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Criteria for the divergence of pairs of Teichmüller geodesics

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Abstract We study the asymptotic geometry of Teichmüller geodesic rays. We show that, when the transverse measures to the vertical foliations of the quadratic differentials determining two different rays are topologically equivalent, but are not absolutely continuous with respect to each other, the rays diverge in Teichmüller space.

Keywords Teichmüller space · Divergent geodesics · Extremal length · Ergodic

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1 Introduction

Let S be an oriented surface of genus g with n punctures. We assume $3g - 3 + n \geq 1$. Let $\mathcal{T}(S)$ denote the Teichmüller space of S with the Teichmüller metric $d(\cdot, \cdot)$. A basic question in geometry is to study the long term behavor of geodesics. In this paper we study the question of when a pair of geodesic rays $X_1(t)$, $X_2(t)$, with possibly distinct basepoints, stay bounded distance apart, and when they diverge in the sense that $d(X_1(t), X_2(t)) \to \infty$ as $t \to \infty$.

Teichmüller's theorem implies that a Teichmüller geodesic ray is determined by a quadratic differential q at the base point and that there are quadratic differentials $q(t)$ on $X(t)$ along the ray found by stretching along the horizontal trajectories of q and contracting along the vertical trajectories.

Many cases of the question of divergence of rays are already known. It is a general principle that the asymptotic behavior of the ray is determined by the properties of the

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vertical foliation of q. The first instance is if the quadratic differentials q_1, q_2 defining the geodesic rays $X_1(t)$, $X_2(t)$ are Strebel differentials. This means that their vertical trajectories are closed and decompose the surface into cylinders. In [7] it was shown that if the homotopy classes of the cylinders for q_1 coincide with those of q_2, then $X_1(t)$, $X_2(t)$ stay bounded distance apart. In particular this showed that the Teichmüller metric was not negatively curved in the sense of Busemann. A second known case is if the vertical foliations of q_1, q_2 are the same uniquely ergodic foliation. In that case the rays also stay bounded distance apart [8].

The next possibility is the vertical foliations of q_1, q_2 are topologically equivalent, have a minimal component and yet are not uniquely ergodic. (It is well-known that for any quadratic differential, the vertical trajectories decompose the surface into cylinders and subsurfaces in which every trajectory is dense). In that case in each minimal component there exist a finite number of mutually singular ergodic measures, and any transverse measure is a convex combination of the ergodic measures. Ivanov [5] showed that if the transverse measures of q_1, q_2 in these minimal components are absolutely continuous with respect to each other, then the rays stay bounded distance apart. In this paper we prove the converse.

Let q_1, q_2 be quadratic differentials on X_1 and X_2 with vertical foliations $\left[F^v_{q_1}, |dx_1|\right]$ and $\left[F^v_{q_2}, |dx_2|\right]$ and determining rays $X_1(t)$, $X_2(t)$. Our main result is then

Theorem A *Suppose $F^v_{q_1}$ and $F^v_{q_2}$ are topologically equivalent. Suppose there is a minimal component Ω of the foliations $F^v_{q_i}$ with ergodic measures $\nu_1, \ldots, \nu_p$ and so that restricted to Ω, $|dx_1| = \sum_{i=1}^p a_i\nu_i$, $|dx_2| = \sum_{i=1}^p b_i\nu_i$ and there is some index i so that either $a_i = 0$ and $b_i > 0$, or $a_i > 0$ and $b_i = 0$. Then the rays $X_1(t)$ and $X_2(t)$ diverge.*

In particular, this holds when the transverse measures are distinct ergodic measures.

The last possibility is that the vertical foliations of q_1, q_2 are not topologically equivalent. If the geometric intersection of the vertical foliations is nonzero, then the rays diverge [5]. We prove

Theorem B *Suppose q_1, q_2 are quadratic differentials such that the vertical foliations $\left[F^v_{q_1}, |dx_1|\right]$ and $\left[F^v_{q_2}, |dx_2|\right]$ are not topologically equivalent, but $i\left(\left[F^v_{q_1}, |dx_1|\right], \left[F^v_{q_2}, |dx_2|\right]\right) = 0$. Then the rays $X_1(t))$ and $X_2(t)$ diverge.*

These theorems together with the previously known results completely answer the question of divergence of rays.

The outline of the proof of Theorem A is as follows. In Proposition 1, we will show that, for the flat metrics defined by the quadratic differentials $q_1(t)$ and $q_2(t)$, for any sufficiently large time t, there is a subsurface $Y(t) \subset \Omega$ with its area small in one metric and bounded away from zero in the other, while its boundary is short in both metrics. This is where we use the assumption that the measures are not absolutely continuous with respect to each other. We will then apply Lemma 6 to find a bounded length curve $\gamma(t) \subset Y(t)$ which is "mostly vertical" with respect to the metric of $q_1(t)$. It has comparable length in the metric of $q_2(t)$. Using the fact that the quadratic differentials give comparable length to $\gamma(t)$ while giving very different areas to $Y(t)$, Lemma 7 will allow us to show that the ratio of the extremal length of $\gamma(t)$ along one ray to the extremal length on the other is large. We then apply Kerckhoff's formula to conclude that the surfaces are far apart in Teichmuller space.

We will also prove

Theorem C *Let $\nu_1, \ldots, \nu_p$ be maximal collection of ergodic measures for a minimal foliation $[F, \mu]$. Then there is a sequence of multicurves $\gamma_n = \left\{\gamma_n^1, \ldots, \gamma_n^k\right\}$ such that $\gamma_n^j \to [F, \nu_j]$ in $\mathcal{PMF}$.*

In other words, any two topologically equivalent measured foliations can be approximated by a sequence of multicurves, with possibly different weights.

This result settles a question asked by Moon Duchin.

2 Background

2.1 Measured foliations

Recall a measured foliation on a surface S consists of a finite set Σ of singular points and a covering of $S \setminus \Sigma$ by open sets $\{U_\alpha\}$ with charts $\phi_\alpha : U_\alpha \to \mathbb{R}^2$ such that the overlap maps are of the form

$$(x, y) \to (\pm x + c, f(x, y)).$$

The leaves of the foliation are the lines $x =$ constant. The points Σ are p-pronged singularities for $p \geq 3$. One allows single pronged singularities at the punctures. A measured foliation comes equipped with a transverse invariant measure which in the above coordinates is given by $\mu = |dx|$. Henceforth we will denote measured foliations by $[F, \mu]$. We will write F to denote a (topological) foliation when we are ignoring the measure.

For the rest of the paper, a curve will always mean a simple closed curve. For any homotopy class of simple closed curves β, let

$$i([F, \mu], \beta) = \inf_{\beta' \sim \beta} \int_{\beta'} d\mu.$$

The intersection with simple closed curves extends to an intersection function

$$i([F_1, \mu_1], [F_2, \mu_2])$$

on pairs of measured foliations. Thurston's space of measured foliations is denoted $\mathcal{M}F$ and the projective space of measured foliations by $\mathcal{P}MF$.

Let Γ_F denote the compact leaves of F joining singularities. It is well-known that each component Ω of $S \setminus \Gamma_F$ is either an annulus swept out by closed leaves or a minimal domain in which every leaf is dense.

Definition 1 We say that two foliations F_1, F_2 on S are topologically equivalent, and we write $F_1 \sim F_2$, if there is a homeomorphism of $S \setminus \Gamma_{F_1} \to S \setminus \Gamma_{F_2}$ isotopic to the identity which takes the leaves of F_1 to the leaves of F_2.

Note that this definition does not refer to the measures.

Definition 2 A foliation $[F, \mu]$ in a minimal domain Ω is said to be *uniquely ergodic* if the transverse measure μ restricted to Ω is the unique transverse measure of the foliation F up to scalar multiplication.

More generally, suppose Ω is a minimal component of $[F, \mu]$. There exist invariant transverse measures $\nu_1 = \nu_1(\Omega), \ldots, \nu_p = \nu_p(\Omega)$ such that

- p is bounded in terms of the topology of Ω.
- ν_i is ergodic for each i.
- any transverse invariant measure ν on Ω can be written as $\nu = \sum_{i=1}^{p} a_i \nu_i$ for $a_i \geq 0$.

Thus the transverse measures are parametrized by a simplex in $\mathbb{R}^p$. Two foliations, $[F, \mu_1]$ and $[F, \mu_2]$, are absolutely continuous with respect to each other if, when the measures are expressed as a convex combination as above, the indices with positive coefficients are identical. Equivalently, they are absolutely continuous with respect to each other if they lie in the same open face of the simplex.

2.2 Quadratic differentials and Teichmüller rays

A meromorphic quadratic differential q on a closed Riemann surface X with a finite number of punctures removed is a tensor of the form $q(z)dz^2$ where q is a holomorphic function and $q(z)dz^2$ is invariant under change of coordinates. We allow q to have at most simple poles at the punctures.

As such there is a metric defined by $|q(z)|^{1/2}|dz|$. The length of an arc β with respect to the metric will be denoted by $|\beta|_q$. There is an area element defined by $|q(z)||dz|^2$. We will denote by $\mathrm{Area}_q\, \Omega$ the area of a subsurface $\Omega \subseteq X$.

Away from the zeroes and poles of q there are *natural* holomorphic coordinates $z = x+iy$ such that in these coordinates $q = dz^2$. The lines $x =$ constant with transverse measure $|dx|$ define the vertical foliation $\left[F_q^v, |dx|\right]$. The lines $y =$ constant with transverse measure $|dy|$ define the horizontal measured foliation $\left[F_q^h, |dy|\right]$. The transverse measure of an arc β with respect to $|dy|$ will be denoted by $v_q(\beta)$ and called the vertical length of β. Similarly, we have the horizontal length denoted by $h_q(\beta)$. The area element in the natural coordinates is given by $dxdy$.

We denote by $\Gamma_q = \Gamma_{F_q^v}$ the vertical critical graph of q. This is the union of the vertical leaves joining the zeroes of q.

The *Teichmüller space* of S denoted by $\mathcal{T}(S)$ is the set of equivalence classes of Riemann surface structures X on S, where X_1 is equivalent to X_2 if there is a conformal map $f : X_1 \to X_2$ isotopic to the identity on S. The *Teichmüller metric* on $\mathcal{T}(S)$ is the metric defined by

$$d(X_1, X_2) := \frac{1}{2} \inf_f \{\log K(f) : f : X_1 \to X_2 \text{ is homotopic to Id }\}$$

where f is quasiconformal and

$$K(f) := ||K_x(f)||_\infty \geq 1$$

is the *quasiconformal dilatation* of f, where

$$K_x(f) := \frac{|f_z(x)| + |f_{\bar{z}}(x)|}{|f_z(x)| - |f_{\bar{z}}(x)|}$$

is the *pointwise quasiconformal dilatation* at x.

Teichmüller's Theorem states that, given any $X_1, X_2 \in \mathcal{T}(S)$, there exists a unique (up to translation in the case when S is a torus) quasiconformal map f, called the *Teichmüller map*, realizing $d(X_1, X_2)$. The Beltrami coefficient $\mu_f := \frac{\bar{\partial} f}{\partial f}$ is of the form $\mu_f = k\frac{\bar{q}}{|q|}$ for a unique unit area quadratic differential q on X_1 and some k with $0 \leq k < 1$. Define t by

$$e^{2t} = \frac{1+k}{1-k}.$$

There is a quadratic differential $q(t)$ on X_2 such that in the natural local coordinates $w = u + iv$ of $q(t)$ and $z = x + iy$ of q the map f is given by

$$u = e^t x \quad v = e^{-t} y.$$

Thus f expands along the horizontal leaves of q by e^t, and contracts along the vertical leaves by e^{-t}.

Conversely, any unit area q on X determines a 1-parameter family of Teichmuller maps f_t defined on X. Namely f_t has Beltrami differential $\mu = k\frac{\bar{q}}{|q|}$ where $e^{2t} = \frac{1+k}{1-k}$. The image surface is denoted by $X(t)$ and $X(t)$; $t \geq 0$ is the *Teichmüller ray* based at X in the direction of q. On each $X(t)$ we have the quadratic differential $q(t)$.

2.3 Extremal length and Annuli

We recall the notion of extremal length. Suppose X is a Riemann surface and Γ is a family of arcs on X. Suppose ρ is a conformal metric on X. For an arc γ, denote by $\rho(\gamma)$ its length and by $A(\rho)$ the area of ρ.

Definition 3

$$\mathrm{Ext}_X(\Gamma) = \sup_\rho \frac{\inf_{\gamma \in \Gamma} \rho^2(\gamma)}{A(\rho)},$$

where the sup is over all conformal metrics ρ.

We will apply this definition when Γ consists of all simple closed curves in a free homotopy class of some α. In that case we will write $\mathrm{Ext}_X(\alpha)$. It is also worth noting that if q is a unit area quadratic differential then $\mathrm{Ext}_X(\alpha) \geq |\alpha|_q^2$ since q gives a competing metric. (Here again $|\alpha|_q$ denotes the length of the geodesic in the homotopy class of α.)

The following formula due to Kerckhoff [6] is extremely useful in estimating Teichmüller distances. For $X_1, X_2 \in \mathcal{T}(S)$

$$d(X_1, X_2) = \frac{1}{2} \log \sup_\alpha \frac{\mathrm{Ext}_{X_2}(\alpha)}{\mathrm{Ext}_{X_1}(\alpha)}. \tag{1}$$

We will also need the following inequalities, comparing hyperbolic and extremal lengths. They are given by Corollary 3 in Maskit [12]. The first says that

$$\mathrm{Ext}_X(\alpha) \leq \frac{1}{2} l_\sigma(\alpha) e^{\frac{1}{2} l_\sigma(\alpha)}$$

where $l_\sigma(\alpha)$ repsesents the length of the geodesic, in the homotopy class of α, with respect to the hyperbolic metric of X. The second says that as $l_\sigma(\alpha) \to 0$ we have

$$\frac{\mathrm{Ext}_X(\alpha)}{l_\sigma(\alpha)} \to 1/\pi.$$

Definition 4 Suppose there is an embedding of a Euclidean cylinder in $\mathbb{R}^3$ into X which is an isometry with respect to the metric of q. The image is called a flat cylinder. The cylinder is maximal if it cannot be enlarged. In that case there are singularities on each boundary component of the cylinder.

We need a definition and estimates found in [3] and [14]. We first adopt the following notation. If two quantites a and b differ by multiplicative and additive constants that depend only on the topology, then we will often write

$$a \asymp b.$$

Definition 5 Given a quadratic differential with its metric q, an expanding annulus A is an annulus where the curvature of each boundary component has constant sign, either positive or negative at each point, the boundary curves are equidistant and there are no zeroes inside A.

Let $d(A)$ be the distance between the boundary components of an expanding annulus. It is universally bounded. The following statement can be found as Corollary 5.4 of [3].

Lemma 1 *Suppose q is a quadratic differential of area 1 on X with its hyperbolic metric σ, and β is a sufficiently short curve.*

(1)

$$\frac{1}{l_\sigma(\beta)} \asymp \max(\mathrm{Mod}(F(\beta)), \mathrm{Mod}(A(\beta)),$$

where $F(\beta)$ is the maximal flat cylinder, $A(\beta)$ is the maximal expanding annulus with one boundary component the q- geodesic in the class of β and

(2)

$$\mathrm{Mod}(A(\beta)) \asymp \log \frac{d(A(\beta))}{|\beta|_q}.$$

(3) *the other boundary component of $A(\beta)$ contains a zero of q.*

We will also need the following result from [13]. Minsky gives a useful estimate of the extremal length of a curve, which uses *collar decomposition*. For $0 < \epsilon_1 < \epsilon_0$ less than the Margulis constant, let $\mathcal{A}$ be the collection of pairwise disjoint annular neighborhoods of the geodesics of hyperbolic length at most ϵ_1, whose internal boundary components have hyperbolic length ϵ_0. Then the union of $\mathcal{A}$ with the collection of components of $X - \mathcal{A}$ is the (ϵ_0, ϵ_1) collar decomposition of X. For a subsurface $Q \subset X$, and α a homotopy class of curves, we denote by $\mathrm{Ext}_Q(\alpha)$ the extremal length of the restriction of the curves in α to Q.

Theorem 1 *[Theorem 5.1 in [13]] Let X be a Riemann surface of finite type with boundary lengths in the hyperbolic metric at most ℓ_0, and let $\mathcal{Q}$ be the set of components of the (ϵ_0, ϵ_1) collar decomposition of X. Then, for any curve α in X, then*

$$\mathrm{Ext}_X(\alpha) \asymp \max_{Q \in \mathcal{Q}} \mathrm{Ext}_Q(\alpha)$$

where the multiplicative factors depends only on $\epsilon_0, \epsilon_1, \ell_0$ and the topological type of X.

2.4 Limits of quadratic differentials

We need the following convergence result for quadratic differentials where one or more curves have extremal length approaching 0. The proof follows more or less immediately from results in [11].

Theorem 2 *Suppose X_n is a sequence of Riemann surfaces, q_n is a sequence of unit area quadratic differentials on X_n, and $\gamma_1, \ldots, \gamma_j$ is a collection of disjoint simple closed curves such that*

- *the extremal length of each γ_i goes to 0 along X_n*
- *the extremal length of every other closed curve is bounded below away from 0 along the sequence.*
- *there is no flat cylinder in the homotopy class of γ_i*

Then by passing to a subsequence, for any subsurface $\Omega_n \subset X_n$ bounded by the geodesic representatives of the γ_i, whose q_n-area is bounded away from 0, there is a surface Ω_∞ with punctures and a nonzero finite area quadratic differential q_∞ on Ω_∞ such that q_n restricted to Ω_n converges uniformly on compact sets to q_∞.

The convergence means that for any neighborhood U of the punctures on Ω_∞

(1) *for large enough n there is a conformal map $F_n : \Omega_\infty \setminus U \to X_n$*
(2) *$F_n^* q_n \to q_\infty$ as $n \to \infty$ uniformly on $\Omega_\infty \setminus U$.*

Proof Using the compactification of the moduli space of Riemann surfaces (see [2]), by passing to a subsequence we can assume X_n converges to a limiting Riemann surface X_∞ with paired punctures corresponding to each γ_i so that (1) holds above. Then again passing to a subsequence we can assume q_n converges to some finite area quadratic differential q_∞ on each component Ω_∞ of X_∞; the convergence as in (2). We need to show that that if Ω_n has q_n-area bounded below then q_∞ is not identically 0 on the corresponding Ω_∞. For each paired punctures on X_∞ pick holomorphic coordinates $0 < |z_i| < 1$ and $0 < |w_i| < 1$ on the corresponding punctured discs. For n sufficiently large, for each i there is a $t_i = t_i(n)$ which goes to 0 as $n \to \infty$ such that X_n can be recovered from X_∞ by removing the punctured discs $0 < |z_i| < |t_i|$ and $0 < |w_i| < |t_i|$ and then gluing the annulus $|t_i| \le |z_i| \le 1$ to the annulus $|t_i| \le |w_i| \le 1$ by the formula

$$z_i w_i = t_i.$$

This produces an annulus in the class of γ_i. In forming X_n, we also allow a small deformation of the complex structure of X_∞ in the complement of the union of the discs. We need to consider the punctured discs $0 < |z_i| < 1$ contained in Ω_∞ and the corresponding annulus

$$A_i = \{z_i : |t_i|^{1/2} < |z_i| < 1\} \subset X_n.$$

In the coordinates of A_i the q_n-geodesic in the class of γ_i lies outside any fixed compact set K for n large enough. Fix now the index i and suppress that subscript. By the Corollary following Lemma 5.1 in [11], we can express q_n in $A = A_i$ as

$$q_n = a_n/z^2 + f_n/z + t g_n/z^3,$$

where a_n, f_n, g_n are uniformly bounded family of holomorphic functions of z. It is easy to see that the last term integrated over A goes to 0 as $t = t_i$ goes to 0. Since there is no flat annulus in the class of γ_i, by Lemma 5.3 of [11], we have

$$-|a_n|^{1/2} \log |t| \le 1.$$

This implies that the first term of the expansion of q_n also has small integral over A. Since we are assuming that the integral of $|q_n|$ is bounded away from 0 on Ω_n we must have that f_n converges to a nonzero function on the disc $0 < |z| < 1$, and so q_∞ is not identically 0. □

3 Lemmas relating length, slope and area

We need to recognize instances when the area of a subsurface is small. As a consequence of the preceeding Theorem we show that if all bounded length curves have small horizontal length, then the area is small.

Lemma 2 *With the same assumption as in Theorem 2 suppose q_n' is another quadratic differential on X_n such that $h_{q_n'}(\alpha) \to 0$ for any fixed homotopy class of curves in a nonannular component Ω_n of the complement of the curves $\gamma_1, \ldots, \gamma_p$. Then* $\mathrm{Area}_{q_n'}(\Omega_n) \to 0$.

Proof By passing to a subsequence we can assume $\Omega_n \to \Omega_\infty$ and $q'_n \to q'_\infty$. Now each geodesic α of q'_∞ has horizontal length equal to 0 which is impossible, since Ω_∞ is not a flat cylinder. □

The next lemma compares areas of flat cylinders with respect to different flat metrics.

Lemma 3 *Suppose q_1, q_2 are quadratic differentials with the same horizontal foliation $|dy|$ and whose vertical foliations are topologically equivalent with transverse measures ν_1, ν_2. For any $B > 0$, there exists ϵ_0, M such that for all $\epsilon < \epsilon_0$, if $C_1 = C_1(\beta)$ is a maximal flat cylinder for q_1 with core curve β with the properties that*

- *The absolute value of the slope of β in C_1 is at least 1.*
- $|\beta|_{q_1} \leq \epsilon_0$.
- *$Area_{q_1}(C_1) \geq B$.*
- *Any horizontal segment I crossing C_1 satisfies $\nu_2(I) \leq \epsilon$*

then, if C_2 is the maximal flat cylinder defined by q_2 in the class of β, we have $Area_{q_2}(C_2) \leq 2\epsilon\epsilon_0$. Moreover the ratio of lengths of vertical arcs crossing the cylinders are comparable.

Proof We may represent C_1 as a parallelogram with a pair of horizontal sides that are glued to each other by a translation. Let I be an oriented horizontal segment crossing C_1 starting at a singularity P_0 on one boundary component. Let Q_0 be the endpoint of I on the other boundary component. Assume without loss of generality that the slope of β in C_1 is negative. This means that there is a vertical leaf through P_0 that enters C_1 in the positive direction and returns to I without leaving C_1 and translated by $h_1 := h_{q_1}(\beta) \leq \epsilon_0$. Starting at P_0, for ϵ_0 sufficiently small compared to B, there will be at least two additional returns for the vertical leaf through P_0 before the leaf leaves the cylinder. Since the vertical foliations of q_1 and q_2 coincide, the same is true for the vertical leaf of q_2 leaving P_0, although now the translation amount, denoted h_2, is different.

Given any three consecutive intersections with I of the leaf starting at P_0, there is a closed geodesic with respect to q_2 homotopic to β through the middle point on I. Thus C_1 contains closed geodesics, with respect to the flat structure of q_2, homotopic to β. That is, there is a maximal flat cylinder C_2 some of whose core curves are contained in C_1. Since maximal cylinders have singularities on their boundaries, either P_0 is on the boundary of C_2 or possibly outside it. It is possible that Q_0 is in the interior of C_2, so if we take the closed geodesic β of q_2 through Q_0 it does not pass through a singularity. However in that case, if we similarly take a horizontal segment I' crossing C_1 starting at a singularity P_1 on the same boundary component of C_1 as Q_0, then β cuts I' in its interior. This implies that the horizontal distance across C_2 is at most $\nu_2(I) + \nu_2(I') \leq 2\epsilon$. Since the height of β is at most ϵ_0, we get the desired area bound for C_2. Since the horizontal foliations coincide, the lengths of corresponding vertical segments coincide and the lengths of vertical segments crossing the cylinders are comparable. □

The next Lemma gives a lower bound of extremal length of a curve family in terms of the area of the surface it is contained in and the length of the boundary.

Lemma 4 *Let X be a Riemann surface. Let q be a unit area quadratic differential on X. Let Ω be a subsurface with geodesic boundary and which does not contain a flat cylinder parallel to a boundary component. If the length $|\partial\Omega|_q$ is small enough, then for any homotopy class of curves $\alpha \subset \Omega$ with geodesic representative α,*

$$\mathrm{Ext}_X(\alpha) \geq \frac{|\alpha|_q^2}{\mathrm{Area}_q(\Omega) + O\left(|\partial\Omega|_q^2\right)}.$$

Proof Note that this lemma does not simply follow from the definition of extremal length since the area of Ω may be smaller than 1. Let $\epsilon = |\partial\Omega|_q$. Define a metric ρ on X as folows. Let ρ coincide with the q-metric on $\mathcal{N}_\epsilon(\Omega)$, the ϵ-neighborhhood of Ω and the q metric multiplied by a small δ on $\Omega' = X \setminus \mathcal{N}_\epsilon(\Omega)$. Let α'' be any curve in the homotopy class of α. If α'' is not contained in Ω then α'' and a segment of $\partial\Omega$ bound a disk. The fact that $d_\rho(\Omega', \Omega) = \epsilon$ and $\partial\Omega$ is a geodesic implies that we can replace an arc of α'' with an arc of $\partial\Omega$ to produce $\alpha''' \subset \overline{\Omega}$ with smaller length. We conclude that the infimum of the length in the metric ρ is realized by the geodesic α in Ω. By definition,

$$\mathrm{Ext}_X(\alpha) \geq \frac{\inf_{\alpha''\sim\alpha} \rho(\alpha'')_q^2}{A(\rho)} \geq \frac{|\alpha|_q^2}{\mathrm{Area}_q(\Omega) + O(\epsilon^2) + \delta\,\mathrm{Area}_q(\Omega')}.$$

The term $O(\epsilon^2)$ in the inequality above comes from $\mathrm{Area}_q(\mathcal{N}_\epsilon(\Omega) \setminus \Omega)$. Since δ is arbitrary, we have the result. □

Definition 6 Given a quadratic differential q and $\delta > 0$, a geodesic γ in the q metric is called *almost* (q, δ)*-vertical* if $v_q(\gamma) \geq \delta h_q(\gamma)$.

Note that δ may be small in the above definition.

Lemma 5 *Let q be a quadratic differential on X a surface without boundary. For any $\delta > 0$ there is a curve β which is almost (q, δ)-vertical.*

Proof If $\Gamma_q \neq \emptyset$ there is a vertical saddle connection which is obviously almost (q, δ)-vertical . If a vertical leaf is dense in a subsurface then the boundary of the subsurface contains a vertical saddle connection. Thus we can assume that the vertical foliation is minimal. Let A be the area of q. The first return map of the foliation to a horizontal transversal I with an endpoint at a singularity defines a generalized interval exchange transformation. Choose a horizontal transversal I of length λ satisfying

$$\lambda^2 < \frac{A}{\delta}. \tag{2}$$

The transversal I determines a decomposition of the surface into rectangles $\{R_i\}$, with heights h_i and widths λ_i, whose horizontal sides are subsets of I. Each rectangle has two horizontal sides on I. Consequently, if we count each λ_i twice we have

$$\sum_i \lambda_i = 2\lambda.$$

Since we count each λ_i twice we have

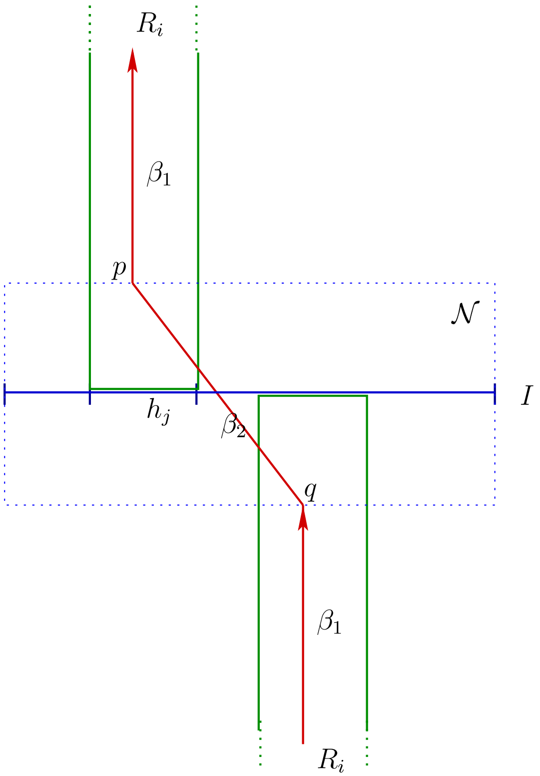
$$\sum_i h_i\lambda_i = 2A.$$

We conclude that

$$\max_i h_i \geq \frac{A}{\lambda} \tag{3}$$

Let h_i realize this maximum. There are two cases. The first case (see Fig. 1) is that the horizontal sides of R_i are on opposite sides of I. Fix a small neighborhood $\mathcal{N}$ of I. We form a simple closed curve $\beta = \beta_1 * \beta_2$. Here β_1 is a vertical segment in R_i whose endpoints p and q are on the boundary of $\mathcal{N}$, and β_2 is an arc transverse to the horizontal foliation in $\mathcal{N}$ joining p and q. Then β is also transverse to the horizontal foliation. Its geodesic representative has

Fig. 1 β is the union of β_1 and β_2



the same vertical length as β, namely, h_i. The horizontal length of β is at most λ. Together with (2) and (3) we have that

$$\frac{v_q(\beta)}{h_q(\beta)} \geq \frac{h_i}{\lambda} \geq \frac{A}{\lambda^2} \geq \delta.$$

In the second case (Fig. 2), both horizontal sides of R_i are on the same side of I (call it I^+). Then there must also be a rectangle R_j with top and bottom on I_-. We may form a simple closed curve β which consists of a vertical segment in R_i, a vertical segment in R_j and a pair of arcs in $\mathcal{N}$ which are transverse to the horizontal foliation. Similar to the case above, the ratio of vertical and horizontal components of β is at least δ. □

Definition 7 Given a unit area quadratic differential q on a surface X without boundary, a subsurface $\Omega \subsetneq X$ is said to be (ϵ, ϵ_0)-thick if the following conditions hold:

- $\partial\Omega$ is a geodesic in the metric of q.
- $\mathrm{Ext}_X(\partial\Omega) \leq \epsilon$
- the shortest non-peripheral curve in Ω has q length at least ϵ_0.

The surface X itself is ϵ_0-thick if it satisfies the third condition above.

The following Lemma says that we can find almost (q, δ) vertical saddle connections in thick surfaces.

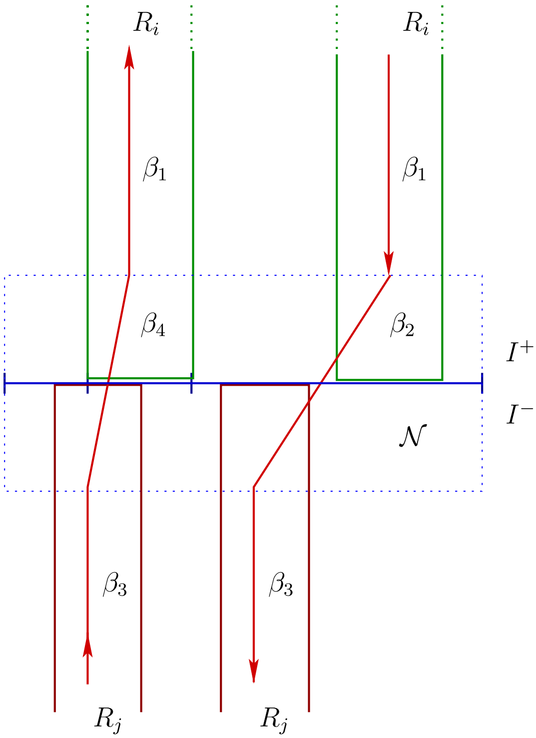


Fig. 2 β is the union of $\beta_1, \beta_2, \beta_3$ and β_4

Lemma 6 *For any $B > 0$, $\epsilon_0 > 0$ there exists $\epsilon > 0$, $\delta > 0$, $D > 0$ and m_0 such that for any (ϵ, ϵ_0)-thick subsurface $\Omega \subset X$ which does not contain a flat annulus isotopic to a boundary component and such that $\mathrm{Area}_q(\Omega) \geq B$, the following two conditions hold:*

(1) *there is an almost (q, δ)-vertical geodesic γ whose interior lies in Ω and such that $|\gamma|_q < D$.*
(2) *For any saddle connection γ which is not vertical or horizontal, there is an $m \leq m_0$ and a collection $\omega_1, \dots, \omega_m$ of disjoint vertical segments so that for every horizontal leaf H*

$$|card(H \cap \gamma) - \sum_{i=1}^{m} card(H \cap \omega_i)| \leq 2.$$

Proof For the proof of the first statement, we argue by contradiction. Suppose the statement is not true. Then there is a sequence X_n of surfaces, a sequence of unit area quadratic differentials q_n on X_n and a sequence of $(1/n, \epsilon_0)$-thick proper subsurfaces Ω_n with q_n-area at least B such that the shortest almost $(q_n, 1/n)$-vertical curve on Ω_n has length at least n. We now apply Theorem 2 to find a subsequence q_n which converges uniformly on compact sets to q_∞ on a limiting surface Ω_∞. The uniform convergence implies that Ω_∞ is $\epsilon_0/\sqrt{B}$-thick.

By Lemma 5, taking $\delta = 1$, there is a simple closed curve β on Ω_∞ such that

$$\frac{v_{q_\infty}(\beta)}{h_{q_\infty}(\beta)} \geq 1.$$

By uniform convergence, $v_{q_n}(\beta) \to v_{q_\infty}(\beta)$, and $h_{q_n}(\beta) \to h_{q_\infty}(\beta)$ and thus for large enough n,

$$\frac{v_{q_n}(\beta)}{h_{q_n}(\beta)} \geq 1/2$$

and furthermore, $|\beta|_{q_n} \leq |\beta|_{q_\infty} + 1$. This is a contradiction to the assumption that the shortest $(q_n, 1/n)$-vertical curve has length at least n, proving the first statement.

We prove the second statement. Begin at one endpont p of γ and take the vertical leaf ℓ_1 leaving p such that γ lies in the $\pi/2$ sector between ℓ_1 and a horizontal leaf leaving p. Move along ℓ_1 as far as possible to a point x_1 in such a way that the segment ω_1 of ℓ_1, from p to x_1, a horizontal segment κ_1 from x_1 to a point $y_1 \in \gamma$ and the segment γ_1 of γ from p to y_1 bounds an embedded triangle Δ_1 with no singularity in its interior. If $\gamma_1 = \gamma$ we are done. We take ω_1 as the desired vertical segment. If not, then there is a singularity p_1 in the interior of κ_1. At p_1 one vertical leaf enters Δ_1. Choose the vertical leaf ℓ_2 at p_1 that makes an angle of π with the vertical leaf that enters Δ_1 and such that the horizontal leaf κ_1 through p_1 on the side of Δ_1 is between them. Then horizontal leaves through points on ℓ_2 near p_1 will intersect γ before returning to ℓ_2. Now repeat the procedure with ℓ_2 in place of ℓ_1 and find a maximal embedded quadrilateral Δ_2 disjoint from Δ_1 in its interior consisting of a pair of horizontal sides, a segment of γ and a segment $\omega_2 \subset \ell_2$. We repeat this procedure, if necessary with a new ℓ_3 until the last segment on γ ends at the other endpoint. There are a fixed number of singularities, hence a fixed number of horizontal sides leaving them and so a bounded number of such embedded quadrilaterals. Say the bound is m_0. The desired vertical segments are $\omega_1, \ldots, \omega_m$, where $m \leq m_0$. $\square$

For the sequel we will need the following result, due to Rafi, [15] relating hyperbolic and flat lengths of curves in a thick subsurface. The first statement is Theorem 1, the second is part of Theorem 4 in [15]

Theorem 3 *For every (ϵ, ϵ_0)-thick subsurface Y of a Riemann surface X with hyperbolic metric σ and quadratic differential q, there exists $\lambda = \lambda(q, Y)$ such that up to multiplicative constants depending only on topology of X*

(1) *For every non-periferal simple closed curve α in Y,*

$$|\alpha|_q \asymp \lambda l_\sigma(\alpha),$$

the multiplicative constants depending only on the topology of Y.

(2) $\text{Area}_q(Y) \leq \lambda^2$

We will now compare extremal lengths of curves that are contained in the "same" subsurface Ω measured with respect to the metrics defined by two different quadratic differentials q_1, q_2 on surfaces X_1, X_2. Specifically, if Ω is a subsurface with geodesic boundary with respect to q_1, and Ω does not contain a flat cylinder isotopic to a boundary component, then we denote by $\Omega \subset X_2$ the subsurface containing the same set of simple closed curves and with geodesic boundary with respect to q_2. If Ω is a flat cylinder with respect to q_1, then we denote by Ω the (possibly empty) maximal flat cylinder in the same homotopy class (with respect to q_2.

The following Lemma allows us to find curves with very different extremal length if a subsurface Ω has very different areas with respect to two quadratic differentials and one of the surfaces is thick.

Lemma 7 *For any $B, M, \delta, \epsilon_0 > 0$, there exist $\epsilon, C, D > 0$ so that the following holds. If q_1 and q_2 are quadratic differentials on X_1, X_2, and Ω is a proper subsurface with geodesic boundary with respect to each quadratic differential, which does not contain a flat cylinder with respect to q_1 parallel to a boundary component and such that Ω satisfies*

(i)

$$\mathrm{Area}_{q_1}(\Omega) \geq B, \ \mathrm{Area}_{q_2}(\Omega) < \epsilon$$

(ii) *for any almost (q_1, δ)-vertical curve $\gamma \subset \Omega$ that satisfies $|\gamma|_{q_1} \leq D$, the vertical components satisfy*

$$\frac{1}{C} \leq \frac{v_{q_1}(\gamma)}{v_{q_2}(\gamma)} \leq C$$

(iii) $|\partial\Omega|_{q_2} < \epsilon$
(iv) *Ω is (ϵ, ϵ_0)- thick with respect to q_1*

then there exists a curve γ in Ω so that

$$\frac{\mathrm{Ext}_{X_2}(\gamma)}{\mathrm{Ext}_{X_1}(\gamma)} \geq M.$$

Proof By Lemma 6, for some δ and D there is an almost (q_1, δ)-vertical curve $\gamma \subset \Omega$ such that

$$\epsilon_0 \leq |\gamma|_{q_1} < D \tag{4}$$

Let σ_i be the hyperbolic metric on X_i and $l_{\sigma_i}(\gamma)$ denote the length of the geodesic γ in the hyperbolic metric. By Theorem 3,

$$l_{\sigma_1}(\gamma) < C_1|\gamma|_{q_1}/\sqrt{\mathrm{Area}_{q_1}(\Omega)} \leq C_1 D/\sqrt{B} \tag{5}$$

where the constant C_1 depends only on the topology of the surface. Also by Maskit's comparison of hyperbolic and extremal lengths,

$$\mathrm{Ext}_{X_1}(\gamma) \leq \frac{1}{2} l_{\sigma_1}(\gamma) e^{l_{\sigma_1}(\gamma)/2} \leq \frac{1}{2} C_1 D/\sqrt{B} e^{C_1 D/2\sqrt{B}}. \tag{6}$$

Set $C_2 = \frac{1}{2} C_1 D/\sqrt{B} e^{C_1 D/2\sqrt{B}}$, so that

$$\mathrm{Ext}_{X_1}(\gamma) \leq C_2 \tag{7}$$

On the other hand, by (4), assumption (ii) and the fact that γ is almost (q_1, δ)-vertical

$$|\gamma|_{q_2} \geq v_{q_2}(\gamma) > \frac{1}{C} v_{q_1}(\gamma) > \frac{\delta}{C(1+\delta)} |\gamma|_{q_1} \geq \frac{\epsilon_0 \delta}{C(1+\delta)}. \tag{8}$$

and by Lemma 4,

$$\mathrm{Ext}_{X_2}(\gamma) \geq \frac{|\gamma|_{q_2}^2}{\mathrm{Area}_{q_2}(\Omega) + O\left(|\partial\Omega|_{q_2}^2\right)} \tag{9}$$

Putting the inequalities (7), (8), (9) together and using assumptions (i) and (iii), we obtain

$$\frac{\mathrm{Ext}_{X_2}(\gamma)}{\mathrm{Ext}_{X_1}(\gamma)} \geq \frac{\epsilon_0^2 \delta^2}{C_2 C^2 (1+\delta)^2 \left(\mathrm{Area}_{q_2}(\Omega) + O\left(|\partial\Omega|_{q_2}^2\right)\right)} \geq \frac{C_3}{\epsilon + O(\epsilon^2)} \tag{10}$$

where $C_3 = \frac{\epsilon_0^2\delta^2}{C_2C^2(1+\delta)^2}$. Now, setting ϵ sufficiently small compared to $\frac{C_3}{M}$ guarantees that the Lemma holds. □

4 Areas of subsurfaces along rays

The proof of the main theorem is now based on the next proposition. We have the following set-up. Suppose q_1, q_2 are quadratic differentials on X_1, X_2 such that the vertical foliations $F^v_{q_1}, F^v_{q_2}$ are topologically equivalent and have a minimal non uniquely ergodic component Ω. Suppose also that with respect to the invariant ergodic measures $\nu_1, \ldots, \nu_p$ on Ω, $|dx_1| = \sum_{k=1}^p a_k\nu_k$, with $a_1 > 0$, while $|dx_2| = \sum_{k=1}^p b_k\nu_k$ with $b_1 = 0$. Suppose $F^h_{q_1} = F^h_{q_2}$. Let $|dy|$ denote the transverse measure to this common horizontal foliation. We normalize so that

$$\int_\Omega a_1 d\nu_1 |dy| = 1. \tag{11}$$

Proposition 1 *With the above assumptions, let $X_1(t), X_2(t)$ be the corresponding rays, and let $q_1(t), q_2(t)$ be the quadratic differentials on $X_1(t), X_2(t)$ respectively. For any sequence of times $t_n \to \infty$, there is a subsequence, again denoted t_n, and constants $\epsilon_0 > 0, c > 0$, so that for sufficiently small $\epsilon > 0$, there is t_0, such that for $t_n \geq t_0$ there is a subsurface $Y_1(t_n) \subset \Omega$ satisfying*

(i) *$Y_1(t_n)$ is (ϵ, ϵ_0) thick with respect to $q_1(t_n)$.*
(ii) $\text{Area}_{q_1(t_n)}(Y_1(t_n)) \geq a_1(1 - c\epsilon)$.
(iii) $\text{Area}_{q_2(t_n)}(Y_1(t_n)) < c\epsilon$.

Proof of Proposition 1 Since $F^v_{q_1}$ is minimal and not uniquely ergodic, we can apply Theorem 1.1 in [9], which says that the ray $X_1(t)$ eventually leaves every compact set in the moduli space as $t \to \infty$. (That theorem was stated in the case when the minimal component was the entire surface. The proof in the case of a minimal non uniquely ergodic component is identical. In fact the main idea of the proof is repeated below in a slightly different context). Passing to a subsequence we conclude that there exist $\gamma_1(t_n), \ldots, \gamma_m(t_n) \subset \Omega$ such that

$$\text{Ext}_{X_1(t_n)}(\gamma_i(t_n)) \to 0$$

and such that the extremal lengths of all other curves are bounded away from 0.

Again, passing to a subsequence, we can apply Theorem 2 to find $\epsilon_0 > 0$ such that for n sufficiently large, there is a nonempty collection $\{Y(t_n)\}$ of disjoint (ϵ, ϵ_0) thick subsurfaces contained in Ω. We can assume that each $Y(t_n)$ is either a flat annulus or it does not contain a flat annulus isotopic to a boundary component. There is a uniform bound N for the number of these surfaces. Let $f_{t_n} : X_1 \to X_1(t_n)$ denote the corresponding Teichmuller map.

Assume first that $Y(t_n)$ is not a flat cylinder. Then by passing to a subsequence, we can assume $Y(t_n)$ converges to a limiting punctured surface Y_∞; the corresponding $q_1(t_n)$ converges to a limiting $q_{1,\infty}$ on Y_∞. Thus for any neighborhood U of the punctures on Y_∞, letting $K := Y_\infty \setminus U$,

(1) for large enough n. there is a conformal map $F_n : K \to Y(t_n)$
(2) $F_n^* q_1(t_n) \to q_{1,\infty}$ as $t_n \to \infty$, uniformly on K.

For each such U, for n large enough, the curves $\gamma_i(t_n)$ whose lengths are approaching 0 satisfy

$$\gamma_i(t_n) \cap F_n(K) = \emptyset.$$

Since $Y(t_n)$ does not contain a flat cylinder in the homotopy class of a component of $\partial Y(t_n)$, we may find U large enough so that

$$\mathrm{Area}_{q_1(t_n)}(Y(t_n) \setminus F_n(K)) \leq \epsilon/2. \tag{12}$$

Now, since ν_i, ν_j are mutually singular measures, there exists $\delta > 0$ and a finite set $\mathcal{I}$ of horizontal transversals I to the vertical foliation in Ω such that for any $\nu_i, \nu_j, i \neq j$ there is a transversal $I_{i,j} \in \mathcal{I}$ such that

$$|\nu_i(I_{i,j}) - \nu_j(I_{i,j})| > \delta. \tag{13}$$

Let Λ_i be the set of generic points for ν_i and the transversals $\mathcal{I}$; that is, Λ_i consists of the set of points x such that, if $l_T(x)$ is the vertical leaf segment of $F^v_{q_1}$ through x of length T, then for each $I \in \mathcal{I}$

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{card}(l_T(x) \cap I) = \nu_i(I). \tag{14}$$

The sets Λ_i are pairwise disjoint. With respect to the measure ν_i, on every transversal almost every point belongs to Λ_i, and, with respect to the area element defined by q_1, almost every point in Ω belongs to $\cup_{i=1}^{p}\Lambda_i$. Let $\Lambda_i(t_n) = f_{t_n}(\Lambda_i)$.

We claim that for n big enough the following holds. Let R be a coordinate rectangle with respect to the flat structure of $q_{1,\infty}$ (i.e. sides are vertical and horizontal) that is contained in K. Then there does *not* exist a pair of indices $j \neq i$ and points $y_{n,i} \in f_{t_n}(\Lambda_i)$, $y_{n,j} \in f_{t_n}(\Lambda_j)$; $i \neq j$ such that

$$z_{n,i} := F_n^{-1}(y_{n,i}) \in R, \quad z_{n,j} := F_n^{-1}(y_{n,j}) \in R.$$

For suppose there were points with this property. There is a coordinate rectangle $R' \subset R$ whose vertical sides L_i, L_j have endpoints at $z_{n,i}, z_{n,j}$. For every horizontal segment H of $q_{1,\infty}$

$$\Big|\mathrm{card}(H \cap L_i) - \mathrm{card}(H \cap L_j)\Big| \leq 2.$$

Let $L_{i,n}, L_{j,n}$ be the vertical leaf segments of $q_1(t_n)$ through $y_{i,n}, y_{j,n}$ of the same length such that $F_n^{-1}(L_{i,n})$ converges to L_i and similarly with $L_{j,n}$. As $t_n \to \infty$, since the length of $I_n = f_{t_n}(I_{i,j})$ goes to infinity, we have

$$\frac{1}{\mathrm{card}(L_{i,n} \cap I_n)}|\mathrm{card}(L_{i,n} \cap I_n) - \mathrm{card}(L_{j,n} \cap I_n))| \to 0.$$

Mapping $L_{i,n}$ back to X_1 by $f_{t_n}^{-1}$, using the fact that

$$\frac{\mathrm{card}\left(f_{t_n}^{-1}(L_{i,n}) \cap I_{i,j}\right)}{|f_{t_n}^{-1}(L_{i,n})|}$$

is bounded, we then have for all large n,

$$\frac{1}{|f_{t_n}^{-1}(L_{i,n})|}|\mathrm{card}(f_{t_n}^{-1}(L_{i,n}) \cap I_{i,j}) - \mathrm{card}(f_{t_n}^{-1}(L_{j,n}) \cap I_{i,j})| \leq \delta/2$$

and we have a contradiction to (13) and (14). Thus for each rectangle R, there is some $i = i(R)$ such that for all $j \neq i$ and for all $x \in R$ we have

$$\chi_{F_n(R)}(F_n(x))\chi_{\Lambda_j(t_n)}(F_n(x)) \to 0. \tag{15}$$

Now we take a covering of K by such rectangles. If any two rectangles R, R' overlap then $i(R) = i(R')$. It follows from the connectedness of K, that there is a single i such that for all R, $i(R) = i$. Thus for n large enough, for all $j \neq i$, (15) holds. ¿From this it follows that for n large enough, for $j \neq i$

$$\int_{Y(t_n)} d\nu_j |dy| \leq \epsilon \tag{16}$$

We would like to prove an estimate similar to (16) in the case that $Y(t_n)$ is a flat cylinder. To do that we need a uniform version of generic points. The reason for that is that there is no natural limiting surface in the case of flat cylinders, and so the previous argument does not quite work. For each T_0, let $\Lambda_i^{T_0}$ consist of those $x \in \Lambda_i$ such that for $T \geq T_0$, for each $I \in \mathcal{I}$,

$$|\frac{1}{T}\mathrm{card}(l_T(x) \cap I) - \nu_i(I)| < \delta/2. \tag{17}$$

Choose T_0 so that with respect to the measure $d\nu_i$, except for a set of measure at most ϵ, every point of I belongs to $\Lambda_i^{T_0}$.

Now suppose $Y(t_n)$ is a flat cylinder with core curve β_n. Set $B = a_1(1 - N\epsilon)$ and let $\epsilon_0 < 1$ be a constant such that Lemma 3 holds. Since f_{t_n} is area preserving, without loss of generality, for n large enough, we can also assume that at time t_n, the core curve β_n is $(q_1(t_n), 2)$-almost vertical and has length smaller than ϵ_0. This means that we can fit coordinate rectangles inside $Y(t_n)$ with vertical sides of length at least $\frac{1}{2}|\beta_n|_{q_1(t_n)}$. We can choose n large enough so that $\frac{1}{2}|\beta_n|_{q_1(t_n)}e^{t_n} \geq T_0$. This means that if we pull back the vertical segment to X_1 its length is at least T_0 so that we can apply (17). Then the same argument given previously shows that for all but at most one j

$$f_{t_n}\left(\Lambda_j^{T_0}\right) \cap Y(t_n) = \emptyset.$$

Otherwise, for a pair $i \neq j$, we can again find a coordinate rectangle contained in the cylinder with one vertical side passing through a point of $f_{t_n}\left(\Lambda_i^{T_0}\right)$ and the other vertical side passing through a point of $f_{t_n}\left(\Lambda_j^{T_0}\right)$, and we find $|\nu_i(I) - \nu_j(I)| < \delta$, and again we have a contradiction. Thus we conclude that for n large enough, for all but one j, for any horizontal segment crossing $Y_1(t_n)$ we have

$$\nu_j(I) < \epsilon. \tag{18}$$

Then except for all but at most one index j, (16) also holds for flat cylinders.

Now let $Z_1(t_n)$ be the union of those $Y(t_n)$ such that (16) holds for the index $j = 1$. Then $Z_1(t) \subsetneq \Omega$ for otherwise we would have

$$\int_{\Omega} d\nu_1 |dy| < N\epsilon,$$

contradicting (11), for ϵ sufficiently small. Let $Y_1(t_n) = \Omega \setminus Z_1(t_n)$, and so we have

$$\mathrm{Area}_{q_1(t_n)}(Y_1(t_n)) \geq a_1 \int_{Y_1(t_n)} d\nu_1 |dy| \geq a_1(1 - N\epsilon) = B.$$

This proves (ii).

We prove (iii). Again the issue is to compare surfaces in different metrics. In the case that $Y_1(t_n)$ is not a flat cylinder, let α be any closed geodesic of $q_{1,\infty}$ in Y_∞. By (15), for $j \neq 1$,

$$\int_{F_n(\alpha)} d\nu_j(t_n) \to 0$$

where $\nu_j(t_n)$ is the push forward measure of ν_j under f_{t_n}. But this is then also true for the geodesic in the class of $F_n(\alpha)$. Then (iii) holds by Lemma 2. If $Y_1(t_n)$ is a flat annulus we know that for $j \neq 1$, the $\nu_j(t_n)$ measure of any horizontal segment crossing $Y_1(t_n)$ is bounded by ϵ. We now apply Lemma 3 to give the desired bound on the area. □

Proof of Theorem A We begin by assuming that the horizontal foliations of q_1 and q_2 coincide. Without loss of generality we can assume that for some minimal component Ω we have $a_1 > 0$ and $b_1 = 0$. By (1) it suffices to show that for any $M > 0$ and for any sequence $t_n \to \infty$, for t_n sufficiently large there is a simple closed curve $\gamma(t_n)$ with

$$\frac{\mathrm{Ext}_{X_2(t_n)}(\gamma(t_n))}{\mathrm{Ext}_{X_1(t_n)}(\gamma(t_n))} > M. \tag{19}$$

For all $\epsilon > 0$ small

we now apply Proposition 1. We find a fixed constant B such that for t_n sufficiently large, the subsurface $Y(t_n)$, given by that Proposition, satisfies

$$\mathrm{Area}_{q_1(t_n)}(Y(t_n)) > B$$
$$\mathrm{Area}_{q_2(t_n)}(Y(t_n)) < \epsilon$$

and

$$\mathrm{Ext}_{X_1(t_n)}(\partial Y(t_n)) \leq \epsilon/M.$$

If $|\partial Y(t_n)|_{q_2(t_n)} \geq \sqrt{\epsilon}$, then $\mathrm{Ext}_{X_2(t_n)}(\partial Y(t_n)) \geq \epsilon$, and we are done; we may choose $\gamma(t_n) = \partial Y(t_n)$. Thus assume

$$|\partial Y(t_n)|_{q_2(t_n)} < \sqrt{\epsilon}.$$

If $Y(t_n)$ is not a flat cylinder, for ϵ small enough, we can apply Lemma 6 to find a bounded length $(q_1(t_n), \delta)$-almost vertical curve $\gamma_n \subset Y(t_n)$ and then Lemma 7, which says that γ_n has the desired property (19).

Thus assume $Y(t_n)$ is a flat cylinder. Let β_n be a core curve of $Y(t_n)$ with t_n chosen so that $|\beta_n|_{q_1(t_n)} < \epsilon$. Fix some $\delta_0 > 0$. Suppose first that β_n is $(q_1(t_n), \delta_0)$-almost vertical. The reciprocal of the modulus of the cylinder is an upper bound for $\mathrm{Ext}_{X_1(t_n)}(\beta_n)$, and we have

$$\mathrm{Ext}_{X_1(t_n)}(\beta_n) \leq \frac{|\beta_n|^2_{q_1(t_n)}}{\mathrm{Area}_{q_1(t_n)}(Y(t_n))} \leq \frac{|\beta_n|^2_{q_1(t_n)}}{B} \tag{20}$$

We now want to estimate $\mathrm{Ext}_{X_2(t_n)}(\beta_n)$. The assumption that β_n is $(q_1(t_n), \delta_0)$-almost vertical, since the vertical lengths coincide, implies by (8) that

$$|\beta_n|_{q_2(t_n)} > \frac{\delta_0|\beta_n|_{q_1(t_n)}}{(1+\delta_0)}.$$

Now there is an annulus $A(t_n)$ which is a union of the flat annulus $Y(t_n)$ and an expanding annulus $Y'(t_n)$. By Proposition 1 the $q_2(t_n)$-area of $Y(t_n)$ is bounded by $c\epsilon$ for some fixed

$c > 0$. The extremal length of a (homotopy class) curve and the hyperbolic length are asymptotically equal as the quantities go to 0. Then by Lemma 1 there are constants $c', c'' > 0$ depending on c and δ_0 but independent of ϵ and t_n such that

$$\mathrm{Ext}_{X_2(t_n)}(\beta_n) \geq c'\ell_{X_2(t_n)}(\beta_n) \geq \frac{c'}{\mathrm{Mod}(Y(t_n)) + \mathrm{Mod}(Y'(t_n))}$$
$$\geq \frac{c'}{\frac{\mathrm{Area}_{q_2}(Y(t_n))}{|\beta_n|^2_{q_2(t_n)}} - \log|\beta_n|_{q_2(t_n)}} \geq \frac{c''}{\frac{\epsilon}{|\beta_n|^2_{q_1(t_n)}} - \log|\beta_n|_{q_1(t_n)}}.$$

Comparing with (20) we see that for ϵ small enough, β_n is a curve that satisfies (19).

Suppose now the core curve β_n of $Y(t_n)$ is not $(q_1(t_n), \delta_0)$ almost vertical. We can find a smaller time at which the slope of the cylinder is at least 1. This means we can apply Lemma 3 to find a corresponding cylinder with respect to the metric $q_2(t_n)$. If $Y(t_n)$ is nonseparating, choose a nontrivial isotopy class of arcs in the complement of $Y(t_n)$ joining the top and bottom of $Y(t_n)$. If $Y(t_n)$ is separating, choose two nontrivial isotopy classes, one that joins the top of $Y(t_n)$ to itself and the other which joins the bottom to itself. These families can be chosen to lie in the thick part of the surface $X_1(t_n)$ and as such have extremal length bounded independently of t_n. In the first case we also take a family of arcs α_n crossing $Y(t_n)$ that intersect any vertical arc crossing $Y(t_n)$ at most once. In the second case we take a pair of (families of) such arcs crossing $Y(t_n)$. These arcs are δ-almost vertical with some uniform constant δ. Now we can form a closed curve γ_n as a concatenation of an arc outside $Y(t_n)$ and an arc α_n, or, in the separating case, a pair of arcs outside and a pair of arcs crossing. By Theorem 1, for some constant c'

$$\mathrm{Ext}_{X_1(t_n)}(\gamma_n) \leq c'\,\mathrm{Ext}_{X_1(t_n)}(\alpha_n) = c'\frac{\inf|\alpha_n|^2_{q_1(t_n)}}{\mathrm{Area}_{q_1(t_n)}(Y(t_n))} \leq c'\frac{\inf|\alpha_n|^2_{q_1(t_n)}}{B} \tag{21}$$

We consider the corresponding arcs α_n crossing the cylinder with respect to $q_2(t_n)$; defined so that they intersect the vertical arcs crossing the cylinder at most once. (These arcs may intersect the arcs perpendicular to the core curve β_n many times). Their lengths are comparable to the lengths with respect to the metric $q_1(t_n)$. The corresponding curves γ_n formed this way are longer than the arcs α_n crossing the cylinder. We have for some constant δ' depending on δ,

$$\mathrm{Ext}_{X_2(t_n)}(\gamma_n) \geq \mathrm{Ext}_{X_2(t_n)}(\alpha_n) = \frac{\inf|\alpha_n|^2_{q_2(t_n)}}{\mathrm{Area}_{q_2(t_n)}(Y(t_n))} \geq \frac{\inf\delta'|\alpha_n|^2_{q_1(t_n)}}{c\epsilon}.$$

Comparing to (21) we are done for ϵ small enough. We have proven the theorem in the case that the horizontal foliations coincide.

Now consider the general case where the horizontal foliations of q_1 and q_2 are distinct. Pick a quadratic differential q_3 with the same vertical foliation as q_1 and the same horizontal foliation as q_2. The rays determined by q_3 and q_2 diverge by what was already proved. The rays determined by q_1 and q_3 stay bounded distance apart as a special case of Ivanov's result [5]. □

Proof of Theorem B Denote the foliations simply by F_1, F_2. The first case is if the minimal components, if any, coincide. Since the foliations are not topologically equivalent, and yet have 0 intersection number, there must be some curve β which is a core curve of a flat cylinder with respect to one quadratic differential, say q_1, but is not the core curve of a flat cylinder of q_2. Since β is isotopic to the core curve of a flat annulus of q_1 we have

$$\mathrm{Ext}_{X_1(t)}(\beta) \leq ce^{-2t},$$

for some c. Since β is not a subset of a minimal component of F_2, we must have $\beta \subset \Gamma_{q_2}$, the critical graph of q_2. Now the length of β in the metric of $q_2(t)$ satisfies $|\beta|_{q_2(t)} = |\beta|_{q_2} e^{-t} \to 0$. Now β determines an expanding annulus. We apply the upper bound for the modulus of that annulus as given in Lemma 1 and hence the lower bound for extremal length to say that

$$\frac{\mathrm{Ext}_{X_2(t)}(\beta)}{\mathrm{Ext}_{X_1(t)}(\beta)} \to \infty.$$

The second case is if one of the foliations, say F_1, has a minimal component Ω_1 which is not a minimal component of F_2. Since $i(F_1, F_2) = 0$, every curve $\beta \subset \Omega_1$ satisfies $i(F_2, \beta) = 0$, so that $\beta \subset \Gamma_{q_2}$, the critical graph of q_2. Since $\beta \subset \Omega_1$, we have $h_{q_1}(\beta) > 0$ and so the flat length of β with respect to q_t satisfies

$$|\beta|_{q_1(t)} \geq h_{q_1}(\beta) e^t.$$

This gives

$$\mathrm{Ext}_{X_1(t)}(\beta) \geq h_{q_1}^2(\beta) e^{2t}.$$

It suffices to find an upper bound for $\mathrm{Ext}_{X_2(t)}(\beta)$. Now β is either on the boundary of a minimal component Ω_2 of F_2 or is on the boundary of a flat cylinder. In either case it determines a maximal expanding annulus A. Since Ω_2 is minimal, the shortest saddle connection $\gamma(t)$ contained in Ω_2 satisfies $|\gamma(t)|_{q_2} \to \infty$ as $t \to \infty$ and hence

$$e^t |\gamma|_{q_2(t)} \to \infty.$$

This means that for a constant $c > 0$, $d(A) \geq c e^{-t}$. Since $|\beta|_{q_2(t)} \asymp e^{-t}$, by Lemma 1 the modulus A is bounded below and so the extremal length of β is bounded above. $\square$

Proof of Theorem C We note that each γ_n^j may itself be a multicurve. Fix a finite set of curves $\alpha_1 \ldots, \alpha_N$ such that the intersection of any measured foliaiton with these curves determines the foliation. Choose a unit area quadratic differential q on some surface X whose vertical foliation is $\left[F, \sum_{i=1}^p \nu_i\right]$. Denote by $|dy|$ the measure on the corresponding horizontal foliation. Let $X(t)$ be the corresponding ray. For any sequence of times $t_n \to \infty$ by Proposition 1 there is $B > 0$ and a collection of disjoint domains $Y_1(t_n), \ldots, Y_p(t_n)$ such that the area of $Y_i(t_n)$ with respect to the measure $d\nu_i |dy|$ is at least B. Suppose first that $Y_i(t_n)$ is not a cylinder. By the first part of Lemma 6 we may pick a $(q(t_n), \delta)$ almost vertical curve $\gamma_i(t_n) \subset Y_i(t_n)$ of length at most D. We claim that $\gamma_i(t_n) \to [F, \nu_i]$. As before, let $\Lambda_i(t_n)$ be the image of the generic points inside $Y_i(t_n)$; generic with respect to the transversals for the set of α_i. The generic points are dense, and $\gamma_i(t_n)$ is a union of a bounded number of saddle connections, so we can find a bounded collection $\left\{\omega_j(t_n)\right\}_{j=1}^m$ of vertical segments beginning at generic points satisfying the second conclusion of Lemma 6. By construction of the $\omega_j(t_n)$, for any fixed α_k,

$$\frac{i(\gamma_i(t_n), \alpha_k)}{\sum_{j=1}^m i(\omega_j(t_n), \alpha_k)} \to 1. \tag{22}$$

Since $\omega_j(t_n)$ is a vertical segment through a generic point,

$$\frac{\mathrm{card}(\omega_j(t_n) \cap \alpha_k)}{|\omega_j(t_n)|_{q(0)}} \to \nu_i(\alpha_k).$$

Summing over all $1 \leq j \leq m$ we have

$$\frac{i(\gamma_i(t_n), \alpha_k)}{\sum_{j=1}^m |\omega_j(n)|_{q(0)}} \to \nu_i(\alpha_k).$$

However

$$\frac{\sum_{j=1}^{m} |\omega_j(n)|_{q(0)}}{v_{q(0)}(\gamma_i(t_n))} \to 1,$$

and so if we let $s_n = \frac{1}{v_{q(0)}(\gamma_i(t_n))}$ then we have for each k,

$$\lim_{n\to\infty} s_n i(\gamma_i(t_n), \alpha_k) \to \nu_i(\alpha_k)$$

and we are done.

Finally suppose $Y_i(t_n)$ is a flat cylinder with core curve $\gamma_i(t_n)$. As in the proof of Proposition 1 we can assume that t_n is chosen so that $\gamma_i(t_n)$ is $(q(t_n), 2)$-almost vertical. As in that argument we find a dense set of generic points $\Lambda_i^{T_0}$, generic for the transversals to the α_i. We then can find vertical segments $\omega_j(t_n)$ through generic points such that (22) holds and the rest of the proof is the same. □

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