Dynamic dependence ordering for Archimedean copulas

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Motivations: dependence and copulas

Definition 1. A copula $C$ is a joint distribution function on $[0, 1]^d$, with uniform margins on $[0, 1]$.

Theorem 2. (Sklar) Let $C$ be a copula, and $F_1, \ldots, F_d$ be $d$ marginal distributions, then $F(x) = C(F_1(x_1), \ldots, F_d(x_d))$ is a distribution function, with $F \in \mathcal{F}(F_1, \ldots, F_d)$.

Conversely, if $F \in \mathcal{F}(F_1, \ldots, F_d)$, there exists $C$ such that $F(x) = C(F_1(x_1), \ldots, F_d(x_d))$. Further, if the $F_i$'s are continuous, then $C$ is unique, and given by

$$C(u) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)) \text{ for all } u_i \in [0, 1]$$

We will then define the copula of $F$, or the copula of $X$. 
Motivations: dependence and copulas

Given a random vector $\mathbf{X}$ with continuous margins. The copula of $C$ satisfies

$$
\mathbb{P} [X_1 \leq x_1, \ldots, X_d \leq x_d] = C \left( \mathbb{P} [X_1 \leq x_1], \ldots, \mathbb{P} [X_d \leq x_d] \right)
$$

and its survival copula $C^*$ satisfies

$$
\mathbb{P} [X_1 > x_1, \ldots, X_d > x_d] = C^* \left( \mathbb{P} [X_1 > x_1], \ldots, \mathbb{P} [X_d > x_d] \right)
$$
Fig. 1 – Graphical representation of a copula, $C(u, v) = \mathbb{P}(U \leq u, V \leq v)$. 
Fig. 2 – Density of a copula, \( c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v} \).
Archimedean copulas

Definition 3. A copula $C$ is called Archimedean if it is of the form

$$C(u_1, \cdots, u_d) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)),$$

where the generator $\phi : [0, 1] \to [0, \infty]$ is convex, decreasing and satisfies $\phi(1) = 0$. A necessary and sufficient condition is that $\phi^{-1}$ is $d$-monotone.
Some examples of Archimedean copulas

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<th>( \phi(t) )</th>
<th>range ( \theta )</th>
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<td>(1) ( \frac{1}{\theta} (t^{-\theta} - 1) )</td>
<td>([-1, 0) \cup (0, \infty))</td>
<td><strong>Clayton</strong>, <strong>Clayton (1978)</strong></td>
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<td>(2) ((1-t)^\theta)</td>
<td>([1, \infty))</td>
<td><strong>Ali-Mikhail-Haq</strong></td>
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<td>(3) (\log \frac{1-\theta(1-t)}{t} )</td>
<td>([-1, 1))</td>
<td><strong>Gumbel</strong>, <strong>Gumbel (1960), Hougaard (1986)</strong></td>
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<td>(4) ((-\log t)^\theta)</td>
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<td><strong>Frank</strong>, <strong>Frank (1979), Nelsen (1987)</strong></td>
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<td>(5) (-\log \frac{e^{-\theta t} - 1}{e^{-\theta} - 1} )</td>
<td>((-\infty, 0) \cup (0, \infty))</td>
<td><strong>Joe</strong>, <strong>Frank (1981), Joe (1993)</strong></td>
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<td>(6) (-\log {1 - (1 - t)\theta} )</td>
<td>([1, \infty))</td>
<td><strong>Barnett (1980), Gumbel (1960)</strong></td>
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<td>(7) (-\log {\theta t + (1 - \theta)} )</td>
<td>((0, 1))</td>
<td><strong>Genest &amp; Ghoudi (1994)</strong></td>
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<td>(8) (\frac{1-t}{1+(\theta-1)t} )</td>
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<td>(9) (\log (1 - \theta \log t) )</td>
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<td><strong>Genest &amp; Ghoudi (1994)</strong></td>
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<td>(10) (\log (2t^{-\theta} - 1) )</td>
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<td><strong>Genest &amp; Ghoudi (1994)</strong></td>
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<td>((0, 1/2))</td>
<td><strong>Genest &amp; Ghoudi (1994)</strong></td>
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<td>(12) ((\frac{1}{t} - 1)^\theta)</td>
<td>([1, \infty))</td>
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<td>(13) ((1 - \log t)^\theta - 1 )</td>
<td>((0, \infty))</td>
<td><strong>Genest &amp; Ghoudi (1994)</strong></td>
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<td>(14) ((t^{-1/\theta} - 1)^\theta)</td>
<td>([1, \infty))</td>
<td><strong>Genest &amp; Ghoudi (1994)</strong></td>
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<td>(15) ((1 - t^{1/\theta})^\theta)</td>
<td>([1, \infty))</td>
<td><strong>Genest &amp; Ghoudi (1994)</strong></td>
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<td>(16) ((\frac{\theta}{t} + 1)(1 - t) )</td>
<td>([0, \infty))</td>
<td><strong>Genest &amp; Ghoudi (1994)</strong></td>
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Why *Archimedean* copulas?

Assume that $X$ and $Y$ are conditionally independent, given the value of an heterogeneous component $\Theta$. Assume further that

$$\mathbb{P}(X \leq x | \Theta = \theta) = (G_X(x))^\theta \quad \text{and} \quad \mathbb{P}(Y \leq y | \Theta = \theta) = (G_Y(y))^\theta$$

for some baseline distribution functions $G_X$ and $G_Y$. Then

$$F(x, y) = \psi(\psi^{-1}(F_X(x)) + \psi^{-1}(F_Y(y))).$$

where $\psi$ denotes the Laplace transform of $\Theta$, i.e. $\psi(t) = \mathbb{E}(e^{-t\Theta})$. 
Fig. 3 – Two classes of risks, $(X_i, Y_i)$ and $(\Phi^{-1}(F_X(X_i)), \Phi^{-1}(F_Y(Y_i)))$. 
Fig. 4 – Three classes of risks, $(X_i, Y_i)$ and $(\Phi^{-1}(F_X(X_i)), \Phi^{-1}(F_Y(Y_i)))$. 
Fig. 5 – Continuous classes of risks, \((X_i, Y_i)\) and \((\Phi^{-1}(F_X(X_i)), \Phi^{-1}(F_Y(Y_i)))\).
Conditioning with Archimedean copulas

Proposition 4. If \((U, V)\) has copula \(C\), with generator \(\phi\). Then the copula of \((U, V)\) given \(U, V \leq u\) is also Archimedean, with generator

\[
\phi_u(x) = \phi(x \cdot C(u, u)) - \phi(C(u, u)), \quad \text{for all } x \in (0, 1].
\]

Ageing with Archimedean copulas

Proposition 5. If \((X, Y)\) is exchangeable, with survival copula \(C\), with generator \(\phi\). Then the survival copula \(C_t\) of \((X, Y)\) given \(X, Y > t\) is also Archimedean, with generator

\[
\phi_t(x) = \phi(x \cdot \gamma_t) - \phi(\gamma_t), \quad \text{for all } x \in (0, 1], t \in [0, \infty),
\]

where \(\gamma_t = C(F(t), \overline{F}(t)) = \mathbb{P}(X > t, Y > t)\).
Fig. 6 – Initial Archimedean generator, $\cdot \mapsto \phi(\cdot)$ on $(0, 1]$. 

Ageing with Archimedean copulas
Ageing with Archimedean copulas

\[ P(X > t, Y > t) \]

**Fig. 7** – Initial Archimedean generator, \( \cdot \mapsto \phi(\cdot) \) on \((0, 1]\).
Ageing with Archimedean copulas

Fig. 8 – Initial Archimedean generator, $\cdot \mapsto \phi(\cdot)$ on $(0, \gamma_t]$, $\gamma_t = C(F(t), \bar{F}(t))$. 
Ageing with Archimedean copulas

\[ \phi(t) \text{ on } (0, \gamma_t) \]

**Fig. 9** – Initial Archimedean generator, \( \cdot \mapsto \phi(\cdot) \) on \( (0, \gamma_t) \).
Fig. 10 – Initial Archimedean generator, \( \cdot \mapsto \phi(\cdot \gamma_t) \) on \([0, 1]\).
Ageing with Archimedean copulas

Fig. 11 – Initial Archimedean generator, $\cdot \mapsto \phi(\gamma_t) - \phi(\gamma_t)$ on $[0, 1]$. 
Ageing with Frailty-Archimedean copulas

Proposition 6. If $(X, Y)$ is exchangeable with a frailty representation (where $\Theta$ has Laplace transform $\psi$). Then $(X, Y)$ given $X, Y > t$ also has a frailty representation, and the factor has Laplace transform

$$\psi_t(x) = \frac{\psi[x + \psi^{-1}(\gamma_t)]}{\gamma_t}, \text{ for all } x \in [0, \infty), t \in [0, \infty).$$

where $\gamma_t = C(\overline{F}(t), \overline{F}(t)).$

Démonstration. The only technical part is to proof that $(X, Y)$ given $X, Y > t$ are conditionally independent.
Ordering tails of Archimedean copulas

Proposition 7. Set $C_1 = C(t_1, t_1)$ and $C_2 = C(t_2, t_2)$. Given $\phi$, define

$$\begin{align*}
f_{12}(x) &= \phi \left( \frac{C_1}{C_2} \phi^{-1}(x + \phi(C_1)) \right) - \phi(C_1) \\
f_{21}(x) &= \phi \left( \frac{C_2}{C_1} \phi^{-1}(x + \phi(C_1)) \right) - \phi(C_2)
\end{align*}$$

Then

$$\begin{align*}
C_{t_2} \preceq C_{t_1} & \text{ if and only if } f_{21} \text{ is subadditive} \\
C_{t_2} \succeq C_{t_1} & \text{ if and only if } f_{12} \text{ is subadditive}
\end{align*}$$

Lemma 8. If $\phi$ is twice differentiable, set $\psi(t) = \log(-\phi'(t))$.

$$\begin{align*}
\text{if } \psi \text{ is concave on } (0, 1), & \quad C_{t_2} \preceq C_{t_1} \text{ for all } 0 < t_1 \leq t_2 \\
\text{if } \psi \text{ is convex on } (0, 1), & \quad C_{t_2} \succeq C_{t_1} \text{ for all } 0 < t_1 \leq t_2
\end{align*}$$
Some examples: Frank copula

Frank copula has generator $\phi(t) = -\log \frac{e^{-\alpha x} - 1}{e^{-\alpha} - 1}$, $\alpha \geq 0$

$$C(u, v) = -\frac{1}{\alpha} \log \left( 1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1} \right) \text{ on } [0, 1] \times [0, 1].$$

$\psi(x) = \log \alpha - \alpha x - \log[1 - e^{-\alpha x}]$ is concave, and thus

$$C \geq C_{t_1} \geq C_{t_2} \geq C^\perp, \text{ for all } 0 \leq t_1 \leq t_2,$$

with $C_t \to C$ as $t \to \infty$. The random pair is less and less positively dependent.
Some examples: Clayton copula

Clayton copula has generator \( \phi(t) = t^{-\alpha} - 1, \alpha \geq 0 \),

\[
C(u, v) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha} \quad \text{on } [0, 1] \times [0, 1].
\]

\( f_{12} \) and \( f_{21} \) are linear (see also Lemma 5.5.8. in Schweizer and Sklar (1983)), hence

\[
C_{t_1} = C_{t_2} = C.
\]
Some examples: Ali-Mikhail-Haq copula

Ali-Mikhail-Haq copula has generator $\phi(t) = \ldots 1, \, \alpha \in [0, 1)$,

$$C(u, v) = \frac{uv}{1 - \alpha(1 - u)}(1 - v) \text{ on } [0, 1] \times [0, 1].$$

$$\psi(x) = \log \left( \frac{1}{x} - \frac{\alpha}{1 - \alpha[1 - x]} \right)$$

is concave, and thus

$$C \succeq C_{t_1} \succeq C_{t_2} \succeq C^\perp,$$

for all $0 \leq t_1 \leq t_2$,

with $C_t \to C$ as $t \to \infty$. 
Some examples : Gumbel copula

Gumbel copula has generator $\phi(t) = (-\log t)^\alpha$, $\alpha \geq 0$,

$$C(u, v) = \exp \left(- [(-\log u)^\alpha + (-\log v)^\alpha]^\frac{1}{\alpha} \right) \text{ on } [0, 1] \times [0, 1].$$

$\psi(x) = \log \alpha - \log x + (\alpha - 1) \log[-\log x]$ is neither concave, nor convex.

But see $C - C_t$ is always positive, i.e.

$$C \succeq C_t \succeq C^\perp, \text{ for all } 0 < t.$$