Lattices with and lattices without spectral gap

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For Fritz Grunewald on his 60th birthday

Abstract

Let $G = G(k)$ be the $k$-rational points of a simple algebraic group $G$ over a local field $k$ and let $\Gamma$ be a lattice in $G$. We show that the regular representation $\rho_{\Gamma \backslash G}$ of $G$ on $L^2(\Gamma \backslash G)$ has a spectral gap, that is, the restriction of $\rho_{\Gamma \backslash G}$ to the orthogonal of the constants in $L^2(\Gamma \backslash G)$ has no almost invariant vectors. On the other hand, we give examples of locally compact simple groups $G$ and lattices $\Gamma$ for which $L^2(\Gamma \backslash G)$ has no spectral gap. This answers in the negative a question asked by Margulis [Marg91, Chapter III, 1.12]. In fact, $G$ can be taken to be the group of orientation preserving automorphisms of a $k$-regular tree for $k > 2$.

1 Introduction

Let $G$ be a locally compact group. Recall that a unitary representation $\pi$ of $G$ on a Hilbert space $\mathcal{H}$ has almost invariant vectors if, for every compact subset $Q$ of $G$ and every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that $\sup_{x \in Q} \| \pi(x)\xi - \xi \| < \varepsilon$. If this holds, we also say that the trivial representation $1_G$ is weakly contained in $\pi$.

Recall that a lattice $\Gamma$ in $G$ is a discrete subgroup such that there exists a finite $G$-invariant regular Borel measure $\mu$ on $\Gamma \backslash G$. Denote by $\rho_{\Gamma \backslash G}$ the unitary representation of $G$ given by right translation on the Hilbert space

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Let \( L^2(\Gamma \backslash G, \mu) \) of the square integrable measurable functions on \( \Gamma \backslash G \). The subspace \( \mathbb{C}1_{\Gamma \backslash G} \) of the constant functions on \( \Gamma \backslash G \) is \( G \)-invariant as well as its orthogonal complement

\[
L^2_0(\Gamma \backslash G) = \left\{ \xi \in L^2(\Gamma \backslash G) : \int_{\Gamma \backslash G} \xi(x) d\mu(x) = 0 \right\}.
\]

Denote by \( \rho^0_{\Gamma \backslash G} \) the restriction of \( \rho_{\Gamma \backslash G} \) to \( L^2_0(\Gamma \backslash G, \mu) \). We say that \( \rho_{\Gamma \backslash G} \) (or \( L^2(\Gamma \backslash G, \mu) \)) has a spectral gap if \( \rho^0_{\Gamma \backslash G} \) has no almost invariant vectors. (In [Marg91, Chapter III., 1.8], \( \Gamma \) is then called weakly cocompact.) It is well-known that \( L^2(\Gamma \backslash G) \) has a spectral gap when \( \Gamma \) is cocompact in \( G \) (see [Marg91, Chapter III, 1.10]). Margulis (op.cit, 1.12) asks whether this result holds more generally when \( \Gamma \) is a subgroup of finite covolume.

The goal of this note is to prove the following results:

**Theorem 1** Let \( G \) be a simple algebraic group over a local field \( k \) and \( G = G(k) \), the group of \( k \)-rational points in \( G \). Let \( \Gamma \) be a lattice in \( G \). Then the unitary representation \( \rho_{\Gamma \backslash G} \) on \( L^2(\Gamma \backslash G) \) has a spectral gap.

**Theorem 2** For an integer \( k > 2 \), let \( X \) be the \( k \)-regular tree and \( G = \text{Aut}(X) \). Then \( G \) contains a lattice \( \Gamma \) for which the unitary representation \( \rho_{\Gamma \backslash G} \) on \( L^2(\Gamma \backslash G) \) has no spectral gap.

So, Theorem 2 answers in the negative Margulis’ question mentioned above.

Theorem 1 is known in case \( k = \mathbb{R} \) ([Bekk98]). It holds, more generally, when \( G \) is a real Lie group ([BeCo08]). Observe also that when \( k - \text{rank}(G) \geq 2 \), the group \( G \) has Kazhdan’s Property (T) (see [BHV]) and Theorem 1 is clear in this case. When \( k \) is non-archimedean with characteristic 0, every lattice \( \Gamma \) in \( G(k) \) is uniform (see [Serr, p.84]) and hence the result holds as mentioned above. By way of contrast, \( G \) has many non uniform lattices when the characteristic of \( k \) is non zero (see [Serr] and [Lubo91]). So, in order to prove Theorem 1, it suffices to consider the case where the characteristic of \( k \) is non-zero and where \( k - \text{rank}(G) = 1 \).

Recall that when \( k \) is non-archimedean and \( k - \text{rank}(G) = 1 \), the group \( G(k) \) acts by automorphisms on the associated Bruhat-Tits tree \( X \) (see [Serr]). This tree is either the \( k \)-regular tree \( X_k \) (in which every vertex has constant degree \( k \)) or is the bi-partite bi-regular tree \( X_{k_0,k_1} \) (where every vertex has either degree \( k_0 \) or degree \( k_1 \) and where all neighbours of a vertex
of degree $k_i$ have degree $k_{1-i}$). The proof of Theorem 1 will use the special structure of a fundamental domain for the action of $\Gamma$ on $X$ as described in [Lubo91] (see also [Ragh89] and [Baum03]).

Theorems 1 and 2 provide a further illustration of the different behaviour of general tree lattices as compared to lattices in rank one simple Lie groups over local fields; for more on this topic, see [Lubo95].

The proofs of Theorems 1 and 2 will be given in Sections 3 and 4; they rely in a crucial way on Proposition 6 from Section 2, which relates the existence of a spectral gap with expander diagrams. In turn, Proposition 6 is based, much in the spirit of [Broo81], on analogues for diagrams proved in [Mokh03] and [Morg94] of the inequalities of Cheeger and Buser between the isoperimetric constant and the bottom of the spectrum of the Laplace operator on a Riemannian manifold (see Proposition 5). This connection between the combinatorial expanding property and representation theory is by now a very popular theme; see [Lubo94] and the references therein. While most applications in this monograph are from representation theory to combinatorics, we use in the current paper this connection in the opposite direction: the existence or absence of a spectral gap is deduced from the existence of an expanding diagram or of a non-expanding diagram, respectively.

2 Spectral gap and expander diagrams

We first show how the existence of a spectral gap for groups acting on trees is related with the bottom of the spectrum of the Laplacian for an associated diagram.

A graph $X$ consists of a set of vertices $V_X$, a set of oriented edges $E_X$, a fix-point free involution $\overline{\cdot} : E_X \to E_X$, and end point mappings $\partial_i : E_X \to V_X$ for $i = 0, 1$ such that $\partial_i(\overline{e}) = \partial_{1-i}(e)$ for all $e \in E_X$. Assume that $X$ is locally finite, that is, for every $x \in V_X$, the degree $\deg(x)$ of $x$ is finite, where $\deg(x)$ is the cardinality of the set

$$\partial_0^{-1}(x) = \{ e \in E_X : \partial_0(e) = x \}.$$ 

The group $\text{Aut}(X)$ of automorphisms of the graph $X$ is a locally compact group in the topology of pointwise convergence on $X$, for which the stabilizers of vertices are compact open subgroups.

We will consider infinite graphs called diagrams of finite volume. An edge-indexed graph $(D, i)$ is a graph $D$ equipped with a function $i : E_D \to \mathbb{R}^+$
A measure \( \mu \) for an edge-indexed graph \((D,i)\) is a function \( \mu : VD \cup ED \to \mathbb{R}^+ \) with the following properties (see [Mokh03] and [BaLu01, 2.6]):

- \( i(e)\mu(\partial_0 e) = \mu(e) \)
- \( \mu(e) = \mu(\overline{e}) \) for all \( e \in VD \), and
- \( \sum_{x \in VD} \mu(x) < \infty \).

Following [Morg94], we will say that \( D = (D,i,\mu) \) is a diagram of finite volume. The in-degree \( \text{indeg}(x) \) of a vertex \( x \in VD \) is defined by

\[
\text{indeg}(x) = \sum_{e \in \partial^{-1}_0(x)} i(e) = \sum_{e \in \partial^{-1}_0(x)} \frac{\mu(e)}{\mu(x)}.
\]

The diagram \( D \) is \( k \)-regular if \( \text{indeg}(x) = k \) for all \( x \in VD \).

Let \( D = (D,i,\mu) \) be a connected diagram of finite volume. Observe that \( \mu \) is determined, up to a multiplicative constant, by the weight function \( i \). Indeed, fix \( x_0 \in VD \) and set \( \Delta(e) = i(e)/i(\overline{e}) \) for \( e \in ED \). Then

\[
\mu(\partial_1 e) = \frac{\mu(\overline{e})}{i(\overline{e})} = \frac{\mu(e)}{i(e)} = \mu(\partial_0 e) \Delta(e)
\]

for every \( e \in ED \). Hence \( \mu(x) = \Delta(e_1)\Delta(e_2)\ldots \Delta(e_n)\mu(x_0) \) for every path \( (e_1,e_2,\ldots,e_n) \) from \( x_0 \) to \( x \in VD \).

Let \( D = (D,i,\mu) \) be a diagram of finite volume. An inner product is defined for functions on \( VD \) by

\[
\langle f, g \rangle = \sum_{x \in VD} f(x)g(x)\mu(x).
\]

The Laplace operator \( \Delta \) on functions \( f \) on \( VD \) is defined by

\[
\Delta f(x) = f(x) - \frac{1}{\text{indeg}(x)} \sum_{e \in \partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)} f(\partial_1(e)).
\]

The operator \( \Delta \) is a self-adjoint positive operator on \( L^2(VD) \). Let

\[
L^2_0(VD) = \{ f \in L^2(VD) : \langle f, 1_{VD} \rangle = 0 \}
\]
and set
\[ \lambda(D) = \inf_{f} \langle \Delta f, f \rangle, \]
where \( f \) runs over the unit sphere in \( L_0^2(VD) \). Observe that
\[ \lambda(D) = \inf \{ \lambda : \lambda \in \sigma(\Delta) \setminus \{0\} \}, \]
where \( \sigma(\Delta) \) is the spectrum of \( \Delta \).

Let now \( X \) be a locally finite tree, and let \( G \) be a closed subgroup of \( \text{Aut}(X) \). Assume that \( G \) acts with finitely many orbits on \( X \). Let \( \Gamma \) be a discrete subgroup of \( G \) acting without inversion on \( X \). Then the quotient graph \( \Gamma \setminus X \) is well-defined. Since \( \Gamma \) is discrete, for every vertex \( x \) and every edge \( e \), the stabilizers \( \Gamma_x \) and \( \Gamma_e \) are finite. Moreover, \( \Gamma \) is a lattice in \( G \) if and only if \( \Gamma \) is a lattice in \( \text{Aut}(X) \) and this happens if and only if
\[ \sum_{x \in D} \frac{1}{|\Gamma_x|} < \infty, \]
where \( D \) is a fundamental domain of \( \Gamma \) in \( X \) (see [Serr]). The quotient graph \( \Gamma \setminus X \cong D \) is endowed with the structure of an edge-indexed graph given by the weight function \( i : ED \to \mathbb{R}^+ \) where \( i(e) \) is the index of \( \Gamma_e \) in \( \Gamma_x \) for \( x = \partial_0(e) \). A measure \( \mu : VD \cup ED \to \mathbb{R}^+ \) is defined by
\[ \mu(x) = \frac{1}{|\Gamma_x|} \quad \text{and} \quad \mu(e) = \frac{1}{|\Gamma_e|} \]
for \( x \in VD \) and \( e \in ED \). Observe that \( \mu(VD) = \sum_{x \in D} 1/|\Gamma_x| < \infty \). So, \( D = (D, i, \mu) \) is a diagram of finite volume.

Let \( G \) be a group acting on a tree \( X \). As in [BuMo00, 0.2], we say that the action of \( G \) on \( X \) is locally \( \infty \)-transitive if, for every \( x \in VX \) and every \( n \geq 1 \), the stabilizer \( G_x \) of \( x \) acts transitively on the sphere \( \{ y \in X : d(x, y) = n \} \).

**Proposition 3** Let \( X \) be either the \( k \)-regular tree \( X_k \) or the bi-partite bi-regular tree \( X_{k_0,k_1} \) for \( k \geq 3 \) or \( k_0 \geq 3 \) and \( k_1 \geq 3 \). Let \( G \) be a closed subgroup of \( \text{Aut}(X) \). Assume that the following conditions are both satisfied:

- \( G \) acts transitively on \( VX \) in the case \( X = X_k \) and \( G \) acts transitively on the set of vertices of degree \( k_0 \) as well as on the set of vertices of degree \( k_1 \) in the case \( X = X_{k_0,k_1} \).
• the action of $G$ on $X$ is locally $\infty$-transitive.

Let $\Gamma$ be a lattice in $G$ and let $D = \Gamma \backslash X$ be the corresponding diagram of finite volume. The following properties are equivalent:

(i) the unitary representation $\rho_{\Gamma \backslash G}$ on $L^2(\Gamma \backslash G)$ has a spectral gap;

(ii) $\lambda(D) > 0$.

For the proof of this proposition, we will need a few general facts. Let $G$ be a second countable locally compact group and $U$ a compact subgroup of $G$. Let $C_c(U \backslash G/U)$ be the space of continuous functions $f : G \to \mathbb{C}$ which have compact support and which are constant on the double cosets $UgU$ for $g \in G$.

Fix a left Haar measure $\mu$ on $G$. Recall that $L^1(G, \mu)$ is a Banach algebra under the convolution product, the $L^1$-norm and the involution $f^*(g) = \overline{f(g^{-1})}$; observe that $C_c(U \backslash G/U)$ is a $*$-subalgebra of $L^1(G, \mu)$. Let $\pi$ be a (strongly continuous) unitary representation of $G$ on a Hilbert space $\mathcal{H}$. A continuous $*$-representation of $L^1(G)$, still denoted by $\pi$, is defined on $\mathcal{H}$ by

$$\pi(f)\xi = \int_G f(x)\pi(x)\xi d\mu(x), \quad f \in L^1(G), \quad \xi \in \mathcal{H}.$$ 

Assume that the closed subspace $\mathcal{H}^U$ of $U$-invariant vectors in $\mathcal{H}$ is non-zero. Then $\pi(f)\mathcal{H}^U \subset \mathcal{H}^U$ for all $f \in C_c(U \backslash G/U)$. In this way, a continuous $*$-representation $\pi_U$ of $C_c(U \backslash G/U)$ is defined on $\mathcal{H}^U$.

**Proposition 4** With the previous notation, let $f \in C_c(U \backslash G/U)$ be a function with the following properties: $f(x) \geq 0$ for all $x \in G$, $\int_G f d\mu = 1$, and the subgroup generated by the support of $f$ is dense in $G$. The following conditions are equivalent:

(i) the trivial representation $1_G$ is weakly contained in $\pi$;

(ii) $1$ belongs to the spectrum of the operator $\pi_U(f)$.

**Proof** Assume that $1_G$ is weakly contained in $\pi$. There exists a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that

$$\lim_n \|\pi(x)\xi_n - \xi_n\| = 0,$$

where $x \in G$. Then

$$\lim_n \|\pi(x)\xi_n - \xi_n\| = 0,$$

for all $x \in \text{supp}(f)$. This implies that $\pi(x)\xi_n \to \xi_n$ weakly. Since $\xi_n$ are $U$-invariant, $\pi_U(f)\xi_n = f(x)\xi_n$. Therefore

$$\lim_n \|\pi(x)\xi_n - f(x)\xi_n\| = 0,$$

for all $x \in \text{supp}(f)$. This shows that $1 \in \sigma(\pi_U(f))$. Conversely, if $1 \in \sigma(\pi_U(f))$, then there exists a sequence of unit vectors $\xi_n \in \mathcal{H}^U$ such that

$$\lim_n \|\pi(x)\xi_n - f(x)\xi_n\| = 0,$$

for all $x \in \text{supp}(f)$. This implies that $\pi(x)\xi_n \to f(x)\xi_n$ weakly. Since $\xi_n$ are $U$-invariant, $\pi_U(f)\xi_n = f(x)\xi_n$. Therefore

$$\lim_n \|\pi(x)\xi_n - f(x)\xi_n\| = 0,$$

for all $x \in \text{supp}(f)$. This shows that $1 \in \sigma(\pi_U(f))$. Hence, $1 \in \sigma(\pi_U(f))$ if and only if $1 \in \sigma(\pi_U(f))$. Therefore

$$\lim_n \|\pi(x)\xi_n - f(x)\xi_n\| = 0,$$

for all $x \in \text{supp}(f)$. This implies that $\pi(x)\xi_n \to f(x)\xi_n$ weakly. Since $\xi_n$ are $U$-invariant, $\pi_U(f)\xi_n = f(x)\xi_n$. Therefore

$$\lim_n \|\pi(x)\xi_n - f(x)\xi_n\| = 0,$$

for all $x \in \text{supp}(f)$. This shows that $1 \in \sigma(\pi_U(f))$. Hence, $1 \in \sigma(\pi_U(f))$ if and only if $1 \in \sigma(\pi_U(f))$. Therefore

$$\lim_n \|\pi(x)\xi_n - f(x)\xi_n\| = 0,$$

for all $x \in \text{supp}(f)$. This implies that $\pi(x)\xi_n \to f(x)\xi_n$ weakly. Since $\xi_n$ are $U$-invariant, $\pi_U(f)\xi_n = f(x)\xi_n$. Therefore

$$\lim_n \|\pi(x)\xi_n - f(x)\xi_n\| = 0,$$
uniformly over compact subsets of $G$. Let

$$\eta_n = \int_U \pi(u) \xi_n du,$$

where $du$ denotes the normalized Haar measure on $U$. It is easily checked that $\eta_n \in \mathcal{H}^U$ and that

$$\lim_n \| \pi(f) \eta_n - \eta_n \| = 0.$$ 

Since

$$\| \eta_n - \xi_n \| \leq \int_U \| \pi(u) \xi_n - \xi_n \| du,$$

we have $\| \eta_n \| \geq 1/2$ for sufficiently large $n$. This shows that 1 belongs to the spectrum of the operator $\pi_U(f)$.

For the converse, assume that 1 belongs to the spectrum of $\pi_U(f)$. Hence, 1 belongs to the spectrum of $\pi(f)$, since $\pi_U(f)$ is the restriction of $\pi(f)$ to the invariant subspace $\mathcal{H}^U$. As the subgroup generated by the support of $f$ is dense in $G$, this implies that $1_G$ is weakly contained in $\pi$ (see [BHV, Proposition G.4.2]).

**Proof of Proposition 3** We give the proof only in the case where $X$ is the bi-regular tree $X_{k_0,k_1}$. The case where $X$ is the regular tree $X_k$ is similar and even simpler.

Let $X_0$ and $X_1$ be the subsets of $X$ consisting of the vertices of degree $k_0$ and $k_1$, respectively. Fix two points $x_0 \in X_0$ and $x_1 \in X_1$ with $d(x_0, x_1) = 1$. So, $X_0$ is the set of vertices $x$ for which $d(x_0, x)$ is even and $X_1$ is the set of vertices $x$ for which $d(x_0, x)$ is odd. Let $U_0$ and $U_1$ be the stabilizers of $x_0$ and $x_1$ in $G$. Since $G$ acts transitively on $X_0$ and on $X_1$, we have $G/U_0 \cong X_0$ and $G/U_1 \cong X_1$.

We can view the normed $*$-algebra $C_c(U_0 \setminus G/U_0)$ as a space of finitely supported functions on $X_0$. Since $U_0$ acts transitively on every sphere around $x_0$, it is well-known that the pair $(G, U_0)$ is a Gelfand pair, that is, the algebra $C_c(U_0 \setminus G/U_0)$ is commutative (see for instance [BLRW09, Lemma 2.1]). Observe that $C_c(U_0 \setminus G/U_0)$ is the linear span of the characteristic functions $\delta_n^{(0)}$ (lifted to $G$) of spheres of even radius $n$ around $x_0$. Moreover, $C_c(U_0 \setminus G/U_0)$ is generated by $\delta_2^{(0)}$; indeed, this follows from the formulas (see
[BLRW09, Theorem 3.3]

\[ \delta^{(0)}_4 = \delta^{(0)}_2 * \delta^{(0)}_2 - k_0(k_1 - 1)\delta^{(0)}_0 - (k_1 - 2)\delta^{(0)}_2 \]

\[ \delta^{(0)}_{2n+2} = \delta^{(0)}_2 * \delta^{(0)}_{2n} - (k_0 - 1)(k_1 - 1)\delta^{(0)}_{2n-2} - (k_1 - 2)\delta^{(0)}_{2n} \quad \text{for} \quad n \geq 2. \]

Let \( f_0 = \frac{1}{\|\delta^{(0)}_2\|_1} \delta^{(0)}_2 \). We claim that \( f_0 \) has all the properties listed in Proposition 4.

Indeed, \( f_0 \) is a non-negative and \( U_0 \)-bi-invariant function on \( G \) with \( \int_G f_0(x)dx = 1 \). Moreover, let \( H \) be the closure of the subgroup generated by the support of \( f_0 \). Assume, by contradiction, that \( H \neq G \). Then there exists a function in \( C_c(U_0 \backslash G/U_0) \) whose support is disjoint from \( H \). This is a contradiction, as the algebra \( C_c(U_0 \backslash G/U_0) \) is generated by \( f_0 \). This shows that \( H = G \).

Let \( \pi \) be the unitary representation of \( G \) on \( L^2_0(\Gamma \backslash G) \) defined by right translations. Observe that the space of \( \pi(U_0) \)-invariant vectors is \( L^2_0(\Gamma \backslash X_0) \).

So, we have a \( * \)-representation \( \pi_{U_0} \) of \( C_c(U_0 \backslash G/U_0) \) on \( L^2(\Gamma \backslash X_0, \mu) \), where \( \mu \) is the measure on the diagram \( D = \Gamma \backslash X \), as defined above.

Similar facts are also true for the algebra \( C_c(U_1 \backslash G/U_1) \) : this is a commutative \( * \)-algebra, it is generated by the characteristic function \( \delta^{(1)}_2 \) of the sphere of radius 2 around \( x_1 \), and the representation \( \pi \) of \( G \) on \( L^2_0(\Gamma \backslash G) \) induces a \( * \)-representation \( \pi_{U_1} \) of \( C_c(U_1 \backslash G/U_1) \) on \( L^2_0(\Gamma \backslash X_1, \mu) \). Likewise, the function \( f_1 = \frac{1}{\|\delta^{(1)}_2\|_1} \delta^{(1)}_2 \) has all the properties listed in Proposition 4.

Let \( A_X \) be the adjacency operator defined on \( \ell^2(X) \) by

\[ A_X f(x) = \frac{1}{\deg(x)} \sum_{e \in \partial^{-1}_0(x)} f(\partial_1(e)), \quad f \in \ell^2(X). \]

Since \( A_X \) commutes with automorphisms of \( X \), it induces an operator \( A_D \) on \( L^2(VD, \mu) \) given by

\[ A_D f(x) = \frac{1}{\indeg(x)} \sum_{e \in \partial^{-1}_0(x)} \frac{\mu(e)}{\mu(x)} f(\partial_1(e)), \quad f \in L^2(VD, \mu), \]

where \( D \) is the diagram obtained from the quotient graph \( \Gamma \backslash X \). So, \( \Delta = I - A_D \), where \( \Delta \) is the Laplace operator on \( D \).
Let $B_D$ denote the restriction of $A_D$ to the space $L^2_0(VD, \mu)$. It follows that $\lambda(\Delta) > 0$ if and only if 1 does not belong to the spectrum of $B_D$.

Proposition 3 will be proved, once we have shown the following

**Claim:** 1 belongs to the spectrum of $B_D$ if and only if 1 is weakly contained in $\pi$.

For this, we consider the squares of the operators $A_X$ and $A_D$ and compute

$$A_X^2f(x) = \frac{1}{k_0k_1}\deg(x)f(x) + \frac{1}{k_0k_1} \sum_{d(x,y)=2} f(y), \quad f \in \ell^2(X).$$

The subspaces $\ell^2(X_0)$ and $\ell^2(X_1)$ of $\ell^2(X)$ are invariant under $A_X^2$ and the restrictions of $A_X^2$ to $\ell^2(X_0)$ and $\ell^2(X_1)$ are given by right convolution with the functions

$$g_0 = \frac{1}{k_0k_1}\delta_e + (1 - \frac{1}{k_0k_1})f_0,$$
$$g_1 = \frac{1}{k_0k_1}\delta_e + (1 - \frac{1}{k_0k_1})f_1,$$

where $\delta_e$ is the Dirac function at the group unit $e$ of $G$.

It follows that the restrictions of $B_D^2$ to the subspaces $L^2_0(\Gamma \backslash X_0, \mu)$ and $L^2_0(\Gamma \backslash X_1, \mu)$ coincide with the operators $\pi_{U_0}(g_0)$ and $\pi_{U_1}(g_1)$, respectively.

For $i = 0, 1$, the spectrum $\sigma(\pi_{U_i}(g_i))$ of $\pi_{U_i}(g_i)$ is the set

$$\sigma(\pi_{U_i}(g_i)) = \left\{ \frac{1}{k_0k_1} + (1 - \frac{1}{k_0k_1})\lambda : \lambda \in \sigma(\pi_{U_i}(f_i)) \right\}.$$

Thus, 1 belongs to the spectrum of $\pi_{U_0}(f_i)$ if and only if 1 belongs to the spectrum of $\pi_{U_0}(g_i)$.

To prove the claim above, assume that 1 belongs to the spectrum of $B_D$. Then 1 belongs to the spectrum of $B_D^2$. Hence 1 belongs to the spectrum of either $\pi_{U_0}(g_0)$ or $\pi_{U_1}(g_1)$ and therefore 1 belongs to the spectrum of either $\pi_{U_0}(f_0)$ or $\pi_{U_1}(f_1)$. It follows from Proposition 4 that $1_G$ is weakly contained in $\pi$.

Conversely, suppose that $1_G$ is weakly contained in $\pi$. Then, again by Proposition 4, 1 belongs to the spectra of $\pi_{U_0}(f_0)$ and $\pi_{U_1}(f_1)$. Hence, 1 belongs to the spectra of $\pi_{U_0}(g_0)$ and $\pi_{U_1}(g_1)$. We claim that 1 belongs to the spectrum of $B_D$. 

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Indeed, assume by contradiction that $1$ does not belong to the spectrum of $B_D$, that is, $B_D - I$ has a bounded inverse on $L^2_0(VD, \mu)$. Since $1$ belongs to the spectrum of the self-adjoint operator $\pi U_0(g_0)$, there exists a sequence of unit vectors $\xi_n^{(0)}$ in $L^2_0(\Gamma \setminus X_0, \mu)$ with

$$\lim_n \|\pi U_0(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| = 0.$$ 

As the restriction of $B^2_D$ to $L^2_0(\Gamma \setminus X_0, \mu)$ coincides with $\pi U_0(g_0)$, we have

$$\|\pi U_0(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| = \|(B^2_D - I)\xi_n^{(0)}\|$$

$$= \|(B_D - I)(B_D + I)\xi_n^{(0)}\|$$

$$\geq \frac{1}{\|(B_D - I)^{-1}\|}(B_D + I)\xi_n^{(0)}\|$$

So, $\lim_n \|B_D\xi_n^{(0)} + \xi_n^{(0)}\| = 0$. On the other hand, observe that $B_D$ maps $L^2_0(\Gamma \setminus X_0, \mu)$ to the subspace $L^2(\Gamma \setminus X_1, \mu)$ and that these subspaces are orthogonal to each other. Hence,

$$\|B_D\xi_n^{(0)} + \xi_n^{(0)}\|^2 = \|B_D\xi_n^{(0)}\|^2 + \|\xi_n^{(0)}\|^2$$

This is a contradiction since $\|\xi_n^{(0)}\| = 1$ for all $n$. The proof of Proposition 3 is now complete. $\blacksquare$

Next, we rephrase Proposition 3 in terms of expander diagrams. Let $(D, i, w)$ be a diagram with finite volume. For a subset $S$ of $VD$, set

$$E(S, S^c) = \{ e \in ED : \partial_0(e) \in S, \partial_1(e) \notin S \}.$$ 

We say that $D$ is an expander diagram if there exists $\varepsilon > 0$ such that

$$\frac{\mu(E(S, S^c))}{\mu(S)} \geq \varepsilon$$

for all $S \subset VD$ with $\mu(S) \leq \mu(D)/2$. The motivation for this definition comes from expander graphs (see [Lubo94]).

We quote from [Mokh03] and [Morg94] the following result which is standard in the case of finite graphs.

**Proposition 5** ([Mokh03], [Morg94]) Let $(D, i, w)$ be a diagram with finite volume. Assume that $\sup_{e \in ED} i(e)/i(e) < \infty$ and that $\sup_{x \in VD} \text{indeg}(x) < \infty$ The following conditions are equivalent:
(i) $D$ is an expander diagram;

(ii) $\lambda(D) > 0$.

As an immediate consequence of Propositions 3 and 5, we obtain the following result which relates the existence of a spectral gap to an expanding property of the corresponding diagram.

**Proposition 6** Let $X$ be either the $k$-regular tree $X_k$ or the bi-partite bi-regular tree $X_{k_0,k_1}$ for $k \geq 3$ or $k_0 \geq 3$ and $k_1 \geq 3$. Let $G$ be a closed subgroup of $\text{Aut}(X)$ satisfying both conditions from Proposition 3. Let $\Gamma$ be a lattice in $G$ and let $D = \Gamma \setminus X$ be the corresponding diagram of finite volume. The following properties are equivalent.

(i) The unitary representation $\rho_{\Gamma \setminus G}$ on $L^2(\Gamma \setminus G)$ has a spectral gap;

(ii) $D$ is an expander diagram.

3 Proof of Theorem 1

Let $G = G(k)$ be the $k$-rational points of a simple algebraic group $G$ over a local field $k$ and let $\Gamma$ be a lattice in $G$. As explained in the Introduction, we may assume that $k$ is non-archimedean and that $k - \text{rank}(G) = 1$. By the Bruhat-Tits theory, $G$ acts on a regular or bi-partite bi-regular tree $X$ with one or two orbits. Moreover, the action of $G$ on $X$ is locally $\infty$-transitive (see [Chou94, p.33]).

Passing to the subgroup $G^+$ of index at most two consisting of orientation preserving automorphisms, we can assume that $G$ acts without inversion. Indeed, assume that $L^2(\Gamma \cap G^+ \setminus G^+)$ has a spectral gap. If $\Gamma$ is contained in $G^+$, then $L^2(\Gamma \setminus G)$ has a spectral gap since $G^+$ has finite index (see [BeCo08, Proposition 6]). If $\Gamma$ is not contained in $G^+$, then $\Gamma \cap G^+ \setminus G^+$ may be identified as a $G^+$-space with $\Gamma \setminus G^+ = \Gamma \setminus G$. Hence, $1_{G^+}$ is not weakly contained in the $G^+$-representation defined on $L_0^2(\Gamma \setminus G)$.

Let $X$ be the Bruhat-Tits tree associated to $G$. It is shown in [Lubo91, Theorem 6.1] (see also [Baum03]) that $\Gamma$ has fundamental domain $D$ in $X$ of the following form: there exists a finite set $F \subset D$ such that $D \setminus F$ is a union of finitely many disjoint rays $r_1, \ldots, r_s$. (Recall that a ray in $X$ is an infinite
path beginning at some vertex and without backtracking.) Moreover, for every ray \( r_j = \{x_0^j, x_1^j, x_2^j, \ldots \} \) in \( D \setminus F \), the stabilizer \( \Gamma_{x^j_i} \) of \( x^j_i \) is contained in the stabilizer \( \Gamma_{x^j_{i+1}} \) of \( x^j_{i+1} \) for all \( i \).

To prove Theorem 1, we apply Proposition 6. So, we have to prove that \( D \) is an expander diagram.

Choose \( i \in \{0, 1, \ldots \} \) such that, with \( D_1 = F \cup \bigcup_{j=1}^s \{x_0^j, \ldots, x_i^j\} \), we have \( \mu(D_1) > 1/2 \).

Let \( S \) be a subset of \( D \) with \( \mu(S) \leq \mu(D)/2 \). Then \( D_1 \not\subseteq S \). Two cases can occur.

- **First case**: \( S \cap D_1 = \emptyset \). Thus, \( S \) is contained in
  \[
  \bigcup_{j=1}^s \{x_{i+1}^j, x_{i+2}^j, \ldots \}.
  \]

  Fix \( j \in \{1, \ldots, s\} \). Let \( i(j) \in \{0, 1, \ldots \} \) be minimal with the property that \( x_{i(j)+1}^j \in S \). Then \( e_j := (x_{i(j)+1}^j, x_{i(j)}^j) \in E(S, S^c) \). Observe that \( |\Gamma_{x_{i+1}^j}| = \deg(x_{i+1}^j) \) for all \( l \geq 0 \). Let \( k \) be the minimal degree for vertices in \( X \) (so, \( k = \min\{k_0, k_1\} \) if \( X = X_{k_0, k_1} \)). Then \( \mu(x_{i+1}^j) \leq \mu(x_{i}^j)/k \) for all \( l \) and
  \[
  \mu(e_j) = \frac{1}{|\Gamma_{e_j}|} \geq \frac{k}{|\Gamma_{x_{i(j)}^j}|} = k\mu(x_{i(j)}^j).
  \]
Therefore, we have

\[
\frac{\mu(E(S, S^c))}{\mu(S)} \geq \frac{\sum_{j=1}^{s} \mu(e_j)}{\sum_{j=1}^{s} \mu(x_{i(j)}^j)} \geq k \frac{\sum_{j=1}^{s} \mu(x_{i(j)}^j)}{\sum_{j=1}^{s} \sum_{l=0}^{\infty} \mu(x_{i(j)}^{l+1})} \geq k \frac{\sum_{j=1}^{s} \mu(x_{i(j)}^j)}{\sum_{j=1}^{s} \mu(x_{i(j)}^j) \sum_{l=0}^{\infty} k^{-l}} = k \frac{1}{1-k^{-1}} = k - 1.
\]

- **Second case:** \(S \cap D_1 \neq \emptyset\). Then there exist \(x \in S \cap D_1\) and \(y \in D_1 \setminus S\). Since \(D_1\) is a connected subgraph, there exists a path \((e_1, e_2, \ldots, e_n)\) in \(ED_1\) from \(x\) to \(y\). Let \(l \in \{1, \ldots, n\}\) be minimal with the property \(\partial_0(e_l) \in S\) and \(\partial_1(e_l) \notin S\). Then \(e_l \in E(S, S^c)\). Hence, with \(C = \min \{\mu(e) : e \in ED_1\} > 0\), we have

\[
\frac{\mu(E(S, S^c))}{\mu(S)} \geq \frac{C}{\mu(D)}.
\]

This completes the proof of Theorem 1. □

### 4 Proof of Theorem 2

Let \((D, i, \mu)\) be a \(k\)-regular diagram. By the “inverse Bass–Serre theory” of groups acting on trees, there exists a lattice \(\Gamma\) in \(G = \text{Aut}(X_k)\) for which \(D = \Gamma \setminus X_k\). Indeed, we can find a finite grouping of \((D, i)\), that is, a graph of finite groups \(D = (D, \mathcal{D})\) such that \(i(e)\) is the index of \(D_e\) in \(\mathcal{D}_{\partial_0 e}\) for all \(e \in ED\). Fix an origin \(x_0\). Let \(\Gamma = \pi_1(D, x_0)\) be the fundamental group of \((D, x_0)\). The universal covering of \((D, x_0)\) is the \(k\)-regular tree \(X_k\) and the diagram \(D\) can be identified with the diagram associated to \(\Gamma \setminus X_k\). For all this, see (2.5), (2.6) and (4.13) in [BaLu01].

In view of Proposition 6, Theorem 2 will be proved once we present examples of \(k\)-regular diagrams with finite volume which are not expanders.
An example of such a diagram appears in [Mokh03, Example 3.4]. For the convenience of the reader, we review the construction.

Fix $k \geq 3$ and let $q = k - 1$. For every integer $n \geq 1$, let $D_n$ be the finite graph with $2n + 1$ vertices:

\begin{align*}
  \circ \quad &- \circ \quad - \circ \quad - \cdots \quad - \circ \quad - \circ \\
  x^{(n)}_1 \quad &- x^{(n)}_2 \quad - x^{(n)}_3 \quad - \cdots \quad - x^{(n)}_{2n} \quad - x^{(n)}_{2n+1}
\end{align*}

Let $D$ be the following infinite ray:

\begin{align*}
  \circ \quad &- \circ \quad - D_1 \quad - \circ \quad - \circ \quad - D_2 \quad - \circ \quad - \circ \quad - \cdots \\
  x_0 \quad &- x_1 \quad - x_2 \quad - x_3 \quad - \cdots \quad - x_{2n-2} \quad - x_{2n-1} \quad - D_n \quad - \circ \quad - \circ \quad - \cdots
\end{align*}

We first define a weight function $i_n$ on $ED_n$ as follows:

- $i_n(e) = 1$ if $e = (x^{(n)}_1, x^{(n)}_2)$ or $e = (x^{(n)}_2, x^{(n)}_1)$
- $i_n(e) = q$ if $e = (x^{(n)}_m, x^{(n)}_{m+1})$ for $m$ even
- $i_n(e) = 1$ if $e = (x^{(n)}_m, x^{(n)}_{m+1})$ for $m$ odd
- $i_n(e) = q$ if $e = (x^{(n)}_{m+1}, x^{(n)}_m)$ for $m$ even
- $i_n(e) = 1$ if $e = (x^{(n)}_{m+1}, x^{(n)}_m)$ for $m$ odd.

Observe that $i_n(e)/i_n(e) = 1$ for all $e \in ED_n$. Define now a weight function $i$ on $ED$ as follows:

- $i(e) = q + 1$ if $e = (x_0, x_1)$
- $i(e) = q$ if $e = (x_1, x_0)$
- $i(e) = 1$ if $e = (x_m, x_{m+1})$ for $m \geq 1$
- $i(e) = q$ if $e = (x_{m+1}, x_m)$ for $m \geq 1$
- $i(e) = i_n(e)$ if $e \in ED_n$.

One readily checks that, for every vertex $x \in D$,

$$
\sum_{e \in \partial x} i(e) = q + 1 = k,
$$

that is, $(D, i)$ is $k$-regular. The measure $\mu : VD \to \mathbb{R}^+$ corresponding to $i$ (see the remark at the beginning of Section 2) is given by
• \( \mu(x_0) = 1/(q + 1) \)
• \( \mu(x_{2m-2}) = 1/q^{m-1} \) for \( m \geq 2 \)
• \( \mu(x_{2m-1}) = 1/q^m \) for \( m \geq 1 \)
• \( \mu(x) = 1/q^n \) if \( x \in D_n \).

One checks that, if we define \( \mu(e) = i(e)\mu(\partial_0 e) \) for all \( e \in ED \), we have \( \mu(\bar{e}) = \mu(e) \). Moreover,

\[
\mu(D_n) = (2n + 1) \frac{1}{q^n}
\]

and hence

\[
\mu(D) \leq \frac{1}{q + 1} + 2 \sum_{n \geq 0} \frac{1}{q^n} + \sum_{n \geq 1} \mu(D_n) < \infty.
\]

We have also

\[
E(D_n, D_n^c) = \{(x_{2n-1}, x_{2n-2}), (x_{2n}, x_{2n+1})\},
\]

so that

\[
\mu(E(D_n, D_n^c)) = q \frac{1}{q^n} + \frac{1}{q^n} = \frac{q + 1}{q^n}.
\]

Hence

\[
\frac{\mu(E(D_n, D_n^c))}{\mu(D_n)} = \frac{\frac{q+1}{q^n}}{(2n+1)\frac{1}{q^n}} = \frac{q + 1}{2n + 1}
\]

and

\[
\lim_{n} \frac{\mu(E(D_n, D_n^c))}{\mu(D_n)} = 0.
\]

Observe that, since \( \lim_n \mu(D_n) = 0 \), we have \( \mu(D_n) \leq \mu(D)/2 \) for sufficiently large \( n \). This completes the proof of Theorem 2.

\section*{References}


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