LOCAL RIGIDITY FOR ACTIONS OF KAZHDAN GROUPS ON NON COMMUTATIVE L_p-SPACES

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Abstract. Let Γ be a discrete group and $\mathcal{N}$ a finite factor, and assume that both have Kazhdan's Property (T). For $p \in [1, +\infty)$, $p \neq 2$, let $\pi : \Gamma \to \mathbf{O}(L_p(\mathcal{N}))$ be a homomorphism to the group $\mathbf{O}(L_p(\mathcal{N}))$ of linear bijective isometries of the L_p-space of $\mathcal{N}$. There are two actions π^l and π^r of a finite index subgroup Γ^+ of Γ by automorphisms of $\mathcal{N}$ associated to π and given by $\pi^l(g)x = (\pi(g)1)^*\pi(g)(x)$ and $\pi^l(g)x = \pi(g)(x)(\pi(g)1)^*$ for $g \in \Gamma^+$ and $x \in \mathcal{N}$. Assume that π^l and π^r are ergodic. We prove that π is locally rigid, that is, the orbit of π under $\mathbf{O}(L_p(\mathcal{N}))$ is open in $\mathrm{H}om(\Gamma, \mathbf{O}(L_p(\mathcal{N})))$. As a corollary, we obtain that, if moreover Γ is an ICC group, then the embedding $g \mapsto \mathrm{A}d(\lambda(g))$ is locally rigid in $\mathbf{O}(L_p(\mathcal{N}(\Gamma)))$, where $\mathcal{N}(\Gamma)$ is the von Neumann algebra generated by the left regular representation λ of Γ.

1. Introduction

Let Γ be a discrete group and G a topological group. A group homomorphism $\pi_0 : \Gamma \to G$ is locally rigid if every sufficiently small deformation of π_0 is trivial, in the sense that it is given by conjugation by elements from G. More precisely, let $\mathrm{H}om(\Gamma, G)$ be the set of all homomorphisms $\pi : \Gamma \to G$ endowed with the topology of pointwise convergence on Γ. The group G acts on $\mathrm{H}om(\Gamma, G)$ by conjugation:

$$\mathrm{Ad}(g)\pi(\gamma) = g\pi(\gamma)g^{-1} \qquad \text{for all} \quad g \in G, \gamma \in \Gamma.$$

We say that π_0 is *locally rigid* if its G-orbit in $\mathrm{H}om(\Gamma, G)$ is open.

Local rigidity was proved for the embedding of a cocompact lattice Γ in a semisimple real Lie group G by Calabi, Vesentini, Selberg, and Weil (see Chapter VII in [17]).

Groups with Kazhdan's Property (T) are defined by a rigidity property of their unitary group representations and play an important role in a large variety of subjects (for an account on Kazhdan's groups, see the monography [3]). It is natural to study local rigidity for homomorphisms from such groups to various topological groups G. As an example, it was shown in [18, Theorem1] that, if Γ is a (discrete) Kazhdan group, then every unitary representation $\Gamma \to U(n)$ is locally rigid in $GL_n(\mathbf{C})$. In recent years, there has been an increasing interest in local rigidity for homomorphisms with "infinite dimensional" groups as targets. For instance, a striking result in [9] shows that every action of a Kazhdan group by isometries on a compact Riemannian manifold is locally rigid in its group of diffeomorphisms. For an overview on local rigidity for actions of groups on various manifolds, see [22] and [8].

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In this paper, we study local rigidity for homomorphisms of discrete Kazhdan groups into the group of linear isometries of non-commutative L_p-spaces, that is, the L_p-spaces associated to a von Neumann algebra. Recently, Property (T) has been studied in the framework of group actions by isometries on Banach spaces and more specifically on $L_p(X,\mu)$ for a measure space (X,μ); see [1]. Some of the results from [1] were extended in [13] to non-commutative L_p-spaces.

Recall that a von Neumann algebra $\mathcal{N}$ is said to be finite if there exists a faithful normal finite trace τ on $\mathcal{N}$. Let $1 \leq p < \infty$; the non-commutative L_p-space $L_p(\mathcal{N})$ is the completion of $\mathcal{N}$ with respect to the norm defined by $\|x\|_p = (\tau(|x|^p))^{1/p}$ for $x \in \mathcal{N}$. For a survey on these spaces, see [15].

The von Neumann algebra $\mathcal{N}$ is a factor if the centre of $\mathcal{N}$ is reduced to the scalar operators. When $\mathcal{N}$ is a finite factor, then either $\mathcal{N}$ is finite dimensional, in which case $\mathcal{N}$ is isomorphic to a matrix algebra $M_n(\mathbf{C})$ equipped with the usual (normalized) trace, or $\mathcal{N}$ is a so-called type II_1 factor.

An important class of examples of type II_1 factors is given by ICC groups. Recall that the group Γ is ICC if its conjugacy classes, except $\{e\}$, are infinite. In this case, the von Neumann algebra $\mathcal{N}(\Gamma)$ of Γ is a (finite) factor. Recall that $\mathcal{N}(\Gamma)$ is the von Neumann algebra generated by the left regular representation λ of Γ on $\ell_2(\Gamma)$; thus, $\mathcal{N}(\Gamma)$ is the closure for the strong operator topology of the linear span of $\{\lambda(g) \ : \ g \in \Gamma\}$ in the algebra $\mathcal{B}(\ell_2(\Gamma))$.

A notion of Kazhdan's property (T) for von Neumann algebras was defined in [7] (see Section 2.3 below) and it was shown there that, for an ICC group Γ, the factor $\mathcal{N}(\Gamma)$ has Property (T) if and only if the group Γ has Kazhdan's property (T).

Let $1 \leq p < \infty$ and $\mathcal{N}$ a finite factor. The orthogonal group $\mathbf{O}(L_p(\mathcal{N}))$ of $L_p(\mathcal{N})$, that is, the group of bijective linear isometries of $L_p(\mathcal{N})$, is a topological group when endowed with the strong operator topology (see Section 2.1 below). Observe that every automorphism or anti-automorphism θ of $\mathcal{N}$ extends to a unique isometry of $L_p(\mathcal{N})$, since $L_p(\mathcal{N})$ contains $\mathcal{N}$ as dense subspace and since θ preserves the trace on $\mathcal{N}$. In this way, we identify the *extended automorphism group* $\mathrm{A}ut_{\mathrm{e}}(\mathcal{N})$, that is the group of automorphisms or anti-automorphisms of $\mathcal{N}$, with a subgroup of $\mathbf{O}(L_p(\mathcal{N}))$.

Let $p \neq 2$. The group $\mathbf{O}(L_p(\mathcal{N}))$ for $p \neq 2$ was described in [23] (see Theorem 2.1 below). It follows from this description that $\mathbf{O}(L_p(\mathcal{N}))$ contains a subgroup $\mathbf{O}^+(L_p(\mathcal{N}))$ of index at most 2 such that, for every U in $\mathbf{O}^+(L_p(\mathcal{N}))$, the mappings

$$U^l : x \mapsto U(1)^* U(x) \ \text{ and } \ U^r : x \mapsto U(x) U(1)^*$$

are automorphisms of $\mathcal{N}$.

If $\pi : \Gamma \to \mathbf{O}(L_p(\mathcal{N}))$ is a group homomorphism, we obtain in this way two actions of a subgroup Γ^+ of index at most 2 by automorphisms of $\mathcal{N}$, given by homomorphisms

$$\pi^l : \Gamma^+ \to \mathrm{A}ut(\mathcal{N}) \text{ and } \pi^r : \Gamma^+ \to \mathrm{A}ut_{\mathrm{e}}(\mathcal{N});$$

we call π^l and π^r the actions of Γ^+ by automorphisms associated to π (see Section 2.2). Recall that an action $\theta : \Gamma \to \mathrm{A}ut(\mathcal{N})$ of a group Γ by automorphisms on a von Neumann algebra $\mathcal{N}$ is *ergodic* if the fixed point algebra

$$\mathcal{N}^\Gamma = \{x \in \mathcal{N} : \theta_g(x) = x \quad \text{for all} \quad g \in \Gamma\}$$

consists only of the scalar multiples of 1.

Here is our main result.

Theorem 1.1. *Let* Γ *be a discrete group and* $\mathcal{N}$ *a finite factor, and assume that both have Property (T). For* $p \in [1, +\infty)$, $p \neq 2$, *let* $\pi : \Gamma \to \mathbf{O}(L_p(\mathcal{N}))$ *be a homomorphism from* Γ *to the group of linear bijective isometries of* $L_p(\mathcal{N})$. *Assume that the associated actions* π^l *and* π^r *of* Γ^+ *by automorphisms on* $\mathcal{N}$ *are both ergodic. Then* π *is locally rigid.*

Thus, there exists a neighbourhood $\mathcal{V}$ of π such that every ρ in $\mathcal{V}$ is conjugate to π by some U in $\mathbf{O}(L_p(\mathcal{N}))$. In fact, we will determine explicitly, in terms of Kazhdan pairs for Γ and $\mathcal{N}$, such a neighbourhood $\mathcal{V}$ for which U can be chosen to be close the identity (see Remark 3.1 below).

As we will see in Section 4, the various assumptions made in the statement of Theorem 1.1 are necessary in one form or another.

Since $\mathcal{N}(\Gamma)$ is a factor when Γ is an ICC group, the action $g \mapsto \mathrm{A}d(\lambda(g))$ of Γ by automorphisms of $\mathcal{N}(\Gamma)$ is ergodic; so, the following corollary is an immediate consequence of the previous theorem.

Corollary 1.2. *Let* Γ *be an ICC group with Kazhdan's Property (T). The embedding* $g \mapsto \mathrm{A}d(\lambda(g))$ *of* Γ *in* $\mathbf{O}(L_p(\mathcal{N}(\Gamma)))$ *is locally rigid, for* $p \in [1, +\infty)$, $p \neq 2$.

Remark 1.3. There is another natural action of Γ by isometries on $L_p(\mathcal{N}(\Gamma))$: it is given by the embedding

$$\pi_0 : \Gamma \to \mathbf{O}(L_p(\mathcal{N}(\Gamma))), g \mapsto \lambda(g),$$

where $\lambda(g)$ denotes the extension from $\mathcal{N}(\Gamma)$ to $L_p(\mathcal{N}(\Gamma))$ of multiplication from the left by the unitary $\lambda(g)$. As will be seen below (Example 4.2), π_0 may fail to be locally rigid when Γ is an ICC group with Property (T). In contrast, it can be shown that, if we view π_0 as a homomorphism in the unitary group $\mathbf{U}(\mathcal{N}(\Gamma))$ of $\mathcal{N}(\Gamma)$, then π_0 is locally rigid.

Apart from Yeadon's description of the group of isometries of $L_p(\mathcal{N})$ for $p \neq 2$, the proof of Theorem 1.1 depends on the following three ingredients: the first one (see Proposition 2.3 below) is that $\mathbf{O}(L_p(\mathcal{N}))$ is isomorphic, as *topological* group, to an appropriate subgroup of the group of isometries of the Hilbert space $L_2(\mathcal{N})$; the second ingredient is the fact (see [7]) that the group of outer automorphisms of a factor with Property (T) is discrete; the third ingredient is an extension of a result from [11] showing that certain 1-cohomology classes associated to actions of a Kazhdan group by automorphisms of a finite factor are open (see Proposition 2.10). For this, we use in a crucial way a rigidity property of projective unitary representations of Kazhdan groups from [12].

This paper is organized as follows. In Section 2, we collect the ingredients necessary for the proof of our main result. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we present counter-examples in relation with the various assumptions made in the statement of Theorem 1.1.

2. Preliminaries

Let $\mathcal{N}$ be a *finite factor*, fixed throughout this section.

2.1. **The group of isometries of $L_p(\mathcal{N})$.** The following result is a corollary of Yeadon's description of the linear (not necessarily surjective) isometries of the non-commutative L_p space of a semi-finite von Neumann algebra for $p \neq 2$ (see [23, Theorem 2]). For an extension of Yeadon's result to arbitrary (not necessarily semi-finite) Neumann algebras, see [19].

Theorem 2.1. ([23]) *Let $1 \leq p < \infty$ and $p \neq 2$. A mapping $U : L_p(\mathcal{N}) \to L_p(\mathcal{N})$ is a linear surjective isometry if and only if there exists a unique pair (u, θ) consisting of a unitary $u \in \mathcal{N}$ and an automorphism or an anti-automorphism θ of $\mathcal{N}$ such that*

$$U(x) = u\theta(x) \qquad \textit{for all} \quad x \in \mathcal{N}.$$

As this result is not explicitly stated in [23], we indicate how it follows from there. Since U is surjective, by [23, Theorem 2], there exist a normal Jordan isomorphism $J : \mathcal{N} \to \mathcal{N}$, a unitary $u \in \mathcal{N}$, and a positive self-adjoint operator B affiliated with $\mathcal{N}$ such that $U(x) = uBJ(x)$ for all $x \in \mathcal{N}$. Since $\mathcal{N}$ is factor, J is either an automorphism or an anti-automorphism (see [4, Proposition 3.2.2]) and $B = 1$.

The group $\mathbf{O}(L_p(\mathcal{N}))$ of linear bijective isometries of $L_p(\mathcal{N})$ is a topological group when equipped with the strong operator topology; this is the topology for which a fundamental system family of neighbourhoods of U in $\mathbf{O}(L_p(\mathcal{N}))$ is given by subsets of the form

$$\{V \in \mathbf{O}(L_p(\mathcal{N})) \,:\, \|V(x_i) - U(x_i)\|_p \leq \varepsilon\}$$

for $x_1, \ldots, x_n \in L_p(\mathcal{N})$ and $\varepsilon > 0$.

We identify the extended automorphism group $\mathrm{A}ut_{\mathrm{e}}(\mathcal{N})$, the automorphism group $\mathrm{A}ut(\mathcal{N})$ and the unitary group $\mathbf{U}(\mathcal{N})$ of $\mathcal{N}$ with subgroups of $\mathbf{O}(L_p(\mathcal{N}))$, endowed with the topology induced by that of $\mathbf{O}(L_p(\mathcal{N}))$. As is easy to show, the topology on $\mathbf{U}(\mathcal{N})$ coincides with the topology induced by its embedding in $L_p(\mathcal{N})$ given by $u \mapsto u1$.

For $U \in \mathbf{O}(L_p(\mathcal{N}))$, we will often write $U = (u, \theta)$ for u in $\mathbf{U}(\mathcal{N})$ and θ in $\mathrm{A}ut_{\mathrm{e}}(\mathcal{N})$ and refer to the pair (u, θ) as the *Yeadon decomposition* of U.

The set of isometries U with Yeadon decomposition (u, θ) for which θ is an automorphism of $\mathcal{N}$ is a closed subgroup of index at most 2 in $\mathbf{O}(L_p(\mathcal{N}))$ and will be denoted by $\mathbf{O}^+(L_p(\mathcal{N}))$.

Observe that $\mathbf{U}(\mathcal{N})$ is normal in $\mathbf{O}^+(L_p(\mathcal{N}))$ but, in general, not normal in $\mathbf{O}(L_p(\mathcal{N}))$.

It follows from Yeadon's result and from [10, Theorem 2] that the subgroup $\mathbf{O}^+(L_p(\mathcal{N}))$ can be intrinsically characterized inside $\mathbf{O}(L_p(\mathcal{N})$ as the subgroup of the complete isometries (or as the subgroup of 2-isometries) of $L_p(\mathcal{N})$ in the sense of operator spaces, that is, the isometries U of $L_p(\mathcal{N})$ such that $\mathrm{id} \otimes U$ is an isometry of $L_p(M_n(\mathbf{C}) \otimes \mathcal{N})$ for every $n \in \mathbf{N}$ (or such that $\mathrm{id} \otimes U$ is an isometry of $L_p(M_2(\mathbf{C}) \otimes \mathcal{N})$) It should be mentioned that completely isometric or, more generally, completely bounded mappings are natural objects to study in the context of operator algebras (see [14]).

Corollary 2.2. *For $1 \leq p < \infty$ and $p \neq 2$, the group $\mathbf{O}^+(L_p(\mathcal{N}))$ is isomorphic as topological group to the topological semi-direct product $\mathbf{U}(\mathcal{N}) \rtimes \mathrm{A}ut(\mathcal{N})$, given by the natural action of $\mathrm{A}ut(\mathcal{N})$ on $\mathbf{U}(\mathcal{N})$.*

Proof The fact that $\mathbf{O}^+(L_p(\mathcal{N}))$ is isomorphic as abstract group to $\mathbf{U}(\mathcal{N}) \rtimes \mathrm{A}ut(\mathcal{N})$ is a consequence of Yeadon's theorem. Moreover, evaluation at $1 \in \mathcal{N}$

shows that the projection

$$\mathbf{O}^+(L_p(\mathcal{N})) \cong \mathbf{U}(\mathcal{N}) \rtimes \mathrm{A}ut(\mathcal{N}) \to \mathbf{U}(\mathcal{N}), \qquad (u,\theta) \mapsto u$$

is continuous. Hence, the projection $\mathbf{O}^+(L_p(\mathcal{N})) \to \mathrm{A}ut(\mathcal{N})$ is also continuous; so, $\mathbf{O}^+(L_p(\mathcal{N}))$ and $\mathbf{U}(\mathcal{N}) \rtimes \mathrm{A}ut(\mathcal{N})$ are isomorphic as topological groups. ■

Every $U = (u,\theta)$ in $\mathbf{O}(L_p(\mathcal{N}))$ defines, for every $1 \leq q < \infty$, a linear bijective isometry of $L_q(\mathcal{N})$ by the same formula: $U(x) = u\theta(x)$ for all x in the dense subspace $\mathcal{N}$ of $L_q(\mathcal{N})$. One obtains in this way a mapping

$$\Phi_{p,q} : \mathbf{O}(L_p(\mathcal{N})) \to \mathbf{O}(L_q(\mathcal{N})).$$

For $q \neq 2$, this mapping is of course surjective; this is not the case for $q = 2$ (if $\mathcal{N}$ is infinite dimensional) and we define the $\mathcal{N}$-unitary groups $\mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$ and $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ of the Hilbert space $L_2(\mathcal{N})$ to be the images of $\mathbf{O}(L_p(\mathcal{N}))$ and $\mathbf{O}^+(L_p(\mathcal{N}))$ under $\Phi_{p,2}$; thus, $\mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$ (respectively $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$) is the group of unitary operators U of $L_2(\mathcal{N})$ which have a Yeadon decomposition $U = (u,\theta)$ for $u \in \mathbf{U}(\mathcal{N})$ and $\theta \in \mathrm{A}ut_{\mathrm{e}}(\mathcal{N})$ (respectively $\theta \in \mathrm{A}ut(\mathcal{N})$).

The next proposition will allow us to transfer representations in $\mathbf{O}(L_p(\mathcal{N}))$ to representations in the $\mathcal{N}$-unitary group $\mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$ of $L_2(\mathcal{N})$. Its proof uses in a crucial way properties of the *Mazur map*, which is the (non linear) mapping $M_{p,q} : L_p(\mathcal{N}) \to L_q(\mathcal{N})$ defined by

$$M_{p,q}(x) = u|x|^{\frac{p}{q}}$$

for $x \in L_p(\mathcal{N})$ with polar decomposition $x = u|x|$.

Proposition 2.3. *Let $1 \leq p,q < \infty$ and $p,q \neq 2$. The mappings*

$$\Phi_{2,p} : \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N})) \to \mathbf{O}(L_p(\mathcal{N})), \Phi_{p,2} : \mathbf{O}(L_p(\mathcal{N})) \to \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N})),$$
$$\text{and} \qquad \Phi_{p,q} : \mathbf{O}(L_p(\mathcal{N})) \to \mathbf{O}(L_q(\mathcal{N}))$$

are continuous. In particular, the groups $\mathbf{O}(L_p(\mathcal{N}))$ for $p \neq 2$ and $\mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$ are mutually isomorphic as topological groups.

Proof It suffices to prove that the mappings $\Phi_{2,p}, \Phi_{p,2}$ and $\Phi_{p,q}$ are continuous on the open subgroups $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ and $\mathbf{O}^+(L_p(\mathcal{N}))$ for $p \neq 2$.

For $U \in \mathbf{O}^+(L_p(\mathcal{N}))$ with Yeadon decomposition (a,θ) and $x \in \mathcal{N}$ with polar decomposition $x = u|x|$, we have

$$\begin{aligned} U(x) &= a\theta(u)\theta(|x|) = a\theta(u)\theta(|x|^{\frac{q}{p}})^{\frac{p}{q}} \\ &= M_{p,q}(a\theta(M_{q,p}(x)) = M_{p,q} \circ U \circ M_{q,p}(x), \end{aligned}$$

so that

$$\Phi_{p,q}(U) = M_{p,q} \circ U \circ M_{q,p} \qquad \text{for all } \ U \in \mathbf{O}^+(L_p(\mathcal{N})).$$

It is known that (even for a general von Neumann algebra $\mathcal{N}$) the restriction $M_{p,q} : B_1(L_p(\mathcal{N})) \to B_1(L_q(\mathcal{N}))$ of $M_{p,q}$ to the unit ball $B_1(L_p(\mathcal{N}))$ of $L_p(\mathcal{N}))$ is uniformly continuous (see [20, Lemma 3.2]; a more precise result is proved in [21]: $M_{p,q}$ is $\min\{p/q, 1\}$–Hölder continuous on $B_1(L_p(\mathcal{N}))$). Since

$$\|\Phi_{p,q}(U)(x) - \Phi_{p,q}(V)(x)\|_q = \|M_{p,q}(U(M_{q,p}(x)) - M_{p,q}(V(M_{q,p}(x))\|_q,$$

for $U, V \in \mathbf{O}^+(L_p(\mathcal{N}))$ and $x \in L_q(\mathcal{N})$, the proposition follows. ■

2.2. **Group representations by linear complete isometries on $L_p(\mathcal{N})$.** Let Γ be a discrete group and $p \in [1, +\infty[$, $p \neq 2$, fixed throughout this section.

For a mapping $\pi : \Gamma \to \mathbf{O}(L_p(\mathcal{N}))$ or $\pi : \Gamma \to \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$, we have corresponding mappings $u : \Gamma \to \mathbf{U}(\mathcal{N})$ and $\theta : \Gamma \to \mathrm{A}ut_{\mathrm{e}}(\mathcal{N})$ given by the Yeadon decomposition $\pi(g) = (u_g, \theta_g)$ for every $g \in \Gamma$. We will refer to $\pi = (u, \theta)$ as the Yeadon decomposition of π. Observe that, if π is a homomorphism, then $\theta : \Gamma \to \mathrm{A}ut_{\mathrm{e}}(\mathcal{N})$ is in general not a homomorphism; however, if π takes its values in $\mathbf{O}^+(L_p(\mathcal{N}))$, then $\theta : \Gamma \to \mathrm{A}ut(\mathcal{N})$ is indeed a homomorphism.

Given a group homomorphism $\theta : \Gamma \to \mathrm{A}ut(\mathcal{N}), g \mapsto \theta_g$, we denote by $Z^1(\Gamma, \theta)$ the set of all corresponding 1-cocycles, that is, the set of mappings $u : \Gamma \to \mathbf{U}(\mathcal{N})$ such that

$$u_{gh} = u_g \theta_g(u_h) \qquad \text{for all} \quad g, h \in \Gamma.$$

Two 1-cocycles u and v are cohomologous, if there exists $w \in \mathbf{U}(\mathcal{N})$ such that

$$v_g = w u_g \theta_g(w^*) \qquad \text{for all} \quad g \in \Gamma.$$

The set of 1-coboundaries $B^1(\Gamma, \theta)$ is the set of 1-cocycles which are cohomologous to the trivial cocycle $g \mapsto 1$.

The proof of following proposition is straightforward.

Proposition 2.4. *Let $\pi : \Gamma \to \mathbf{O}^+(L_p(\mathcal{N}))$ or $\pi : \Gamma \to \mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ be a mapping with Yeadon decomposition $\pi = (u, \theta)$. The following conditions are equivalent.*

(i) *π is a group homomorphism;*
(ii) *$\theta : \Gamma \to \mathrm{A}ut(\mathcal{N})$ is a group homomorphism and $u : \Gamma \to \mathbf{U}(\mathcal{N})$ is a 1-cocycle with respect to θ.* ■

Given $\pi \in \mathrm{H}om(\Gamma, \mathbf{O}^+(L_p(\mathcal{N}))$, with Yeadon decomposition (u, θ), there are two associated actions π^l and π^r in $\mathrm{H}om(\Gamma, \mathrm{A}ut(\mathcal{N})$ defined by

$$\pi^l(g) = \theta_g \text{ and } \pi^r(g) = \mathrm{A}d(u_g)\theta_g \qquad \text{for all} \quad g \in \Gamma.$$

(For $u \in \mathbf{U}(\mathcal{N})$, $\mathrm{A}d(u)$ denotes the automorphism of $\mathcal{N}$ given by $\mathrm{A}d(u)x = uxu^*$ for $x \in \mathcal{N}$.)

Recall that $G = \mathbf{O}^+(L_p(\mathcal{N}))$ or $G = \mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ acts on the set of all mappings $\pi : \Gamma \to G$ by conjugation $\mathrm{Ad}(U)\pi(g) = U\pi(g)U^{-1}$ for $U \in G$, $g \in \Gamma$.

Proposition 2.5. *Let π and ρ be homomorphisms from Γ to $G = \mathbf{O}^+(L_p(\mathcal{N}))$ or $G = \mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$, with Yeadon decompositions $\pi = (u, \theta)$ and $\rho = (v, \alpha)$. The following conditions are equivalent:*

(i) *ρ belongs to the G-orbit of π;*
(ii) *there exists $\varphi \in \mathrm{Aut}(\mathcal{N})$ such that $\alpha = \mathrm{Ad}(\varphi)(\theta)$ and such that v is cohomologous to $\mathrm{Ad}(\varphi)(u) : g \mapsto \varphi(u_g)$ in $Z^1(\Gamma, \mathrm{Ad}(\varphi)(\theta))$.*

Proof For an element $U = (w, \varphi)$ in G, one computes that the Yeadon decomposition (v, α) of $\mathrm{A}d(U)\pi$ is given by

$$\alpha_g = \mathrm{Ad}(\varphi)(\theta_g) = \varphi \theta_g \varphi^{-1}, \quad v_g = w\varphi(u_g)(\mathrm{A}d(\varphi)(\theta_g))(w^*)$$

for every $g \in \Gamma$ and the claim follows.■

2.3. **Groups and factors with Kazhdan's Property (T).** We recall (see [3]) that a (discrete) group Γ has Kazhdan's Property (T) if there exist a finite subset S of Γ and $\varepsilon > 0$ with the following property: if a unitary representation $\pi : \Gamma \to \mathbf{U}(\mathcal{H})$ of Γ in a Hilbert space $\mathcal{H}$ has a (S, ε)-invariant unit vector, that is, a unit vector $v \in \mathcal{H}$ with

$$\|\pi(s)v - v\| \leq \varepsilon \qquad \text{for all} \quad s \in S,$$

then there exists a non-zero Γ-invariant vector in $\mathcal{H}$. The pair (S, ε) is called a *Kazhdan pair* for Γ. Moreover, if this is the case, then for every $\delta > 0$ and every $(S, \delta\varepsilon)$-invariant unit vector v, there exists a Γ-invariant vector $w \in \mathcal{H}$ with $\|v - w\| \leq \delta$ (see Proposition 1.1.9 in [3]).

We shall need the extension from [12] of the previous result to projective unitary representations; recall that a projective unitary representation of Γ in a Hilbert space $\mathcal{H}$ is a mapping π from Γ to the unitary group $\mathbf{U}(\mathcal{H})$ of $\mathcal{H}$ such that, for every $g, h \in G$, there exists a scalar $\mu_{g,h} \in \mathbf{S}^1 = \{\lambda \in \mathbf{C} : \ |z| = 1\}$ with

$$\pi(g)\pi(h) = \mu_{g,h}\pi(gh) \qquad \text{for all} \quad g, h \in \Gamma.$$

A projective unitary representation π determines a homomorphism $\widetilde{\pi} : \Gamma \to \mathbf{PU}(\mathcal{H})$ to the projective unitary group $\mathbf{PU}(\mathcal{H}) = \mathbf{U}(\mathcal{H})/\mathbf{S}^1$ of $\mathbf{U}(\mathcal{H})$, where $\mathbf{S}^1$ is identified with the subgroup of scalar multiples of the identity operator $\mathrm{Id}_{\mathcal{H}}$; conversely, every lift $\pi : \Gamma \to \mathbf{U}(\mathcal{H})$ of a homomorphism $\widetilde{\pi} : \Gamma \to \mathbf{PU}(\mathcal{H})$ is a projective unitary representation of Γ. The mapping $\mu : \Gamma \times \Gamma \to \mathbf{S}^1, (g, h) \mapsto \mu_{g,h}$ is a 2-cocycle, that is, it satisfies the identity

$$\mu_{h,k}\mu_{g,hk} = \mu_{g,h}\mu_{gh,k} \text{ for all}\, g, h, k \in \Gamma.$$

If $\mu : \Gamma \times \Gamma \to \mathbf{S}^1$ is a 2-coboundary, that is, if there exists a mapping $\lambda : \Gamma \to \mathbf{S}^1$ such that

$$\mu_{g,h} = \lambda_g\lambda_h\overline{\lambda_{gh}} \qquad \text{for all} \quad g, h \in \Gamma,$$

then π gives rise to a genuine representation $\overline{\pi} : \Gamma \to \mathbf{U}(\mathcal{H})$, defined by

$$\overline{\pi}(g) = \overline{\lambda_g}\pi(g) \text{ for all}\, g \in \Gamma$$

and inducing the same homomorphism $\Gamma \to \mathbf{PU}(\mathcal{H})$ as π.

Given a projective unitary representation $\pi : \Gamma \to \mathbf{U}(\mathcal{H})$ and a subset S of Γ and $\varepsilon > 0$, we will say that a unit vector $v \in \mathcal{H}$ is *projectively* (S, ε)*-invariant* if, for every $s \in S$, there exists $\alpha_s \in \mathbf{C}$ such that $\|\pi(s)v - \alpha_s v\| \leq \varepsilon$.

The following result is proved in [12] in the more general situation of a pair of groups with the relative Property (T). When Γ has Property (T), with a Kazhdan pair (S, ε), one checks easily that the proof of Lemma 1.1 of [12] yields exactly the following result.

Theorem 2.6. ([12]) *Let Γ be a Kazhdan group, with a Kazhdan pair (S, ε). Fix δ with $0 < \delta < 1$. Let $\pi : \Gamma \to \mathbf{U}(\mathcal{H})$ be a projective unitary representation of π, with corresponding 2-cocycle $\mu : \Gamma \times \Gamma \to \mathbf{S}^1$, and let $v \in \mathcal{H}$ be unit vector which is projectively $(S, \varepsilon\delta^2/56)$-invariant. Then there exists a mapping $\lambda : \Gamma \to \mathbf{S}^1$ with $\mu_{g,h} = \lambda_g\lambda_h\overline{\lambda}_{gh}$ for all $g, h \in \Gamma$ and a vector $v_0 \in \mathcal{H}$ such that*

$$\|v - v_0\| \leq \delta \qquad \textit{and} \qquad \pi(g)v_0 = \lambda_g v_0$$

for all $g \in \Gamma$. In particular, μ is a coboundary.

We now recall Property (T) for von Neumann algebras as defined in [7].

Let $\mathcal{N}$ be finite factor. A *Hilbert bimodule* over $\mathcal{N}$ is a Hilbert space $\mathcal{H}$ carrying two commuting normal representations, one of $\mathcal{N}$ and one of the opposite algebra $\mathcal{N}^0$; we will write

$$v \mapsto xvy \qquad \text{for all } v \in \mathcal{H},\ x, y \in \mathcal{N}.$$

The factor $\mathcal{N}$ is said to have Property (T) if there exist a finite subset F of $\mathcal{N}$ and $\varepsilon' > 0$ such that the following property holds: if a Hilbert bimodule $\mathcal{H}$ for $\mathcal{N}$ contains a unit vector v which is (F, ε')-central, that is, which is such that

$$\|xv - vx\| \leq \varepsilon' \qquad \text{for all} \quad x \in F,$$

then $\mathcal{H}$ has a non-zero central vector, that is, a non-zero vector $w \in \mathcal{H}$ such that $xw = wx$ for all $x \in \mathcal{N}$. Moreover, one can choose (F, ε') such that for every $\delta > 0$ and every $(F, \delta\varepsilon')$-central unit vector v, there exists a central vector $w \in \mathcal{H}$ with $\|v - w\| \leq \delta$ (see Proposition 1 in [7]). We call such a pair (F, ε') a *Kazhdan pair* for $\mathcal{N}$.

It was shown in [7] (see Proposition 12.1.19 in [5]) that the subgroup $\mathrm{I}nn(\mathcal{N})$ of inner automorphisms of $\mathcal{N}$ (that is, the subgroup of automorphisms of the form $\mathrm{A}d(u)$ for $u \in \mathbf{U}(\mathcal{N})$) is open in $\mathrm{A}ut(\mathcal{N})$. Here, $\mathrm{A}ut(\mathcal{N})$ is endowed with the topology of pointwise L_2-convergence (we could also take the induced topology from $\mathbf{O}^+(L_p(\mathcal{N}))$ as above for any $1 \leq p < \infty$). We will need a quantitative estimate, in terms of a Kazhdan pair (F, ε'), for the distance to 1 of the unitary operators defining the appropriate inner automorphisms.

Proposition 2.7. *Let $\mathcal{N}$ be a finite factor with Property (T). Let (F, ε') be a Kazhdan pair for $\mathcal{N}$. Let $0 < \delta < 1$ and let $\mathcal{V}_\delta$ be the neighbourhood of the trivial automorphism* $\mathrm{id}_{\mathcal{N}}$ *given by*

$$\mathcal{V}_\delta = \{\theta \in \mathrm{A}ut(\mathcal{N}) \,:\, \|\theta(x) - x\|_2 \leq \varepsilon'\delta/2 \textit{ for all } x \in F\}.$$

Then $\mathcal{V}_\delta$ is contained in $\mathrm{Inn}(\mathcal{N})$. *More precisely, for every θ in $\mathcal{V}_\delta$, there exists u in $\mathbf{U}(\mathcal{N})$ with $\theta = \mathrm{A}d(u)$ and $\|u - 1\|_2 \leq \delta$.*

Proof We follow the standard proof that $\mathrm{I}nn(\mathcal{N})$ is open in $\mathrm{A}ut(\mathcal{N})$ as given, for instance, in the proof of Proposition 12.1.19 in [5].

Let $\theta \in \mathcal{V}_\delta$. We define a bimodule structure on $L_2(\mathcal{N})$ over $\mathcal{N}$ by

$$v \mapsto \theta(x)vy \ \text{ for all } \ v \in L_2(\mathcal{N}),\ x, y \in \mathcal{N}.$$

Then $1 \in L_2(\mathcal{N})$ is a unit vector which is $(F, \varepsilon'\delta/2)$-central. Hence, there exists a central vector $w \in L_2(\mathcal{N})$ with $\|w - 1\|_2 \leq \delta/2$. Let $w = u|w|$ be the polar decomposition of w, viewed as a densely defined operator on $L_2(\mathcal{N})$ affiliated to $\mathcal{N}$. Then, $|w|$ is in the center of $\mathcal{N}$ and hence $|w| = \lambda 1$ for some $\lambda > 0$. It follows that u is a unitary element in $\mathcal{N}$ such that $\theta = \mathrm{A}d(u)$. As $\|w\|_2 = \lambda$, we have $|1 - \lambda| \leq \|w - 1\|_2 \leq \delta/2$ and therefore

$$\|u - 1\|_2 \leq \|u - w\|_2 + \|w - 1\|_2 = |1 - \lambda| + \|w - 1\|_2 \leq \delta. \ \blacksquare$$

2.4. **Projective 1-cocycles for actions of Kazhdan groups.** In the sequel, we will need to deal with mappings $\Gamma \to \mathbf{U}(\mathcal{N})$ which are 1- cocycles for an action of Γ on $\mathcal{N}$ modulo scalars in the following sense (cocycles of this type appear in Section 1.3 of [16], where they are called weak 1-cocycles).

Definition 2.8. Let Γ be a group, $\mathcal{N}$ a von Neumann algebra, and $\theta : \Gamma \to \mathrm{A}ut(\mathcal{N})$ a homomorphism. A *projective* 1*-cocycle* for θ is a mapping $u : \Gamma \to \mathbf{U}(\mathcal{N})$ such that, for every $g, h \in \Gamma$, there exists a scalar $\mu_{g,h} \in \mathbf{S}^1$ with

$$u_g\theta_g(u_h) = \mu_{g,h}u_{gh}.$$

Two projective 1-cocycles u and v are cohomologous if there exist w in $\mathbf{U}(\mathcal{N})$ and a mapping $\lambda : \Gamma \to \mathbf{S}^1$ such that

$$v_g = \lambda_g w u_g \theta_g(w^*) \qquad \text{for all} \quad g \in \Gamma.$$

A *projective coboundary* is a projective 1-cocycle which is cohomologous to the trivial cocycle $g \mapsto 1$.

The following lemma, which can be checked by a straightforward computation, shows that projective cocycles appear naturally.

Lemma 2.9. *Let Γ be a group, $\mathcal{N}$ a factor, and $\theta : \Gamma \to \mathrm{A}ut(\mathcal{N})$ a homomorphism. For a mapping $u : \Gamma \to \mathbf{U}(\mathcal{N})$, the following properties are equivalent.*

(i) *u is a projective 1-cocycle for θ;*
(ii) *the mapping $g \mapsto \mathrm{Ad}(u_g)\theta_g$ is a homomorphism from Γ to $\mathrm{A}ut(\mathcal{N})$.*■

We denote by $Z^1_{\rm proj}(\Gamma, \theta)$ and by $B^1_{\rm proj}(\Gamma, \theta)$ the set of projective 1-cocycles and coboundaries for θ. We equip $Z^1_{\rm proj}(\Gamma, \theta)$ with the topology of pointwise L_2-convergence: a sequence $(u^{(n)})_n$ in $Z^1_{\rm proj}(\Gamma, \theta)$ converges to $u \in Z^1_{\rm proj}(\Gamma, \theta)$ if $\lim_n \|u^{(n)}_g - u_g\|_2 = 0$ for every $g \in \Gamma$.

Assume now that Γ has Property (T). We will need to know that cohomology classes in $Z^1_{\rm proj}(\Gamma, \theta)$ are open. This is not true in general even for classes in $Z^1(\Gamma, \theta)$ and even when θ is ergodic (see Examples 4 and 8 in [11]). However, the following result was shown in [11, Theorem 7]. Let $u \in Z^1(\Gamma, \theta)$ be such that the action of Γ on $\mathcal{N}$ given by $g \mapsto \mathrm{A}d(u_g)\theta_g$ is ergodic; then the equivalence class of u is open in $Z^1(\Gamma, \theta)$. Following the same proof and making crucial use of Theorem 2.6, we now show that a quantitative version of this result is true for projective 1-cocycles.

Proposition 2.10. *Let Γ be a group with Kazhdan's Property (T), with a Kazhdan pair (S, ε). Let $\mathcal{N}$ be a finite factor and $\theta : \Gamma \to \mathrm{A}ut(\mathcal{N})$ a homomorphism. Let $u : \Gamma \to \mathbf{U}(\mathcal{N})$ be a projective 1-cocycle for θ. Assume that the action of Γ on $\mathcal{N}$ given by $g \mapsto \mathrm{Ad}(u_g)\theta_g$ is ergodic. Fix $0 < \delta < 1$ and let $\mathcal{U}_\delta$ be the neighbourhood of u in $Z^1_{\rm proj}(\Gamma, \theta)$ defined by*

$$\mathcal{U}_\delta = \{v \in Z^1_{\rm proj}(\Gamma, \theta) : \|v_s - u_s\|_2 \le \varepsilon\delta^2/224 \text{ for all } s \in S\}.$$

Then every $v \in \mathcal{U}_\delta$ is cohomologous to u. More precisely, for every $v \in \mathcal{U}_\delta$, there exists $w \in \mathbf{U}(\mathcal{N})$ with $\|w - 1\|_2 \le \delta$ and a mapping $\lambda : \Gamma \to \mathbf{S}^1$ such that $v_g = \lambda_g w u_g \theta_g(w^*)$ for all $g \in \Gamma$.

Proof We adapt the proof from [11, Theorem 7], making it quantitative at the appropriate places. Let $v \in \mathcal{U}_\delta$. For every $g \in \Gamma$, let $\pi(g)$ be the unitary operator on $L_2(\mathcal{N})$ given by

$$\pi(g)x = u_g\theta_g(x)v_g^* \qquad \text{for all} \quad x \in \mathcal{N}.$$

Since u and v are projective 1-cocycles for θ, the mapping $\pi : g \mapsto \pi(g)$ is a projective unitary representation of Γ, as is easily checked. Let $w : \Gamma \times \Gamma \to \mathbf{S}^1$ be the corresponding 2-cocycle. Observe that $1 \in L_2(\mathcal{N})$ is a unit vector which is $(S, \varepsilon\delta^2/224)$-invariant. Hence, it follows from Theorem 2.6 that there exists a mapping $\lambda : \Gamma \to \mathbf{S}^1$ with

$$\mu_{g,h} = \lambda_g \lambda_h \overline{\lambda}_{gh} \qquad \text{for all} \quad g, h \in \Gamma$$

and a vector $b \in L_2(\mathcal{N})$ such that $\|b - 1\| \leq \delta/2$ and $\pi(g)b = \lambda_g b$ for all $g \in \Gamma$. Thus, $b \neq 0$ and $u_g \theta_g(b) v_g^* = \lambda_g b$ for every $g \in \Gamma$. We view b as a densely defined operator on $L_2(\mathcal{N})$ affiliated to $\mathcal{N}$. Taking adjoints, we see that the positive operator $bb^* \in L^1(\mathcal{N})$ is fixed by the extension to $L^1(\mathcal{N})$ of $\mathrm{A}d(u_g)\theta_g$ for every $g \in \Gamma$. Since $g \mapsto \mathrm{A}d(u_g)\theta_g$ is ergodic, it follows that $bb^* = \beta 1$ for some $\beta > 0$ Then $w := b^*/\sqrt{\beta}$ is a unitary element in $\mathcal{N}$ such that

$$v_g = \overline{\lambda_g} w u_g \theta_g(w^*) \qquad \text{for all} \quad g \in \Gamma.$$

Moreover, as in the proof of Proposition 2.7, we have $\|w - 1\|_2 \leq \delta$. ■

One can improve upon the constant defining $\mathcal{U}_\delta$ in the previous proposition, when one deals with genuine 1-cocycles instead of projective ones; indeed, in this case, the projective unitary representation appearing in the proof is a true unitary representation and one checks that the following statement holds.

Proposition 2.11. *Let Γ, (S, ε), $\mathcal{N}$ and θ be as in Proposition 2.10. Let $u : \Gamma \to \mathbf{U}(\mathcal{N})$ be a 1-cocycle for θ such that $g \mapsto \mathrm{A}d(u_g)\theta_g$ is ergodic. For $0 < \delta < 1$, set*

$$\mathcal{U}_\delta = \{v \in Z^1(\Gamma, \theta) : \|v_s - u_s\|_2 \leq \varepsilon\delta/2 \text{ for all } s \in S\}.$$

Then, for every $v \in \mathcal{U}_\delta$, there exists $w \in \mathbf{U}(\mathcal{N})$ with $\|w - 1\|_2 \leq \delta$ such that

$$v_g = w u_g \theta_g(w^*) \qquad \text{for all} \quad g \in \Gamma. \blacksquare$$

3. Proof of Theorem 1.1

Let $\pi : \Gamma \to \mathbf{O}(L_p(\mathcal{N}))$ be a group homomorphism for $p \neq 2$. Then $\Phi_{p,2} \circ \pi$ is a group homomorphism from Γ to the $\mathcal{N}$-unitary group $\mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$ as defined in Section 2.1, where $\Phi_{p,2} : \mathbf{O}(L_p(\mathcal{N})) \to \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$ is the identity mapping. By Proposition 2.3, $\mathbf{O}(L_p(\mathcal{N}))$ and $\mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$ are topologically isomorphic groups. Hence, to prove that $\pi : \Gamma \to \mathbf{O}(L_p(\mathcal{N}))$ is locally rigid amounts to prove that $\Phi_{p,2} \circ \pi : \Gamma \to \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N}))$ is locally rigid. So, we can replace π by $\Phi_{p,2} \circ \pi$.

Set $\Gamma^+ := \pi^{-1}(\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$; then Γ^+ is a normal subgroup of index at most 2 in Γ.

Let $\pi = (u, \theta)$ be the Yeadon decomposition of π. Recall that the associated homomorphisms $\pi^l, \pi^r \in \mathrm{H}om(\Gamma^+, \mathrm{A}ut(\mathcal{N}))$ are given by

$$\pi^l(g) = \theta_g \qquad \text{and} \qquad \pi^r(g) = \mathrm{A}d(u_g)\theta_g$$

for every $g \in \Gamma^+$.

Assume now that Γ and $\mathcal{N}$ have both Property (T) and that π^l and π^r are ergodic.

We first prove Theorem 1.1 in the case where π takes its values in $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ and will then reduce the general case to this situation.

3.1. **The case $\Gamma = \Gamma^+$.** In this case, $\theta \in \mathrm{H}om(\Gamma, \mathrm{A}ut(\mathcal{N}))$ and $u \in Z^1(\Gamma, \theta)$; see Section 2.2. Recall that, by Corollary 2.2 and Proposition 2.3, $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ is topologically isomorphic to the topological semi-direct product $\mathbf{U}(\mathcal{N}) \rtimes \mathrm{A}ut(\mathcal{N})$.

Let (S, ε) be a Kazhdan pair for Γ and (F, ε') a Kazhdan pair for $\mathcal{N}$. We can assume that S is a generating set for Γ, since Γ is finitely generated (see Theorem 1.3.1 in [3]). Moreover, since $\mathrm{A}ut(\mathcal{N})$ is open in $\mathrm{A}ut_{\mathrm{e}}(\mathcal{N})$, upon enlarging F and reducing ε' if necessary, we can also assume that

$$\max_{x\in F} \|\tau(x) - x\|_2 > \varepsilon'$$

for every anti-automorphism τ of $\mathcal{N}$.

Fix $0 < \delta < 1$ and define $\mathcal{V}_\delta$ to be the neighbourhood of π in $\mathrm{H}om(\Gamma, \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N})))$ consisting of all $\rho \in \mathrm{H}om(\Gamma, \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N})))$ with Yeadon decomposition $\rho = (v, \alpha)$ such that

(1) $\|v_s - u_s\|_2 \le \delta\varepsilon/4$ for all $s \in S$ and

(2) $\|\alpha_s(x) - \theta_s(x)\|_2 \le \dfrac{\delta^2\varepsilon^3\varepsilon'}{28672} = \dfrac{\delta^2\varepsilon^3\varepsilon'}{2^7\cdot 224}$ for all $s \in S$ and all $x \in F$.

Let $\rho \in \mathcal{V}_\delta$ with Yeadon decomposition $\rho = (v, \alpha)$. Since, by (2),

$$\max_{x\in F} \|(\theta_s^{-1}\alpha_s)(x) - x\|_2 = \|\alpha_s(x) - \theta_s(x)\|_2 \le \varepsilon',$$

it follows that $\alpha_s \in \mathrm{A}ut(\mathcal{N})$ for all $s \in S$ and hence α takes its values in $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$. Hence, $\alpha \in \mathrm{H}om(\Gamma, \mathrm{A}ut(\mathcal{N}))$ and $v \in Z^1(\Gamma, \alpha)$.

Claim. There exist $a, b \in \mathbf{U}(\mathcal{N})$ with

$$\|a - 1\|_2 \le \delta \text{ and } \|b - 1\|_2 \le \delta$$

such that

$$\alpha_g = \mathrm{A}d(a)\theta_g\mathrm{A}d(a^*) \;\text{ and }\; v_g = bau_g a^*(\mathrm{A}d(a)\theta_g\mathrm{A}d(a^*))(b^*)$$

for all $g \in \Gamma$. In particular, once proved, this claim will show that ρ is in the $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$-orbit of π (see Proposition 2.5). The proof will be carried out in four steps.

• *First step:* We claim that there exists a projective 1-cocycle w in $Z^1_{\mathrm{proj}}(\Gamma, \theta)$ with the following properties:

$$\alpha_g = \mathrm{A}d(w_g)\theta_g \quad \text{for all} \quad g \in \Gamma \quad \text{and}$$

$$\|w_s - 1\|_2 \le \frac{\delta^2\varepsilon^3}{2^6\cdot 224} \quad \text{for all} \quad s \in S.$$

Indeed, by (2) above, for every $s \in S$ and $x \in F$, we have

$$\|\theta_s^{-1}(\alpha_s(x)) - x\|_2 = \|\alpha_s(x) - \theta_s(x)\|_2 \le \frac{\varepsilon'\delta^2\varepsilon^3}{2^7\cdot 224}.$$

Hence, it follows from Proposition 2.7 that, for every $s \in S$, there exists w_s in $\mathbf{U}(\mathcal{N})$ with

$$\|w_s - 1\|_2 \le \frac{\delta^2\varepsilon^3}{2^6\cdot 224}$$

and such that $\alpha_s = \mathrm{A}d(w_s)\theta_s$.

Now, α and θ are group homomorphisms from Γ to $\mathrm{A}ut(\mathcal{N})$ and $\mathrm{I}nn(\mathcal{N})$ is a normal subgroup in $\mathrm{A}ut(\mathcal{N})$. Moreover, we have just shown that the homomorphisms $p \circ \alpha$ and $p \circ \theta$ from Γ to the quotient group $\mathrm{A}ut(\mathcal{N})/\mathrm{I}nn(\mathcal{N})$ agree on the generating set S, where

$$p : \mathrm{A}ut(\mathcal{N}) \to \mathrm{A}ut(\mathcal{N})/\mathrm{I}nn(\mathcal{N})$$

is the canonical projection. It follows that $p \circ \alpha = p \circ \theta$ on Γ. Hence, we can extend $S \mapsto \mathbf{U}(\mathcal{N}), s \mapsto w_s$ to a mapping $w : \Gamma \mapsto \mathbf{U}(\mathcal{N})$ such that $\alpha_g = \mathrm{A}d(w_g)\theta_g$ for all $g \in \Gamma$. By Lemma 2.9, w is a projective 1-cocycle for θ.

• *Second step:* We claim that there exist $a \in \mathbf{U}(\mathcal{N})$ and a mapping $\lambda : \Gamma \to \mathbf{S}^1$ such that such that

$$w_g = \lambda_g a \theta_g(a^*) \qquad \text{for all} \quad g \in \Gamma$$

(that is, w is in $B^1_{\mathrm{proj}}(\Gamma, \theta)$) and such that

$$\|a - 1\|_2 \leq \frac{\delta\varepsilon}{2^3}.$$

Indeed, the action of Γ given by $\pi^l = \theta$ is ergodic and

$$\|w_s - 1\|_2 \leq \frac{\delta^2\varepsilon^3}{2^6 \cdot 224} = \left(\frac{\delta\varepsilon}{2^3}\right)^2 \frac{\varepsilon}{224} \qquad \text{for all} \quad s \in S.$$

for every $s \in S$. Hence, the claim follows from Proposition 2.10 applied to the trivial cocycle $u : g \mapsto 1$.

• *Third step:* Let $a \in \mathbf{U}(\mathcal{N})$ be as in the second step. We claim that $\alpha = \theta^{\mathrm{A}d(a)}$, that is,

$$\alpha_g = \mathrm{A}d(a)\theta_g \mathrm{A}d(a^*) \qquad \text{for all} \quad g \in \Gamma.$$

Indeed, this follows from the fact that $\alpha_g = \mathrm{A}d(w_g)\theta_g$ and $w_g = \lambda_g a \theta_g(a^*)$ for every $g \in \Gamma$.

Let $u' = \mathrm{A}d(a)u$ be the cocycle in $Z^1(\Gamma, \alpha) = Z^1(\Gamma, \theta^{\mathrm{A}d(a)})$ defined by

$$u'_g = a u_g a^* \qquad \text{for all} \quad g \in \Gamma.$$

• *Fourth step:* We claim that there exists $b \in \mathbf{U}(\mathcal{N})$ with $\|b - 1\|_2 \leq \delta$ such

$$v_g = b u'_g \theta_g^{\mathrm{A}d(a)}(b^*) \qquad \text{for all} \quad g \in \Gamma.$$

Indeed, by (1) above and the choice of a, we have

$$\begin{aligned}
\|v_s - u'_s\|_2 &= \|v_s - a u_s a^*\|_2 \leq \|v_s - u_s\|_2 + \|a u_s - u_s a\|_2 \\
&\leq \|v_s - u_s\|_2 + \|(a-1)u_s\|_2 + \|u_s(a-1)\|_2 \\
&\leq \|v_s - u_s\|_2 + 2\|a - 1\|_2 \\
&\leq \frac{\delta\varepsilon}{4} + 2\frac{\delta\varepsilon}{8} = \frac{\delta\varepsilon}{2},
\end{aligned}$$

for every $s \in S$. Moreover, the action of Γ given by $\pi^r(g) = \mathrm{A}d(u_g)\theta_g$ for $g \in \Gamma$ is ergodic. Hence, the action given by $g \mapsto \mathrm{A}d(u'_g)\alpha_g = \mathrm{A}d(a)\pi^r(g)\mathrm{A}d(a^*)$ is also ergodic. The claim follows now from Proposition 2.11.

3.2. **The case $\Gamma \neq \Gamma^+$.** We assume now that $\Gamma^+ = \pi^{-1}(\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ is a proper subgroup and hence a normal subgroup of index 2 in Γ. Observe that Γ^+ has also Property (T). Let (S, ε) be a Kazhdan pair for Γ^+.

Since $\mathrm{I}nn(\mathcal{N})$ is open in $\mathrm{A}ut(\mathcal{N})$, the set

$$\mathcal{U} = \{\varphi = (u, \mathrm{Ad}(v)) \in \mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N})) \ : \ \|u-1\|_2 < \sqrt{3}/4 \quad \text{and} \quad \|v-1\|_2 < \sqrt{3}/4\}$$

is an open neighbourhood of the identity in $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$.

Fix $s_0 \in \Gamma \setminus \Gamma^+$. Let $0 < \delta < \sqrt{3}/20$. Define $\mathcal{V} = \mathcal{V}_\delta$ to be the neighbourhood of π in $\mathrm{H}om(\Gamma, \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N})))$ consisting of all $\rho = (v, \alpha)$ in $\mathrm{H}om(\Gamma, \mathbf{O}_{\mathcal{N}}(L_2(\mathcal{N})))$ such that the conditions (1) and (2) from above hold and such that, moreover,

$$(3) \qquad \rho(s_0) \in \mathcal{U}\pi(s_0).$$

Let $\rho \in \mathcal{V}$. We can apply the conclusion of the first case to the restrictions $\pi|_{\Gamma^+}$ and $\rho|_{\Gamma^+}$ of π and ρ to Γ^+ and conclude that there exists $U = (a, \mathrm{Ad}(b))$ in $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ with unitaries a and b in $\mathbf{U}(\mathcal{N})$ which are δ-close to 1 in the L^2-norm and such that $\pi|_{\Gamma^+} = \mathrm{Ad}(U)(\rho|_{\Gamma^+})$.

We claim that $\pi = \mathrm{Ad}(U)\rho$. Indeed, set $\beta := \mathrm{Ad}(U)\rho$. By (3), there exists $\varphi_1 = (a_1, \mathrm{Ad}(b_1)) \in \mathcal{U}$ such that $\rho(s_0) = \varphi_1\pi(s_0)$. Hence,

$$\beta(s_0) = U\varphi_1\pi(s_0)U^{-1} = U\varphi_1(\pi(s_0)U^{-1}\pi(s_0)^{-1})\pi(s_0).$$

Set $\varphi_2 := \pi(s_0)U^{-1}\pi(s_0)^{-1}$. One checks that $\varphi_2 = (a_2, \mathrm{Ad}(b_2))$ for unitaries a_2 and b_2 which are 4δ-close to 1, since a and b are δ-close to 1 in the L^2-norm. Set $\varphi := U\varphi_1\varphi_2$, so that $\beta(s_0) = \varphi\pi(s_0)$. Then $\varphi = (c, \mathrm{Ad}(d))$ for unitaries c and d. Since a_1 and b_1 are $\sqrt{3}/4$-close to 1, since a_2 and b_2 are 4δ-close to 1, and since $\delta < \sqrt{3}/20$, one checks that c and d are $\sqrt{3}/2$-close to 1 in the L^2-norm.

Using the fact that β and π are homomorphisms on Γ and coincide on the normal subgroup Γ^+, we have, for every $g \in \Gamma^+$,

$$\pi(s_0gs_0^{-1}) = \beta(s_0gs_0^{-1}) = \beta(s_0)\beta(g)\beta(s_0^{-1}) = \varphi\pi(s_0gs_0^{-1})\varphi^{-1}.$$

So, φ commutes with $\pi(g)$ for all $g \in \Gamma^+$.

The condition that $\varphi = (c, \mathrm{Ad}(d))$ commutes with $\pi(g) = (u_g, \theta_g)$ means that

$$cdu_g\theta_g(x)d^* = u_g\theta_g(cdxd^*) \ \text{ for all } x \in \mathcal{N}. \ (*)$$

Taking adjoints, we deduce that

$$d\theta_g(x^*x)d^* = \theta_g(dx^*xd^*),$$

that is, $\theta_g(d^*)d$ commutes with $\theta_g(xx^*)$ for every $x \in \mathcal{N}$. Since $\mathcal{N}$ is a factor, it follows that, for every $g \in \Gamma^+$, we have $\theta_g(d^*)d = \lambda_g 1$ for some scalar λ_g with $|\lambda_g| = 1$. Using the fact that $g \mapsto \theta_g$ is a group homomorphism, we see that $g \mapsto \lambda_g$ is a unitary character of Γ^+.

Since d is $\sqrt{3}/2$-close to 1, we have $|\lambda_g - 1| < \sqrt{3}$ for all $g \in \Gamma^+$. As is well-known, this implies that $\lambda_g = 1$ for all $g \in \Gamma^+$ (indeed, the only subgroup G of the unit circle with $|z-1| < \sqrt{3}$ for all $z \in G$ is the trivial subgroup).

So, d^* is fixed by the automorphisms θ_g for $g \in \Gamma^+$ and hence $d = \lambda 1$ for some scalar λ with $|\lambda| = 1$, by ergodicity of π^l. Hence, $\mathrm{Ad}(d)$ is the identity and we can assume that $d = 1$.

From $(*)$, we then obtain that c is fixed by the automorphisms $\pi^r(g) = u_g\theta_g u_g^*$ for $g \in \Gamma^+$ and so $c = \lambda 1$ for some scalar λ with $|\lambda| = 1$, by ergodicity of π^r. Hence, $\beta(s_0) = \lambda\pi(s_0)$. Since $s_0^2 \in \Gamma^+$ and therefore $\beta(s_0^2) = \pi(s_0^2)$, we see that $\lambda^2 = 1$. As

$\|c-1\|_2 < 2$, it follows that $\lambda = 1$, that is, $\beta(s_0) = \pi(s_0)$. Hence, $\beta = \pi$ and the proof of the theorem is complete. ■

Remark 3.1. Let $\pi : \Gamma \to \mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ be a homomorphism. For a fixed $0 < \delta < 1$, the set $\mathcal{V}_\delta$ given above is a neighbourhood of π in $\mathrm{H}om(\Gamma, \mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N})))$ such that every $\rho \in \mathcal{V}_\delta$ is conjugate to π by some $U = (b, \mathrm{Ad}(a))$ in $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ for which a and b are δ-close to the identity in the L_2-norm.

Using estimates for the Mazur map $M_{p,q}$, we can determine a neighbourhood $\mathcal{V}_{\delta,p}$ of π in $\mathrm{H}om(\Gamma, \mathbf{O}^+(L_p(\mathcal{N})))$ for $p \neq 2$, with the same properties. Indeed, by [21], there exist constants $C = C_p$ (of order p) and D (independent of p) such that

$$\|M_{p,2}(x) - M_{p,2}(y)\|_2 \leq C\|x-y\|_p^\alpha$$
$$\|M_{2,y}(x') - M_{2,p}(y')\|_p \leq D\|x'-y'\|_2^\beta,$$

where $\alpha = \min\{p/2, 1\}$ and $\beta = \min\{2/p, 1\}$, for x, y in the unit ball in $L_p(\mathcal{N})$ and x', y' in the unit ball in $L_2(\mathcal{N})$.

We can clearly assume that, for every x in the Kazhdan set F for $\mathcal{N}$, we have $\|x\|_2 = 1$ and hence $\|M_{2,p}(\theta(x))\|_p = 1$ for every $\theta \in \mathrm{A}ut(\mathcal{N})$, since $\|M_{2,p}(\theta(x))\|_p = \|\theta(M_{2,p}(x))\|_p = \|x\|_2^{2/p}$.

Set $\delta_p = \left(\frac{\delta}{D}\right)^{1/\beta}$ and

$$\varepsilon_p = \left(\frac{\delta_p \varepsilon}{4C}\right)^{1/\alpha}, \quad \varepsilon'_p = \left(\frac{\delta_p^2 \varepsilon^3 \varepsilon'}{C \cdot 2^7 \cdot 224}\right)^{1/\alpha}.$$

Let $\mathcal{V}_{\delta_p}$ to be the neighbourhood of $\pi = (u, \theta)$ in $\mathrm{H}om(\Gamma, \mathbf{O}^+(L_p(\mathcal{N}))$ consisting of all $\rho = (v, \alpha)$ in $\mathrm{H}om(\Gamma, \mathbf{O}^+(L_p(\mathcal{N}))$ such that

$$\|v_s - u_s\|_p \leq \varepsilon_p \quad \text{for all } s \in S \text{ and}$$
$$\|\alpha_s(M_{2,p}(x)) - \theta_s(M_{2,p}(x))\|_p \leq \varepsilon'_p \quad \text{for all } s \in S \text{ and all } x \in F.$$

Then every $\rho \in \mathcal{V}_{\delta_p}$ is conjugate to π by some $U = (b, \mathrm{A}d(a))$ in $\mathbf{O}^+(L_p(\mathcal{N}))$ for which a and b are δ-close to the identity in the L_p-norm.

4. On the assumptions in the statement of Theorem 1.1

We present counterexamples in relation with the various assumptions made in Theorem 1.1.

Example 4.1. If the finite factor $\mathcal{N}$ does not have Property (T), the conclusion of Theorem 1.1 may not be true. Indeed, let $\mathcal{R}$ be the hyperfinite type II_1-factor. M. Choda constructed in [6] a continuous family $(\theta_t)_{t\in[0,1]}$ of actions of the group $\Gamma = SL_n(\mathbf{Z})$ for $n \geq 2$ (recall that $SL_n(\mathbf{Z})$ has Property (T) for $n \geq 3$) by automorphisms on $\mathcal{R}$, which are ergodic and mutually non conjugate in $\mathrm{Aut}(\mathcal{R})$ for irrational t. It follows from Proposition 2.5 that, for any $1 \leq p < \infty, p \neq 2$ and any irrational t, the homomorphisms $\pi_t : \Gamma \to \mathbf{O}^+(L_p(\mathcal{R}))$ defined by these actions are mutually non conjugate in $\mathbf{O}^+(L_p(\mathcal{R}))$ and hence are not locally rigid. In fact, a more general result in [16, Corollary 0.2] implies that any Kazhdan group admits a continuous family of actions by automorphisms on $\mathcal{R}$ which are mutually non conjugate.

Example 4.2. The conclusion of Theorem 1.1 might also fail, if any one of the associated actions π^l and π^r by automorphisms of $\mathcal{N}$ is not ergodic. A counter-example may be obtained by a slight modification of Example 8 in [11] as follows.

Let H be an ICC group with Kazhdan's property (for instance $H = SL_3(\mathbf{Z})$). Set $\Gamma = H \times H$ and $\mathcal{N} = \mathcal{N}(\Gamma)$. Then Γ is an ICC group with Kazhdan's property and $\mathcal{N}$ can be identified with the tensor product $\mathcal{N}(H)\overline{\otimes}\mathcal{N}(H)$ of von Neumann algebras, with trace $\tau = \tau_H \otimes \tau_H$, where τ_H is the canonical trace on $\mathcal{N}(H)$. For $1 \le p < \infty, p \neq 2$, let π_0 denote the embedding $\Gamma \to \mathbf{O}^+(L_p(\mathcal{N}))$ given by $g \mapsto \lambda(g)$; observe that the associated action π_0^l is the trivial action, while the action π_0^r, which is given by $g \mapsto \mathrm{A}d(\lambda(g))$, is ergodic.

Since $\mathcal{N}(H)$ is a factor of type II_1, for every $t \in [0,1]$, there exists a projection p_t in $\mathcal{N}(H)$ with $\tau_H(p_t) = t$. For $g = (h_1, h_2) \in \Gamma$ and $t \in [0,1]$, let $u_g^{(t)} \in \mathcal{N}(\Gamma)$ be defined by

$$u_g^{(t)} = \lambda(h_1) \otimes p_t + 1 \otimes (1 - p_t).$$

Then $u_g^{(t)}$ is unitary and

$$u^{(t)} : \Gamma \to \mathbf{U}(\mathcal{N}),\ g \mapsto u_g^{(t)}$$

is a group homomorphism. For $g \in \Gamma$, let $\pi_t(g)$ denote the isometry of $L_p(\mathcal{N})$ with Yeadon decomposition $(\lambda(g)u_{g^{-1}}^{(t)}, \mathrm{A}d(u_g^{(t)}))$, that is,

$$\pi_t(g)x = \lambda(g)u_{g^{-1}}^{(t)}\mathrm{A}d(u_g^{(t)})(x) = \lambda(g)xu_{g^{-1}}^{(t)} \qquad \text{for all} \quad x \in L_p(\mathcal{N}).$$

Then $\pi_t : \Gamma \to \mathbf{O}^+(L_p(\mathcal{N}))$ is a group homomorphism.

We claim that π_0 is not locally rigid. For this, it suffices to show (see Proposition 2.3) that π_0 is not locally rigid when viewed as homomorphism with values in the $\mathcal{N}$-unitary group $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N})$.

For $g = (h_1, h_2) \in \Gamma$ and $t \in [0,1]$, we have

$$\|u_g^{(t)} - 1\|_2^2 = 2(1 - \tau(u_g^{(t)})) = 2t(1 - \tau_H(\lambda(h_1)) \le 2t$$

and hence $\lim_{t\to 0} \|u_g^{(t)} - 1\|_2 = 0$. It follows that $\lim_{t\to 0} \pi_t(g) = \pi_0(g)$ in $\mathbf{O}^+_{\mathcal{N}}(L_2(\mathcal{N}))$ for every $g \in \Gamma$.

Assume, by contradiction, that π_0 is locally rigid. Then, by Proposition 2.5, for $t > 0$ sufficiently small, $\mathrm{A}d(u_g^{(t)})$ is conjugate to the trivial automorphism $x \mapsto x$ in $\mathrm{A}ut(\mathcal{N})$ and hence $\mathrm{A}d(u_g^{(t)})$ is the trivial automorphism for every $g \in \Gamma$. This is a contradiction, as $u_g^{(t)}$ is not a scalar multiple of the identity for $g = (h_1, h_2)$ with $h_1 \neq e$.

Similarly, one can show that the embedding $\rho_0 : \Gamma \to \mathbf{O}^+(L_p(\mathcal{N}))$, given by

$$\rho_0(g) : x \mapsto x\lambda(g^{-1}) = \lambda(g^{-1})\mathrm{A}d(\lambda(g))(x),$$

is not locally rigid; here, it is the associated action ρ_0^l which is ergodic, while ρ_0^r is the trivial action.

Example 4.3. The conclusion of Theorem 1.1 does not hold in general for actions by isometries on the classical (commutative) L_p-spaces. We give a counter-example for actions on the sequence space ℓ_p (more involved counter-examples can be found for actions on the space $L_p[0,1]$).

Let Γ be an arbitrary group which is not a torsion group; thus Γ contains a subgroup A isomorphic to $\mathbf{Z}$. There exists a family $(\chi_t)_{t\in[0,1]}$ of unitary characters χ_t of A with $\chi_t \neq 1$ for $t \neq 0$, $\chi_0 = 1$ and such that $\lim_{t\to 0} \chi_t(a) = 1$ for all $a \in A$. Choose a set of representatives X for the left cosets of Γ modulo A with $e \in X$; so, $\Gamma = XA$. Let $c : \Gamma \times X \to A$ be the cocycle defined by

$$gx \in Xc(g,x) \qquad \text{for all} \quad g \in \Gamma,\, x \in X.$$

We transfer the natural Γ-action on Γ/A to an action $(g,x) \mapsto g(x)$ of Γ on $X \cong \Gamma/A$, by setting

$$g(x) = gxc(g,x)^{-1} \qquad \text{for all} \quad g \in \Gamma,\, x \in X.$$

For every $t \in \mathbf{R}$ and every $p \in [1, +\infty[$, the operator $\pi_t(g)$ on $\ell_p(X)$, defined by

$$\pi_t(g)f(x) = \chi_t(c(g^{-1},x))f(g^{-1}(x)) \qquad \text{for all} \quad f \in \ell_p(X), x \in X,$$

is an isometry and $\pi_t : \Gamma \to \mathbf{O}(\ell_p(X))$ is a homomorphism. (For $p = 2$, π_t is the unitary representation of Γ induced by the character χ_t of A. Observe also that π_0 is the quasi-regular representation of Γ in $\ell_p(X) \cong \ell_p(\Gamma/A)$.)

We have $\lim_{t\to 0} \pi_t(g) = \pi_0(g)$ in $\mathbf{O}(\ell_p(X))$, for every $g \in \Gamma$. Indeed, by linearity and density, it suffices to check that

$$\lim_{t\to 0} \|\pi_t(g)\delta_x - \pi_0(g)\delta_x\|_p = 0,$$

where δ_x is the Dirac function at $x \in X$. This is the case, since

$$\|\pi_t(g)\delta_x - \pi_0(g)\delta_x\|_p = |\chi_t(c(g^{-1}, g(x)) - 1|.$$

Let $t \neq 0$ and $p \neq 2$. We claim that π_t does not belong to the $\mathbf{O}(\ell_p(X))$-orbit of π_0. Indeed, assume, by contradiction, that there exists $U = U_t$ in $\mathbf{O}(\ell_p(X))$ such that

$$\pi_t(g) = U\pi_0(g)U^{-1} \qquad \text{for all} \quad g \in \Gamma.$$

By Banach characterization of the isometries of $\ell_p(X)$ from [2, Chap. XI], there exists a function $\alpha : X \to \mathbf{S}^1$ and a bijective mapping $\varphi : X \to X$ such that

$$(Uf)(x) = \alpha(x)f(\varphi(x)) \qquad \text{for all} \quad f \in \ell_p(X), x \in X.$$

One computes that $U\pi_t(g^{-1})U^{-1} = \pi_0(g^{-1})$ amounts to the equation

$$\alpha(x)\alpha(\varphi^{-1}g\varphi(x))^{-1}\chi_t(c(g,\varphi(x)))f(\varphi^{-1}g\varphi(x)) = f(g(x)),$$

for all $f \in \ell_p(X)$ and $x \in X$. It follows from this that $\varphi^{-1}g\varphi = g$ on X and, consequently,

$$\alpha(x)\alpha(g(x))^{-1}\chi_t(c(g,\varphi(x)) = 1 \qquad \text{for all} \quad g \in \Gamma,\, x \in X. \qquad (*)$$

Let $x = \varphi^{-1}(e)$ and $g \in A$. Then, $c(g,e) = g$ and $g(e) = e$. Hence, we have

$$g(x) = g(\varphi^{-1}(e)) = \varphi^{-1}(g(e)) = \varphi^{-1}(e) = x$$

If follows from $(*)$ that $\chi_t(g) = 1$ for all $g \in A$ and this is a contradiction.

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References

[1] U. Bader, A. Furman, T. Gelander, and N. Monod. Property (T) and rigidity for actions on Banach spaces. *Acta Math.* **198**, 57–105, 2007.

[2] S. Banach. *Théorie des opérations linéaires*, Hafner Publishing Company, 1932.

[3] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property (T)*, Cambridge Univ. Press, 2008.

[4] O. Bratteli and D. Robinson. *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag 1979.

[5] N. Brown and N. Ozawa. *C^*–algebras and finite-dimensional approximations*, American Mathematical Society 2008.

[6] M. Choda. A continuum of non-conjugate property T actions of $SL(n, \mathbf{Z})$ on the hyperfinite II_1-factor. *Math. Japon.***30**, 133–150, 1985.

[7] A. Connes and V. Jones. Property (T) for von Neumann algebras. *Bull. London Math. Soc.* **17**, 51-62 (1985).

[8] D. Fisher. Local rigidity of group actions: past, present, future. In: Dynamics, Ergodic Theory and Geometry, p.45–98, Cambridge Univ. Press, 2007.

[9] D. Fisher and G. Margulis. Almost isometric actions, property (T), and local rigidity. *Invent. Math.* **162**, 19–80, 2005.

[10] M. Junge, Z.J. Ruan, and D. Sherman. A classification for 2-isometries of noncommutative L_p-spaces. *Israel J. Math.***150**, 285–314, 2005.

[11] Y. Kawahigashi. Cohomology of actions of discrete groups on factors of type II_1. *Pac. J. Math.* **149**, 303–317, 1991.

[12] R. Nicoara, S. Popa and R. Sasyk. On II_1-factors arising from 2-cocycles of w-rigid groups. *J. Funct. Anal.* **242**, 230–246, 2007.

[13] B. Olivier. Kazhdan's property (T) with respect to non-commutative L_p-spaces. *Proc. Amer. Math. Soc.* **140**, 4259–4269, 2012.

[14] V. Paulsen. *Completely bounded maps and operator algebras*, Cambridge University Press, 2002.

[15] G. Pisier and Q. Xu. Non-commutative L_p-spaces. Handbook of the geometry of Banach spaces, Vol. 2, pp. 1459–1517, North-Holland 2003.

[16] S. Popa. Some rigidity results for non-commutative Bernoulli shifts. *J. Funct. Anal.* **230** , 273–328, 2006.

[17] M.S. Raghunathan. *Discrete subgroups of Lie groups*, Springer-Verlag, 1972.

[18] A. Rapinchuk. On the finite dimensional unitary representations of Kazhdan groups. *Proc. Amer. Math. Soc.* **127**, 1557–1562, 1999.

[19] D. Sherman. Noncommutative L_p structure encodes exactly Jordan structure. *J. Funct. Anal.* **221** , 150–166, 2005.

[20] Y. Raynaud. On ultrapowers of non commutative L_p spaces. *J. Operator Theory* **48** , 41–68, 2002.

[21] E. Ricard. Hölder estimates for the non commutative Mazur maps. *Preprint 2014*, arXiv:1407.8334v3.

[22] R.J. Spatzier. An invitation to rigidity theory. In: Modern dynamical systems and applications, p.211–231, Cambridge Univ. Press, 2004.

[23] F. J. Yeadon. Isometries of noncommutative L^p-spaces. *Math. Proc. Cambridge Philos. Soc.* **90**, 41–50, 1981.

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