal{C}} c_{[h]}\mathbf{1}_{[h]}$ for a sequence $(c_{[h]})_{[h]\in\mathcal{C}}$ of complex numbers with $\sum_{[h]\in\mathcal{C}} \#[h]|c_{[h]}|^2 < \infty$, where $\mathcal{C}$ is a set of representatives for the Γ-conjugacy classes in Γ_{fc}. Let ρ_Γ be the right regular representation of Γ. Since $T \in \lambda_\Gamma(\Gamma)''$ and $\rho_\Gamma(\Gamma) \subset \lambda_\Gamma(\Gamma)'$, we have, for every $x \in \Gamma$,

$$T(\delta_x) = T\rho_\Gamma(x)(\delta_e) = \rho_\Gamma(x)(f) = \rho_\Gamma(x)\left(\sum_{[h]\in\mathcal{C}} c_{[h]}\mathbf{1}_{[h]}\right) = \sum_{[h]\in\mathcal{C}} c_{[h]}T_{[h]}(\delta_x),$$

where the last sum is convergent in $\ell^2(\Gamma)$. We also have $T^*(\delta_x) = \sum_{[h]\in\mathcal{C}} \overline{c_{[h]}} T_{[h^{-1}]}(\delta_x)$.

Let $S \in \{T_{[h]} \mid h \in \Gamma_{\mathrm{fc}}\}'$. For every $x, y \in \Gamma$, we have

$$\begin{aligned}\langle ST(\delta_x) \mid \delta_y\rangle &= \langle S \left(\sum_{[h]\in\mathcal{C}} c_{[h]} T_{[h]}(\delta_x)\right) \mid \delta_y\rangle = \langle \sum_{[h]\in\mathcal{C}} c_{[h]} S T_{[h]}(\delta_x) \mid \delta_y\rangle \\ &= \sum_{[h]\in\mathcal{C}} c_{[h]} \langle T_{[h]} S(\delta_x) \mid \delta_y\rangle = \langle S(\delta_x) \mid \sum_{[h]\in\mathcal{C}} \overline{c_{[h]}} T_{[h^{-1}]}(\delta_y)\rangle \\ &= \langle S(\delta_x) \mid T^*(\delta_y)\rangle = \langle TS(\delta_x) \mid \delta_y\rangle\end{aligned}$$

and it follows that $ST = TS$. □

The following well-known corollary shows that the Plancherel measure is the Dirac measure at δ_e in the case where Γ is ICC group, that is, when $\Gamma_{\mathrm{fc}} = \{e\}$.

Corollary 5. *Assume that Γ is ICC. Then $L(\Gamma) = \lambda_\Gamma(\Gamma)''$ is a factor.*

3. Proof of Theorem A

Consider a direct integral decomposition $\int_X^\oplus \pi_x d\mu(x)$ of λ_Γ associated to the centre $\mathcal{Z}$ of $L(\Gamma) = \lambda_\Gamma(\Gamma)''$ (see [8.3.2][Dix77]); so, X is a standard Borel space equipped with a probability measure μ and $(\pi_x, \mathcal{H}_x)_{x\in X}$ is measurable field of representations of Γ over X, such that there exists an isomorphism of Hilbert spaces

$$U : \ell^2(\Gamma) \to \int_X^\oplus \mathcal{H}_x d\mu(x)$$

which transforms λ_Γ into $\int_X^\oplus \pi_x d\mu(x)$ and for which $U\mathcal{Z}U^{-1}$ is the algebra of diagonal operators on $\int_X^\oplus \mathcal{H}_x d\mu(x)$. (Recall that a diagonal operator on $\int_X^\oplus \mathcal{H}_x d\mu(x)$ is an operator of the form $\int_X^\oplus \varphi(x) I_{\mathcal{H}_x} d\mu(x)$ for an essentially bounded measurable function $\varphi : X \to \mathbf{C}$.)

Then, upon disregarding a subset of X of μ-measure 0, the following holds (see [8.4.1][Dix77]):

(1) π_x is a factor representation for every $x \in X$;
(2) π_x and π_y are disjoint for every $x, y \in X$ with $x \neq y$;
(3) we have $U\lambda_\Gamma(\Gamma)''U^{-1} = \int_X^\oplus \pi_x(\Gamma)'' d\mu(x)$.

Let ρ_Γ be the right regular representation of Γ. Let $\gamma \in \Gamma$. Then $U\rho_\Gamma(\gamma)U^{-1}$ commutes with every diagonalisable operator on $\int_X^\oplus \mathcal{H}_x d\mu(x)$, since $\rho_\Gamma(\gamma) \in L(\Gamma)'$. It follows (see [Dix69, Chap. II, §2, No 5, Théorème 1]) that $U\rho_\Gamma(\gamma)U^{-1}$ is a decomposable operator, that is, there exists a measurable field of unitary operators $x \mapsto \sigma_x(\gamma)$ such that

$U\rho_\Gamma(\gamma)U^{-1} = \int_X^\oplus \sigma_x(\gamma)d\mu(x)$. So, we have a measurable field $x \mapsto \sigma_x$ of representations of Γ in $\int_X^\oplus \mathcal{H}_x d\mu(x)$ such that

$$U\rho_\Gamma(\gamma)U^{-1} = \int_X^\oplus \sigma_x(\gamma)d\mu(x) \quad \text{for all } \gamma \in \Gamma.$$

Let $(\xi_x)_{x\in X} \in \int_X^\oplus \mathcal{H}_x d\mu(x)$ be the image of $\delta_e \in \ell^2(\Gamma)$ under U. We claim that ξ_x is a cyclic vector for π_x and σ_x, for μ-almost every $x \in X$. Indeed, since $\delta_e \in \ell^2(\Gamma)$ is a cyclic vector for both λ_Γ and ρ_Γ,

$$\{(\pi_x(\gamma)\xi_x)_{x\in X} \mid \gamma \in \Gamma\} \quad \text{and} \quad \{(\sigma_x(\gamma)\xi_x)_{x\in X} \mid \gamma \in \Gamma\}$$

are countable total subsets of $\int_X^\oplus \mathcal{H}_x d\mu(x)$ and the claim follows from a general fact about direct integral of Hilbert spaces (see Proposition 8 in Chap. II, §1 of [Dix69]).

Since $\lambda_\Gamma(\gamma)\delta_e = \rho_\Gamma(\gamma^{-1})\delta_e$ for every $\gamma \in \Gamma$ and since Γ is countable, upon neglecting a subset of X of μ-measure 0, we can assume that

(4) $\pi_x(\gamma)\xi_x = \sigma_x(\gamma^{-1})\xi_x$;
(5) $\pi_x(\gamma)\sigma_x(\gamma') = \sigma_x(\gamma')\pi_x(\gamma)$;
(6) ξ_x is a cyclic vector for both π_x and σ_x,

for all $x \in X$ and all $\gamma, \gamma' \in \Gamma$.

Let $x \in X$ and let φ_x be the function of positive type on Γ defined by

$$\varphi_x(\gamma) = \langle \pi_x(\gamma)\xi_x \mid \xi_x\rangle \quad \text{for every} \quad \gamma \in \Gamma.$$

We claim that $\varphi_x \in \mathrm{Ch}(\Gamma)$. Indeed, using (4) and (5), we have, for every $\gamma_1, \gamma_2 \in \Gamma$,

$$\begin{aligned}\varphi_x(\gamma_2\gamma_1\gamma_2^{-1}) &= \langle \pi_x(\gamma_2\gamma_1\gamma_2^{-1})\xi_x \mid \xi_x\rangle = \langle \pi_x(\gamma_2\gamma_1)\sigma_x(\gamma_2)\xi_x \mid \xi_x\rangle \\ &= \langle \sigma_x(\gamma_2)\pi_x(\gamma_2\gamma_1)\xi_x \mid \xi_x\rangle = \langle \pi_x(\gamma_1)\xi_x \mid \pi_x(\gamma_2^{-1})\sigma_x(\gamma_2^{-1})\xi_x\rangle \\ &= \langle \pi_x(\gamma_1)\xi_x \mid \xi_x\rangle = \varphi_x(\gamma_1).\end{aligned}$$

So, φ_x is conjugation invariant and hence $\varphi_x \in \mathrm{Tr}(\Gamma)$. Moreover, φ_x is an extreme point in $\mathrm{Tr}(\Gamma)$, since π_x is factorial and ξ_x is a cyclic vector for π_x.

Finally, since $U \colon \ell^2(\Gamma) \to \int_X^\oplus \mathcal{H}_x d\mu(x)$ is an isometry, we have for every $f \in \mathbf{C}[\Gamma]$,

$$\begin{aligned}\|f\|^2 &= f^* * f(e) = \langle \lambda_\Gamma(f^* * f)\delta_e \mid \delta_e\rangle \\ &= \|\lambda_\Gamma(f)\delta_e\|^2 = \|U(\lambda_\Gamma(f)\delta_e)\|^2 \\ &= \int_X \|\pi_x(f)\xi_x\|^2 d\mu(x) = \int_X \varphi_x(f^* * f)d\mu(x).\end{aligned}$$

The measurable map $\Phi : X \to \mathrm{Ch}(\Gamma)$ given by $\Phi(x) = \varphi_x$ is injective, since π_x and π_y are disjoint by (2) and hence $\varphi_x \neq \varphi_y$ for $x, y \in X$ with

$x \neq y$. It follows that $\Phi(X)$ is a Borel subset of $\mathrm{Ch}(\Gamma)$ and that Φ is a Borel isomorphism between X and $\Phi(X)$ (see [Mac57, Theorem 3.2]). Pushing forward μ to $\mathrm{Ch}(\Gamma)$ by Φ, we can therefore assume without loss of generality that $X = \mathrm{Ch}(\Gamma)$ and that μ is a probability measure on $\mathrm{Ch}(\Gamma)$. With this identification, it is clear that Items (i), (ii) and (iii) of Theorem A are satisfied and that the Plancherel formula holds.

It remains to show the uniqueness of μ. Let ν any probability measure on $\mathrm{Ch}(\Gamma)$ such that the Plancherel formula. By polarization, we have then $\delta_e = \int_{\mathrm{Ch}(\Gamma)} t d\nu(t)$, which is an integral decomposition of $\delta_e \in \mathrm{Tr}(\Gamma)$ over extreme points of the convex set $\mathrm{Tr}(\Gamma)$. The uniqueness of such a decomposition implies that $\nu = \mu$.

Remark 6. (i) For μ-almost every $t \in \mathrm{Ch}(\Gamma)$, we have $t = 0$ on $\Gamma \setminus \Gamma_{\mathrm{fc}}$. Indeed, let $\gamma \notin \Gamma_{\mathrm{fc}}$. Then $\langle \lambda_\Gamma(\gamma)\lambda_\Gamma(h)\delta_e \mid \delta_e\rangle = 0$ for every $h \in \Gamma_{\mathrm{fc}}$ and hence

$$(*) \qquad \langle \lambda_\Gamma(\gamma) T \delta_e \mid \delta_e \rangle = 0 \qquad \text{for all} \quad T \in \lambda_\Gamma(\Gamma_{\mathrm{fc}})''.$$

With the notation as in the proof above, let E be a Borel subset of X. Then $T_E := U^{-1} P_E U$ is a projection in $\mathcal{Z}$, where P_E is the diagonal operator $\int_X^{\oplus} \mathbf{1}_E(x) I_{\mathcal{H}_x} d\mu(x)$. It follows from Lemma 4 and $(*)$ that

$$\int_E \varphi_x(\gamma) d\mu(x) = \langle T_E \lambda_\Gamma(\gamma)\delta_e \mid \delta_e\rangle = \langle \lambda_\Gamma(\gamma) T_E \delta_e \mid \delta_e \rangle = 0.$$

Since this holds for every Borel subset E of X, this implies that $\varphi_x(\gamma) = 0$ for μ-almost every $x \in X$.

As Γ is countable, for μ-almost every $x \in X$, we have $\varphi_x(\gamma) = 0$ for every $\gamma \notin \Gamma_{\mathrm{fc}}$.

(ii) Let $\lambda_\Gamma = \int_{\mathrm{Ch}(\Gamma)}^{\oplus} \pi_t d\mu(t)$, $\rho_\Gamma = \int_{\mathrm{Ch}(\Gamma)}^{\oplus} \sigma_t d\mu(t)$, and $\delta_e = (\xi_t)_{t \in \mathrm{Ch}(\Gamma)}$ be the decompositions as above. For μ-almost every $t \in \mathrm{Ch}(\Gamma)$, the linear map

$$\pi_t(\Gamma)'' \to \mathcal{H}_t,\ T \mapsto T\xi_t$$

is injective. Indeed, this follows from the fact that ξ_t is cyclic for σ_t and that $\sigma_t(\Gamma) \subset \pi_t(\Gamma)'$.

4. Proof of Theorem B

Set $N := \Gamma_{\mathrm{fc}}$ and $X := \mathrm{Ch}(N)$. Consider the direct integral decomposition $\int_X^{\oplus} \pi_t d\nu(t)$ of λ_N into factor representations $(\pi_t, \mathcal{K}_t)$ of N with corresponding traces $t \in X$, as in Theorem A.

Let K_Γ be the compact group which is the closure in $\mathrm{Aut}(N)$ of $\mathrm{Ad}(\Gamma)|_N$. Since the quotient space X/K_Γ is a standard Borel space, there exists a Borel section $s : X/K_\Gamma \to X$ for the projection map $X \to X/K_\Gamma$. Set $\Omega := s(X/K_\Gamma)$.Then Ω is a Borel transversal for

X/K_Γ. The Plancherel measure ν can accordingly be decomposed over Ω : we have

$$\nu(f) = \int_\Omega \int_{\mathcal{O}_\omega} f(t) dm_\omega(t) d\dot{\nu}(\omega)$$

for every bounded measurable function f on $\mathrm{Ch}(\Gamma_{\mathrm{fc}})$, where m_ω be the unique normalized K_Γ-invariant probability measure on the K_Γ-orbit $\mathcal{O}_\omega$ of ω and $\dot{\nu}$ is the image of ν under s.

Let $\omega \in \Omega$ and set

$$\pi_{\mathcal{O}_\omega} := \int_{\mathcal{O}_\omega}^{\oplus} \pi_t dm_\omega(t),$$

which is a unitary representation of N on the Hilbert space

$$\mathcal{K}_\omega := \int_{\mathcal{O}_\omega}^{\oplus} \mathcal{K}_t dm_\omega(t).$$

For $g \in \mathrm{Aut}(N)$, let $\pi_{\mathcal{O}_\omega}^g$ be the conjugate representation of N on $\mathcal{K}_\omega$ given by $\pi_{\mathcal{O}_\omega}^g(h) = \pi_{\mathcal{O}_\omega}(g(h))$ for $h \in N$.

Step 1 There exists a unitary representation $U_\omega : g \mapsto U_{\omega,g}$ of K_Γ on $\mathcal{K}_\omega$ such that

$$U_{\omega,g} \pi_{\mathcal{O}_\omega}(h) U_{\omega,g}^{-1} = \pi_{\mathcal{O}_\omega}(g(h)) \qquad \text{for all} \quad g \in K_\Gamma, h \in N;$$

in particular, $\pi_{\mathcal{O}_\omega}^g$ is equivalent to $\pi_{\mathcal{O}_\omega}$ for every $g \in K_\Gamma$.

Indeed, observe that the representations π_t for $t \in \mathcal{O}_\omega$ are conjugate to each other (up to equivalence) and may therefore be considered as defined on the same Hilbert space.

Let $g \in K_\Gamma$. Then $\pi_{\mathcal{O}_\omega}^g$ is equivalent to $\int_{\mathcal{O}_\omega}^{\oplus} \pi_{t^g} dm_\omega(t)$. Define a linear operator $U_{\omega,g} : \mathcal{K}_\omega \to \mathcal{K}_\omega$ by

$$U_{\omega,g}\left((\xi_t)_{t \in \mathcal{O}_\omega}\right) = (\xi_{t^g})_{t \in \mathcal{O}_\omega} \qquad \text{for all} \quad (\xi_t)_{t \in \mathcal{O}_\omega} \in \mathcal{K}_\omega.$$

Then $U_{\omega,g}$ is an isometry, by K_Γ-invariance of the measure m_ω. It is readily checked that U_ω intertwines $\pi_{\mathcal{O}_\omega}$ and $\pi_{\mathcal{O}_\omega}^g$ and that U_ω is a homomorphism. To show that U_ω is a representation of K_Γ, it remains to prove that $g \mapsto U_{\omega,g}\xi$ is continuous for every $\xi \in \mathcal{K}_\omega$.

For this, observe that $\mathcal{K}_\omega$ can be identified with the Hilbert space $L^2(\mathcal{O}_\omega, m_\omega) \otimes \mathcal{K}$, where $\mathcal{K}$ is the common Hilbert space of the π_t's for $t \in \mathcal{O}_\omega$; under this identification, U_ω corresponds to $\kappa \otimes I_{\mathcal{K}}$, where κ is the Koopman representation of K_Γ on $L^2(\mathcal{O}_\omega, m_\omega)$ associated to the action $K_\Gamma \curvearrowright \mathcal{O}_\omega$ (for the fact that κ is indeed a representation of K_Γ, see [BHV08, A.6]) and the claim follows.

Next, let

$$\widetilde{\pi_{\mathcal{O}_\omega}} := \operatorname{Ind}_N^\Gamma \pi_{\mathcal{O}_\omega}$$

be the representation of Γ induced by $\pi_{\mathcal{O}_\omega}$.

We recall how $\widetilde{\pi_{\mathcal{O}_\omega}}$ can be realized on $\ell^2(R, \mathcal{K}_\omega) = \ell^2(R) \otimes \mathcal{K}_\omega$, where $R \subset \Gamma$ is a set of representatives for the cosets of N with $e \in R$. For every $\gamma \in \Gamma$ and $r \in R$, let $c(r, \gamma) \in N$ and $r \cdot \gamma \in R$ be such that $r\gamma = c(r, \gamma) r \cdot \gamma$. Then $\widetilde{\pi_{\mathcal{O}_\omega}}$ is given on $\ell^2(R, \mathcal{K}_\omega)$ by

$$(\widetilde{\pi_{\mathcal{O}_\omega}}(\gamma)F)(r) \;=\; \pi_{\mathcal{O}_\omega}(c(r,\gamma))(F(r\cdot\gamma)) \qquad \text{for all} \quad F \in \ell^2(R, \mathcal{K}_\omega).$$

Step 2 We claim that there exists a unitary map

$$\widetilde{U_\omega} : \ell^2(R, \mathcal{K}_\omega) \to \ell^2(R, \mathcal{K}_\omega)$$

which intertwines the representation $I_{\ell^2(R)} \otimes \pi_{\mathcal{O}_\omega}$ and the restriction $\widetilde{\pi_{\mathcal{O}_\omega}}|_N$ of $\widetilde{\pi_{\mathcal{O}_\omega}}$ to N; moreover, $\omega \to \widetilde{U_\omega}$ is a measurable field of unitary operators on Ω.

Indeed, we have an orthogonal decomposition

$$\ell^2(R, \mathcal{K}_\omega) = \oplus_{r\in R}(\delta_r \otimes \mathcal{K}_\omega)$$

into $\widetilde{\pi_{\mathcal{O}_\omega}}(N)$-invariant; moreover, the action of N on every copy $\delta_r \otimes \mathcal{K}_\omega$ is given by $\pi^r_{\mathcal{O}_\omega}$. For every $r \in R$, the unitary operator $U_{\omega,r} : \mathcal{K}_\omega \to \mathcal{K}_\omega$ from Step 1 intertwines $\pi_{\mathcal{O}_\omega}$ and $\pi^r_{\mathcal{O}_\omega}$. In view of the explicit formula of $U_{\omega,r}$, the field $\omega \to U_{\omega,r}$ is measurable on Ω.

Define a unitary operator $\widetilde{U_\omega} : \ell^2(R, \mathcal{K}_\omega) \to \ell^2(R, \mathcal{K}_\omega)$ by

$$\widetilde{U_\omega}(\delta_r \otimes \xi) = \delta_r \otimes U_{\omega,r}(\xi) \qquad \text{for all} \quad \xi \in \mathcal{K}_\omega.$$

Then $\widetilde{U_\omega}$ intertwines $I_{\ell^2(R)} \otimes \pi_{\mathcal{O}_\omega}$ and $\widetilde{\pi_{\mathcal{O}_\omega}}|_N$; moreover, $\omega \to \widetilde{U_\omega}$ is a measurable field on Ω.

Observe that the representation λ_N is equivalent to $\int_\Omega^\oplus \pi_{\mathcal{O}_\omega} d\dot\nu(\omega)$. Since λ_Γ is equivalent to $\operatorname{Ind}_N^\Gamma \lambda_N$, it follows that λ_Γ is equivalent to $\int_\Omega^\oplus \widetilde{\pi_{\mathcal{O}_\omega}} d\dot\nu(\omega)$.

In the sequel, we will identify the representations λ_N on $\ell^2(N)$ and λ_Γ on $\ell^2(\Gamma)$ with respectively the representations

$$\int_\Omega^\oplus \pi_{\mathcal{O}_\omega} d\dot\nu(\omega) \quad \text{on} \quad \mathcal{K} := \int_\Omega^\oplus \mathcal{K}_\omega d\dot\nu(\omega)$$

and

$$\int_\Omega^\oplus \widetilde{\pi_{\mathcal{O}_\omega}} d\dot\nu(\omega) \quad \text{on} \quad \mathcal{H} := \int_\Omega^\oplus \ell^2(R, \mathcal{K}_\omega) d\dot\nu(\omega).$$

Step 3 The representations $\widetilde{\pi_{\mathcal{O}_\omega}}$ are factorial and are mutually disjoint, outside a subset of Ω of $\dot{\nu}$-measure 0.

To show this, it suffices to prove (see [Dix77, 8.4.1]) that the algebra $\mathcal{D}$ of diagonal operators in $\mathcal{L}(\mathcal{H})$ coincides with the center $\mathcal{Z}$ of $\lambda_\Gamma(\Gamma)''$.

Let us first prove that $D \subset \mathcal{Z}$. For this, we only have to prove that $\mathcal{D} \subset \lambda_\Gamma(\Gamma)''$, since it is clear that $\mathcal{D} \subset \lambda_\Gamma(\Gamma)'$.

By Step 2, for every $\omega \in \Omega$, there exists a measurable field $\omega \to \widetilde{U_\omega}$ of unitary operators $\widetilde{U_\omega} : \ell^2(R, \mathcal{K}_\omega) \to \ell^2(R, \mathcal{K}_\omega)$ intertwining $I_{\ell^2(R)} \otimes \pi_{\mathcal{O}_\omega}$ and $\pi_{\mathcal{O}_\omega}|_N$. So,

$$\widetilde{U} := \int_\Omega^\oplus \widetilde{U_\omega} d\dot{\nu}(\omega)$$

is a unitary operator on $\mathcal{H}$ which intertwines $I_{\ell^2(R)} \otimes \lambda_N$ and $\lambda_\Gamma|_N$; it is obvious that $\widetilde{U}$ commutes with the diagonal operators on $\mathcal{H}$.

Let $\varphi : \Omega \to \mathbf{C}$ be a measurable essential bounded function on Ω. By Theorem A.ii, the corresponding diagonal operator

$$T = \int_\Omega^\oplus \varphi(\omega) I_{\mathcal{K}_\omega} d\dot{\nu}(\omega)$$

on $\mathcal{K}$ belongs to $\lambda_N(N)''$. For the corresponding diagonal operator

$$\widetilde{T} = \int_\Omega^\oplus \varphi(\omega) I_{\ell^2(R, \mathcal{K}_\omega)} \dot{\nu}(\omega)$$

on $\mathcal{H}$, we have $\widetilde{T} = I_{\ell^2(R)} \otimes T$. So, $\widetilde{T}$ belongs to $(I_{\ell^2(R)} \otimes \lambda_N)(N)''$. Since, $\widetilde{U}$ commutes with $\widetilde{T}$ and intertwines $I_{\ell^2(R)} \otimes \lambda_N$ and $\lambda_\Gamma|_N$, it follows that

$$\widetilde{T} = \widetilde{U}(I_{\ell^2(R)} \otimes T)\widetilde{U}^{-1} \in \lambda_\Gamma(N)'' \subset \lambda_\Gamma(\Gamma)''.$$

So, we have shown that $\mathcal{D} \subset \mathcal{Z}$. Observe that this implies (see Théorème 1 in Chap. II, §3 of [Dix69]) that $L(\Gamma)$ is the direct integral $\int_\Omega^\oplus \widetilde{\pi_{\mathcal{O}_\omega}}(\Gamma)'' d\dot{\nu}(\omega)$ and that $\mathcal{Z}$ is the direct integral $\int_\Omega^\oplus \mathcal{Z}_\omega d\dot{\nu}(\omega)$, where $\mathcal{Z}_\omega$ is the center of $\widetilde{\pi_{\mathcal{O}_\omega}}(\Gamma)''$.

Let $\widetilde{T} \in \mathcal{Z}$. Then $\widetilde{T} \in \lambda_\Gamma(N)''$, by Lemma 4. So, $\widetilde{T} = \int_\Omega^\oplus T_\omega d\dot{\nu}(\omega)$, where T_ω belongs to the center of $\widetilde{\pi_{\mathcal{O}_\omega}}(N)''$, for $\dot{\nu}$-almost every ω. Since $\widetilde{\pi_{\mathcal{O}_\omega}}|_N$ is equivalent to $I_{\ell^2(R)} \otimes \pi_{\mathcal{O}_\omega}$ and since $\pi_{\mathcal{O}_\omega}$ and hence $I_{\ell^2(R)} \otimes \pi_{\mathcal{O}_\omega}$ is a factor representation, it follows that T_ω is a scalar operator, for $\dot{\nu}$-almost every ω. So, $\widetilde{T} \in \mathcal{D}$.

As a result, we have a decomposition

$$\lambda_\Gamma = \int_\Omega^\oplus \widetilde{\pi_{\mathcal{O}_\omega}} d\dot{\nu}(\omega)$$

of λ_Γ as a direct integral of pairwise disjoint factor representations. By the argument of the proof of Theorem A, it follows that, for $\dot{\nu}$-almost every $\omega \in X$, there exists a cyclic unit vector $\xi_\omega \in \ell^2(R, \mathcal{K}_\omega)$ for $\widetilde{\pi_{\mathcal{O}_\omega}}$ so that

$$\varphi_\omega := \langle \widetilde{\pi_{\mathcal{O}_\omega}}(\cdot)\xi_\omega \mid \xi_\omega \rangle$$

belongs to $\mathrm{Ch}(\Gamma)$. In particular,

$$\widetilde{\mathcal{M}_\omega} := \widetilde{\pi_{\mathcal{O}_\omega}}(\Gamma)''$$

is a factor of type II_1 and its normalized trace is the extension of φ_ω to $\widetilde{\mathcal{M}_\omega}$, which we again denote by φ_ω. Our next goal is to determine φ_ω in terms of the character $\omega \in \mathrm{Ch}(N)$.

Fix $\omega \in \Omega$ such that $\widetilde{\mathcal{M}_\omega}$ is a factor. We identity the Hilbert space $\mathcal{K}_\omega$ of $\pi_{\mathcal{O}_\omega}$ with the subspace $\delta_e \otimes \mathcal{K}_\omega$ and so $\pi_{\mathcal{O}_\omega}$ with a subrepresentation of the restriction of $\widetilde{\pi_{\mathcal{O}_\omega}}$ to N.

Let $\eta_\omega \in \mathcal{K}_\omega$ be a cyclic vector for $\pi_{\mathcal{O}_\omega}$ such that

$$\omega = \langle \pi_{\mathcal{O}_\omega}(\cdot)\eta_\omega \mid \eta_\omega \rangle.$$

For $g \in K_\Gamma$, define a normal state $\psi_{\omega,g}$ on $\widetilde{\mathcal{M}_\omega}$ by the formula

$$\psi_{\omega,g}(\widetilde{T}) = \langle \widetilde{T} U_{g,\omega}^{-1}\eta_\omega \mid U_{g,\omega}^{-1}\eta_\omega \rangle \qquad \text{for all} \quad \widetilde{T} \in \widetilde{\mathcal{M}_\omega},$$

where $U_{g,\omega}$ is the unitary operator on $\mathcal{K}_\omega$ from Step 1.

Consider the linear functional $\psi_\omega : \widetilde{\mathcal{M}_\omega} \to \mathbf{C}$ given by

$$\psi_\omega(\widetilde{T}) = \int_{K_\Gamma} \psi_{\omega,g}(\widetilde{T}) dm(g) \qquad \text{for all} \quad \widetilde{T} \in \widetilde{\mathcal{M}_\omega},$$

where m is the normalized Haar measure on K_Γ.

Step 4 We claim that ψ_ω is a normal state on $\widetilde{\mathcal{M}_\omega}$

Indeed, it is clear that ψ_ω is a state on $\widetilde{\mathcal{M}_\omega}$. Let $(\widetilde{T}_n)_n$ be an increasing sequence of positive operators in $\widetilde{\mathcal{M}_\omega}$ with $\widetilde{T} = \sup_n \widetilde{T}_n \in \widetilde{\mathcal{M}_\omega}$.

For every $g \in K_\Gamma$, the sequence $(\psi_{\omega,g}(\widetilde{T}_n))_n$ is increasing and its limit is $\psi_{\omega,g}(\widetilde{T})$. It follows from the monotone convergence theorem that

$$\lim_n \psi_\omega(T_n) = \lim_n \int_{K_\Gamma} \psi_{\omega,g}(T_n) dm(g) = \int_{K_\Gamma} \psi_{\omega,g}(T) dm(g) = \psi_\omega(T).$$

So, ψ_ω is normal, as $\widetilde{\mathcal{M}_\omega}$ acts on a separable Hilbert space.

Step 5 We claim that, writing γ instead of $\widetilde{\pi_{\mathcal{O}_\omega}}(\gamma)$ for $\gamma \in \Gamma$, we have

$$\psi_\omega(\gamma) = \begin{cases} \int_{K_\Gamma} \omega^g(\gamma) dm(g) & \text{if} \qquad \gamma \in N \\ 0 \qquad \text{if} \qquad \gamma \notin N. \end{cases}$$

Moreover, ψ_ω coincides with the trace φ_ω on $\widetilde{\mathcal{M}_\omega}$ from above.

Indeed, Let $\gamma \in N$. Since

$$U_{\omega,g}\pi_{\mathcal{O}_\omega}(\gamma)U_{\omega,g}^{-1} = \pi_{\mathcal{O}_\omega}(g(\gamma)),$$

we have

$$\begin{aligned}\psi_\omega(\gamma) &= \int_{K_\Gamma} \langle \widetilde{\pi}_{\mathcal{O}_\omega}(g(\gamma))\eta_\omega \mid \eta_\omega\rangle dm(g) = \int_{K_\Gamma} \langle \pi_{\mathcal{O}_\omega}(g(\gamma))\eta_\omega \mid \eta_\omega\rangle dm(g)\\ &= \int_{K_\Gamma} \omega^g(\gamma) dm(g).\end{aligned}$$

Let $\gamma \in \Gamma \setminus N$. Then, by the usual properties of an induced representation, $\widetilde{\pi}_{\mathcal{O}_\omega}(\gamma)(\mathcal{K}_\omega)$ is orthogonal to $\mathcal{K}_\omega$. It follows that

$$\psi_{\omega,g}(\gamma) = \langle \widetilde{\pi}_{\mathcal{O}_\omega}(\gamma)U_{g,\omega}^{-1}\eta_\omega \mid U_{g,\omega}^{-1}\eta_\omega\rangle = 0$$

and hence $\psi_\omega(\gamma) = 0$.

In particular, this shows that ψ_ω is a Γ-invariant state on $\widetilde{\mathcal{M}_\omega}$; since ψ_ω is normal (Step 4), it follows that ψ_ω is a trace on $\widetilde{\mathcal{M}_\omega}$. As $\widetilde{\mathcal{M}_\omega}$ is a factor (see Step 3), the fact that $\widetilde{\psi_\omega} = \varphi_\omega$ follows from the uniqueness of normal traces on factors (see Corollaire p. 92 and Corollaire 2 p.83 in [Dix69, Chap. I, §6]).

Step 6 The Plancherel measure μ on Γ is the image of ν under the map Φ as in the statement of Theorem B.ii.

Indeed, for $f \in \mathbf{C}[\Gamma]$, we have by Step 5

$$\begin{aligned}\|f\|_2^2 &= \int_\Omega \|\widetilde{\pi_{\mathcal{O}_\omega}}(f)\|^2 d\dot{\nu}(\omega) = \int_\Omega \psi_\omega\left(f^* * f\right) d\dot{\nu}(\omega)\\ &= \int_\Omega \int_{K_\Gamma} \omega^g((f^* * f)|_N) dm(g) d\dot{\nu}(\omega)\\ &= \int_\Omega \int_{\mathcal{O}_\omega} \left(\int_{K_\Gamma} \omega^g((f^* * f)|_N) dm(g)\right) dm_\omega(t) d\dot{\nu}(\omega)\\ &= \int_{\mathrm{Ch}(\Gamma_{\mathrm{fc}})} \int_{K_\Gamma} t^g((f^* * f)|_N) dm(g) d\nu(t)\\ &= \int_{\mathrm{Ch}(\Gamma)} \Phi(t)(f^* * f) d\nu(t)\end{aligned}$$

and the claim follows.

Remark 7. The map Φ in Theorem B.ii can be described without reference to the group K_Γ as follows. For $t \in \mathrm{Ch}(\Gamma_{\mathrm{fc}})$ and $\gamma \in \Gamma$, we have

$$\Phi(t)(\gamma) = \begin{cases} \dfrac{1}{\#[\gamma]} \sum_{x\in[\gamma]} t(x) & \text{if} \quad \gamma \in \Gamma_{\mathrm{fc}} \\ 0 \quad \text{if} \quad \gamma \notin \Gamma_{\mathrm{fc}} \end{cases},$$

where $[\gamma]$ denotes the Γ-conjugacy class of γ. Indeed, it suffices to consider the case where $\gamma \in \Gamma_{\mathrm{fc}}$. The stabilizer K_0 of γ in K_Γ is an open and hence cofinite subgroup of K_Γ; in particular, the K_Γ-orbit of γ coincides with the $\mathrm{Ad}(\Gamma)$-orbit of γ and so $\{\mathrm{Ad}(x) \mid x \in [\gamma]\}$ is a system of representatives for K_Γ/K_0. Let m_0 be the normalized Haar measure on K_0. The normalized Haar measure m on K_Γ is then given by $m(f) = \dfrac{1}{[\gamma]} \sum_{x\in[\gamma]} \int_{K_0} f(\mathrm{Ad}(x)g)dm_0(g)$ for every continuous function f on K_Γ. It follows that

$$\Phi(t)(\gamma) = \int_{K_\Gamma} t(g(\gamma))dm(g) = \frac{1}{[\gamma]} \sum_{x\in[\gamma]} t(x).$$

5. Proofs of Corollary C and Corollary D

Let Γ be a countable linear group. So, Γ is a subgroup of $GL_n(\mathbf{k})$ for a field $\mathbf{k}$, which may be assumed to be algebraically closed. Let $\mathbf{G}$ be the closure of Γ in the Zariski topology of $GL_n(\mathbf{k})$ and let $\mathbf{G}_0$ be the irreducible component of $\mathbf{G}$. As is well-known, $\mathbf{G}_0$ has finite index in $\mathbf{G}$ and hence $\Gamma_0 := \mathbf{G}_0 \cap \Gamma$ is a normal subgroup of finite index in Γ

Let $\gamma \in \Gamma_{\mathrm{fc}}$. On the one hand, the centralizer Γ_γ of γ in Γ is a subgroup of finite index of Γ; therefore, the irreducible component of the Zariski closure of Γ_γ coincides with $\mathbf{G}_0$. On the other hand, the centralizer $\mathbf{G}_\gamma$ of γ in $\mathbf{G}$ is clearly a Zariski-closed subgroup of $\mathbf{G}$. It follows that the irreducible component of $\mathbf{G}_\gamma$ contains (in fact coincides with) $\mathbf{G}_0$ and hence

$$\Gamma_0 = \mathbf{G}_0 \cap \Gamma \subset \mathbf{G}_\gamma \cap \Gamma = \Gamma_\gamma.$$

As a consequence, we see that Γ_0 acts trivially on Γ_{fc} and hence $\mathrm{Ad}(\Gamma)|_{\Gamma_{\mathrm{fc}}}$ is a finite group. In particular, $\Gamma_0 \cap \Gamma_{\mathrm{fc}}$ is contained in the center $Z(\Gamma_{\mathrm{fc}})$ of Γ_{fc}; so $Z(\Gamma_{\mathrm{fc}})$ has finite index in Γ_{fc} which is therefore a central group. This proves Items (i) and (ii) of Corollary C. Item (iii) follows from Theorem B.ii.

Assume now that $\mathbf{G}$ is connected, that is $\mathbf{G} = \mathbf{G}_0$. Then $\Gamma = \Gamma_0$ acts trivially on Γ_{fc} and so Γ_{fc} coincides with the center $Z(\Gamma)$ of Γ. This proves the first part of Corollary D.

It remains to prove that the assumption $\mathbf{G} = \mathbf{G}_0$ is satisfied in Cases (i) and (ii) of Corollary D:

(i) Let $\mathbf{G}$ be a connected linear algebraic group over a countable field $\mathbf{k}$ of characteristic 0. Then $\Gamma = \mathbf{G}(\mathbf{k})$ is Zariski dense in $\mathbf{G}$, by [Ros57, Corollary p.44]).

(ii) Let $\mathbf{G}$ be a connected linear algebraic group $\mathbf{G}$ over a local field $\mathbf{k}$. Assume that $\mathbf{G}$ has no proper $\mathbf{k}$-subgroup $\mathbf{H}$ such that $(\mathbf{G}/\mathbf{H})(\mathbf{k})$ is compact. Then every lattice Γ in $\mathbf{G}(\mathbf{k})$ is Zariski dense in $\mathbf{G}$, by [Sha99, Corollary 1.2].

6. The Plancherel Formula for some countable groups

6.1. **Restricted direct product of finite groups.** Let $(G_n)_{n\geq 1}$ be a sequence of finite groups. Let $\Gamma = \prod'_{n\geq 1} G_n$ be the **restricted direct product** of the G_n's, that is, Γ consists of the sequences $(g_n)_{n\geq 1}$ with $g_n \in G_n$ for all n and $g_n \neq e$ for at most finitely many n. It is clear that Γ is an FC-group.

Set $X_n := \mathrm{Ch}(G_n)$ for $n \geq 1$ and let $X = \prod_{n\geq 1} X_n$ be the cartesian product equipped with the product topology, where each X_n carries the discrete topology. Define a map $\Phi : X \to \mathrm{Tr}(\Gamma)$ by

$$\Phi((t_n)_{n\geq 1})((g_n)_{n\geq 1}) = \prod_{n\geq 1} t_n(g_n) \qquad \text{for all} \quad (t_n)_{n\geq 1} \in X, (g_n)_{n\geq 1} \in \Gamma$$

(observe that this product is well-defined, since $g_n = e$ and hence $t_n(g_n) = 1$ for almost every $n \geq 1$). Then $\Phi(X) = \mathrm{Ch}(\Gamma)$ and $\Phi : X \to \mathrm{Ch}(\Gamma)$ is a homeomorphism (see [Mau51, Lemma 7.1])

For every $n \geq 1$, let ν_n be the measure on X_n given by

$$\nu_n(\{t\}) = \frac{d_t^2}{\#G_n} \qquad \text{for all} \quad t \in X_n,$$

where d_t is the dimension of the irreducible representation of G_n with t as character; observe that ν_n is a probability measure, since $\sum_{t\in X_n} d_t^2 = \#G_n$.

Let $\nu = \otimes_{n\geq 1}\nu$ be the product measure on the Borel subsets of X. The Plancherel measure on Γ is the image of ν under Φ (see Equation (5.6) in [Mau51]). The regular representation λ_Γ is of type II if and only if infinitely many G_n's are non abelian (see *loc.cit.*, Theorem 1 or Theorem E below).

6.2. **Infinite dimensional Heisenberg group.** Let $\mathbf{F}_p$ be the field of order p for an odd prime p and let $V = \oplus_{i\in\mathbf{N}}\mathbf{F}_p$ be a vector space over $\mathbf{F}_p$ of countable infinite dimension. Denote by ω the symplectic form on $V \oplus V$ given by

$$\omega((x, y), (x', y')) = \sum_{i\in\mathbf{N}}(x_i y'_i - y_i x'_i) \quad \text{for } (x, y), (x', y') \in V \oplus V.$$

The "infinite dimensional" Heisenberg group over $\mathbf{F}_p$ is the group Γ with underlying set $V \oplus V \oplus \mathbf{F}_p$ and with multiplication defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \omega((x, y), (x', y')))$$

for $(x, y, z), (x', y', z') \in \Gamma$.

The group Γ is an FC-group; since $p \geq 3$, its center Z coincides with $[\Gamma, \Gamma]$ and consists of the elements of the form $(0, 0, z)$ for $z \in \mathbf{F}_p$. Observe that Γ is not virtually abelian.

Let z_0 be a generator for the cyclic group Z of order p. The unitary dual $\widehat{Z}$ consists of the characters defined by $\chi_\omega(z_0^j) = \omega^j$ for $j \in \{0, 1, \ldots, p-1\}$ and $\omega \in C_p$, where C_p is the group of p-th roots of unity in $\mathbf{C}$.

For $\omega \in C_p$, the subspace

$$\mathcal{H}_\omega = \{f \in \ell^2(\Gamma) \mid f(z_0 x) = \omega f(x) \text{ for every } x \in \Gamma\}.$$

is left and right translation invariant and we have an orthogonal decomposition of $\ell^2(\Gamma) = \bigoplus_{\omega \in C_p} \mathcal{H}_\omega$. The orthogonal projection P_ω on $\mathcal{H}_\omega$ belongs to the center of $L(\Gamma)$ and is given by

$$P_\omega(f)(x) = \frac{1}{p} \sum_{i=0}^{p-1} \omega^{-i} f(z_0^i x) \qquad \text{for all} \quad f \in \ell^2(\Gamma), x \in \Gamma.$$

One checks that $\|P_\omega(\delta_e)\|^2 = 1/p$.

Let π_ω be the restriction of λ_Γ to $\mathcal{H}_\omega$. Observe that $\mathcal{H}_1$ can be identified with $\ell^2(\Gamma/Z)$ and π_1 with $\lambda_{\Gamma/Z}$.

For $\omega \neq 1$, the representation π_ω is factorial of type II_1 and the corresponding character is $\widetilde{\chi_\omega}$ (for more details, see the proof of Theorem 7.D.4 in [BH]).

The integral decomposition of δ_e is

$$\delta_e = \frac{1}{p} \int_{\widehat{\Gamma/Z}} \chi d\nu(\chi) + \frac{1}{p} \sum_{\omega \in C_p \setminus \{1\}} \widetilde{\chi_\omega},$$

with the corresponding Plancherel formula given for every $f \in \mathbf{C}[\Gamma]$ by

$$\|f\|_2^2 = \frac{1}{p} \int_{\widehat{\Gamma/Z}} |\mathcal{F}(P_1(f))(\chi)|^2 d\nu(\chi) + \frac{1}{p} \sum_{\omega \in C_p \setminus \{1\}} \sum_{j=0}^{p-1} (f^* * f)(z_0^j) \omega^j,$$

where ν is the normalized Haar measure of the compact abelian group $\widehat{\Gamma/Z}$ and $\mathcal{F}$ the Fourier transform. In particular, $\lambda_\Gamma(\Gamma)''$ is a direct sum of an abelian von Neumann algebra and $p-1$ factors of type II_1. For a more general result, see [Kap51, Theorem 2].

6.3. **An example involving $SL_d(\mathbf{Z})$.** Let $\Lambda = SL_d(\mathbf{Z})$ for an odd integer d. Fix a prime p and for $n \geq 1$, let $G_n = SL_3(\mathbf{Z}/p^n\mathbf{Z})$, viewed as (finite) quotient of Λ. Let Γ be the semi-direct product $\Lambda \ltimes \prod'_{n\geq 1} G_n$, where Λ acts diagonally in the natural way on the restricted direct product $G := \prod'_{n\geq 1} G_n$ of the G_n's.

Since Λ is an ICC-group, it is clear that $\Gamma_{\rm fc} = G$. The group K_Γ as in Theorem B can be identified with the projective limit of the groups G_n's, that is, with $SL_d(\mathbf{Z}_p)$, where $\mathbf{Z}_p$ is the ring of p-adic integers. Since Λ acts trivially on $\mathrm{Ch}(G)$, the same is true for the action of K_Γ on $\mathrm{Ch}(G)$.

Let $\lambda_G = \int_{\mathrm{Ch}(G)}^{\oplus} \pi_t d\nu(t)$ be the Plancherel decomposition of λ_G (see Example 6.1). It follows from Theorem B that the Plancherel decomposition of λ_Γ is

$$\lambda_\Gamma = \int_{\mathrm{Ch}(G)}^{\oplus} \mathrm{Ind}_G^\Gamma \pi_t d\nu(t).$$

Appendix A. Proof of Theorem E

A.1. **Easy implications.** The implications $(iv') \Rightarrow (iv)$ and $(i) \Rightarrow (iii)$ are obvious; if (iv) holds then Γ is a so-called CCR group and so (i) holds, by a general fact (see [Dix77, 5.5.2]).

We are going to show that $(ii) \Rightarrow (iv')$, Assume that Γ contains an abelian normal subgroup N of finite index. Let $(\pi, \mathcal{H})$ be an irreducible representation of Γ.

Denote by $\mathcal{B}$ the set of Borel subsets of the dual group $\widehat{N}$ and by $\mathrm{Proj}(\mathcal{H})$ the set of orthogonal projections in $\mathcal{L}(\mathcal{H})$. Let $E\colon \mathcal{B}(\widehat{N}) \to \mathrm{Proj}(\mathcal{H})$ be the projection-valued measure on $\widehat{N}$ associated with the restriction $\pi|_N$ by the SNAG Theorem (see [BHV08, D.3.1]); so, we have

$$\pi(n) = \int_{\widehat{G}} \chi(n) dE(\chi) \quad \text{for all } n \in N.$$

The dual action of Γ on $\widehat{N}$, given by $\chi^\gamma(n) = \chi(\gamma^{-1} n \gamma)$ for $\chi \in \widehat{N}$ and $\gamma \in \Gamma$, factorizes through Γ/N. Moreover, the following covariance relation holds

$$\pi(\gamma)E(B)\pi(\gamma^{-1}) \ = \ E(B^\gamma) \qquad \text{for all} \quad B \in \mathcal{B}(\widehat{N}),$$

where $B^\gamma = \{\chi^\gamma \mid \chi \in B\}$.

Let $S \in \mathcal{B}(\widehat{N})$ be the support of E, that is, S is the complement of the largest open subset U of $\widehat{N}$ with $E(U) = 0$. We claim that S consists of a single Γ-orbit.

Indeed, let $\chi_0 \in S$ and let $(U_n)_{n\geq 1}$ be a sequence of open neighbourhoods of χ with $\bigcap_{n\geq 1} U_n = \{\chi_0\}$. Fix $n \geq 1$. The set U_n^Γ is Γ-

invariant and hence $E(U_n^\Gamma) \in \pi(\Gamma)'$, by the covariance relation. Since π is irreducible and $E(U_n) \neq 0$, we have therefore $E(U_n^\Gamma) = I_{\mathcal{H}}$. By the usual properties of a projection-valued measure, this implies that

$$E(\chi_0^\Gamma) = E(\bigcap_{n\geq 1} U_n^\Gamma) = I_{\mathcal{H}}.$$

and the claim is proved.

Since S is finite, we have $\mathcal{H} = \bigoplus_{\chi\in S} \mathcal{H}^\chi$, where

$$\mathcal{H}^\chi := \{\xi \in \mathcal{H} \mid \pi(n)\xi = \chi(n)\xi \qquad \text{for all} \quad n \in N\};$$

moreover, since N is a normal subgroup, we have $\pi(\gamma)\mathcal{H}^\chi = \mathcal{H}^{\chi^\gamma}$ for every $\chi \in S$ and every $\gamma \in \Gamma$.

Let H be the stabilizer of χ_0 and let $T \subset \Gamma$ be a set of representatives of the right T-cosets of H. Then $\mathcal{H}^{\chi_0}$ is invariant under $\pi(H)$ and we have

$$\mathcal{H} = \bigoplus_{t\in T} \mathcal{H}^{\chi_0^t} = \bigoplus_{t\in T} \pi(t)\mathcal{H}^{\chi_0}.$$

This shows that π is equivalent to the induced representation $\mathrm{Ind}_H^\Gamma \sigma$, where σ is the subrepresentation of $\pi|_H$ defined on $\mathcal{H}^{\chi_0}$

We claim that $\mathrm{Ind}_H^\Gamma \sigma$ is contained in $\mathrm{Ind}_N^\Gamma \chi_0$. Indeed, as is well-known (see [BHV08, E.2.5]), $\mathrm{Ind}_N^H(\sigma|_N)$ is equivalent to the tensor product representation $\sigma \otimes \lambda_{H/N}$, where $\lambda_{H/N}$ is the quasi-regular representation on $\ell^2(\Gamma/H)$. Since H/N is finite, 1_H is contained in $\lambda_{H/N}$ and therefore σ is contained in $\mathrm{Ind}_N^H(\sigma|_N)$. Notice that $\sigma|_N$ is a multiple $n\chi_0$ of χ_0, for some cardinal n. We conclude that $\pi = \mathrm{Ind}_H^\Gamma \sigma$ is contained in $\mathrm{Ind}_H^\Gamma(\mathrm{Ind}_N^H n\chi_0) = n \,\mathrm{Ind}_N^\Gamma \chi_0$. Since π is irreducible, it follows that π is contained in $\mathrm{Ind}_N^\Gamma \chi_0$.

Now, $\mathrm{Ind}_N^\Gamma \chi_0$ has dimension $[\Gamma : N]$; hence, $\dim \pi \leq [\Gamma : N]$ and so (iv') holds.

A.2. **Proof of the other implications.** We have to give the proof of the implication $(iii) \Rightarrow (i)$ and the equivalence $(v) \Leftrightarrow (iv)$.

In the sequel, Γ will be a countable group and $\lambda_\Gamma = \int_{\mathrm{Ch}(\Gamma)}^\oplus \pi_t d\mu(t)$ the direct integral decomposition given by the Plancherel Theorem A. Recall (see Section 3) that, if we write $\delta_e = \int_{\mathrm{Ch}(\Gamma)}^\oplus \xi_t d\mu(t)$, then ξ_t is a cyclic vector in the Hilbert space $\mathcal{H}_t$ of π_t and $t = \langle \pi_t(\cdot)\xi_t \mid \xi_t\rangle$, for μ-almost every $t \in \mathrm{Ch}(\Gamma)$.

A.2.1. *Case where the FC-centre of* Γ *has infinite index.* We assume that $[\Gamma\colon \Gamma_{\mathrm{fc}}]$ is infinite; we claim that λ_Γ is of type II.

By Remark 6.i, there exists a subset X of $\mathrm{Ch}(\Gamma)$ with $\mu(X) = 1$ such that $t = 0$ outside Γ_{fc} for every $t \in X$.

Let $t \in X$. Then the factor $\pi_t(\Gamma)''$ is infinite dimensional. Indeed, since $[\Gamma\colon \Gamma_{\rm fc}]$ is infinite, we can find a sequence $(\gamma_n)_{n\geq 1}$ in Γ with $\gamma_m^{-1}\gamma_n \notin \Gamma_{\rm fc}$ for every m, n with $m \neq n$. Then

$$\langle \pi_t(\gamma_n)\xi_t \mid \pi_t(\gamma_m)\xi_t\rangle = \langle \pi_t(\gamma_m^{-1}\gamma_n)\xi_t \mid \xi_t\rangle = t(\gamma_m^{-1}\gamma_n) = 0,$$

for $m \neq n$; so, $(\pi_t(\gamma_n)\xi_t)_{n\geq 1}$ is an orthonormal sequence in $\mathcal{H}_t$. This implies that $(\pi_t(\gamma_n))_{n\geq 1}$ is a linearly independent sequence in $\pi_t(\Gamma)''$ and the claim is proved.

Observe that we have proved, in particular, that Γ is not of type I.

A.2.2. *Reduction to FC-groups.* Let ${\rm Ch}(\Gamma)_{\rm fd}$ be the set of $t \in {\rm Ch}(\Gamma)$ for which $\pi_t(\Gamma)''$ is finite dimensional. Then

$${\rm Ch}(\Gamma)_{\rm fd} = \bigcup_{n\geq 1} {\rm Ch}(\Gamma)_n,$$

where ${\rm Ch}(\Gamma)_n$ is the set of t such that $\dim \pi_t(\Gamma)'' = n$. We claim that ${\rm Ch}(\Gamma)_n$ and hence ${\rm Ch}(\Gamma)_{\rm fd}$ is a measurable subset of ${\rm Ch}(\Gamma)$.

Indeed, let $\mathcal{F}$ be collection of finite subsets of Γ. For every $F \in \mathcal{F}$, let C_F be the set of $t \in {\rm Ch}(\Gamma)$ such that the family $(\pi_t(\gamma))_{\gamma\in F}$ is linearly independent, equivalently (see Remark 6.ii), such that $(\pi_t(\gamma)\xi_t)_{\gamma\in F}$ is linearly independent. Since $t = \langle \pi_t(\cdot)\xi_t \mid \xi_t\rangle$, it follows that

$$C_F := \left\{ t \in {\rm Ch}(\Gamma) \mid \det(t(\gamma^{-1}\gamma')) \neq 0 \qquad \text{for all} \quad (\gamma, \gamma') \in F \times F \right\}$$

and this shows that C_F is measurable. Since

$$(*) \qquad {\rm Ch}(\Gamma)_n = \bigcup_{F\in\mathcal{F}:\#F=n} C_F \setminus \left(\bigcup_{F'\in\mathcal{F}:\#F'>n} C_{F'} \right)$$

and $\mathcal{F}$ is countable, it follows that ${\rm Ch}(\Gamma)_n$ is measurable.

Assume now that $[\Gamma\colon \Gamma_{\rm fc}]$ is finite. Observe that, for a cyclic representation π of $\Gamma_{\rm fc}$, the induced representation ${\rm Ind}_{\Gamma_{\rm fc}}^{\Gamma} \pi$ is cyclic and so $({\rm Ind}_{\Gamma_{\rm fc}}^{\Gamma} \pi)(\Gamma)''$ is finite dimensional if and only if $\pi(\Gamma_{\rm fc})''$ is finite dimensional.

let ν be the Plancherel measure of $\Gamma_{\rm fc}$. It follows from Theorem B that $\mu({\rm Ch}(\Gamma)_{\rm fd}) = \nu({\rm Ch}(\Gamma_{\rm fc})_{\rm fd})$; in particular, we have $\mu({\rm Ch}(\Gamma)_{\rm fd}) = 0$ (or $\mu({\rm Ch}(\Gamma)_{\rm fd}) = 1$) if and only if $\nu({\rm Ch}(\Gamma_{\rm fc})_{\rm fd}) = 0$ (or $\nu({\rm Ch}(\Gamma_{\rm fc})_{\rm fd}) = 1$), that is, λ_Γ is of type I (or of type II) if and only if $\lambda_{\Gamma_{\rm fc}}$ is of type I (or of type II).

Observe also that Γ is virtually abelian if and only if $\Gamma_{\rm fc}$ is virtually abelian. As a consequence, we see that it suffices to prove the implication $(iii) \Rightarrow (i)$ and the equivalence $(v) \Leftrightarrow (iv)$ in the case where $\Gamma = \Gamma_{\rm fc}$.

A.2.3. *Case of an FC-group.* We will need the following lemma of independent interest, which is valid for an arbitrary countable group Γ. Let $r : \mathrm{Ch}(\Gamma) \to \mathrm{Tr}([\Gamma, \Gamma])$ be the restriction map. We will identify $\mathrm{Ch}(\Gamma/[\Gamma, \Gamma])$ with the set $\{s \in \mathrm{Ch}(\Gamma) \mid r(s) = 1_{[\Gamma,\Gamma]}\}$, that is, with the set of unitary characters of Γ. Observe that, for every $s \in \mathrm{Ch}(\Gamma/[\Gamma, \Gamma])$ and $t \in \mathrm{Ch}(\Gamma)$, we have $st \in \mathrm{Ch}(\Gamma)$.

Lemma 8. *Let Γ be a countable group and $t, t' \in \mathrm{Ch}(\Gamma)$ be such that $r(t) = r(t')$. Then there exists $s \in \mathrm{Ch}(\Gamma/[\Gamma, \Gamma])$ such that $t' = st$.*

Proof. The integral decomposition of $\mathbf{1}_{[\Gamma,\Gamma]} \in \mathrm{Tr}(\Gamma)$ into characters is given by

$$\mathbf{1}_{[\Gamma,\Gamma]} = \int_{\mathrm{Ch}(\Gamma/[\Gamma,\Gamma])} s d\nu(s),$$

where ν is the Haar measure of $\Gamma/[\Gamma, \Gamma]$. By assumption, we have $t\mathbf{1}_{[\Gamma,\Gamma]} = t'\mathbf{1}_{[\Gamma,\Gamma]}$ and hence

$$t\mathbf{1}_{[\Gamma,\Gamma]} = \int_{\mathrm{Ch}(\Gamma/[\Gamma,\Gamma])} ts d\nu(s) = \int_{\mathrm{Ch}(\Gamma/[\Gamma,\Gamma])} t's d\nu(s).$$

By uniqueness of integral decomposition, it follows that the images ν_t and $\nu_{t'}$ of ν under the maps $\mathrm{Ch}(\Gamma/[\Gamma, \Gamma]) \to \mathrm{Ch}(\Gamma)$ given respectively by multiplication with t and t' coincide. In particular, the supports of ν_t and $\nu_{t'}$ are the same, that is, $t \,\mathrm{Ch}(\Gamma/[\Gamma, \Gamma]) = t' \,\mathrm{Ch}(\Gamma/[\Gamma, \Gamma])$ and the claim follows. □

We assume from now on that $\Gamma = \Gamma_{\mathrm{fc}}$.

Step 1 We claim that the regular representation λ_Γ is of type II if and only if $[\Gamma, \Gamma]$ is infinite.

We have to show that $\mu(\mathrm{Ch}(\Gamma)_{\mathrm{fd}}) > 0$ if and only if $[\Gamma, \Gamma]$ is finite.

Assume first that $[\Gamma, \Gamma]$ is finite. The representation $\lambda_{\Gamma/[\Gamma,\Gamma]}$, lifted to Γ, is a subrepresentation of λ_Γ, since $\ell^2(\Gamma/[\Gamma, \Gamma])$ can be viewed in an obvious way as Γ-invariant subspace of $\ell^2(\Gamma)$. As $\Gamma/[\Gamma, \Gamma]$ is abelian, $\lambda_{\Gamma/[\Gamma,\Gamma]}$is of type I and so $\mu(\mathrm{Ch}(\Gamma)_{\mathrm{fd}}) > 0$.

Conversely, assume that $\mu(\mathrm{Ch}(\Gamma)_{\mathrm{fd}}) > 0$. Since Γ is an FC-group, it suffices to show that Γ has a subgroup of finite index with finite commutator subgroup (see [Neu55, Lemma 4.1]).

As $\mu(\mathrm{Ch}(\Gamma)_{\mathrm{fd}}) > 0$ and $\mathrm{Ch}(\Gamma)_{\mathrm{fd}} = \bigcup_{n\geq 1} \mathrm{Ch}(\Gamma)_n$, we have $\mu(\mathrm{Ch}(\Gamma)_n) > 0$ for some $n \geq 1$. It follows from (*) that there exists $F \in \mathcal{F}$ with $|F| = n$ such that $\mu(C_F \cap \mathrm{Ch}(\Gamma)_n) > 0$.

Let Λ be the subgroup of Γ generated by F. Since Γ is an FC-group and Λ is finitely generated, the centralizer $H := \mathrm{Cent}_\Gamma(\Lambda)$ of Λ in Γ has finite index.

Let $t \in C_F \cap \mathrm{Ch}(\Gamma)_{\mathrm{n}}$ and $\gamma_0 \in H$. On the one hand, since $(\pi_t(\gamma))_{\gamma\in F}$ is a basis of the vector space $\pi_t(\Gamma)$, we have $\pi_t(\Lambda)'' = \pi_t(\Gamma)''$. On the other hand, as γ_0 centralizes Λ, we have $\pi_t(\gamma_0) \in \pi_t(\Lambda)'$. Hence, $\pi_t(\gamma_0)$ belongs to the center $\pi_t(\Gamma)' \cap \pi_t(\Gamma)''$ of the factor $\pi_t(\Gamma)''$ and so $\pi_t(\gamma_0)$ is a scalar multiple of $I_{\mathcal{H}_t}$. It follows in particular that π_t is trivial on $[H, H]$. As a result, the subrepresentation $\int_{C_F\cap \mathrm{Ch}(\Gamma)_{\mathrm{n}}}^{\oplus} \pi_t d\mu(t)$ of λ_Γ is trivial on $[H, H]$. Since the matrix coefficients of λ_Γ vanish at infinity, it follows that $[H, H]$ is finite and the claim is proved.

In view of what we have shown so far, we may and will assume from now on that $[\Gamma, \Gamma]$ is finite and that λ_Γ is of type I. We are going to show that Γ is a virtually abelian (in fact, a central) group and this will finish the proof of Theorem E.

Set $N := [\Gamma, \Gamma]$ and let $r : \mathrm{Ch}(\Gamma) \to \mathrm{Tr}(N)$ be the restriction map.

Step 2 We claim that there exist finitely many functions $s_1, \dots, s_m$ in $\mathrm{Tr}(N)$ such that $r(t) \in \{s_1, \dots, s_m\}$, for μ-almost every $t \in \mathrm{Ch}(\Gamma)$.

Indeed, since λ_Γ is of type I, there exists a subset X of $\mathrm{Ch}(\Gamma)$ with $\mu(X) = 1$ such that $\pi_t(\Gamma)''$ is finite dimensional for every $t \in X$.

Let $t \in X$. The Hilbert space $\mathcal{H}_t$ of π_t is finite dimensional and π_t is a (finite) multiple of an irreducible representation σ_t of Γ. As π_t and σ_t have the same normalized character, we may assume that π_t is irreducible.

Let $\mathcal{K}$ be an irreducible N-invariant subspace of $\mathcal{H}_t$ and let ρ be the corresponding equivalence class of representation of N. For $g \in \Gamma$, the subspace $\pi_t(g)\mathcal{K}$ is N-invariant with ρ^g as corresponding representation of N. Since π_t is irreducible, we have $\mathcal{H}_t = \sum_{g\in\Gamma} \pi_t(g)\mathcal{K}$.

Let L be the stabilizer of ρ; observe that L has finite index in Γ, since L contains the centralizer of N and Γ is an FC-group. Let $g_1, \dots, g_r$ be a set of representatives for the coset space Γ/L. Then $\mathcal{H}_t = \oplus_{j=1}^r \pi_t(g_j)\mathcal{K}_\rho$, where $\mathcal{K}_\rho$ is the sum of all N-invariant subspaces of $\mathcal{H}_t$ with corresponding representation equivalent to ρ. The normalized trace of the representation of N on $\pi_t(g_j)\mathcal{K}_\rho$ is $\chi_\rho^{g_j}$, where χ_ρ is the normalized character of ρ. It follows that, for every $g \in N$, we have

$$t(g) = \frac{1}{r} \sum_{j=1}^{r} \chi_\rho^{g_j}(g).$$

Since $[\Gamma, \Gamma]$ is finite, $[\Gamma, \Gamma]$ has only finitely many equivalence classes of irreducible representations and the claim follows.

Step 3 We claim that the center $Z(\Gamma)$ has finite index in Γ.

Indeed, by Step 2, there exists a subset X of $\mathrm{Ch}(\Gamma)$ with $\mu(X) = 1$ and finitely many $t_1, \ldots, t_m \in X$ such that $r(t) \in \{r(t_1), \ldots, r(t_m)\}$ and such that $\mathcal{H}_t$ is finite dimensional for every $t \in X$.

It follows from Lemma 8 that, for every $t \in X$, there exists $s \in \mathrm{Ch}(\Gamma/[\Gamma, \Gamma])$ such that $t = st_i$ for some $i \in \{1, \ldots, m\}$ and hence $\dim \pi_t(\Gamma)'' = \dim \pi_{t_i}(\Gamma)''$. As a result, we can find a finitely generated normal subgroup M of Γ such that $\dim \pi_t(\Gamma)'' = \dim \pi_t(M)''$ for every $t \in X$.

Since the centralizer C of M in Γ has finite index, it suffices to show that C contains $Z(\Gamma)$.

For $g \in C, \gamma \in \Gamma$ and $x \in M$, we have $t(x^{-1}g\gamma g^{-1}) = t(x^{-1}\gamma)$, that is,

$$\langle \pi_t(g\gamma g^{-1})\xi_t \mid \pi_t(x)\xi_t\rangle = \langle \pi_t(\gamma)\xi_t \mid \pi_t(x)\xi_t\rangle.$$

Since $\dim \pi_t(\Gamma)'' = \dim \pi_t(M)''$, the linear span of $\pi_t(M)\xi_t$ is dense in $\mathcal{H}_t$ and this implies that $\pi_t(g\gamma g^{-1})\xi_t = \pi_t(\gamma)\xi_t$ for all $g \in C, \gamma \in \Gamma$, and $t \in X$. It follows that $\lambda_\Gamma(g\gamma g^{-1})\delta_e = \lambda_\Gamma(\gamma)\delta_e$ and hence $g\gamma g^{-1} = \gamma$ for all $g \in C$ and $\gamma \in \Gamma$; so, $C \subset Z(\Gamma)$.

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