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On the History of Souslin's Problem

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To the memory of ANTONIO BUENROSTRO

Contents

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1. Introduction

According to K. Kuratowski,[1] the first volume of the Polish mathematical journal FUNDAMENTA MATHEMATICAE marked the birth in 1920 of the Polish School of mathematics. One important feature of this journal was its novel section of unsolved problems. Kuratowski noted that in this section the editors wanted to give the mathematical community an idea of the activities and problems under discussion in set theory and the foundations of mathematics.

The third problem of the first issue of this journal states:

> Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?[2]

This problem was stated by the Russian mathematician Michel Souslin, who did not formulate any conjecture about the answer. Nevertheless, in the literature on set theory the hypothesis of a positive answer became known as *Souslin's hypothesis*. The issue remained unresolved until 1967, when S. Tennenbaum and T. Jech showed[3] the non-provability of Souslin's hypothesis; *i.e*. the impossibility of providing a proof for an affirmative answer to the question formulated by Souslin. Two years later, in 1969, M. Solovay, D.A. Martin and S. Tennenbaum showed[4] that Souslin's hypothesis is independent of the Zermelo-Fraenkel axioms, and that it was impossible therefore to answer the problem within the framework of these axioms.

It is difficult to reconstruct how Souslin came to formulate the problem, since the few details that are known about his mathematical career are mostly related to his famous achievement in descriptive set theory: his definition of *analytic sets* which generalize *Borel sets*. Souslin was a brilliant pupil of Nicholas Luzin. Under Luzin's influence he began in 1916 to study Lebesgue's memoir "Sur les fonctions représentables analytiquement".[5] Souslin found a counter-example to Lebesgue's statement that the projection (and even any continuous image) of a *B*-measurable set is also a *B*-measurable set; this remark gave rise to a new class of point sets, the analytic sets (also known as *Souslin sets*) defined as the continuous images of *B*-measurable sets. In 1917, Souslin published a short note concerning this new class of sets in the COMPTES RENDUS DE L'ACADEMIE DES SCIENCES DE PARIS[6] establishing their primary properties. One important property proved by Souslin states that the continuum hypothesis holds for this new class of sets: any infinite analytic set is either countable, or has the power of the continuum.

In fact, Souslin's approach to Lebesgue's memoir was very close to the main topics of research investigated by Luzin and his pupils, known as the "Luzitanian group".

[1] Kuratowski 1980, p. 32.

[2] "Is a linearly ordered set, with no jumps and no gaps, and such that any collection of non overlapping intervals is at most countable, necessarily an (ordinary) linear continuum?". Souslin 1920.

[3] Tennenbaum 1968; Jech 1967.

[4] Solovay, Tennenbaum 1970; Martin, Solovay 1970.

[5] Lebesgue 1905.

[6] Souslin 1917.

Between 1914 and 1917 this group studied the theory of functions of a real variable and set theory. A good example of how Souslin's work on analytic sets relates to these topics can be found in a note published in 1916 by P. Aleksandrov (an eminent "Luzitanian"), who studied the power of *B*-measurable sets[7] and proved that the continuum hypothesis holds for these sets. This theorem is analogous to the one proved by Souslin one year later and published in the COMPTES RENDUS, and even their proofs followed the same pattern: they showed that any infinite and non-countable subset of a *B*-measurable set, or of an analytic set, contains a *perfect* subset.[8]

After 1918, when Luzin and Souslin moved from Moscow to the Polytechnic Institute in Ivanovo, Voznesensk, the activity of the Luzitanian group decreased. Although Souslin was still very young and had no experience in teaching, he became a professor at Ivanovo. But, according to P. Aleksandrov, quoted by A.P. Youshkevich, "Souslin didn't get on in Ivanovo and soon lost his job there".[9] The main reason seems to have been that his research activity was not substantial; in fact, his published work consists only of the small note in the COMPTES RENDUS of 1917, the text of his "problem", published in 1920 (one year after his death) and an article written by Kuratowski from a posthumous memoir.[10]

It seems that the first to remark on the importance and the difficulty of Souslin's problem was W. Sierpiński in his book *Leçons sur les nombres transfinis* of 1928.[11] In Sierpiński's view, Souslin's problem concerned mainly the theory of ordered sets and order types. Following F. Hausdorff, Sierpiński characterized the order type λ of the set of real numbers of the open interval $(0, 1)$ by the following three conditions:

1. The open interval has neither a first nor a last element.
2. It is *continuous* in Dedekind's sense.
3. It contains a countable subset N with the property that between any two elements of the set there is a member of the subset N (*i.e.* the subset N is *order-dense* in $(0, 1)$).

Thus any set satisfying these conditions is order-isomorphic to the open interval $(0, 1)$, and its order type is λ. But Sierpiński remarked that any continuously ordered set satisfying condition 3 also satisfies the following condition:

4. Any family of non-overlapping intervals is at most countable.

Since any ordered set of type λ also satisfies this new property, Sierpiński could explain the sense of Souslin's problem:

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7 Aleksandrov 1916.

8 A *perfect set* P (of the real line) is a set which is closed and dense in itself, so $P = P'$. Cantor proved the important property that if P is a perfect set then $|P| = 2^{\aleph_0}$.

9 Yushkevich 1991, p. 13.

10 Souslin 1923. In this paper Souslin provided an example of a non-countable algebraic field of real numbers, which is different from the set of all real numbers. With this example Souslin gave an answer to another problem also published in the first issue of the Polish journal (problem 8, stated by M. Mazurkiewicz).

11 Sierpiński 1928, pp. 151–153.

> Or, on ne sait pas si tout ensemble ordonné, jouissant des propriétés 1, 2 et 4 est nécessairement du type λ, et ce problème (dû à M. Souslin) semble très difficile.[12]

Souslin's problem is equivalent, according to Sierpiński, to the question of whether the order type λ of the open interval $(0, 1)$ could be characterized, in a slightly different way, through properties 1, 2, and 4. It seems that Souslin merely asked whether a linearly ordered continuous set X, having the property that every family of non-overlapping open intervals is at most countable, contains an *order-dense* countable subset N. In other words, Souslin asked about the possible equivalence of two properties of the ordinary linear continuum: that of having a countable order-dense subset – the *separability condition* – and the property that any family of non-overlapping intervals is at most countable – the *countable chain condition*. The proof of one implication is immediate and was seen by Sierpiński when he commented on the problem,[13] but it was for the converse implication that the difficulties appeared. As we have already said, the proofs given by Solovay, Tennenbaum and Martin between 1967 and 1969 show that indeed this implication cannot be proved within the frame of the Z-F axioms.

Even though the deduction of the countable chain condition from the separability condition is immediate, it was not until Souslin's problem arose that the logical relation between them became the heart of a set-theoretical question. In fact, when these two properties were characterized for the first time as holding for the "ordinary continuum", no remark was made concerning their possible logical relation. Since our aim is to follow the theoretical debates and mathematical strategies which aimed at the solution of Souslin's problem, we must first understand the context in which these two conditions were seen as "properties" for the "ordinary continuum". With this in mind we will begin our study of the history of Souslin's problem by considering the origins of the theory of order types, emphasizing aspects related to linearly ordered continuous sets and the role of the separability condition. Next, we will examine the context in which the countable chain condition was first studied.

The history of Souslin's problem is that of the progressive recognition of the difficulties and the discovery of logical links between apparently different domains, and we cannot avoid explaining the mathematical conditions in which the problem was stated. Mathematics and history are linked here in a very interesting way, which we hope to express correctly.

As we stated above, Souslin's hypothesis is an independent proposition for Z-F set theory, but long before methods for "independence proofs" in set theory were developed in the sixties, significant attempts to prove it had been made. We will analyze some of these attempts, concentrating on the emergence of different methods to find equivalent versions of the problem. Our study will thus focus on the variety of attempts to solve Souslin's problem, and the concomitant lack of a thorough understanding of this problem.

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[12] "But we don't know if any ordered set satisfying properties 1, 2 and 4 is necessarily of type λ, and this problem (stated by M. Souslin) seems very difficult". *Ibid*., p. 153.

[13] For any order-dense and separable set, a family of non-overlapping intervals satisfies the condition that for each interval at least one element of the countable dense subset lies on it; so the family is at most countable.

We will, however, pay special attention to the work developed by D. Kurepa between 1934 and 1937; we claim that in some sense he established in this period the theoretical framework for all the further studies.

Subsequently, we will analyze the emergence of some other propositions related to Souslin's hypothesis, some of them equivalent to it and others which are sufficient conditions for its proof. Our aim is to understand the role which these propositions played in the later proof of the independence of Souslin's hypothesis from the Z-F axioms.

2. The context of the problem: order types, separability and countable chain conditions

2.1. Early contributions to the theory of continuous order types

As we stated above, Sierpiński thought that Souslin's problem was mainly related to the theory of order types. This theory was created by G. Cantor in 1883, but his first paper on the subject, submitted for publication to ACTA MATHEMATICA in 1885, was rejected by the editor of the journal, G. Mittag-Leffler and it remained unpublished until 1970.[14] The first published paper by Cantor on the subject appeared in two different issues of the MATHEMATISCHE ANNALEN – in 1895 and 1897 – under the general title of "Beiträge zur Begründung der transfiniten Mengenlehre".[15] At the turn of the century some remarkable studies on the subject were published by Huntington, Veblen and above all Hausdorff, [16] so by 1928 when Sierpiński published his book (and even in 1920 when Souslin stated his problem) the theory of order types was a well-known branch of set theory.

Cantor defined in 1895 the order type of a set M as

> den Allgemeinbegriff, welcher sich aus M ergibt, wenn wir nur der Beschaffenheit der Elemente m abstrahieren, die Rangordnung unter ihnen aber beibehalten.[17]

According to this definition, two ordered sets X and Y have the same order type χ if a one-to-one function from X onto Y, preserving the order relation, can be defined. In Cantor's notation, $\chi = \overline{X} = \overline{Y}$. When two sets have the same order type, then, regardless of the nature of their elements, they can be considered as "the same" set; this means that the properties characterizing the order type of the set determine in a *categorical way* the set itself. In this context Cantor gave three conditions to completely characterize the order type η of the set **Q** of rational numbers, with the obvious conclusion that if a set X

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[14] Cantor 1885.

[15] Cantor 1895–1897.

[16] Huntington 1905–1906; Veblen 1905; Hausdorff 1908 and Hausdorff 1914.

[17] "The general concept which results from M if we only abstract from the nature of the elements m, and retain the order of precedence among them". *Op. cit.* Cantor 1932, p. 297.

satisfies these three conditions, then an order isomorphism can be defined from X onto $\mathbf{Q}$.[18] The ordered set $\mathbf{Q}$ satisfies:

1. It is a countable set.
2. It has no first and no last element.
3. It is an *everywhere-dense* set (*überalldichte Menge*). This means that given any two distinct elements of the set, another element of the set lies between them.

After this characterization for the order type $\eta = \overline{\mathbf{Q}}$, Cantor looked for a definition of the continuity property in terms of the order relation; he looked for those conditions that could describe the linear continuum $[0, 1]$ in a unique way and as an ordered set. Even if the definition of the continuity property given in his fifth memoir on trigonometric series[19] provided him with the necessary background for this new characterization of the linear continuum, previous references to the metrical properties had to be avoided. An important concept introduced for this purpose is that of an increasing *fundamental sequence* of elements in an infinite linearly ordered set X: it is a subset $\{b_\nu\}$ of X such that $b_\nu < b_\mu$ whenever $\nu < \mu$.[20] A limit point of an increasing fundamental sequence $\{b_\nu\}$ is an element $b_0 \in X$ with the following properties:

i) For every element b_ν of the sequence, $b_\nu < b_0$.
ii) For any element $x \in X$, such that $x < b_0$, there exists an element b_μ of the sequence such that $x < b_\mu$.

Clearly if a limit point exists for a fundamental sequence, then it is unique. Two increasing fundamental sequences $\{a_\nu\}$ and $\{b_\nu\}$ are said to be *coherent* (*zusammengehörig*), $\{a_\nu\} \| \{b_\nu\}$, if for every element a_ν of the first sequence there is an element b_μ of the second such that $a_\nu < b_\mu$; and also, for every b_λ of the second sequence there is an element a_ι of the first such that $b_\lambda < a_\iota$. Analogous definitions are made for decreasing fundamental sequences. The following two statements are immediate consequences:

Proposition. *If a fundamental sequence in X has a limit point in X, any fundamental sequence which is coherent with it has the same limit point.*

Proposition. *If two fundamental sequences have one and the same limit point in X, they are coherent.*

Whenever a set X has the property that any of its elements is the limit point of a fundamental increasing or decreasing sequence, X is said to be *dense in itself* (*insichdichte Menge*). On the other hand, if every fundamental sequence in X has a limit point in X, then X is a *closed* set (*abgeschlossene Menge*). If X is both closed and dense in itself, then X is a *perfect* set (*perfekte Menge*).

Cantor stated that the order type (the type θ) of the closed interval $[0, 1]$ is given through the following two properties:

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[18] The same argument is used in stating the *universal property* of the order type η: any countable ordered set is linearly isomorphic to a subset of $\mathbf{Q}$.

[19] Cantor 1872.

[20] In the original definition, Cantor only considered sequences whose index runs through the set $\mathbf{N}$ of natural numbers. Further developments on the theory of linearly ordered sets stated that the index could run over some transfinite ordinal number.

(C1) The interval $[0, 1]$ is a *perfect ordered set*: every point in $[0, 1]$ is the limit of an increasing or decreasing fundamental sequence, and any increasing or decreasing fundamental sequence contained in the interval $[0, 1]$ has a limit point in this interval.

(C2) The interval $[0, 1]$ contains a denumerable subset which is order-dense in it: there is a subset $S \subset [0, 1]$, whose order type is η (the order type of the set **Q**), such that for any two points $x, y \in [0, 1]$ $(x < y)$, there is always a point $s \in S$ such that $x < s < y$.

These two properties characterize the closed interval $[0, 1]$ as a *continuous* ordered set without making any reference to metrical properties for convergent sequences.[21] The limit of a fundamental sequence $\{x_\nu\}$ contained in the closed interval $[0, 1]$ is clearly the least upper bound (sup) (if the sequence is increasing) or the greatest lower bound (inf) (if it is decreasing) of the sequence, considered as an ordered subset contained in $[0, 1]$. From this point of view Cantor's characterization is equivalent to the continuity condition for an ordered set given by R. Dedekind in his famous *Stetigkeit und irrationale Zahlen*[22] of 1872, which Sierpiński and Hausdorff used to characterize the order type λ of the open interval $(0, 1)$. Dedekind's condition for the *essential quality of continuity* states that:

(D) An ordered set X is a *continuous* set if, whenever A and B are two disjoint subsets whose union is equal to X and such that every element $x \in A$ is smaller than any $y \in B$ $(A < B)$, then A has a least upper bound and B has a greatest lower bound.

By 1877 Dedekind had explained to Cantor[23] in what sense his condition D should be understood as giving the essential quality of continuity for an ordered set X: it is a condition that should be added to the condition of *dense order*. Without condition D, an order-dense set might have *gaps*, like the set **Q** of rational numbers; without the condition of dense order, the set might have *jumps*.

These two characterizations given by Cantor and Dedekind ensure the continuity of the set; nevertheless, there is a slight difference between them. In Cantor's characterization for a *perfect* linearly ordered set, a first and a last element should be included in it. The open interval $(0, 1)$ and the closed interval $[0, 1]$ both satisfy Dedekind's condition D, but the first one is not a *perfect set* in Cantor's sense since neither a decreasing sequence whose limit is 0 nor an increasing sequence whose limit is 1, defined in the set, have limit points within the set.

Despite this difference, both the closed interval $[0, 1]$ and the open interval $(0, 1)$ satisfy condition *C2* given by Cantor; they are both *separable sets*. Clearly a separable ordered set is also order-dense, and we have already remarked how Dedekind intended that this last condition should be added to condition D (and the same could be said for

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[21] This means that the limit of a sequence $\{x_\nu\}$ is not defined as a point having the property that for any positive real number ε, there exists a member of the sequence whose distance to the limit point is smaller than ε.

[22] Dedekind 1872.

[23] Letter sent to Cantor on May 18, 1877. In *Correspondance Cantor-Dedekind*, edited by Jean Cavaillès and Emmy Noether, published in Cavaillès 1962.

condition *C1*) in order to give a complete characterization for the continuity property. But for Cantor, condition *C2* says something more: not only does it state that the set is order-dense, but it also states that the set contains a *countable* order-dense subset which is indeed a subset of type η. According to the original concept of the *order type* of a set, any linearly ordered set L whose type is λ (as characterized by Sierpiński) or θ (as characterized by Cantor) is linearly isomorphic to $(0, 1)$ or to $[0, 1]$. The only way to show the existence of an isomorphism between the set L and the open or closed interval is by defining it as an extension of the already existing isomorphism between their countable and order-dense subsets of type η. As we already said, the condition that the set be order-dense should be added to conditions *C1* of Cantor and *D* of Dedekind; only in this case the ordered set is a *continuous* ordered set. But only when the *separability condition* holds for the continuous ordered set is this set isomorphic to the closed interval $[0, 1]$ or the open interval $(0, 1)$.

Since for any continuous ordered set to be isomorphic to the "ordinary continuum" it is necessary and sufficient that it be *separable*, any definition for this ordinary continuum should state, either directly or through an equivalent condition, the existence of a countable and order-dense subset. According to Sierpiński this is the core of Souslin's problem.

2.2. Veblen. The deduction of the separability condition

The question raised by Souslin was not the first search for an equivalence for the separability condition; an important study on this subject had already been made in the years 1904–1905 by the American mathematician Oswald Veblen. Despite the role of the separability condition for the characterizations of the order types λ and θ, Veblen remarked that it involved the concept of an infinite cardinal number which he considered external to the theory of ordered sets, since for an ordered set to be separable it must contain an order-dense subset which is *countable*. On December 30, 1904, Veblen delivered a communication to the American Mathematical Society under the general title "Non-Metrical Definition of the Linear Continuum". One month later, he submitted a paper based on this talk.[24] Veblen looked for a complete characterization of the ordinary linear continuum only in terms of the order relation; to achieve this, he introduced five groups of postulates in order to define a linear continuum X. The separability condition was not included, but he deduced it from these groups of postulates. These groups of postulates for the set X are:

I) General postulates of order.
II) The postulate of *continuity* (in Dedekind's sense).
III) The postulate of density.
IV) A slight variation of the Archimedean postulate (the group of *pseudo-Archimedean* postulates):
   IV.*i*) There exists an increasing sequence $\{p_n\}$ in X ($n = 1, 2, 3, \ldots$) such that if $p \in X$ and $p_1 < p$, there exists a number ν such that $p_\nu > p$.

[24] Veblen 1905.

IV.*ii*) There exists a decreasing sequence $\{p_n\}$ in X $(n = 1, 2, 3, \ldots)$ such that if $p \in X$ and $p_1 > p$, there exists a number ν such that $p_\nu < p$.

V) A group of postulates for *uniformity*: for every point $p \in X$ and for every integer n, there exists an open interval $I_{n,p}$ containing p such that

V.*i*) For a fixed point p, the family $\{I_{n,p}\}$ $(n = 1, 2, 3, \ldots)$ forms a nested sequence of intervals.

V.*ii*) $\{p\} = \bigcap_{n=1}^{\infty} I_{n,p}$.

V.*iii*) For any open interval V, there exists a number n_V such that $V \not\subseteq I_{n_V,p}$ for every p.

Veblen first proved that for any linearly ordered set X satisfying the postulate of continuity the following three conditions hold:

a) Any bounded subset has a least upper bound and a greatest lower bound.
b) Every infinite bounded subset has at least one limit point.
c) *Heine-Borel* property: if every element of a (closed) interval $[p, q]$ belongs to at least one (open) interval of a family of open intervals $\{U_\alpha\}$, then there exists a finite collection of intervals of the family: $U_1, U_2, \ldots, U_n$, such that every point of $[p, q]$ belongs to at least one of the intervals $U_1, U_2, \ldots, U_n$. In other words, every open covering for a closed interval has a finite subcovering.

A few months earlier, Veblen had already shown that this *Heine-Borel* property could be considered as a continuity axiom since it is equivalent to Dedekind's postulate of continuity. More precisely, in a note published in 1904 Veblen had proved[25] the following:

Theorem ***(V.1).*** *Assuming the ordinal relations of the real number system, the Heine-Borel property is a consequence of Dedekind's postulate and the latter is a consequence of the Heine-Borel property.*

This theorem is to be understood as stating that for the set of real numbers the continuity property holds if and only if any closed interval $[a, b]$ satisfies the *Heine-Borel* property.[26] Once Veblen obtained the equivalence of Dedekind's continuity principle and the *Heine-Borel* property, he was ready to show (in his text of 1905) the logical relations between the different groups of postulates. It was his aim to prove that the separability condition, which is not included in these groups of postulates, could be

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[25] Veblen 1904, p. 437.

[26] Besides this theorem, Veblen introduced a generalization of the *Heine-Borel* property; the so-called *H-B* property: a set of real numbers X (or any linearly ordered set) has the *H-B* property if given any family $\{U_\alpha\}$ of intervals, with the property that every element of X belongs to at least one U_α, a finite collection $U_1, U_2, \ldots, U_n$ can be selected such that every element of X belongs to at least one U_k $(k = 1, \ldots, n)$. Now, after defining a closed set as one which includes all its limit points, he stated the following theorem (*Ibid*., p. 438.):

Theorem ***(V.2).*** *A necessary and sufficient condition for a bounded set X of real numbers to have the H-B property is that X be closed.*

deduced from the five groups of postulates given by him: if $[a, b]$ is any (closed) interval contained in a linear continuum X satisfying the groups of postulates I-V, then the family $\{I_{\nu_0,p}\}$ constitutes an open cover for this interval, for any fixed value $n = \nu_0$, when p runs through $[a, b]$. Because of the *Heine-Borel* property, a finite collection of points $\{p_1^{\nu_0}, p_2^{\nu_0}, \ldots, p_n^{\nu_0}\}$ exists, such that the finite family of open intervals

$$\left\{I_{\nu_0, p_i^{\nu_0}}\right\}_{i=1}^{n} \tag{2.I}$$

is also a cover for $[a, b]$. The extreme points of these intervals ($2n$ points), which clearly depend upon the fixed number ν_0, become a countable collection when making $n = 1, 2, \ldots$. Let $Y_{[a,b]} \subset X$ be this countable collection of points. By property (V.*iii*), for any two points $x, y \in [a, b]$ an integer number n_0 exists such that no interval $I_{n_0,p}$ contains the interval (x, y); this means that at least one of the extreme points of the finite cover of intervals (2.I), belonging to the countable family of points $Y_{[a,b]}$, lies between x and y. This proves that the countable collection $Y_{[a,b]}$ is order-dense in the closed interval $[a, b]$.

Based on the *pseudo-Archimedean* postulates, it is possible to say that there exist two countable sequences $\{p_n\}$ and $\{q_n\}$ contained in X and an integer number n, such that $q_n < p_n$[27] and such that any point $p \in X$ lies in a closed interval $[q_{n+k}, p_{n+k}]$ for which an order-dense countable subset $Y_{[q_{n+k}, p_{n+k}]}$ exists. Since a countable order-dense set $Y_{[q_{n+k}, p_{n+k}]}$ exists for each closed interval $[q_{n+k}, p_{n+k}]$ ($k = 1, 2, \ldots$), the set

$$Y = \bigcup_{k=1}^{\infty} Y_{[q_{n+k}, p_{n+k}]} \tag{2.II}$$

is a countable subset of points which is order-dense in X.

As far as we know, this proof by Veblen constitutes the first attempt to deduce the separability condition from other conditions of the linear continuum.[28]

According to this proof, it is possible to obtain the separability condition for a continuous ordered set whenever the pseudo-Archimedean and the uniformity conditions are added to the first three groups of postulates. But it is possible for a continuously ordered set to satisfy the groups of postulates I–III, and still not be separable; *i.e.* this condition is independent of the postulates I–III. Veblen showed this by considering the example of a linearly ordered continuous set satisfying the groups of postulates I–III, but

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[27] $\{p_n\}$ is an increasing sequence, while $\{q_n\}$ is a decreasing sequence.

[28] At the end of his paper, Veblen introduced two new postulates that are substitutes for the uniformity condition. Therefore the separability condition can also be deduced from them: by taking Cantor's concept of a fundamental (increasing) sequence, where clearly the limit point is the least upper bound of the sequence, Veblen stated that

I) Every point of the set is a limit point.

II) For any limit point P', a family of sequences $\{P^{\lambda}_{\nu,P'}\}$ exists such that:

- *i*) Each sequence is increasing and has P' as its limit point.
- *ii*) If $P''$and P''' are two limit points ($P'' < P'''$), there exists a number ν_1 such that there is no point P' for which $P_{\nu_1} < P''$, $P''' < P'$, whenever P_{ν_1} is an element of a sequence whose limit is P'.

which is not separable: the set $X = \{(x, y); x \in \mathbf{R}, y \in [0, 1]\}$ with the *lexicographical* ordering.[29] This is a continuous and densely ordered set, but it is not separable (an element of an order-dense subset should exist for every $x \in \mathbf{R}$).

According to what we said before, a continuously ordered set should satisfy the postulates of density and the postulates of continuity, but the above example shows that the separability condition could fail for a continuous set. A continuous ordered set may be separable, as in the case of the "ordinary continuum", or may be not separable, as in the above example.[30]

We can be sure that the independence of the separability condition for the continuous sets, and its necessary role for the "ordinary continuum", were well known by 1920 when Souslin formulated his question about the linear continuum. In this sense, Sierpiński's understanding of the essence of Souslin's problem seems right: the question is if it is possible to deduce the separability condition from another different condition. But, whereas Veblen's aim was to deduce this condition from other postulates where only the order relations were considered, Souslin seems to have asked whether this condition could be deduced from another condition which is known to be an immediate consequence of it.

As we said before, a peculiar aspect of this question is that the countable chain condition, despite the fact that it could be considered as a possible substitute for the separability condition, was also a very well known property of the "ordinary continuum". But such a property had first been known and characterized in a set-theoretic context independent of the theory of order types.

2.3. Cantor. The first characterization of a family of non-overlapping intervals

The fact that the countable chain condition can be deduced from the separability condition is clear enough, but none of the studies on the linear continuum we have talked about so far ever mentioned this possible deduction. When the countable chain property was stated for the first time for the linear continuum $(0, 1)$, it was not conceived as a consequence of the separability condition. This happened in 1882, in the third memoir on the infinite linear set of points, "Über unendliche lineare Punktmannigfaltigkeiten 3"[31] when Cantor gave a first step towards an "abstract" theory of sets by stating that the concept of the *power* of a set should not be restricted to linear sets of points:

> Auch der *Mächtigkeitsbegriff*, welcher den Begriff der ganzen Zahl, dieses Fundament der Größenlehre, als Spezialfall in sich faßt und als das allgemeinste genuine Moment bei Mannigfaltigkeiten angesehen werden dürfte, ist so wenig auf die linearen Punktmengen beschränkt, daß er vielmehr als Attribut einer jeglichen *wohldefinierten* Mannigfaltigkeit

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[29] This order relation is defined in the following way: given two elements of this set (x_1, y_1) and (x_2, y_2), $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$, or if $x_1 = x_2$ and $y_1 < y_2$.

[30] This example also shows that this postulate of uniformity is independent of the other postulates, since it supports the proof of the separability condition: for the continuous set $[0, 1]$ the postulate of uniformity holds, but for the continuous set X just defined, this postulate fails.

[31] Cantor 1882.

betrachtet werden kann, welche begriffliche Beschaffenheit ihre Elemente auch haben mögen.[32]

The existence of two different infinite powers for the linear sets of points, together with the hypothesis, only partially formulated by that time, that only these two infinite powers could exist for sets of points, led Cantor to the question of whether these two infinite powers could also exist for other kinds of sets. Cantor first claimed that given any infinite set, a countable subset exists (if X is an infinite set, then a sequence $\{x_n\}$ in X always exists). According to the order relation provided in 1878 for the powers of sets,[33] this statement ensured that the power of the sequence of positive integer numbers is the "smallest" infinite power that can be found for infinite sets (*unendlichen Mengen*).

Cantor tried to formulate the foundation for the study of this first infinite power in two other statements related to countable sets:

1) Any infinite subset of a countable subset is also countable.
2) Given any countable set whose elements are also countable sets, the set obtained by the union (*Zusammenfassung*) of the elements belonging to all these sets is in its turn a countable set.[34]

Cantor considered that arithmetic and algebra provide several examples of countable sets. Yet in his inquiry of the basic property of countable sets he proved the following theorem which in his opinion provided a geometrical example of a countable set:[35]

Theorem ***(C.1).*** *If in an infinitely extended and continuous n-dimensional space B^n, infinitely many n-dimensional continuous partial domains are submitted to the condition that they do not intersect each other, or that they intersect each other at most at the boundaries, then the set of these partial domains is always countable.*

Since it was considered as a geometrical example, the proof for this theorem was given through a geometric transformation: a projective transformation of the n-dimensional space B^n to an n-dimensional unitary sphere S^n contained in an $(n+1)$-dimensional space B^{n+1}. Through this transformation, each n-dimensional subset of the space B^n has a corresponding n-dimensional subset of S^n whose "size" or "measure" (*Rauminhalt*) is bounded, since it is contained in the unit sphere.[36] Under this transfor-

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[32] "Also the concept of *power*, which comprises as a special case the concept of an integer number, the very foundation of the theory of quantities, and which may be considered as the most general and genuine moment of sets, is not limited to linear sets of points; it can be considered as an attribute of any well defined set, no matter which might be the conceptual nature of its elements". *Ibid*. Cantor 1932, p. 150.

[33] Cantor 1878. In this memoir Cantor introduced the following definition: for two sets A and B the power of A is smaller than the power of B (in formula: $|A| < |B|$) if a one to one function from A to B can be defined, but no one to one function from B to A can be defined.

[34] In his memoir on the infinite and linear set of points Cantor does not provide a proof for this statement. The proof is not difficult, but it requires the axiom of choice.

[35] Cantor 1882; Cantor 1932, p. 153.

[36] It is important to notice that at this stage of his work, Cantor had no precise definition for the *measure* of a set. His definition of the measure was first introduced in his sixth memoir on the infinite linear sets of points [Cantor 1884*a*] and in his memoir on the power of perfect sets [Cantor 1884*b*].

mation the family of non-overlapping partial domains of B^n is transformed into a family of non-overlapping domains of the unitary sphere S^n. For any given number $\gamma > 0$, the number of domains in the sphere whose *measure* is greater than γ is necessarily finite since their sum ought to be less than the *measure* of the sphere ($2^n\pi$ for $n \geqq 2$). The family of non-overlapping domains of the sphere can be arranged in a countable sequence according to their decreasing *measure*.

Cantor's conclusion from this general theorem, in the case $n = 1$, is the following:

> Der Fall $n = 1$ liefert folgenden Satz, welcher für die weitere Ausbildung der Theorie der linearen Punktmengen wesentlich ist: *jeder Inbegriff von getrennten, höchstens in ihren Endpunkten zusammenfallenden Intervallen* (α, β), *welche in einer unendlichen geraden Linie definiert sind, ist notwendig ein abzählbarer Inbegriff*.[37]

In this third memoir on the infinite linear point sets, the countable chain condition was not conceived by Cantor as a consequence of the fact that the linear continuum possesses an order-dense countable subset. However six months later, in his fourth memoir,[38] this condition became the keystone in proving that an infinite *discrete* subset $M \subset B^n$ is always countable. For each point $m \in M$, Cantor considered the existence of a neighborhood $V_{\rho_m}(m)$ of m which contains no other point of M; the neighborhoods $V_{\rho_m}(m)$ can be defined so that they do not intersect each other, or they intersect at most at their boundaries.

3. Attempts to prove Souslin's hypothesis. Kurepa's work on ordered and ramified sets

3.1. First step: the acknowledgment of the difficulty

As we have said, besides the remark made by Sierpiński in his *Leçons sur les Nombres Transfinis*, no serious study was published concerning Souslin's problem until D. Kurepa started to publish his notes related to it in 1934. The core of his studies is found in his memoir *Ensembles Ordonnés et Ramifiés*, written in 1935 as a doctoral thesis. In the introduction to this work Kurepa pointed out what he considered to be the three main problems of the theory of sets:

1. The possibility of well-ordering any set.
2. The question of the existence of a cardinal number between κ and 2^κ.
3. Souslin's problem.[39]

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[37] "The case $n = 1$ gives rise to the following theorem, which is very important for the future development of the theory of linear sets of points: *every collection of disjoint intervals* (α, β), *defined in an infinite straight line and intersecting each other at most in the extreme points, is necessarily a countable collection*". Cantor 1882; Cantor 1932, p. 153.

[38] Cantor 1883.

[39] It is a remarkable fact that these three problems, which Kurepa considered as the main problems for set theory, are indeed related to three *independent* propositions in Z-F axioms: the axiom of choice, the generalized continuum hypothesis and Souslin's hypothesis. The independence proofs for the first two problems was given by P. Cohen in 1963.

For Kurepa, Souslin's problem merely raised the following question:

> si tout ensemble ordonné continu E, avec la condition de la chaîne dénombrable, est nécessairement identique, au point de vue de l'ordre, au continu mathématique.[40]

In this memoir Kurepa gave no definitive answer to Souslin's problem. He did not prove that a continuous linearly ordered set satisfying the countable chain condition is necessarily separable, nor could he give an example of a continuous ordered set for which the countable chain condition holds, but which is not separable. Kurepa's achievement was to state equivalent conditions for a positive answer to the problem.

Kurepa started his study in 1934, with four successive notes presented to the Academy of Science and published in the COMPTES RENDUS in February, March, April and July of 1934.[41] Without any doubt these four notes constitute an important background for his 1935 memoir, even if, according to the style of the COMPTES RENDUS, results were only claimed and no proof was provided for the main theorems. In these notes Kurepa outlined his approach to Souslin's problem by introducing the main concepts that support the logical structure of his 1935 memoir. But we must point out that there is an important difference between these four notes and *Ensembles Ordonnés et Ramifiés*. The difference concerns his own confidence in the possibility of giving a positive answer to the problem; in the notes of 1934 Kurepa claimed that he had obtained a proof for Souslin's hypothesis, while, as we said, no definite answer was provided one year later.

Already in his February note Kurepa claimed to have a proof for the positive answer to Souslin's problem; this was given through the following theorem:

Theorem ***(CR.I.1).*** *Theorem on the linear continuum. The following 7 propositions are equivalent for a linearly ordered set E and any one of them makes this set equivalent to the set* $\mathbf{R}$ *of real numbers.*

A_1 *The set E is continuous, homogeneous*[42] *and is not the (Cartesian) product of two continuous sets.*

A_2 *The set E is continuous, it has neither a first nor a last element, and every continuous subset $F \subseteq E$ has the same order type as E (E is irreducible).*

A_3 *The set E is continuous, it has neither a first nor a last element, and any family of non-overlapping intervals of E is at most countable.*

A_4 *The set E is continuous, it has neither a first nor a last element, and is perfectly separable (in the sense of Fréchet).*[43]

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[40] "whether any continuous ordered set E, satisfying the countable chain condition, is necessarily equal, according to the order relation, to the mathematical continuum". Kurepa 1935, p. 1.

[41] The February note, Kurepa 1934*a*; the March note, Kurepa 1934*b*; the April note, Kurepa 1934*c*; and the July note, Kurepa 1934*d*.

[42] Kurepa defined a set E to be *homogeneous* if it is homeomorphic to any of its intervals.

[43] In his book of 1928 (Fréchet 1928), Fréchet gave the following definitions: a set E is separable if there exists a countable subset N such that any point $x \in E$ is the limit of a sequence of distinct points of N. Further on, he stated that a separable set E is one having a countable subset N such that any point of E is a point of N or an accumulation point of N. For a set E where

A_5 *The set E is continuous, it has neither a first nor a last element, and is metrizable (a set $\mathfrak{D}$ of Fréchet).*

A_6 *The set E is connected, it has neither a first nor a last element, and is metrizable.*

A_7 *The set E is dense, it has neither a first nor a last element, is metrizable and is complete.*[44]

The affirmative answer to Souslin's problem came from the conclusion $A_3 \rightarrow A_4$; as Kurepa said:

> L'inclusion $A_3 \rightarrow A_4$ donne la solution (affirmative) d'un problème de Souslin.[45]

Through the different characterizations for the linear continuum given by these seven propositions, it is possible to see the source of Kurepa's confidence regarding a positive answer to the problem. Even if he emphasized the implication $A_3 \rightarrow A_4$, he certainly did not have a direct proof at hand; but through these propositions Kurepa tried to connect the property of being "the smallest" continuous ordered set (conditions A_1 and A_2) with the *countable chain condition*, and the property of being a metrizable set (conditions A_5, A_6 and A_7) with the *separability condition*. So for the chain of implications $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \cdots A_7 \rightarrow A_1$ of this theorem, Kurepa took into account not only the properties of the set of real numbers conceived as an ordered set, but also those properties of the set of real numbers conceived as a metrizable set as studied by Fréchet in *Les Espaces Abstraits*.[46] It appears from his 1935 memoir that one year later Kurepa was still searching for the relation between the countable chain condition and the condition for a continuous set to be metrizable.[47]

3.1.1. First strategy: cardinal functions associated with continuous sets. Besides the possible relation between separable and metrizable sets, Kurepa began to explore two characteristic properties in relation to the continuous ordered sets. The first one deals with the possibility of defining some cardinal functions, the values of which depend on the properties of the set. This idea was explored for the first time in his second note of March 1934[48] where he stated that Souslin's problem could be formulated through the relation between two (infinite) cardinal numbers: the cardinal number of a family of non-overlapping intervals of a continuous ordered set, and the cardinal number of a dense subset of the same set; if the first cardinal number were at most $\aleph_0$, Souslin's

no *distance* between its points is defined but only a system of neighborhoods is defined for each point, Fréchet said that it is perfectly separable if there exists a countable family $\mathscr{F}$ of subsets such that for any point $a \in E$, the family of sets belonging to $\mathscr{F}$ and containing a as an interior point is equivalent to the family of neighborhoods of a defined in the set E. Fréchet showed that both conditions, being separable and perfectly separable, are equivalent for a metrizable set. This is not true in general.

[44] Being a metrizable set, the terms connected and complete are well defined.

[45] "The implication $A_3 \rightarrow A_4$ gives the (affirmative) solution to a problem of Souslin". Kurepa 1934*a*.

[46] Fréchet *Op. cit*.

[47] *Cf.* theorem *EOR*.1 below.

[48] Kurepa 1934*b*.

hypothesis would say that the second cardinal number should also be $\aleph_0$. These two cardinal numbers are defined by Kurepa for any continuous ordered set E:

$p_1E = \inf\{|F|\,;\, F \subset E$ is a dense subset of $E\}$

$p_2E = \sup\{|\mathscr{F}|\,;\, \mathscr{F}$ is a family of disjoint and non-empty open intervals of $E\}$

Once these cardinal numbers were formally introduced, the truth of Souslin's hypothesis was claimed in the following terms:

> Pour tout ensemble ordonné infini, $p_2E \leqq p_1E$; si E est dense alors $p_2E = p_1E$ [...]. Le fait que $p_1E = \aleph_0$ si E est continu et $p_2E = \aleph_0$ constitue la solution d'un problème de Souslin.[49]

According to this last statement, Souslin's hypothesis would be merely a particular case (the countable case) of a more general equality between the cardinal numbers p_1E and p_2E. But this renewed conviction regarding Souslin's hypothesis, which was based on the belief that $p_2E = p_1E$ for any continuous set E, had to be reconsidered one year later. The relation that Kurepa established without any doubt in his memoir *Ensembles Ordonnés et Ramifiés* is that if E is an ordered set, it is possible to conclude from the equality $p_1E = \kappa$ that $p_2E \leqq \kappa$. In other words, the inequality $p_2E \leqq p_1E$ can be stated between the two cardinal numbers.[50] Concerning the equality $p_2E = p_1E$ claimed in his March note, Kurepa stated in 1935 the following question, which is equivalent to Souslin's problem:

> Le problème si, pour des E continus, la borne supérieure p_2E est atteinte et si elle est égale à p_1E, se réduit, comme on le voit, à cette question: soient E un ensemble ordonné continu et $\mathscr{F}$ une famille d'intervalles déterminant E; existe-t-il nécessairement une sous-famille disjonctive de $\mathscr{F}$ ayant la puissance p_1E?[51]

This more prudent attitude towards the truth of Souslin's hypothesis can also be clearly seen by looking at the new version that he provided in this memoir for theorem *CR.I.1*:[52]

Theorem ***(EOR.1).*** *For an ordered set E the following statements are equivalent and each of them characterizes the order type of a closed interval of real numbers*

B_0 *E has the order type θ .*

B_1 *E is continuous and irreducible (which means that every continuous subset of E is similar to E).*

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[49] "For every infinite ordered set, $p_2E \leqq p_1E$; if E is dense then $p_2E = p_1E$ [...]. The fact that $p_1E = \aleph_0$ if E is continuous and $p_2E = \aleph_0$ gives the solution for a problem of Souslin". *Ibid.* In this note the symbols m_1E and m_2E are used in place of p_1E and p_2E respectively.

[50] As we will see hereafter, Kurepa proved that if $p_2E < p_1E$ then there is no cardinal number between them.

[51] "The problem, for a continuous set E, of knowing if the upper bound p_2E has been reached and if it is equal to p_1E, reduces itself, as we can see, to the following question: let E be an ordered continuous set and let $\mathscr{F}$ be a family of determining intervals of E. Is there necessarily a disjoint subfamily of $\mathscr{F}$ with the power p_1E?" Kurepa 1935, p. 66.

[52] *Ibid.*, p. 123. In this memoir Kurepa makes no reference to his theorem *CR.I.1*.

$B_{2,1}$ *E is a continuous set such that* $p_1 E = \aleph_0$.
B_3 *E is continuous and metric.*
B_4 *E is dense, with first and last elements, metric and complete.*
B_5 *E is dense, metric and compact in itself.*

Propositions B_1, $B_{2,1}$, B_3, B_4, were already stated in theorem *CR.I.1* (A_2, A_4, A_5 and A_7), but the important difference is that proposition A_3, which in this new theorem would be a proposition $B_{2,2}$ (E is a continuous set such that $p_2 E = \aleph_0$), is absent. Kurepa provided a proof for theorem *EOR.1* which he had been unable to give one year before because of this proposition A_3; but this time he stated clearly and precisely the relation, which in theorem *CR.I.1* he had only suggested, between the conditions involved in Souslin's problem and the necessary fact that the continuous set E is metric:

> A cause de l'inclusion $B_3 \rightarrow B_0$, il suffirait, pour obtenir la réponse affirmative au problème de Souslin, de prouver que, sous l'hypothèse $B_{2,2}$, E est distanciable.[53]

If in February 1934 Kurepa claimed that a continuous ordered set satisfying the countable chain condition is *separable*, one year later he stated clearly that a suitable way to prove this implication would be to show that this set is metrizable. In view of the difficulty of this proof, between 1934 and 1935 he confirmed Sierpiński's opinion on Souslin's problem: this seemed indeed to be a very difficult problem to solve.

Proposition $B_{2,2}$ is related to the cardinal number of a maximal family of subsets of a linear continuum. The fact that it can be deduced from proposition $B_{2,1}$ ($B_{2,1} \rightarrow B_{2,2}$) follows from the well known implication that a *separable* set satisfies the countable chain condition. But the difficulty of answering whether the equality $p_1 E = p_2 E$ always holds – and so of answering positively whether the implication $B_{2,2} \rightarrow B_{2,1}$ holds – indeed arose from the fact that while $p_1 E$ is the cardinal number of a subset of the continuous set E, $p_2 E$ is the cardinal number of a family of subsets of the same set E, and so it is the cardinal number of a subset of the power set, 2^E, of E. Kurepa thought that besides the well known relation implying $p_1 E \geqq p_2 E$, no other possible relation between these two cardinal numbers could be obtained within the frame of the theory of order types. A new insight into the nature and the structure of continuous ordered sets was necessary to overcome this difficulty.

3.1.2. Second strategy: the complete development of a continuous set. The solution proposed by Kurepa derived from a concept that was introduced in his first note of February 1934 and which is related to a second characteristic property of a continuous set E: the possibility of giving a *complete development* for any continuous set. The February note opened with a definition that would become very important for his investigations in the subsequent notes and in his 1935 memoir: that of a (*complete*) *development for a continuous set*. Following Hausdorff's theory of the general product and power of a set, Kurepa considered the case when an ordered continuous set E can be seen as the

[53] "Because of the implication $B_3 \rightarrow B_0$, it would be enough, in order to obtain the affirmative answer to Souslin's problem, to prove that under the hypothesis $B_{2,2}$, E is metrizable". *Ibid*., p. 125. Kurepa calls a metrizable set "distanciable".

union of a family $\{E_a\}_{a\in F}$ of pairwise disjoint intervals of E, with $F \subset E$ and $a \in E_a$. In general, if F is a subset of E and $\{E_a\}_{a\in F}$ is a disjoint family of intervals of E, the union $\bigcup_{a\in F} E_a$ is an ordered subset of E, where every point $x \in E_a$ is smaller than any point $y \in E_b$ whenever $a < b$ ($E_a < E_b$ if $a < b$). When the equality

$$E = \bigcup_{a\in F} E_a \tag{3.I}$$

holds, a *development* for E is obtained according to its subset F. Each subset E_a is a closed interval of E, with the possible existence of at most two semi-closed intervals. Each E_a can reduce to a single point $\{a\}$. Kurepa stated that when F is a continuous subset, any continuous set E accepts a development (according to the continuous subset F):

> Pour qu'un continu E soit développable suivant son sous-ensemble F, il faut et il suffit que F soit sans lacunes [...] dans le cas ou F est continu, le développement $E = \bigcup_{a\in F} E_a$ est unique.[54]

This concept of a development for a set E is related to propositions A_1 and A_2 of theorem *CR.I.1*. Kurepa's purpose in that theorem was to show that the countable chain condition holds for a continuous set E if no development, according to a proper subset of type θ, is possible for this set (with the only exception of the "trivial" development, where each $E_a = \{a\}$).[55] Since theorem *CR.I.1* stated that the type λ is in some sense the smallest continuous type (proposition A_2), and that a separable continuous order set (whose type is λ or θ) *could not* be obtained as a non-trivial development according to a continuous subset of type θ (proposition A_1), a proof for Souslin's hypothesis would reduce to proving that any continuous ordered set whose type is different from λ or θ accepts a non-trivial development according to a subset of type θ. In that case the only continuous and ordered sets with the countable chain condition would be those sets whose type is λ or θ.

In the March note, the complete development for a continuous set E made it possible to state that whenever the equality 3.I holds, each E_a is a (continuous) interval which can be developed according to a (continuous) subset F_a of E_a. In this way, a sequence of developments is obtained:

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[54] "In order for a continuous set E to have a development according to a subset F, it is necessary and sufficient that F has no gaps [...] if F is continuous, the development $E = \bigcup_{a\in F} E_a$ is unique". Kurepa 1934*a*.

[55] This idea was already clear with Cantor's definition of the product of two order types. According to the definition, the type $\theta^2 = \theta \times \theta$, which is the order type of the square $[0, 1] \times [0, 1]$ with the lexicographic order, is perfect, thus continuous. But the countable chain condition does not hold, and so this set is not separable. The ordered set $E = [0, 1] \times [0, 1]$, whose order type is θ^2, has a development according to the subset $F = [0, 1] \times \{0\}$: for each $a \in F$ let

$$E_a = \{(a, x); x \in [0, 1]\} \quad ,$$

then, for $a \neq b$, $E_a \cap E_b = \emptyset$ and 3.I holds.

As we saw in section 2.2, Veblen used a similar model to prove the independence of the separability condition.

$$E_a = \bigcup_{b \in F_a} E_{ab} \tag{3.II}$$

is a development for E_a and, recursively, it is possible to define:

$$E_{a_0} \cdots {}_{a_\alpha} = \bigcup_{a_{\alpha+1} \in F_{a_0 \cdots a_\alpha}} E_{a_0} \cdots {}_{a_{\alpha+1}} \quad . \tag{3.III}$$

Each subset with index $\alpha + 1$ that is to be developed is an interval of a previously developed subset with index α. But a subset whose index is a limit ordinal number α is defined through the previous subsets:

$$E_\alpha = E_{a_0 \ldots a_\zeta \ldots} = \bigcap_{\zeta < \alpha} E_{a_0 \ldots a_\zeta} \quad . \tag{3.IV}$$

For this sequence of developments, Kurepa claimed that it is possible to write:

$$E = \bigcup_{a \in F} E_a = \bigcup_{a \in F; b \in F_a} E_{ab} = \cdots = \bigcup_{\alpha} E_\alpha \quad . \tag{3.V}$$

All the sets $F, F_a, \ldots, F_{[\alpha]}$ are called the *arguments* of the development. Their type is θ or 1 and they are subsets of the sets $E, E_a, \ldots, E_{[\alpha]}$, which are the *terms* of the development. Now, there is always an ordinal number β such that the set E_β reduces to a single point. E is then equal to the union of its points:

$$E = \bigcup_{\beta} E_\beta \quad . \tag{3.VI}$$

The order type (θ or 1) of each *argument* depends on whether the respective *term* is a continuous subset or reduces to a single point of E; all the indexes run over the *arguments*.

In order to understand the source of the central concept introduced by Kurepa in his memoir *Ensembles Ordonnés et Ramifiés,* let us take a closer look at the developments described through equations 3.I to 3.IV. Through these developments a family $\mathfrak{T}$ of subsets of E appears that is organized in the following way: first a development, according to a subset F of E, is obtained for the set E itself; this gives rise to a disjoint family $\{E_a\}_{a \in F}$ of subsets of E. When each subset E_a is developed according to a subset F_a of E_a, another family $\{E_{ab}\}_{b \in F_a}$ of subsets of E_a is obtained for each subset E_a. The family of sets $\mathfrak{T}$ is formed by the set E, by all the subsets E_a of E, by the subsets E_{ab} of each E_a, etc. Clearly the family $\mathfrak{T}$ obtained in this way has the property that any two of its sets are disjoint or one of them is a proper subset of the other; and since the set E itself belongs to the family, it is clear that any member of $\mathfrak{T}$ is a proper subset of this first set. Kurepa called a family of sets with this property a *ramified table of sets*.[56] According to this definition the family of subsets of E obtained from a (complete) development, the *terms* of the development, form a ramified table of sets. Kurepa's aim in 1935 was to obtain a new comparison between $p_1 E$ and $p_2 E$ through this ramified table of sets.

[56] A *Tableau Ramifié d'Ensembles*.

In this 1935 memoir the following recursive definition for the ramified table $\mathfrak{T}$, which reproduces the steps 3.I–3.IV, is introduced[57]

1. The first member of the family $\mathfrak{T}$ is the set E itself,
$$R_0\mathfrak{T} = E \quad .$$
2. If the sets forming the subfamily $R_\alpha\mathfrak{T}$ have been defined, the sets forming the subfamily $R_{\alpha+1}\mathfrak{T}$ are obtained from those of $R_\alpha\mathfrak{T}$ when for each $S \in R_\alpha\mathfrak{T}$ a partition $\mathscr{P}(S) = \{S_a\}_{a\in I}$[58] is defined,
$$R_{\alpha+1}\mathfrak{T} = \{S_a \in \mathscr{P}(S);\, S \in R_\alpha\mathfrak{T}\} \quad .$$
3. For a limit ordinal α, each set of the subfamily $R_\alpha\mathfrak{T}$ is defined by taking the intersection of a nested family of subsets, one (and only one) for each level $R_\zeta\mathfrak{T}$ $(\zeta < \alpha)$.

$$R_\alpha\mathfrak{T} = \left\{ \bigcap_{\zeta<\alpha} E_\zeta;\ E_\zeta \in R_\zeta\mathfrak{T} \text{ and } E_\zeta \subseteq \bigcap_{\eta<\zeta} E_\eta \right\} \quad .$$

Denoting by $\gamma\mathfrak{T}$ the order type of the set of all ordinals α such that $R_\alpha\mathfrak{T} \neq \emptyset$, the ordinal number $\gamma\mathfrak{T}$ is defined as the *height* of the ramified table of sets $\mathfrak{T}$.[59] $\mathfrak{T}$ is then the family of sets belonging to all the levels $R_\alpha\mathfrak{T}$ $\left(\mathfrak{T} = \bigcup_{\alpha<\gamma\mathfrak{T}} R_\alpha\mathfrak{T}\right)$.[60] In this way it becomes clear that the family $\mathfrak{T}$ is partially ordered by the inverse relation for contention: for two sets of the family $\mathfrak{T}$, E_α and E_β ($E_\alpha \in R_\alpha\mathfrak{T}$ and $E_\beta \in R_\beta\mathfrak{T}$), it is said that $E_\alpha < E_\beta$ whenever $E_\beta \subset E_\alpha$.[61] In the notes he wrote in 1934, Kurepa studied the *complete* development of a continuous set. Now, the ramified table of sets $\mathfrak{T}$ is complete when the following conditions hold:

i) $\bigcup\mathfrak{T} = E \in \mathfrak{T}$.
ii) If $A \in \mathfrak{T}$ and if $|A| > 1$, then $A = \bigcup R_0(A, .)$.[62]
iii) If $\mathfrak{G} \subset \mathfrak{T}$ is a monotone subfamily, then $\bigcap\mathfrak{G} \in \mathfrak{T}$.
iv) For every $x \in \bigcup\mathfrak{T} = E$, $\{x\} \in \mathfrak{T}$.

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[57] This new definition was introduced for the first time in the July note of 1934.
[58] This means that $S_a \cap S_b = \emptyset$ $(a \neq b)$ and that $\bigcup_{a\in I} S_a = S$.
[59] This definition for $\gamma\mathfrak{T}$ was given in the memoir of 1935; in his July note, the height of $\mathfrak{T}$, $\gamma\mathfrak{T}$, was defined as the sup $\{\alpha;\, R_\alpha\mathfrak{T} \neq \emptyset\}$.
[60] Each α-level, $R_\alpha\mathfrak{T}$, is just the 0-level (the "first level") of $\left(\mathfrak{T} - \bigcup_{\beta<\alpha} R_\beta\mathfrak{T}\right)$.
[61] And clearly, according to the recursive definition this is possible only if $\alpha < \beta$.
[62] The symbol $R_0(A, .)$ is to be understood in the obvious way: it is the first level of the subfamily of sets, contained in $\mathfrak{T}$, which are properly contained in A. Property (*ii*) then states that a partition of A is made up by the disjoint family $\{A_\alpha\}$ of subsets of A that forms the first level of $(A, .)$.

For a complete ramified table of sets $\mathfrak{T}$, the set E is the only set belonging to the first level $R_0\mathfrak{T}$. The ordinal number $\nu(x) = \alpha$ states that $\{x\}$ is an element of the α-level, $R_\alpha\mathfrak{T}$, of $\mathfrak{T}$.

Kurepa defined $\mathfrak{T}$ as a ν-complete partition of E if for any $A \in \mathfrak{T}$ $(A \subset E)$, the family of sets forming the first level of A, $\{A_\alpha\} = R_0(A, .)$, has order type ν.[63]

The questions of how far it is possible to go for the complete development of a continuous set E (in other words, whether or not the ordinal number $\nu(x)$ is the same for all the points $x \in E$), and what is the height of the ramified table of sets $\mathfrak{T}$, depend in some sense on the nature of the set E itself. In the memoir *Ensembles Ordonnés et Ramifiés* the following theorem provided a partial answer:[64]

Theorem ***(EOR.2).*** *If $\mathfrak{T}$ is a complete development of any ordered set E such that $p_0E \leqq \aleph_\beta$ ($p_0E = \sup\{|F|\,;\, F \subset E$ is a well-ordered subset of $E\}$[65]), then $\gamma\,\mathfrak{T} \leqq \omega_{\beta+1}$.*

3.2. *Second step: the conditions for an answer*

The ramified table of sets obtained through a complete development of the set E whose arguments are subsets either of type θ or 1, is clearly a θ-complete partition. With this particular ramified table of sets, Kurepa intended to establish the conditions under which the cardinal number p_2E could be equal to p_1E. This was done as follows: every interval S of a continuous ordered set E contains a closed interval $\varphi(S)$ whose order type is θ; $\{S_a\}_{a\in\varphi(S)}$ is the disjoint family of subsets of S giving the development of S according to $\varphi(S)$ (the family $\mathscr{P}(S) = \{S_a\}_{a\in\varphi(S)}$ is a partition for S so that $S = \bigcup_{a\in\varphi(S)} S_a$). Kurepa considered the ramified table of sets $\mathfrak{T}$, which comes out from the complete development of E, but also the following subfamily $\Psi\mathfrak{T}$ contained in $\mathfrak{T}$, which is also a ramified table of sets:

$$\Psi\mathfrak{T} = \{X \in \mathfrak{T};\, |X| > 1\} \qquad (D.3.1)$$

If $\mathfrak{T}$ is a θ-complete development of E, then for every $a \in E$, $\{a\} \in R_{\nu(a)}\mathfrak{T}$, and so two disjoint subsets can be defined:

$$E_1 = \{a \in E;\, \nu(a) \text{ is a successor ordinal}\} \qquad (D.3.2)$$

and

$$E_2 = \{a \in E;\, \nu(a) \text{ is a limit ordinal}\}\ . \qquad (D.3.3)$$

Since the order type of $\varphi(S)$ is θ, $p_1\,[\varphi(S)] = \aleph_0$, and there exists a denumerable subset $\varphi_0(S)$ which is order-dense in $\varphi(S)$ $\left(\overline{\varphi_0(S)} = \eta\right)$. By defining the following subset of E:

[63] Clearly, for A_i and $A_j \in \{A_\alpha\} = R_0(A, .)$, it is said that A_i is smaller than A_j $(A_i < A_j)$ whenever every point of A_i is smaller, according to the order relation in A, than any point of A_j. Of course the order type ν of $\{A_\alpha\} = R_0(A, .)$ is smaller than the order type of A.

[64] Kurepa 1935, p. 113.

[65] F could also be a reversed well-ordered subset of E.

$$F_1 = \bigcup_{S \in \Psi\mathfrak{T}} \varphi_0(S) \quad , \qquad (D.3.4)$$

it is clear that F_1 is dense in $\bigcup_{S\in\Psi\mathfrak{T}} \varphi(S)$. Now, for every $a \in E_1$, $\{a\} \in R_{\nu(a)}\mathfrak{T}$, where, according to the definition of the set E_1, $\nu(a) = \beta + 1$, this means that a is a point of a subset S of E whose order type is θ ($S = \varphi(S)$), which is a member of the β-level of $\mathfrak{T}$ ($a \in S \in R_\beta\mathfrak{T}$; $S \in \Psi\mathfrak{T}$). This shows that $E_1 \subset \bigcup_{S\in\Psi\mathfrak{T}} \varphi(S)$; it follows then that F_1 is dense in E_1.

Kurepa defined another subset of E:

$$F_2 = \{a \in E; a \text{ is an extreme point of a portion } S \in \Psi\mathfrak{T}\} \quad . \qquad (D.3.5)$$

Two important properties have to be stated for this subset:

i) When $\Psi\mathfrak{T}$ is infinite then

$$|\Psi\mathfrak{T}| = |F_2| \quad . \qquad (3.\text{VII})$$

ii) F_2 is dense in E_2: since $\mathfrak{T}$ is a complete development, if $a \in E_2$ and I is an interval of E containing a, $a = \bigcap \mathfrak{F}$, where $\mathfrak{F} = \{A \in \Psi\mathfrak{T};\ a \in A\}$; so there is at least one element of $\mathfrak{F}$ which is completely contained in I.

Kurepa defined in this way the set $F = F_1 + F_2$ (which is dense in $E = E_1 + E_2$). Clearly

$$|F_1| = \aleph_0\, |\Psi\mathfrak{T}| \quad , \qquad (3.\text{VIII})$$

and from 3.VII and 3.VIII follows

$$|F| = \aleph_0\, |\Psi\mathfrak{T}| \quad . \qquad (3.\text{IX})$$

From the fact that F is dense in E, it follows that

$$p_1 E \leqq |F| = \aleph_0\, |\Psi\mathfrak{T}| \quad . \qquad (3.\text{X})$$

Kurepa then introduced a new cardinal number:

$$b\,(\Psi\mathfrak{T}) = \sup\{|\mathscr{F}|; \mathscr{F} \text{ is a disjoint or a monotone family of } (\Psi\mathfrak{T})\} \quad . \qquad (D.3.6)$$

He easily proved the following equality:

$$p_2 E = \aleph_0 b\,(\Psi\mathfrak{T}) \quad . \qquad (3.\text{XI})$$

From 3.X and 3.XI, follows

$$\aleph_0 b\,(\Psi\mathfrak{T}) = p_2 E \leqq p_1 E \leqq \aleph_0\, |\Psi\mathfrak{T}| \quad . \qquad (3.\text{XII})$$

But Kurepa went farther and he proved that indeed

$$p_1 E = |F| = \aleph_0\, |\Psi\mathfrak{T}| \quad . \qquad (3.\text{XIII})$$

The main conclusion from 3.XII (and 3.XIII) is that whenever the following equality holds

$$b\,(\Psi\mathfrak{T}) = |\Psi\mathfrak{T}| \quad , \tag{3.XIV}$$

then $p_2 E = p_1 E$.

A proof for Souslin's hypothesis would then be provided if for the ramified table of sets $\mathfrak{T}$ obtained from a continuous ordered set E satisfying $p_2 E = \aleph_0$, the equality 3.XIV were to hold.

Kurepa's aim in *Ensembles Ordonnés et Ramifiés* was precisely to analyze, within a new framework, the necessary and sufficient conditions for a ramified table of sets $\mathfrak{T}$ so that 3.XIV holds. Concerning this new framework, we have underlined the important contributions made in his 1934 notes. A remarkable fact is that in the note published in the COMPTES RENDUS in July 1934, Kurepa claimed the truth of Souslin's hypothesis precisely in this form: he stated that 3.XIV holds.

Theorem ***(CR.IV.1).*** *Fundamental theorem: Let $\mathfrak{T}$ be a ramified table of sets; if $\Psi\mathfrak{T}$ is infinite, there exists a subfamily $\Psi'\mathfrak{T}$ of pairwise disjoint sets which has the same power as $\Psi\mathfrak{T}$.*

Remarkably this theorem is the only one for which Kurepa provided a sketch of a proof ("une esquisse de preuve"). In the next section we will analyze the gist of his arguments; but here we stress that with this theorem Kurepa claimed for the last time the truth of Souslin's hypothesis. As theorem *CR.IV.1* shows, in the last note he sent to the *Académie des Sciences* before his 1935 memoir, Kurepa knew exactly the condition that should be proved for a ramified table of sets in order to prove Souslin's hypothesis. But while this condition was claimed to hold in July 1934, in *Ensembles Ordonnés et Ramifiés* it was only stated as an *equivalent* condition to be proved. Certainly, as we have said, in 1935 he had not abandoned his faith, but he realized that the question whether 3.XIV holds had to be answered not only for the particular ramified table of sets $\Psi\mathfrak{T}$, but for any partially ordered set T having the same (partial) order relation as $\Psi\mathfrak{T}$. This became the main subject of research in his memoir of 1935.

3.3. Third step: the redefinition of the problem

3.3.1. Partially ordered sets and ramified tables. *Ensembles Ordonnés et Ramifiés* is devoted to the study of a particular kind of partially ordered sets. A *ramified table* is a partially ordered set T satisfying the following conditions:

1. For any three elements $a, b, c \in T$, if $a < c$ and $b < c$ then $a \sim b$ (which means $a = b$ or $a < b$ or $a > b$).
2. For any $a \in T$ the ordered subset $(\cdot, a)_T = \{x \in T; x < a\}$ is well ordered.[66]

[66] The fact that $(\cdot, a)$ is an ordered subset of T is an immediate consequence of the first condition. This definition makes clear that a *ramified table* T is what we call a *tree* in modern set theory.

The *height* of an element $a \in T$ is the order type of the well ordered set $(\cdot, a)_T$. The first level of points $R_0 T \subset T$ of a ramified table T is a disjoint subset of T having the property that any point of $(T - R_0 T)$ is preceded by one (and only one) point of $R_0 T$. The α-level $R_\alpha T \subset T$ is also a disjoint subset of T.[67]

The *height* of T, γT, is now the order type of the set of ordinals $\{\alpha;\ R_\alpha T \neq \emptyset\}$, and the *width* of T is the cardinal number $mT = \sup\{m_\alpha T\}_{\alpha<\gamma T}$ $(m_\alpha T = |R_\alpha T|)$.

From these new definitions it is clear that

$$|T| = \sum_{\alpha<\gamma T} m_\alpha T \tag{3.XV}$$

and that

$$|T| = |\gamma T| \cdot mT \quad . \tag{3.XVI}$$

By stating that any monotone or disjoint subset of T is called a *degenerate* subset, another cardinal number, analogous to the cardinal number defined in *D*.3.6, is defined:

$$bT = \sup\{|F|\,;\ F \text{ is a degenerate subset of } T\} \quad . \tag{D.3.7}$$

Under these general definitions the question is to find the relation between the numbers bT and $|T|$. Another question closely related to this one is whether the number bT is reached (in other words, whether there always exists a degenerate subset F of T such that $|F| = bT$). As for the relation between the two cardinal numbers, clearly it can be said that

$$bT \leqq |T| \quad . \tag{3.XVII}$$

But Kurepa proved that if $bT < |T|$, no cardinal number exists between them.

A *normal table* is defined as one where bT is always reached and is equal to $|T|$. In other words, T is *normal* if there exists a degenerate subset T' of T such that $\left|T'\right| = |T|$.

With all these general definitions, Kurepa stated that a proof for Souslin's hypothesis would be attained by showing that the ramified table of sets $\mathfrak{T}$ obtained from the complete development of a continuous ordered set E with the countable chain condition (*i.e.* such that $p_2 E = \aleph_0$) is a *normal* ramified table.

Now Kurepa noticed two important implications of the equality $p_2 E = \aleph_0$:

i) Since $p_0 E \leqq p_2 E = \aleph_0$, it follows from theorem *EOR.2* that $\gamma\mathfrak{T} \leqq \omega_1$.
ii) If $\mathfrak{T}$ is a ramified table of sets obtained from E through a ν-complete partition, then $|\nu| \leqq \aleph_0$.

From 3.XVI and these two remarks it follows that $|\mathfrak{T}| \leqq \aleph_1$.[68] This means that the question that becomes relevant in relation to Souslin's problem is whether a ramified table T such that $|T| \leqq \aleph_1$ is a normal table.

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[67] As in the case of a ramified table of sets, the α-level of T, $R_\alpha T$, is the first level $R_0\left(T - \bigcup_{\xi<\alpha} R_\xi T\right)$.

[68] If a ramified table T is such that $\gamma T \leqq \omega_1$ and each of its nodes is at most countable then $|T| \leqq \aleph_1$.

The proof that if $|T| = \aleph_0$ then T is normal presents no problem, since one of the following two conditions hold: there exists a denumerable level in T, which is a disjoint subset, or else there exists a denumerable monotone subset having one point at each level (*i.e.* a "countable chain"). This was stated by Kurepa through the following theorem:[69]

Theorem ***(EOR.3).*** *When $|T| = \aleph_0$, there always exists an infinite degenerate subset T_d (T is normal).*

This is the well-known theorem of König stating that a *tree* whose height is ω and whose levels are always finite has at least one *cofinal branch*. If T has an infinite level (which is necessarily the case when $\gamma T < \omega$), this one will be the degenerate subset of T. If, contrary to this first possibility, every level of T is finite, then by induction it is possible to define a monotone subset: there always exists a point $a_0 \in R_0 T$ such that $|[a_0, \cdot)| = |T|$; if up to the level n a point $a_n \in R_n T$ has been taken such that $a_0 < a_1 < \cdots < a_n$ (and such that $|[a_n, \cdot)| = |T|$), then a_{n+1} is a point in $R_{n+1} T$ such that $a_n < a_{n+1}$.

After theorem *EOR.3*, the only remaining question for Kurepa was to determine whether a ramified table T, such that $|T| = \aleph_1$, is normal:

> Le problème de savoir *si tout T ayant la puissance $\aleph_1$ est normal* est d'une importance considérable parce que [...] il est intimement lié au problème bien connu de Souslin. Il s'agit donc de voir si T est normal, c'est-à-dire *s'il contient un ensemble dégéneré non-dénombrable T_d*.[70]

Since the problem is now reduced to the study of a ramified table T of power $\aleph_1$, from equality 3.XVI, two cases ought to be considered:

I. The case $\gamma T < \omega_1$ ($|\gamma T| = \aleph_0$).
II. The case $\gamma T = \omega_1$ ($|\gamma T| = \aleph_1$).

In the first case T is a normal ramified table, since $mT = \aleph_1$, and there exists an ordinal number $\alpha < \gamma T$ such that the α-level $R_\alpha T$ has power $\aleph_1$. In this case T is said to be a *wide ramified table* (for some $\alpha < \gamma T$, $m_\alpha T \geqq |cf(\gamma T)|$).[71]

In the second case, if $\gamma T = \omega_1$, three cases are to be considered:

II.1 $mT = \aleph_1$.
II.2 $mT < \aleph_0$.
II.3 $mT = \aleph_0$.

For the first two cases, if $mT = \aleph_1$ or $mT < \aleph_0$, then T is again a normal ramified table: in the first one, for the same argument as in case I, T is *wide*: an α-level $R_\alpha T$ has power $\aleph_1$. In the second case, since $mT < \aleph_0$ and $cf(\gamma T) = \omega_1$, T is said to be a *narrow*

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69 Kurepa *Op. cit.*, p. 105.

70 "The question of *whether every T with power $\aleph_1$ is normal* is very important since [...] it is closely related to the well known problem of Souslin. The question is whether T is normal, which means whether *it contains a degenerate and non-countable subset T_d*". Kurepa 1935, p. 106.

71 For any ordinal number α, its cofinality, $cf(\alpha)$, is the least ordinal number such that there exists an increasing sequence of length $cf(\alpha)$ whose limit is precisely α.

table (T is narrow if $mT < |cf(\gamma T)|$ and whenever $cf(\gamma T) = \omega_{\beta+1}$, $mT < \aleph_\beta$); in this case the conclusion follows from the following theorem[72]:

Theorem ***(EOR.4).*** *Each narrow ramified table has the same cardinal number as one of its monotone subsets.*

For a narrow ramified table T, a non-denumerable monotone subset having just one point at each level of T exists; in this case, Kurepa says that T accepts a *monotone descent*.

For the case II.3 ($|\gamma T| = \aleph_1$ and $mT = \aleph_0$), T is neither wide nor narrow, but an *ambiguous* ramified table (for every ordinal $\alpha < \gamma T$, $m_\alpha T < |cf(T)|$ and if $cf(\gamma T) = \omega_{\beta+1}$ then $mT \geqq \aleph_\beta$).[73] At first sight, the case of an ambiguous table T of height ω_1 just seemed a generalization of the case of a table T such that $|\gamma T| = \aleph_0$ and $mT < \aleph_0$, which, according to theorem *EOR 3*, is normal. But an important example found by N. Aronszajn and communicated to Kurepa[74] showed that this property cannot be generalized when $\gamma T = \omega_1$ and $mT = \aleph_0$. The example showed a ramified table S of height ω_1, all of whose levels are countable, but having no monotone subset of length ω_1.[75]

In accordance with the example given by Aronszajn, Kurepa defined the set

$$\sigma_0 = \{X \subset \mathbf{Q};\ X \neq \emptyset \text{ is a well-ordered and bounded subset}\},$$

which is a ramified table of non-reached height ω_1.[76] And now the ambiguous ramified table S is defined recursively:

1. The first level of S is defined as the first level of σ_0

$$R_0 S = R_0 \sigma_0 \ .$$

2. If, for every $\xi < \alpha$ ($\alpha < \omega_1$), the level $R_\xi S$, which is a subset of σ_0, has been defined in such a way that the following conditions hold:

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72 *Ibid.* p. 80.

73 This classification for *wide*, *narrow* and *ambiguous* ramified tables was already introduced in the July note, although with some slight differences.

74 In his memoir Kurepa says that he received the example of Aronszajn at the end of June 1934.

75 The existence of a ramified table of height ω_1 whose levels are all countable, but for which there is no monotone descent, was the first published construction of an *Aronzajn tree*. In modern set theory, a κ-tree T is such that $\gamma T = \kappa$ and for any $\alpha < \gamma T$, $m_\alpha T < \kappa$. König's theorem proves that for a $\aleph_0$-tree there always exists a monotone descent (it has always a "cofinal branch"). An Aronzajn tree is a $\aleph_1$-tree which has no cofinal branch. This fact makes clear the introduction of *ambiguous* ramified tables, besides *wide* and *narrow* ramified tables.

76 The set σ_0 is seen as a set of complexes; *i.e.* the elements of σ_0 are well ordered sequences $A = (a_0, a_1, \ldots a_\nu, \ldots)$ of rational numbers whose length is smaller than ω_1. The partial order relation is defined in σ_0 in the following way: for two elements of σ_0, A and B, $A < B$ if the sequence A is an initial segment of the sequence B (clearly in this case the length of B is larger than that of A). $A = B$ only if $A \equiv B$. When $A \not< B$, $B \not< A$ and $A \neq B$, it is said that $A \not\sim B$. With this partial order σ_0 is a ramified sequence. Cantor's theorem assures us that $\gamma\sigma_0 = \omega_1$, and since $p_0\mathbf{Q} = \aleph_0$, this height is not reached.

i) $\left|R_\xi S\right| = \aleph_0$ for every $\xi < \alpha$.
ii) If $\xi + 1 < \alpha$ and $A \in R_\xi S$, then the node $R_0(A, \cdot)_S = \left\{B \in R_{\xi+1} S;\, A \subset B\right\}$ is a countable set and an order relation is defined on it such that it has no first element.

Then the α-level $R_\alpha S$ is defined in the following way:

2.1. If $\alpha = \beta + 1$,

$$R_\alpha S = \bigcup_{A \in R_\beta S} R_0(A, \cdot)_{\sigma_0} \ .$$

2.2. If α is a limit ordinal number, $R_\alpha S$ can be any countable subset $F \subset R_\alpha \sigma_0$ dense in the segment $(\cdot, \alpha)_S = \bigcup_{\xi < \alpha} R_\xi S$, and such that this segment is in its turn dense in F. Since $|(\cdot, \alpha)_S| = \aleph_0$, $R_\alpha \sigma_0$ is an ordered subset which is dense in $(\cdot, \alpha)_{\sigma_0}$ and so is also dense in $(\cdot, \alpha)_S$, this makes possible the definition of $R_\alpha S$ in this case.[77]

The ambiguous table S is defined as $S = \bigcup_{\xi < \omega_1} R_\xi S$. The fact that S is an ambiguous table whose height ω_1 is not reached is easy to prove: by construction for every limit ordinal number α between 0 and γS, the subsets $(\cdot, \alpha)_S$ and $R_\alpha S$ are dense one over the other. S is an ambiguous table and clearly $\gamma S = \gamma \sigma_0$; therefore this height is not reached.

3.3.2. Aronszajn tables and distinctive ramified sequences. It is not clear why Aronszajn did not himself publish this first example of an *Aronszajn ramified table*, but certainly it had a great influence on Kurepa's further studies related to Souslin's problem. This example exhibited an ambiguous and normal table S with an uncountable disjoint subset S_d having just one point at each level –S admits a *disjoint descent*. But this example also showed an ambiguous table for which the following conditions hold:

i) $\forall a \in S$, and $\forall \alpha < \gamma S$, $\exists b \in R_\alpha S$ such that $b \sim a$ (whenever a ramified table S satisfies this condition it is called a *ramified sequence*).[78]
ii) S is ambiguous and admits no monotone descent.
iii) $\forall a \in S$, $|a|_S = \{x \in S; (\cdot, x) = (\cdot, a)\}$ has the power $\aleph_0$; the set $|a|_S$ is called the *node* of a.

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[77] Once the ramified relation was defined for the set σ_0, Kurepa defined a *complete order*: for any two elements A and B of σ_0, $A < B$ in the complete order if $A < B$ according to the ramified relation, but when $A \not\sim B$, $A < B$ if $a_\kappa < b_\kappa$, where the κ-level is the first one where $a_\kappa \neq b_\kappa$. This generalization to a complete order can be given for any ramified table S whose elements are *complexes*. When α is a limit ordinal number, and A and B are two complexes of S belonging to $R_\alpha S$, such that $A < B$ according to this complete order, then at a certain level $\beta < \alpha$, the element a_β is smaller than b_β (with this we assume that $a_i = b_i \ \forall i < \beta$). Since each node is infinite and has no first element, a complex C of length ν ($\beta < \nu < \alpha$) such that $c_i = a_i \ \forall i \leqq \beta$ but such that $c_{\beta+1} > a_{\beta+1}$ lies between A and B. This proves that the subset $(\cdot, \alpha)_S$ is dense in $R_\alpha S$. From this fact it follows clearly that the subset $R_\alpha S$ is dense in $(\cdot, \alpha)_S$ if every complex of length $\beta < \alpha$ can be continued up to the limit ordinal α.

[78] Kurepa defined the *portion* of a, $[a]_S = \{x \in S; x \sim a\}$. This condition (*i*) says that $S = \bigcup_{a \in R_\alpha S} [a]_S$.

Kurepa defined a *distinctive ramified sequence* S as one satisfying conditions (i)–(iii). The example given by Aronszajn and Kurepa is then a distinctive ramified sequence accepting a disjoint descent.

The important role that these *Aronszajn tables* played for Souslin's problem was stressed by Kurepa himself in *Ensembles Ordonnés et Ramifiés* and also in an important memoir published two years later and devoted to these ambiguous tables accepting no monotone descent:[79] Souslin's hypothesis would be proved if every Aronszajn table accepts a disjoint descent:

> La condition nécessaire et suffisante pour que la réponse au problème de Souslin soit affirmative c'est que tout tableau ramifié de M. Aronszajn contient un sous-ensemble non denombrable de points deux à deux incomparables.[80]

The importance of this remark lies in the fact that it shows the precise theoretical point at which Kurepa had to recognize that he was unable to provide a definite answer to Souslin's problem: it was not possible for him to prove that every Aronszajn table is *normal*. The difficulty came from the following fact. Given an ambiguous table T of height ω_1 with no monotone descent, Kurepa considered the subset

$$T_0 = \{a \in T; |[a, \cdot)_T| \leqq \aleph_0\} \quad . \tag{D.3.8}$$

Since $|T_0| \leqq |T| = \aleph_1$, if the equality holds, then $m_0 T_0 = \aleph_1$ and so $T_d = R_0 T_0 \subset T$ is a disjoint subset and T is *normal*. But when $|T_0| \leqq \aleph_0$, the existence of an uncountable degenerate subset T_d cannot be proved in general. However, in this case Kurepa found that a *distinctive* ramified sequence $S \subset T$ can be defined such that whenever T is an *abnormal table*, S is also *abnormal*.[81] Despite the impossibility of providing a definite answer to Souslin's problem, the existence of an abnormal distinctive sequence contained in an abnormal table is one of the most important facts established by Kurepa. This importance will be underlined in the following section, since no proof of the equivalent conditions for Souslin's hypothesis is possible without this construction. We think that it is very enlightening to compare this construction of Kurepa's with the similar procedures which Miller and Sierpiński introduced several years later and which we will analyze below.

The definition of the distinctive ramified sequence $S \subset T$ goes as follows. Making $T_1 = T - T_0$, it turns out that$|T_1| = |T| = \aleph_1$ and $\left|[a, .)_{T_1}\right| = |T_1|\ \forall a \in T_1$, so that T_1 is an ambiguous sequence whose height ω_1 is not reached.[82] Now for every $a \in T_1$, the set $T'_a = [a, \cdot)_{T_1}$ is a ramified (ambiguous) sequence having at least one infinite level (since otherwise, if $a_0 \in T_1$ is such that $T'_{a_0} = [a_0, .)_{T_1}$ has no infinite levels, then T'_{a_0} is a narrow ramified table and it must accept a monotone descent, contrary to

[79] Kurepa 1937*d.*

[80] "The necessary and sufficient condition in order that the answer to Souslin's problem be affirmative is that any Aronszajn ramified table contains a non-countable subset of pairwise incomparable points". *Ibid*., p. 134.

[81] This means that no disjoint or monotone subset of T has the same power of T.

[82] Indeed T_1 is an ambiguous sequence of non-reached height ω_1, as is T itself, but T_1 has the property that for every point, the power of the set of points above it is not countable.

the hypothesis). It is possible then to define recursively a distinctive ramified sequence $S \subset T_1$ of height ω_1:[83]

1. The first level of S is equal to the first level of T_1,

$$R_0 S = R_0 T_1 \ .$$

2. If for every $\nu < \alpha$ ($\alpha < \omega_1$), the countable level $R_\nu S$ has been defined, the disjoint set $R_\alpha S$ is defined in the following way:
   2.1. For $\alpha = \beta + 1$,

$$R_\alpha S = \bigcup_{i \in R_\beta S} R_{\alpha_i} \, [i, \cdot)_{T_1} \quad ,$$

   where α_i gives the index of the first infinite level of the ambiguous sequence $[i, \cdot)_{T_1}$.
   2.2. For a limit ordinal α

$$R_\alpha S = R_\beta T_1 \ ,$$

   where $\beta = \sup \left\{ \eta ; \, R_\eta T_1 \cap \left(\bigcup_{\xi < \alpha} R_\xi S \right) \neq \emptyset \right\}$.

   With these two steps, the distinctive sequence

$$S = \bigcup_{\alpha < \omega_1} R_\alpha S$$

is defined.

At this point we can take a closer look at Kurepa's sketch of a proof for theorem *CR.IV.1*, which we mentioned at the end of section 3.2. As we have said, at the end of June 1934, a few weeks before he presented his July note at the *Académie des Sciences*, Kurepa received the first example of an *Aronszajn ramified table*. But even if this example made clear to him the possible existence of an ambiguous table not reaching its height, he still believed in the possibility of finding a *disjoint descent* for such tables. For an ambiguous table T of height ω_1 with no monotone descent, Kurepa introduced the subtable T_0 as in D.3.8, and its complement $T_1 = T - T_0$, and he defined recursively the sequence $T' = \{a_\alpha\}_{\alpha < \omega_1}$ as a disjoint subset of T:

1. The first element of T' is any element of the first level of T_1,

$$a_0 \in R_0 T_1 \ .$$

2. For any ordinal $\alpha < \omega_1$, the element a_α is any element of the α-level of T_1 which is not comparable to any of the previous elements a_ξ ($\xi < \alpha$) already defined:

$$a_\alpha \in R_\alpha \left(T_1 - \bigcup_{\xi < \alpha} \left[a_\xi, \cdot \right)_{T_1} \right) \quad .$$

[83] Being a distinctive sequence, its height is not reached.

One year later Kurepa became aware of the problems related to this argument: this procedure defines a disjoint descent only if for every $\alpha < \omega_1$ the α-level $R_\alpha\big(T_1 - \bigcup_{\xi<\alpha} [a_\xi, \cdot)_{T_1}\big)$ is not empty.

So in *Ensembles Ordonnés et Ramifiés* the only possible conclusion was that the ramified table T contained a distinctive sequence S of height ω_1, so that if S is normal then T_1 and T are normal; but if T is abnormal, the distinctive sequence S is abnormal. This is formulated through the following theorem *EOR.5*,[84] which became a meaningful device that allowed Kurepa to prove the *equivalence* between Souslin's hypothesis and the property that every ramified table of power $\aleph_1$ is normal (the equivalence $P_2 \longleftrightarrow P_5$ of the theorem *EOR.7* hereafter).

Theorem ***(EOR.5).*** *If T is an abnormal table of height ω_1 it contains an abnormal distinctive sequence S of height ω_1.*

As we have already seen, the hypothesis that every ramified table of power $\aleph_1$ is normal implies Souslin's hypothesis:

> D'après les résultats des derniers paragraphes, *l'hypothèse que tout tableau ramifié de puissance $\aleph_1$ est normal* (ou ce qui revient au même, que tout suite distinguée de rang ω_1 admet une descente disjonctive) *entraîne la réponse affirmative au problème de Souslin*.[85]

As a consequence of this last statement, from the negation of Souslin's hypothesis (the possible existence of a continuous ordered set E such that $p_2E = \aleph_0$, but $p_1E = \aleph_1$) the existence of an abnormal table of height ω_1, and so the existence of an abnormal sequence of power $\aleph_1$, could be obtained.

The proof of the converse relation reduced to showing that if an abnormal ramified table T of power $\aleph_1$ exists, then there is an ordered set E such that $p_2E = \aleph_0$ but $p_1E = \aleph_1$. As we have claimed, for this converse relation theorem *EOR.5* played a crucial role.

First, given a ramified table T it is necessary to define an ordered set E; for this a complete order relation, which is a generalization of the partial order relation (the "ramified relation"), is introduced for the set T: if a and b are two comparable points of T, they keep the same order as in the partial order relation, but if $a \not\sim b$, there is an ordinal number $\alpha < \gamma T$, and two points $a', b' \in R_\alpha T$, such that

i) $(\cdot, a') = (\cdot, b')$ (a' and b' belong to the same node),
ii) $a' \in (\cdot, a)$ but $a' \notin (\cdot, b)$,
iii) $b' \in (\cdot, b)$ but $b' \notin (\cdot, a)$.

By defining a linear order for each node of T, it follows that the order relation $a < b$ can be defined whenever $a' < b'$ according to the linear order given to the node containing a' and b'. Clearly this linear order defined for the ramified table T depends

[84] Kurepa 1935, p. 109.
[85] "According to the results given in the last paragraphs, *the hypothesis that every ramified table of power $\aleph_1$ is normal* (or, equivalently, that every distinguished sequence of height ω_1 admits a disjoint descent) *implies the positive answer to Souslin's problem*". *Ibid*., p. 124.

on the linear order defined on each node. For the particular case of a distinctive sequence S of height $\gamma S = \omega_1$, each node of S has the cardinal number $\aleph_0$; $S(\omega)$ is the ordered set obtained from S when each node has the order type ω and, in general, $S(\tau)$ is the ordered set obtained when each node has the countable order type τ.

For a distinctive sequence S of height $\gamma S = \omega_1$, Kurepa stated four important properties:

(I) Any ordered set oS defined from the distinctive sequence S has the property that $p_1[oS] = \aleph_1$. So $p_1[S(\omega)] = p_1\left[S(\omega^*)\right] = p_1\left[S(1+\omega^*)\right] = \aleph_1$.

Another important property, which can be deduced from the obvious fact that the set $S(\omega^*)$ is densely ordered, is that S is normal whenever $p_2\left[S(\omega^*)\right] = |S|$:

(II) If $p_2\left[S(\omega^*)\right] = \aleph_1$, there exists a disjoint subset R, contained in the distinctive sequence S, such that $|R| = |S| = \aleph_1$.

But another important consequence, which is also obtained from the fact that $S(\omega^*)$ is densely ordered, is that if $p_2\left[S(\omega^*)\right] = \aleph_1$, this cardinal number is the same for any order relation given to the nodes of S:

(III) If $p_2\left[S(\omega^*)\right] = \aleph_1$, then $p_2[oS] = \aleph_1$ for any linear order oS.

Finally, Kurepa proved the converse of II:

(IV) If S is a normal ramified sequence, then $p_2\left[S(\omega^*)\right] = \aleph_1$.

From these statements Kurepa easily concluded the following theorem:[86]

Theorem ***(EOR.6).*** *In order that a distinctive ramified sequence S be normal, it is necessary and sufficient that for every natural order of S, $p_1[oS] = p_2[oS]$.*

Kurepa's proof is now complete: if there exists an abnormal ramified table T of power $\aleph_1$, then there exists an *abnormal distinctive sequence* $S \subset T$. According to property I, $p_1[oS] = \aleph_1$, but according to theorem *EOR.6*, $p_2[oS] < \aleph_1$.

So, between the first question about the decomposition of a continuous ordered set E, and the construction of a ramified table of sets $\mathfrak{T}$ obtained through a complete development for E, he found several conditions that allowed him to give an equivalent formulation of Souslin's hypothesis. As a conclusion to his whole research, Kurepa stated the following theorem, which includes the equivalence between several conditions, each of which could give a positive answer to Souslin's problem:[87]

Theorem ***(EOR.7).*** *The following propositions are all equivalent:*

P_1 *For any ramified table T the cardinal number bT is always reached. (Ramification Hypothesis) (Hypothèse de Ramification).*

P_2 *Every infinite ramified table has the same cardinal number as one of its degenerate subsets (Reduction Principle) (every ramified table is normal).*

86 *Ibid.*, p. 129.
87 *Ibid.*, pp. 130–132.

P_3 *For any infinite ordered set E there is a family of disjoint and non empty intervals with the cardinal number p_1E.*
P_4 *If S is a distinctive ramified sequence, any ordered set oS defined from it is a normal linearly ordered set (i.e. $p_1(oS) = p_2(oS)$).*
P_5 *If S is a distinctive ramified sequence, the degree of cellularity for any ordered set oS is always the same.*
P_6 *Every distinctive ramified sequence S has the same cardinal number as one of its disjoint subsets.*
P_7 *Every ramified sequence admits a disjoint descent.*
P_8 *If S is a ramified sequence, there exists a subtable T of S such that $|T| = |S|$ and T contains no distinctive sequence of height γS.*
P_9 *If T is a ramified table and γT is an initial regular ordinal, and if for every disjoint subtable F of T its power $|F|$ is smaller than $|\gamma T|$, then T accepts a monotone descent.*

A few months after this achievement in *Ensembles Ordonnés et Ramifiés*, Kurepa sent another note to the *Académie des Sciences*[88] in which he clearly established his contribution to the solution of Souslin's problem: he enunciated a hypothesis which, in case it were true, would imply the truth of Souslin's hypothesis. This is his *Ramification Hypothesis*:

> Quel que soit le tableau ramifié T, la borne supérieure bT est atteinte dans T, c'est-à-dire qu'il existe un sous-tableau dégénéré de T ayant la puissance bT.[89]

Kurepa remarked that this hypothesis is equivalent to the proposition stating that any infinite table has the same power as one of its degenerate subtables. An immediate consequence of this *ramification hypothesis* is that

> Tout tableau infini non dénombrable contient un sous-tableau infini dégénéré non dénombrable, *proposition que nous ne savons ni prouver ni réfuter*, et qui est équivalente a l'hypothèse que la réponse au problème bien connu de Souslin est affirmative.[90]

After the initial optimism of his 1934 notes, Kurepa had to accept that he was not able to prove Souslin's hypothesis, because he could not prove that every Aronszajn table admits a disjoint descent. Nevertheless, with his research on the subject he introduced an equivalent proposition which gave the framework within which all further studies on Souslin's problem were developed.

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[88] Presented by Emile Borel in January 20, 1936. Kurepa 1936.

[89] "For any ramified table T, the upper bound bT is reached within T, which means that there exists a degenerate subtable of T having the power bT". *Ibid.*,

[90] "Every non-countable infinite table contains a non-countable infinite degenerate subtable, *a proposition which we can neither prove nor refute*, and which is equivalent to the hypothesis that the answer to the well-known Souslin's problem is affirmative". *Ibid.*, (Italics ours).

4. Equivalent statements

4.1. The rediscovery of an equivalence

Certainly the two most famous papers related to Souslin's problem are E. Miller's paper "A note on Souslin's Problem", published in 1943,[91] and Sierpiński's paper "Sur un Problème de la Théorie des Ensembles Equivalent au Problème de Souslin", published in 1948 in the Polish journal FUNDAMENTA MATHEMATICAE.[92] These two articles are clear examples of how this problem, which in its original form dealt with the theory of order types, was "translated" into a problem about partially ordered sets.

Even if these two papers closely resemble Kurepa's work, no reference is made by Miller or Sierpiński to any of his notes or articles. It is not clear if their authors even knew any of Kurepa's texts. At the beginning of his paper of 1948, Sierpiński refers to the book of A. Denjoy *L'Enumération Transfinie*[93] as a source of extensive information related to Souslin's problem. In the bibliographical notes, Denjoy includes all of Kurepa's papers published by that time; so Sierpiński knew, at least from a bibliographical reference, about the existence of Kurepa's works related to Souslin's problem.

The papers of Miller and Sierpiński are certainly better known than Kurepa's. Miller is frequently considered as having been the first one to prove that a necessary and sufficient condition for the existence of a continuous ordered set L, such that $p_1 L = \aleph_1$ and $p_2 L = \aleph_0$, is the existence of an *abnormal* ramified table $\mathfrak{L}$ of power $\aleph_1$ or, as the standard mathematical terminology states, that a *Souslin line* exists if and only if there exists a *Souslin tree*.[94]

We must state that even if we can analyze these two papers simultaneously since they refer to the same problem and we claim that their approach is completely equivalent, we must keep in mind that the authors' respective backgrounds, and certainly their interest in Souslin's problem, arose from different sources.

4.1.1. Miller's theorem: Souslin lines and Souslin trees. Two years before he wrote his paper on Souslin's problem, Miller had published with B. Dushnik a short but profound study on some important properties of partially ordered sets.[95] The two main problems analyzed in that paper are:

1. To find the powers of the linear subsets of a partially ordered set P: *i.e.* to answer how large a linear subset $F \subset P$ can be.

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[91] Miller 1943.

[92] Sierpinski 1948. This is the only article on Souslin's problem that was ever published by the journal that had published Souslin's problem.

[93] Denjoy 1946.

[94] This is the case in two important books on set theory written by Devlin (Devlin 1974) and Kuratowski (Kuratowski 1976).

[95] Miller, Dushnik 1941.

2. The possibility of having a *representation* for a partially ordered set P; *i.e*. to find a family of sets $\mathscr{P}$, partially ordered by inclusion,[96] such that P and $\mathscr{P}$ are similar.[97]

Concerning the first problem, Miller and Dushnik were interested in the relation between the size of the monotone subsets of a partially ordered set P, and the size of its disjoint subsets. First, they proved that when a partially ordered set P of power $\aleph_1$ admits no disjoint subsets of power $\aleph_1$, then almost all of its elements are comparable with $\aleph_1$ elements. A second important fact – a generalization of theorem *EOR.4* for partially ordered sets – gives a clearer relation between monotone and disjoint subsets: if a partially ordered set P of power $\aleph_1$ admits only finite disjoint subsets, then there exists a non-countable monotone (linear) subset F of P.

These facts were stated through the following theorems[98]:

Theorem ***(M-D.1).*** *If P is a partially ordered set of power $\aleph_1$ and if every subset $G \subset P$ of power $\aleph_1$ contains two comparable elements, then there exists an element $x \in P$ which is comparable to $\aleph_1$ elements of P.*

Theorem ***(M-D.2).*** *If P is a partially ordered set of power $\aleph_1$ and if every subset $G \subset P$ of power $\aleph_0$ contains two comparable elements, then there exists a linear subset $F \subset P$ of power $\aleph_1$.*

For the second problem, the representation of a partially ordered set P, Miller and Dushnik showed first that a *canonical* representation always exists for P: by taking for every $a \in P$ the subset $\mathscr{X}_a = \{x \in P; x \leqq a\}$, the family $\mathscr{P} = \{\mathscr{X}_a\}_{a \in P}$, ordered by inclusion, is similar to P: $a < b$ in P if and only if $\mathscr{X}_a < \mathscr{X}_b$ in $\mathscr{P}$.

The question that arises immediately is related to the possible existence of another representation for P, besides this *canonical* representation. A family $\mathscr{U}$ of intervals defined on a linearly ordered set is an example of a partially ordered set which is represented by the same set $\mathscr{U}$; but Miller and Dushnik tried to find the answer for the converse problem: under which conditions a partially ordered set P could be represented by a family of intervals defined on a linearly ordered set. They found that such a representation is possible whenever the (partial) order relation can be *inverted* in such a way that the disjoint subsets become monotone subsets and the monotone subsets become disjoint:[99]

Theorem ***(M-D.3).*** *A necessary and sufficient condition for a partially ordered set P to have a representation by means of a family $\mathscr{F}$ of intervals on some linearly ordered set L is that there can be defined on P another (partial) order relation, defining a partially ordered set Q, such that $x \sim y$ in P if and only if $x \not\sim y$ in Q.*

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[96] This means that for $\mathscr{X}, \mathscr{Y} \in \mathscr{P}$, the order $\mathscr{X} < \mathscr{Y}$ is defined whenever $\mathscr{X} \subseteq \mathscr{Y}$. So if $\mathscr{X} \not\subseteq \mathscr{Y}$ and $\mathscr{Y} \not\subseteq \mathscr{X}$, then $\mathscr{X} \not\sim \mathscr{Y}$.

[97] This means that there exists a one to one function from P onto $\mathscr{P}$ preserving the order relation.

[98] Miller, Dushnik 1941, pp. 606–608. For theorems *MD.1–MD.4* we have modified slightly the terminology employed by the authors in their paper.

[99] *Ibid.*, p. 602.

It is important to look closely at the proof provided by Miller and Dushnik, because it can give us some insight about why, two years later, Miller turned to Souslin's problem, and how he found the equivalence between *Souslin trees* and *Souslin lines*.

In order to prove that the existence of a representation for a partially ordered set P is sufficient to define on P another partial order relation satisfying the desired condition, let a representation for P be given through a family $\mathscr{F} = \{I_x\}_{x \in P}$ of intervals on a linear set L, such that for $x \in P$, $I_x \subset L$ is the corresponding interval. If x and y are two non-comparable elements of P, then none of their corresponding intervals contains the other, so for I_x and I_y their left-hand extreme points must be different – since if these extreme points coincide then one of these intervals contains the other or else they are equal. The partially ordered set Q is defined as follows: if $x \not\sim y$ in P, then the relation $x <_Q y$ holds (and so $x \sim y$ in Q) if the extreme left point of I_x precedes in L the extreme left point of I_y. Under this definition P and Q satisfy the required condition.

Conversely, let P and Q exist as (partial) order relations on the same set S satisfying the condition that for any two elements $x, y \in S$ one, and only one, of these two conditions holds: $x \sim_Q y$ or $x \sim_P y$. Let S_P and S_Q denote the partially ordered set S with the order relations P or Q.[100] Two linear order relations can be defined on S:

1. A relation $\sim_A$: for any two elements $x, y \in S$, $x \sim_A y$ if $x \sim_P y$ or $x \sim_Q y$; in this case it is said that $A = P + Q$.
2. A relation $\sim_B$: for any two elements $x, y \in S$, $x \sim_B y$ if $x \sim_P y$ or $x \sim_{Q^*} y$; in this case it is said that $B = P + Q^*$, (where Q^* is the inverse relation of Q: $x <_Q y$ if and only if $x >_{Q*} y$).

Let again S_A and S_B be two such linearly ordered sets, and let L be a linearly ordered set whose order type is the same as that of S_{B^*}, and such that $L \cap S = \emptyset$. If $C = L \cup S_A$ is defined in such a way that (L, S_A) is a *cut* in C, and if S is given the order relation B^*, then for each $x \in S$, $x \in S_{B^*}$ and $x' \in L$ is the corresponding element of L. Since at the same time it is possible to consider that every $x \in S$ is an element of S_A, for $x \in S$ let $I_x = [x', x] \subset C$. If for $x, y \in S$, $x <_P y$, then clearly $x <_A y$ in S_A and $y' <_L x'$ so that $y' < x' < x < y$ in C and $I_x \subset I_y$. On the other hand, if $x, y \in S$ are such that $x \not\sim_P y$ then $x \sim_Q y$, let us suppose that $x <_Q y$; in this case $x <_A y$ in S_A and $x' <_L y'$ in L and so $x' < y' < x < y$ in C and so I_x does not contain I_y nor is it contained in it ($I_x \not\sim I_y$).

From this proof, it is clear that for the family $\mathscr{F} = \{I_x\}_{x \in P}$ of intervals on a linearly ordered set C which represent the partially ordered set P, the non-comparative relation between two intervals reduces to the fact that none of them contains the other, but not necessarily to the fact that they are disjoint.

Besides theorem *M-D.3*, Miller and Dushnik showed that for any set of power $\aleph_1$ a partial order relation can be defined on it, so that it can be represented by a family of intervals defined on a linearly ordered set. But they also showed that any set of power $\aleph_1$ could be represented by a family of intervals with no uncountable monotone or disjoint families: if N is any set of power $\aleph_1$ and $\Omega = sg(\omega_1) = \{\alpha; \alpha$ is an ordinal number $< \omega_1\}$, two (one to one) functions can be defined:

[100] Two elements x and y are comparable in S_P if and only if they are incomparable in S_Q.

1. $f : N \to \mathbf{R}$.
2. $g : N \to \Omega$.

A partial order can be defined on N: for $x, y \in N$, $x < y$ if $f(x) < f(y)$ and $g(x) < g(y)$ ($f(x)$ and $f(y)$ are always comparable in $\mathbf{R}$, and $g(x)$ and $g(y)$ are comparable in Ω, so if $f(x) < f(y)$ and $g(x) > g(y)$ then $x \not\sim y$ in N). For a subset $M \subset N$ of power $\aleph_1$, the subset $f(M)$ has power $\aleph_1$ and a point $x_0 \in M$ exists such that $f(x_0)$ is a *condensation point* for $f(M)$; so there are two points $y, z \in M$ such that $f(y) < f(x_0) < f(z)$, and also such that $g(x_0) < g(y)$ and $g(x_0) < g(z)$. This makes $x_0 < z$ and $x_0 \not\sim y$ in N. So for any set N of power $\aleph_1$ it is always possible to define a partial order relation where every subset $M \subset N$ of power $\aleph_1$ contains two comparable elements and two non-comparable elements.

With this partial order relation defined on N it is easy to see that it satisfies the condition of reversibility of theorem *M-D.3*, so N can be represented by a family $\mathscr{F}$ of intervals defined on a linearly ordered set having the following properties:

1. Every non-denumerable subfamily $\mathscr{F}' \subset \mathscr{F}$ contains two comparable intervals.
2. Every monotone subfamily of $\mathscr{F}$ is at most countable.

$\mathscr{F}$ is a non-denumerable family of sets with the property that every subfamily of disjoint sets is at most countable and also every subfamily of nested sets is at most countable; but for this family the non-comparability of its sets does not correspond to the fact that they are disjoint.

Let us call G this last condition:

(G) A family of sets $\mathscr{F}$ is said to satisfy condition G if any two of its elements are either disjoint or one is a subset of the other.

The next step was to find the conditions for a partially ordered set P so that the family of sets representing it satisfies condition G. A partial answer was given for the family $\mathscr{P} = \{\mathscr{X}_a\}_{a \in P}$, the "canonical representation": if the partial order in P does not "split", then the non-comparable relation in $\mathscr{P}$ corresponds to the fact that the subsets are disjoint[101]:

Theorem ***(M-D.4).*** *If the partially ordered set P is such that, for any three elements a, b and c, if $a < b$ and $a < c$ then $b \sim c$, then for the canonical representation $\mathscr{P}$ any two of its sets are disjoint or else one is contained in the other.*

For two non-comparable elements of P, $x \not\sim y$, if $\mathscr{X}_x \cap \mathscr{X}_y \neq \emptyset$, then for $z \in \mathscr{X}_x \cap \mathscr{X}_y$, $z < x$ and $z < y$, but, according to the hypothesis, it should be concluded that $x \sim y$, contrary to the assumption that $x \not\sim y$.

A completely equivalent theorem could be obtained by considering, for the partially ordered set P, the family of sets $\mathscr{R} = \{\mathscr{Y}_a\}_{a \in P}$ where each $\mathscr{Y}_a = \{x \in P; x \geqq a\}$, and defining the (partial) order relation in $\mathscr{R}$ by stating that $\mathscr{Y}_a < \mathscr{Y}_b$ if $\mathscr{Y}_b \subset \mathscr{Y}_a$. In this case, for the family $\mathscr{R}$, representing the partially ordered set P, the non-comparable relation between two sets is equivalent to the fact that they are disjoint whenever the

101 *Ibid.*, p. 604.

following condition holds: if x and y are two non-comparable elements of P, then there exists no $z \in P$ such that $x < z$ and $y < z$. In this case, the condition for the order relation in P so that the family $\mathscr{R}$ satisfies condition G is that it would not "converge".

The problem was then to state the conditions that a partially ordered set P should satisfy so that it could be represented through a family of intervals defined on a linearly ordered set, and satisfying condition G. As we noticed, Miller and Dushnik had already proved that a set N of power $\aleph_1$ could be considered as a partially ordered set satisfying the (double) condition that it accepts no uncountable chain and no uncountable anti-chain, and that it could be represented through a family $\mathscr{F}$ of intervals defined on a linearly ordered set L. If the family $\mathscr{F}$ satisfies condition G then the set L should admit at least a countable family of non-overlapping intervals.

Two years later Miller found that a representation is possible for a partially ordered set P not accepting any uncountable disjoint or monotone subsets, through a family of intervals of a linearly ordered set that satisfies condition G, if P satisfies the "non-splitting" condition stated in theorem *M-D.4* (or the "non-convergent" condition). Two facts were immediately remarked by Miller: the first one is that the linearly ordered set L on which the family of intervals is taken should satisfy the countable chain condition. The second one is that after the conditions stated for the partially ordered set P (not accepting any uncountable disjoint or monotone subsets), only two possibilities seemed acceptable for its power (and so for the power of the family of intervals): this power could be $\aleph_1$ or $\aleph_0$. Miller remarked immediately that the most interesting case is that of an uncountable partially ordered set P; the problem for him, as it was for Kurepa, was to find out if such conditions for a partially ordered set P of power $\aleph_1$ could exist.

Miller's paper of 1943 showed that the conditions which he found for a partially ordered set P of power $\aleph_1$ to obtain the desired representation are equivalent to the existence of a continuous linear ordered set L with no first and no last element, which satisfies the countable chain condition, but is not separable (a *Souslin line*). The core of his paper was the following theorem:[102]

Theorem ***(M.1).*** *In order that there exist a linear order L which possesses the properties:*

1. *It has no first and no last element,*
2. *It is continuous,*
3. *Any set of non-overlapping intervals on L is at most countable, but which does not satisfy the following condition:*
4. *There exists a denumerable subset D of L such that between any two elements of L there is an element of D,*

   it is necessary and sufficient that there exist a partially ordered set P of power $\aleph_1$ such that

   - *m.i) if $Q \subset P$ and $|Q| = \aleph_1$, then Q contains two comparable elements and two non-comparable elements,*
   - *m.ii) if x and y are non-comparable elements of P, then there exists no $z \in P$ such that $x < z$ and $y < z$.*

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[102] Miller 1943, p. 673.

In some sense we can say that the course followed by Miller is the opposite of the one followed by Kurepa. There are two reasons for saying this:

1. Kurepa started studying the development of a continuous ordered set; this problem led him to study a family of intervals of the set, obtained from the (complete) development of the set, which constituted a partially ordered family of sets satisfying condition *G*. In contrast, Miller tried to find the conditions that a partially ordered set *P* should satisfy in order to be represented by a family of intervals of a linearly ordered set satisfying the same condition *G*.
2. Kurepa started his study on the development for a continuous set, and so the study of partially ordered sets, by trying to give an answer to Souslin's problem. Miller found that an answer to his problem of the representation for a partially ordered set *P* of power $\aleph_1$ could be obtained from a negative answer to Souslin's problem.

4.1.2. Sierpiński's search for an equivalent statement to a non-proved hypothesis. We have already said that in his book published in 1928 W. Sierpiński was the first ever to remark on the importance and the difficulty of Souslin's problem, but he only published one paper related to this problem which appeared in the journal FUNDAMENTA MATHEMATICAE in 1948. Sierpiński knew Souslin personally and he was present when Souslin talked with Luzin about his discovery of the "error" in Lebesgue's memoir.[103] Sierpiński was captivated by Souslin's discovery, which, as we said in the Introduction, gave rise to his important contribution in descriptive set theory, *i.e.* the class of *analytic sets*, and he published several papers on this topic.[104]

It seems to us that Sierpiński's paper on Souslin's problem belongs to the group of inquiries concerning the quest for equivalent conditions to some unsolved set-theoretic problems. After the first issue of FUNDAMENTA MATHEMATICAE, Sierpiński embarked on this kind of research by proving equivalences or consequences of the continuum hypothesis; the theorems that establish equivalent statements to the continuum hypothesis form the kernel of the first chapter of his book *L'Hypothèse du Continu* of 1938.

A remarkable fact is that in 1928 Sierpiński identified Souslin's problem as a problem related to the theory of ordered sets and order types; but twenty years later, he gave an equivalent statement to Souslin's problem in the general theory of sets.

Sierpiński posed the following *problem P* of set theory:
Let $\mathscr{F}$ be an infinite family of sets having the following properties:

s.i) Any two sets belonging to $\mathscr{F}$ are disjoint or one is a subset of the other,
s.ii) Every subfamily of $\mathscr{F}$ of disjoint sets is at most countable,
s.iii) Every subfamily of nested sets is at most countable and has a maximal element.
Under these conditions, is the family $\mathscr{F}$ necessarily countable?

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103 This is related by K. Kuratowski in Kuratowski 1980, p. 68.
104 Some of these paper are: "Sur quelques propriétés des ensembles (*A*) (with Luzin) (1918). "Sur un ensemble non mesurable *B*" (with Luzin) (1923). "Sur une propriété des ensembles (*A*)" (1926). "Sur une propriété caractéristique des ensembles analytiques" (1927). "Sur la puissance des ensembles analytiques" (1930). "Le théorème de Souslin dans la théorie générale des ensembles" (1935).

Souslin's problem (*problem S*) is stated by Sierpiński in the following terms:

> Un ensemble L ordonné linéairement, dense et tel que toute famille d'intervalles de L n'empiétant pas les uns sur les autres est au plus dénombrable, contient-il nécessairement un sous-ensemble au plus dénombrable dense dans L?[105]

Sierpiński stated the following theorem:[106]

Theorem ***(S.1).*** *A positive answer to problem P is equivalent to a positive answer to problem S.*

It is clear that the family of sets $\mathscr{F}$ is partially ordered and that it indeed forms a ramified table of sets. Conditions (*s.ii*) and (*s.iii*) state that every monotone or disjoint subfamily is at most countable, so the question of whether $|\mathscr{F}| = \aleph_0$ becomes completely equivalent to Kurepa's formulation: a positive answer to Souslin's problem can be obtained if (and only if) under the above conditions $|\mathscr{F}| = \aleph_0$; *i.e.* if no *abnormal* ramified table of sets of power $\aleph_1$ exists.

Miller's formulation for his theorem seems more general than Sierpiński's since it refers to a partial ordered set P of power $\aleph_1$ and not to the special case of a ramified table of sets. But in any case, they both proved the existence of a ramified table (of sets) P of power $\aleph_1$ having no uncountable monotone or disjoint subsets (a *Souslin tree*), under the assumption that a continuously ordered set L exists, which satisfies the countable chain condition but which is not separable (a *Souslin line*).

We follow here the construction of a *Souslin tree* given by Miller; it is equivalent, except for minor details, to Sierpiński's construction. Proceeding by induction Miller defined:

1. I_1 is any (open) interval of a *Souslin line* L.
2. For $\alpha < \omega_1$ it is assumed that an interval I_β has been defined for every $\beta < \alpha$, in such a way that I_β contains no extreme point of any of the intervals $I_\gamma, \gamma < \beta$. Since the extreme points of all these intervals $\left\{I_\beta\right\}_{\beta<\alpha}$ form a countable subset of L, there exists an interval I which contains no point of this subset; I_α is any interval properly contained in I.

By construction it is clear that whenever $\beta < \alpha$, then $I_\alpha \subset I_\beta$ or $I_\alpha \cap I_\beta = \emptyset$. Let $\mathfrak{L} = \{I_\alpha; \alpha < \omega_1\}$, and for this set the following partial order is defined, $I_\beta < I_\alpha$ if $I_\alpha \subset I_\beta$, and $I_\alpha \not\sim I_\beta$ if $I_\alpha \cap I_\beta = \emptyset$; clearly $|\mathfrak{L}| = \aleph_1$. Now if $\mathfrak{Q}$ is an uncountable subset of $\mathfrak{L}$, then $\mathfrak{Q}$ is not a disjoint subset (formed only by incomparable elements) since, in this case, $\mathfrak{Q}$ would be an uncountable collection of non-overlapping intervals of L, contrary to the assumption that L satisfies the countable chain condition. But the same subset $\mathfrak{Q}$ is not a monotone subset, since if it were, it would be an uncountable increasing sequence $\{I_\alpha\}_{\alpha<\omega_1} \subset \mathfrak{L}$. If x_α is the extreme left point of the interval I_α, then the collection of intervals $\{(x_\alpha, x_{\alpha+1})\}_{\alpha<\omega_1}$ is an uncountable and non-overlapping

[105] "Does a linearly ordered set L, which is dense and is such that any family of non-overlapping intervals of L is at most countable, necessarily contain a countable dense subset in L?" Sierpinski 1848, p.165.

[106] *Ibid.*, p. 165.

family of intervals; again contrary to the assumption that the countable chain condition is satisfied.

With this construction Miller proved not only the first part of theorem *M.1, i.e.* that if a *Souslin line L* exists, then a partially ordered set $P = \mathfrak{L}$ satisfying conditions (*m.i*)–(*m.ii*) exists; he also proved that for the ordered set L the family of intervals $\mathfrak{L}$ satisfies condition *G* and is a representation of the partially ordered set P. Sierpiński proved with an equivalent construction that a negative answer to *problem S* gives a negative answer to *problem P*: the family of sets $\mathfrak{L}$ satisfies conditions (*s.i*)–(*s.iii*) but is an uncountable family of sets.

We must point out that according to this construction of Miller and Sierpiński, the fact that L satisfies the countable chain condition implies that no uncountable disjoint or monotone subset of $\mathfrak{L}$ can exist; and it is the assumption that L is not separable which makes the *height* of $\mathfrak{L}$ equal to ω_1. Kurepa's approach is now clearly stated: the coexistence of the two conditions for a *Souslin line L*, satisfying the countable chain condition and not being separable, is possible if and only if for a *ramified table* (a tree) $\mathfrak{L}$ of *height* ω_1 it is possible that all of its monotone and disjoint subsets are at most countable.

Concerning the sufficiency proof, Miller and Sierpiński showed that given a partially ordered set P of power $\aleph_1$ satisfying the conditions (*m.i*) and (*m.ii*) of theorem *M.1*, or given a non-denumerable family of sets $\mathscr{F}$ satisfying conditions (*s.i*)–(*s.iii*) of theorem *S.1*, a *Souslin line L* can be defined. But this proof first requires the definition of a subset $A \subset P$ (and a completely equivalent argument makes it possible to prove the existence of a subfamily $\mathscr{B} \subset \mathscr{F}$), which, as we stated above, is a *distinctive sequence*, as defined by Kurepa.[107]

Although Miller gave in theorem *M.1* a weaker condition than the one Sierpiński gave for his theorem *S.1*,[108] with the help of theorems *M-D.1* and *M-D.2* he showed the desired equivalence. For a partially ordered set P of power $\aleph_1$ satisfying condition (*m.i*), theorem *M-D.1* implies that almost every element of P is comparable to $\aleph_1$ of its elements. From the two conditions (*m.i*) and (*m.ii*) it is possible to remove a countable subset from P to obtain a partially ordered subset $P' \subset P$ of power $\aleph_1$. P' satisfies the following conditions:

i) Conditions (*m.i*) and (*m.ii*),
ii) For every element $x \in P'$, at most $\aleph_0$ elements of P' are smaller than x,
iii) For every element $x \in P'$, $\aleph_1$ elements of P' are greater than x.

The subset $A \subset P$ is defined recursively in the following terms:

1. First a subset $A_1 \subset P'$ is defined, whose existence is guaranteed by Miller from theorem *M-D.2* and condition (*m.i*), which is a maximal and countable subset, all of whose elements are mutually incomparable. From this subset A_1 another countable subset Q_1 of P' is defined:

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107 The definition of these subsets given by Miller and Sierpinski shows that theorem *EOR.5* is a necessary previous lemma for the complete proof of theorems *M.1* and *S.1*.

108 This fact would be remarked by T. Jech in 1967. (Jech 1967).

$$Q_1 = \{x \in P'; x < y, y \in A_1\} \quad . \tag{4.I}$$

Q_1 is countable by virtue of the condition (*ii*).

2. If the subsets A_β and Q_β have already been defined for any ordinal number $\beta < \alpha$, ($\alpha < \omega_1$), in such a way that they satisfy the conditions:
   i) $|A_\beta| = \aleph_0$ and $|Q_\beta| \leqq \aleph_0$,
   ii) The elements of A_β are mutually incomparable and

$$A_\beta \subset P' - \bigcup_{\mu<\beta} (A_\mu + Q_\mu) \quad , \tag{4.II}$$

   besides this, A_β is maximal relative to

$$P' - \bigcup_{\mu<\beta} (A_\mu + Q_\mu) \quad .$$

   iii) Q_β is the subset of elements of P' that are smaller than some element of A_β:

$$Q_\beta = \{x \in P'; x < y, y \in A_\beta\} \quad . \tag{4.III}$$

   Then the subsets A_α and Q_α are defined:

2.1. For $\alpha = \beta + 1$, if $x \in A_\beta$ let

$$B_x = \left\{ y \in P' - \bigcup_{\mu<\beta} (A_\mu + Q_\mu) \, ; y > x \right\} \quad . \tag{4.IV}$$

It is clear that $|B_x| = \aleph_1$ and it is possible to define a subset $C_x \subset B_x$ which is a maximal denumerable subset whose elements are all mutually incomparable. Miller then defined

$$A_\alpha = \bigcup_{x \in A_\beta} C_x \tag{4.V}$$

and

$$Q_\alpha = \{x \in P'; x < y, y \in A_\alpha\} \quad . \tag{4.VI}$$

Conditions (*i*)–(*iii*) hold for A_α and Q_α.

2.2. For a limit ordinal α, the set A_α is defined as a maximal and denumerable subset of

$$P' - \bigcup_{\beta<\alpha} (A_\beta + Q_\beta) \quad ,$$

all of whose elements are mutually incomparable. The set Q_α is defined as in 4.VI.

The set

$$A = \bigcup_{\alpha<\omega_1} A_\alpha \tag{4.VII}$$

satisfies the following properties:

1. $|A| = \aleph_1$ and $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$,
2. If $x \in A_\alpha$ then x is smaller than $\aleph_0$ elements of $A_{\alpha+1}$,
3. If $x \in A_\alpha$ and $y \in A_\beta$ $(\alpha > \beta)$ then $x > y$ or $x \not> y$,
4. If $x \in A_\alpha$ there is only one element $y \in A_\beta$ $(\alpha > \beta)$, such that $x > y$. Indeed, by writing in this case $y = x_\beta$, a sequence $x_1 < x_2 < \cdots < x_\beta < \cdots < x$ is defined.

From the partially ordered set A a linear extension L' is defined by giving to each "node" of A, which is denumerable, the order type ζ of the set $\mathbf{Z}$ of integer numbers. For an element $x \in A$, if $x \in A_\alpha$, a sequence $x_1 < x_2 < \cdots < x_\beta < \cdots$ of all the elements of A which are smaller than x is obtained by property 4. This element x can be written in the form $x = x_{n_1 n_2 n_3 \ldots n_\alpha}$ to denote its "path" (each $n_i \in \mathbf{Z}$), it states that x_1 is the element that takes the place n_1 in A_1, x_2 takes the place n_2 in the node formed by the immediate successors of x_1 in A_2, and so on. For two elements $x, y \in A$ the order for L' is defined by stating $x <_{L'} y$ if $x <_A y$; but if $x \not> y$, with $x = x_{n_1 n_2 n_3 \ldots}$ and $y = y_{m_1 m_2 m_3 \ldots}$, $x <_{L'} y$ if the sequence $n_1 n_2 n_3 \ldots$ is smaller than the sequence $m_1 m_2 m_3 \ldots$ according to the lexicographic order. The set L' has no first and no last elements because of the order type ζ given to each node of A, and from this condition L' is densely ordered. No denumerable subset is order-dense in L' since for any denumerable subset D of A, there exists an ordinal $\alpha < \omega_1$ such that $D \subset \bigcup_{\beta<\alpha} A_\beta$. So it cannot be an order-dense subset, since the order-dense subsets of L' should have points at each level of A. Finally, any collection of non-overlapping intervals in L' is denumerable: for any set of non-overlapping intervals it is always possible to choose an element in each one of them in such a way that these elements are mutually incomparable in A. The set L is defined as the completion that fills the gaps of L'; in this way L is the *Souslin line* whose existence is proved from the assumed existence of a *Souslin tree*.

We have seen how Souslin's problem was first analyzed by Kurepa in the general context of continuous ordered sets. Thereafter, Kurepa defined a ramified table of sets from a complete partition of the continuous ordered set, and in this way he translated Souslin's problem into an equivalent question in the field of partial ordered sets. The conclusion is that Souslin's hypothesis claims the non-existence of an *abnormal* continuous set E such that $p_2 E = \aleph_0$ (a *Souslin line*), and is equivalent to the non-existence of an *abnormal* ramified table T of height ω_1 (a *Souslin tree*). So Kurepa, as did Miller and Sierpiński after him, translated the original question concerning the possibility of deducing the *separability condition* for a linearly ordered set from the fact that it satisfies the countable chain condition into a question concerning the possible existence of a particular linear ordered set and a particular partial ordered set. The approach to Souslin's hypothesis shifted from a question concerning the possibility of proving an implication to the question concerning the existence or non-existence of a *Souslin tree*.

4.2. Souslin's hypothesis and measure Boolean algebras

Besides the equivalence between *Souslin trees* and *Souslin lines*, some open questions related to the theory of Boolean algebras gave rise to another equivalent version of

Souslin's problem. In 1947 D. Maharam published a paper[109] where she studied the "purely algebraic conditions" required for a non-atomic Boolean σ-algebra $\mathfrak{A}$ so that it be a *measure algebra*. The question came from the work of M. Stone, particularly from his profound article on the representation of Boolean algebras,[110] and the theorems included in that paper concerning the fact that every Boolean algebra is isomorphic to a field of sets.[111] The question concerning the necessary and sufficient conditions for the existence of a *measure* in a field of sets became for Maharam equivalent to the question concerning the necessary and sufficient algebraic conditions for the existence of a real valued non-negative function, defined on a Boolean σ-algebra $\mathfrak{A}$, $\mu : \mathfrak{A} \rightarrow \mathbf{R}$ which is:

1. Countably additive: if $\{\mathfrak{a}_n\}$ is any countable sequence of pairwise disjoint elements of $\mathfrak{A}$, (two elements $\mathfrak{a}$ and $\mathfrak{b}$ are disjoint when $\mathfrak{a} \wedge \mathfrak{b} = \mathfrak{o}$) then

$$\mu\left(\bigvee_{n=0}^{\infty} \mathfrak{a}_n\right) = \sum_{n=0}^{\infty} \mu(\mathfrak{a}_n) \ ,$$

2. $\mu(\mathfrak{o}) = 0$, $\mu(\mathfrak{a}) > 0$ whenever $\mathfrak{a} \neq \mathfrak{o}$ and $\mu(\mathfrak{l}) = 1$,
3. If $\mathfrak{a} \leqq \mathfrak{b}$ then $\mu(\mathfrak{a}) \leqq \mu(\mathfrak{b})$ ($\mathfrak{a} \leqq \mathfrak{b}$ if $\mathfrak{a} - \mathfrak{b} = \mathfrak{o}$).

Two facts make this paper important for our study of Souslin's problem. The first one is that Maharam found that among the necessary and sufficient conditions for the existence of a real measure on $\mathfrak{A}$, one of them could be weakened only if Souslin's hypothesis is true. The second fact, which is a consequence of the first one, is that a new version of Souslin's problem could be stated in terms of Boolean algebras. This algebraic version became a very useful tool for the studies related to the independence of Souslin's hypothesis.[112]

4.2.1. A deduction of Souslin's hypothesis. First Maharam introduced the definition of a weaker measure on $\mathfrak{A}$, an *outer measure*, which satisfies conditions 2 and 3 stated above for the real valued and non-negative function μ defined on $\mathfrak{A}$, and also the following two conditions:

1'. $\mu(\mathfrak{x} \vee \mathfrak{y}) \leqq \mu(\mathfrak{x}) + \mu(\mathfrak{y})$,

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[109] Maharam 1947.
[110] Stone 1936.
[111] This means that any Boolean algebra $\mathfrak{A}$ can be identified with a collection $\mathscr{F}$ of subsets of a non-empty set S which satisfies:

i) $S \in \mathscr{F}$,
ii) if $X, Y \in \mathscr{F}$, then $X \cup Y \in \mathscr{F}$, $X \cap Y \in \mathscr{F}$, and $X - Y \in \mathscr{F}$.

So the algebraic operations defined on the Boolean algebra $\mathfrak{A}$: $\vee$ and $\wedge$, correspond to the union and intersection of sets. If $\mathfrak{A}$ is a Boolean σ-algebra, the field of sets $\mathscr{F}$ representing it is closed under countable unions and intersections.
[112] This is clear in Martin, Solovay 1970 and Solovay, Tennenbaum 1971.

4. If $\{\mathfrak{x}_n\}_{n=1}^{\infty}$ is an increasing sequence, $\mu(\mathfrak{x}_n) \;\to\; \mu\left(\bigvee_{n=1}^{\infty} \mathfrak{x}_n\right)$.

Moreover, the outer measure μ becomes a *continuous outer measure* if whenever $\{\mathfrak{x}_n\} \;\to\; \mathfrak{x}$, which means that

$$\limsup \{\mathfrak{x}_n +_2 \mathfrak{x}\} = \bigwedge_n \left[\bigvee_{m \geqq n} ((\mathfrak{x}_m - \mathfrak{x}) \vee (\mathfrak{x} - \mathfrak{x}_m)) \right] = \mathfrak{o} \ ,$$

then $\mu(\mathfrak{x}_n) \;\to\; \mu(\mathfrak{x})$.

Clearly a continuous outer measure is a measure if and only if condition (1') becomes an equality in the case $\mathfrak{x} \wedge \mathfrak{y} = \mathfrak{o}$.

The main theorem of the paper states that:[113]

Theorem ***(MA.1).*** *$\mathfrak{A}$ admits a continuous outer measure if and only if it satisfies the following two conditions:*

I. A distributive law: For any double sequence $\{\mathfrak{x}_{pn}\}$ which, for each fixed p, decreases monotonically to $\mathfrak{o}$ as $n \;\to\; \infty$, i.e.

$$\bigwedge_n \mathfrak{x}_{pn} = \mathfrak{o} \ ,$$

there exists a positive integer-valued function n(i, p) such that

$$\limsup \left\{\mathfrak{x}_{pn(i,p)}\right\} = \bigwedge_i \left[\bigvee_p \mathfrak{x}_{pn(i,p)} \right] = \mathfrak{o}. \tag{4.VIII}$$

II. There exists a countable family $\mathscr{F} = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \ldots\}$ of subsets of $\mathfrak{A}$, where each $\mathfrak{C}_i$ satisfies the condition that for every sequence $\{\mathfrak{x}_n\} \subset \mathfrak{C}_i$,

$$\limsup \{\mathfrak{x}_n\} = \bigwedge_n \left[\bigvee_{m \geqq n} (\mathfrak{x}_m) \right] \neq \mathfrak{o}$$

and such that if $\mathfrak{Y} \subset \mathfrak{A}$ is any other countable subset satisfying this same condition, then $\mathfrak{Y}$ is contained in one subset $\mathfrak{C}_n$ of the family $\mathscr{F}$.

If besides these two conditions the following condition III holds, then a continuous measure on $\mathfrak{A}$ becomes a measure.

*III. The sets $\mathfrak{C}_i$ of the family $\mathscr{F}$ can be chosen so as to satisfy the requirements that the set $\mathfrak{C}_i +_2 \left(\mathfrak{C}_{i+1}\right)' = \left\{\mathfrak{x}_i +_2 \mathfrak{y}_{i+1};\, \mathfrak{x}_i \in \mathfrak{C}_i, \mathfrak{y}_{i+1} \notin \mathfrak{C}_{i+1}\right\} \subset \mathfrak{C}_{i+1}$.
And also that if $\mathfrak{x}, \mathfrak{y} \in \mathfrak{C}_{i+1}$ and $\mathfrak{x} \wedge \mathfrak{y} = \mathfrak{o}$, then $\mathfrak{x} \vee \mathfrak{y} \in \mathfrak{C}_i$.*

A remarkable fact is that if a Boolean σ-algebra $\mathfrak{A}$ satisfies condition II, it also satisfies the "countable chain condition" for a Boolean σ-algebra:

[113] Maharam 1947, p. 159.

ccc $\mathfrak{A}$ satisfies the *ccc* condition if $|\mathfrak{X}| \leqq \aleph_0$ for any $\mathfrak{X} \subset \mathfrak{A}$ satisfying $\mathfrak{x} \wedge \mathfrak{y} = \mathfrak{o}$ whenever $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$ and $\mathfrak{x} \neq \mathfrak{y}$.

From theorem *MA.1* it appears that this *ccc* condition should be necessarily satisfied by $\mathfrak{A}$ when a continuous outer measure is defined on it. But in a way completely analogous to Souslin's when he formulated his problem, and closely related to this problem (as we will see), Maharam asked whether condition II could be replaced by the weaker *ccc* condition in order to prove, together with condition I, the existence of a *non-trivial*[114] outer measure on $\mathfrak{A}$. Maharam proved that this substitution is possible *only if* Souslin's hypothesis is true:[115]

Theorem ***(MA.2).*** *If conditions (I) and* ccc *always suffice for the existence of a non-trivial outer measure on a Boolean σ-algebra $\mathfrak{A}$, then Souslin's hypothesis is true.*

For this theorem Maharam described her version of Souslin's hypothesis: A family of sets $\mathscr{A}$ is a *Souslin system* if it satisfies the three conditions:

ma.i) If $a, b \in \mathscr{A}$, then $a \subset b$ or $b \subset a$ or $a \cap b = \emptyset$,
ma.ii) For $\mathscr{B} \subset \mathscr{A}$ such that if $a, b \in \mathscr{B}$ $(a \neq b)$, $a \cap b = \emptyset$, then $|\mathscr{B}| \leqq \aleph_0$,
ma.iii) For $\mathscr{B} \subset \mathscr{A}$ such that if $a, b \in \mathscr{B}$ $(a \neq b)$, $a \cap b \neq \emptyset$, then $|\mathscr{B}| \leqq \aleph_0$.

It is easy to see the close relation between these three conditions and the conditions (*s.i*)–(*s.iii*) that Sierpiński would give one year later in his theorem *S*.1. Of course Maharam could not make any reference to Sierpiński's work,[116] but it is remarkable that, earlier than Sierpiński's proof, she already identified Souslin's hypothesis with the assumption that every *Souslin system* $\mathscr{A}$ should be countable:

> It was conjectured by Souslin that every *Souslin system* is countable.[117]

With this version of the problem, the proof proceeds by assuming first that Souslin's hypothesis is false, and then proving from this assumption the impossibility of deriving the existence of an outer measure on a Boolean σ-algebra $\mathfrak{E}$, satisfying conditions I and *ccc*, defined from a *Souslin system* $\mathscr{A}$. The most important contribution in this sense is precisely the definition of the Boolean σ-algebra which satisfies condition *ccc*: first, if $\mathscr{A}$ is a non-countable *Souslin system* it is possible to conceive that $\mathscr{A} = \bigcup_{\alpha<\omega_1} \mathscr{A}_\alpha$, where each $\mathscr{A}_\alpha$ is a family of pairwise disjoint subsets (which is at most countable); thus $\mathscr{A}$ is a *Souslin tree*, whose elements are sets, and $\mathscr{A}_\alpha$ is its α-level. For each set

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114 A measure μ on $\mathfrak{A}$ is *non-trivial* if whenever $\mathfrak{x} \in \mathfrak{A} - \{\mathfrak{o}\}$, there exists an element $\mathfrak{y} < \mathfrak{x}$ such that $0 < \mu(\mathfrak{y}) < \mu(\mathfrak{x})$. Maharam stated that this condition cannot be omitted, since every Boolean algebra admits a "trivial" measure: by taking $\mu(\mathfrak{x}) = 1$ for every $\mathfrak{x} \neq 0$ and $\mu(\mathfrak{o}) = 0$.

115 *Ibid.*, p. 164.

116 In contrast, Sierpinski declared in a note at the end of his paper of 1948 that while he was reading and correcting the first printings of his text, he had had the opportunity to read Maharam's paper of 1947.

117 *Ibid.*, p. 164. Certainly this version of Souslin's hypothesis was obtained by her from Miller's paper of 1943, which is not quoted in this 1947 paper, but is quoted in her paper of 1948 (Maharam 1948).

$a_\alpha \in \mathscr{A}_\alpha$ two or more disjoint subsets $a_{\alpha+1} \subset a_\alpha$ belong to $\mathscr{A}_{\alpha+1}$.[118] It is also possible to assume that for each $a \in \mathscr{A}$ there are uncountably many subsets $b \in \mathscr{A}$ such that $b \subset a$. Maharam used three properties of this *Souslin system* $\mathscr{A}$:

1. if $\alpha > \beta$ each $a_\alpha \in \mathscr{A}_\alpha$ is contained in a unique $a_\beta \in \mathscr{A}_\beta$,
2. if $\alpha > \beta$ each a_β contains some a_α,
3. if $\alpha > \beta$ each a_β contains at least two disjoint sets a_α.

Now a *maximal decreasing* sequence $\mathscr{S} = \{a_1, a_2, \ldots, a_\nu, \ldots\} \subset \mathscr{A}$ (the fact that $\mathscr{S}$ is decreasing means that $a_1 \supset a_2 \supset \cdots \supset a_\nu \supset \cdots$) is called a *point*, and $\mathfrak{S}$ is the set of all these *points*; $\mathfrak{c}_{a_\alpha} = \{\mathscr{S} \in \mathfrak{S};\, a_\alpha \in \mathscr{S}\}$ is the set of all the *points* "passing through a_α", and $\mathfrak{C}_\alpha = \left\{\mathfrak{c}_{a_\alpha}\right\}_{a_\alpha \in \mathscr{A}_\alpha}$, for a fixed $\alpha < \omega_1$, is a set of subsets of $\mathfrak{S}$.

$$\mathfrak{C} = \bigcup_{\alpha<\omega_1} \mathfrak{C}_\alpha$$

is a new uncountable *Souslin system* defined from $\mathscr{A}$.

Without any reference to the published works of Kurepa or Miller, Maharam defined from the *Souslin system* $\mathscr{A}$ a set of *points* $\mathfrak{S}$ and the *Souslin system* $\mathfrak{C}$ as a partially ordered family of subsets of $\mathfrak{S}$; *i.e.* if $\mathfrak{x} \in \mathfrak{C}$ then $\mathfrak{x} \in \mathfrak{C}_\alpha$, for some $\alpha < \omega_1$, and so $\mathfrak{x} = \mathfrak{c}_{a_\alpha}$ is a subset of $\mathfrak{S}$ ($a_\alpha \in \mathscr{A}_\alpha$). Up to this point this procedure introduced just one innovation in relation to the previous constructions of a *Souslin tree*: from the tree $\mathscr{A}$ a new set satisfying the *ccc* condition, the set of "maximal branches" of $\mathscr{A}$, is taken.[119] But the real innovation appeared when an algebraic structure was introduced for a family of subsets of $\mathfrak{S}$. Clearly no algebraic structure (for the set operations of union and intersection) is satisfied by the *Souslin system* $\mathfrak{C}$, so Maharam defined first the set of *points* reaching the α-level:

$$\mathfrak{S}_\alpha = \bigcup_{a_\alpha \in \mathscr{A}_\alpha} \mathfrak{c}_{a_\alpha}$$

and its complement:

$$\mathfrak{N}_\alpha = \mathfrak{S} - \mathfrak{S}_\alpha \quad .$$

Then she defined two sets of subsets of $\mathfrak{S}$:

$$\mathfrak{N} = \{\mathfrak{n} \subset \mathfrak{S};\ \text{such that } \mathfrak{n} \subset \mathfrak{N}_\alpha \text{ for some } \alpha\}$$

and, for each $\alpha < \omega_1$, the set

$$\mathfrak{D}_\alpha = \left\{\mathfrak{x};\, \mathfrak{x} \text{ is any union of subsets } \mathfrak{c}_{a_\alpha} \text{ for a fixed } \alpha\right\},$$

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[118] If $a_{\alpha+1} \subset a_\alpha$ as sets of the family $\mathscr{A}$ then $a_\alpha < a_{\alpha+1}$ in the partial order relation of the tree $\mathscr{A}$.

[119] This procedure was already used when an ordered set of *complexes* was defined from a tree. The relation between a tree and the set of its maximal branches was studied in relation to some properties of ordered sets and continuous ordered sets by W. Sierpiński, J. Novák and M. Novotný in a series of papers published in FUNDAMENTA MATHEMATICAE between 1949 and 1952.

all of which give rise to

$$\mathfrak{D} = \bigcup_{\alpha<\omega_1} \mathfrak{D}_\alpha \quad .$$

For the sets $\mathfrak{D}$ and $\mathfrak{N}$ the symmetric difference is defined:

$$\mathfrak{D} +_2 \mathfrak{N} = \{(\mathfrak{d} - \mathfrak{n}) \cup (\mathfrak{n} - \mathfrak{d});\, \mathfrak{n} \in \mathfrak{N}, \mathfrak{d} \in \mathfrak{D}\} \quad .$$

The major properties of these sets are given through the following proposition:[120]

Lemma ***(MA.3).*** *The set $\mathfrak{D} +_2 \mathfrak{N}$ is a Boolean σ-algebra of sets and $\mathfrak{N}$ is a σ-ideal of $\mathfrak{D} +_2 \mathfrak{N}$.*

From this lemma the Boolean σ-algebra

$$\mathfrak{E} = (\mathfrak{D} +_2 \mathfrak{N})\,/\mathfrak{N} \tag{4.IX}$$

is defined (for $\mathfrak{x}, \mathfrak{y} \in \mathfrak{D} +_2 \mathfrak{N}$, $\mathfrak{x} \sim \mathfrak{y}$ whenever $(\mathfrak{x} - \mathfrak{y}) \cup (\mathfrak{y} - \mathfrak{x}) \in \mathfrak{N}$).

$\mathfrak{E}$ is a non-atomic[121] Boolean σ-algebra which satisfies the *ccc* condition as well as the distributive law (I).

The proof for theorem *MA.2* now proceeds as follows: The negation of Souslin's hypothesis implies the existence of a Boolean σ-algebra $\mathfrak{E}$, defined through 4.IX, satisfying the distributive law (I) and the *ccc* condition, but Maharam proved that $\mathfrak{E}$ cannot admit a non-trivial outer measure μ, because in this case the Souslin system $\mathfrak{C}$ would be *countable*.[122] Clearly $\mathfrak{C}$ is a subset of $\mathfrak{D}$ and if an outer measure μ is defined on $\mathfrak{E}$, then $\mu(\mathfrak{c}_{a_\alpha}) > 0$ for any element $\mathfrak{c}_{a_\alpha} \in \mathfrak{C}$.[123] For any rational number $q \in \mathbf{Q}$ consider the set $\mathfrak{C}_q = \left\{\mathfrak{c}_{a_\alpha} \in \mathfrak{C};\, \mu(\mathfrak{c}_{a_\alpha}) \leqq q\right\}$. Each element of $\mathfrak{C}_q$ is contained in a maximal element $\mathfrak{g} = \mathfrak{g}_{\mathfrak{d}_\beta} \in \mathfrak{C}$ $(\beta < \alpha)$ having the same property (this means that $\mathfrak{c}_{a_\alpha} \subset \mathfrak{g}_{\mathfrak{d}_\beta}$ and $\mathfrak{g}_{\mathfrak{d}_\beta} \in \mathfrak{C}_q$); let $\mathfrak{G}_q$ be the set of all these maximal elements, this subset is countable since all these $\mathfrak{g}_{\mathfrak{d}_\beta}$'s are pairwise disjoint and thus $\mathfrak{G} = \bigcup_{q\in\mathbf{Q}} \mathfrak{G}_q$ is also countable. Now since $\mu(\mathfrak{c}_{a_\alpha}) > 0$ there exists a rational number $q \in \mathbf{Q}$ such that $\mu(\mathfrak{c}_{a_\alpha}) > q$; there also exists a set $\mathfrak{d} \in \mathfrak{D}$ such that $0 < \mu(\mathfrak{d}) < q$ and $\mathfrak{d} \subset \mathfrak{c}_{a_\alpha}$ (μ is a non-trivial measure). For any $\mathfrak{c}_{a'_\gamma} \subset \mathfrak{d}$ $(\gamma > \alpha)$ clearly $\mathfrak{c}_{a'_\gamma} \subset \mathfrak{c}_{a_\alpha}$ and $\mu(\mathfrak{c}_{a'_\gamma}) < q$. The set $\mathfrak{c}_{a'_\gamma}$ is thus contained in a set $\mathfrak{g}_{\mathfrak{d}_\beta}$ and so $\mathfrak{c}_{a_\alpha} \cap \mathfrak{g}_{\mathfrak{d}_\beta} \neq \emptyset$. $\mathfrak{C}$ is a Souslin system, so $\mathfrak{g}_{\mathfrak{d}_\beta} \subset \mathfrak{c}_{a_\alpha}$ since $\mu(\mathfrak{g}_{\mathfrak{d}_\beta}) \leqq q < \mu(\mathfrak{c}_{a_\alpha})$. At most $\aleph_0$ sets $\mathfrak{c}_{a_\alpha}$ contain the set $\mathfrak{g}_{\mathfrak{d}_\beta}$, and there are at most $\aleph_0$ sets $\mathfrak{g} = \mathfrak{g}_{\mathfrak{d}_\beta}$; this proves that $\mathfrak{C}$ is countable.

A more simple way to define a non-atomic Boolean σ-algebra satisfying the *ccc* condition from a *Souslin system* of sets is gived by Solovay and Tennenbaum:[124] given the family $\mathscr{A}$ a topology is defined by defining, for any $a \in \mathscr{A}$, the set $O_a = \{b \in \mathscr{A};\, b \subseteq a\}$ and by taking the sets O_a as a basis for the open sets. Now by taking $\mathfrak{B}_1$ as the Boolean σ-algebra generated by the open sets of $\mathscr{A}$, $\mathfrak{I}_1 = \{\mathscr{D} \in$

120 *Ibid.*, p. 165.

121 This means that for any set $\mathfrak{c} \in \mathfrak{E}$ there exists another $\mathfrak{n} \in \mathfrak{E}$ such that $\mathfrak{n} \subset \mathfrak{c}$.

122 Maharam says in her paper that the idea for this proof was based on an observation made by Gödel and communicated orally.

123 For any $\mathfrak{d} \in \mathfrak{D}$, $\mu(\mathfrak{d})$ is defined as $\mu\left(\overline{\mathfrak{d}}\right)$, where $\overline{\mathfrak{d}}$ is the equivalent class of $\mathfrak{d}$ in $\mathfrak{E}$.

124 Solovay, Tennenbaum 1971.

$\mathfrak{B}_1$; $\mathscr{D}$ is a nowhere-dense subset of $\mathscr{A}\}$ is a σ-ideal and $\mathfrak{B} = \mathfrak{B}_1/\mathfrak{I}_1$ is the non-atomic Boolean σ-algebra with the *ccc* condition.[125]

4.2.2. An algebraic equivalence. In any case, from theorem *MA.2* it is possible to assert that a Boolean σ-algebra satisfying the distributive law (I) and the condition *ccc* exists whenever a non-countable *Souslin system* exists. A few months later Maharam tried to find out whether the converse condition also holds;[126] with this she gave a complete argument to prove a "purely algebraic property equivalent to Souslin's hypothesis":

Theorem ***(MA.4).*** *Souslin's hypothesis is true if and only if each non-atomic Boolean σ-algebra $\mathfrak{A}$ satisfying the countable chain condition contains a double sequence of elements $\{\mathfrak{k}_{ni}\}$ such that:*

$$\bigvee_i \mathfrak{k}_{ni} = \mathfrak{l} \tag{4.X}$$

and for every function $j(n)$,

$$\bigwedge_n \mathfrak{k}_{nj(n)} = \mathfrak{o} \ . \tag{4.XI}$$

To prove that Souslin's hypothesis is a necessary condition for the existence of a double sequence $\{\mathfrak{k}_{ni}\}$ contained in a non-atomic Boolean σ-algebra satisfying the countable chain condition, Maharam followed the same argument used in theorem *MA.2*. First, if the existence of an uncountable Souslin system $\mathscr{A}$ is assumed, then, as it was stated in 4.IX, a non-atomic Boolean σ-algebra $\mathfrak{E} = (\mathfrak{D} +_2 \mathfrak{N})/\mathfrak{N}$ satisfying the *ccc* condition can be defined such that it contains another uncountable Souslin system $\mathfrak{C} \subset \mathfrak{E}$. The new Souslin system $\mathfrak{C}$ has, besides the properties (*ma.i*)–(*ma.iii*), the following properties:[127]

ma.iv) $\mathfrak{C} = \bigcup_{\alpha<\omega_1} \mathfrak{C}_\alpha$,

ma.v) If $\alpha < \beta$, then for each $\mathfrak{c}_\beta \in \mathfrak{C}_\beta$ there exists an $\mathfrak{c}_\alpha \in \mathfrak{C}_\alpha$ such that $\mathfrak{c}_\beta < \mathfrak{c}_\alpha$,

ma.vi) If $\alpha < \beta$, $\mathfrak{c}_\alpha = \bigvee \left\{\mathfrak{c}_\beta; \mathfrak{c}_\beta < \mathfrak{c}_\alpha\right\}$,

ma.vii) For each $\mathfrak{x} \in \mathfrak{E}$, there exists an $\alpha < \omega_1$ such that

$$\mathfrak{x} = \bigvee \{\mathfrak{c}_\alpha; \mathfrak{c}_\alpha < \mathfrak{x}\} \ .$$

According to the hypothesis there exists in $\mathfrak{E}$ a double sequence $\{\mathfrak{k}_{ni}\}$ satisfying conditions 4.X and 4.XI. The proof for this part is obtained by showing again that $\mathfrak{C}$ is countable. For a given positive integer n, each $\mathfrak{c} \in \mathfrak{C}$ for which $\mathfrak{c} < \mathfrak{k}_{ni}$ (for some i) is contained in a maximal element of $\mathfrak{C}$ satisfying the same condition. This means that there exists an element $\mathfrak{q}_i(\mathfrak{c}, n) \in \mathfrak{C}$ such that $\mathfrak{c} < \mathfrak{q}_i(\mathfrak{c}, n) < \mathfrak{k}_{ni}$: if $\mathfrak{c} = \mathfrak{c}_\beta$ then for each $\alpha < \beta$,

[125] With the topology given for the set $\mathscr{A}$ it is possible to prove that $\mathfrak{B}$ is isomorphic to the Boolean algebra of *regular open* sets of $\mathscr{A}$.

[126] Maharam 1948, p. 590.

[127] These properties state that the partial order relation of $\mathfrak{C}$ as a Souslin tree is the inverse order relation of $\mathfrak{C}$ as a subset of $\mathfrak{E}$.

according to (*ma.v*), there is a unique $\mathfrak{c}_\alpha$ such that $\mathfrak{c}_\beta < \mathfrak{c}_\alpha$, in this case $\mathfrak{q}_i(\mathfrak{c}, n) = \mathfrak{c}_\alpha$ for the smallest α for which $\mathfrak{c}_\alpha < \mathfrak{k}_{ni}$. For a fixed n let $\mathfrak{Q}_n = \{\mathfrak{q}_i(\mathfrak{c}, n)\}_{i\in\mathbf{N}}$ and clearly if $\mathfrak{q}_i(\mathfrak{c}, n), \mathfrak{q}_j(\mathfrak{c}', n) \in \mathfrak{Q}_n$ then either $\mathfrak{q}_i(\mathfrak{c}, n) = \mathfrak{q}_j(\mathfrak{c}', n)$ or $\mathfrak{q}_i(\mathfrak{c}, n) \wedge \mathfrak{q}_j(\mathfrak{c}', n) = 0$. So each $\mathfrak{Q}_n$ is countable as well as $\mathfrak{Q} = \bigcup_{n\in\mathbf{N}} \mathfrak{Q}_n$. The desired contradiction showing that $\mathfrak{C}$ is countable is obtained from the fact that each $\mathfrak{c} \in \mathfrak{C}$ is greater or equal to some $\mathfrak{q}_i(\mathfrak{c}, n)$, for some n. Now if $\mathfrak{K}_j = \left\{\mathfrak{k}_{nj(n)}\right\}_{n\in\mathbf{N}}$, property 4.XI states that $\inf \mathfrak{K}_j = \mathfrak{o}$ (for each function $j : \mathbf{N} \to \mathbf{N}$), so given $\mathfrak{c} \in \mathfrak{C}$ there is a positive integer n_0 and a function $i : \mathbf{N} \to \mathbf{N}$ such that $\mathfrak{k}_{n_0 i(n_0)} < \mathfrak{c}$. Since $\mathfrak{C}$ is a non-atomic Boolean algebra there is an element $\mathfrak{x} \in \mathfrak{C}$ such that $\mathfrak{o} < \mathfrak{x} < \mathfrak{k}_{n_0 i(n_0)}$; by virtue of (*ma.vii*) there also exists another element $\mathfrak{c}' = \mathfrak{c}_\alpha$ such that $\mathfrak{c}_\alpha < \mathfrak{x} < \mathfrak{k}_{n_0 i(n_0)}$ and, as stated before, an element $\mathfrak{q}_{i(n_0)}(\mathfrak{c}', n_0) \in \mathfrak{C}$ exists such that $\mathfrak{q}_{i(n_0)}(\mathfrak{c}', n_0) < \mathfrak{k}_{n_0 i(n_0)} < \mathfrak{c}$.

For the proof of the converse statement the axiom of choice is required, for if $\mathfrak{A}$ is a non-atomic Boolean σ-algebra which satisfies the *ccc* condition, Maharam deduced the existence of a maximal subset $\mathfrak{S} \subset \mathfrak{A}$ which satisfies conditions (*ma.i*)–(*ma.iii*) of a *Souslin system*. If Souslin's hypothesis is assumed, then $\mathfrak{S} = \{\mathfrak{s}_n\}_{n=1}^{\infty}$ is a countable system and $\mathfrak{l} \in \mathfrak{S}$; let $\mathfrak{l} = \mathfrak{s}_1$.

It is proved that:

I. For any $\mathfrak{s}_k \in \mathfrak{S}$ there exists another element $\mathfrak{s}_m \in \mathfrak{S}$ such that $\mathfrak{s}_m < \mathfrak{s}_k$: $\mathfrak{A}$ is a non-atomic algebra and there exists an element $\mathfrak{x} \in \mathfrak{A}$ such that $\mathfrak{o} < \mathfrak{x} < \mathfrak{s}_k$; if $\mathfrak{x} \in \mathfrak{S}$ there is nothing to be proved, but if $\mathfrak{x} \notin \mathfrak{S}$ then because $\mathfrak{S}$ is maximal, there exists an element $\mathfrak{s}_m \in \mathfrak{S}$ such that $\mathfrak{s}_m \wedge \mathfrak{x} \neq \mathfrak{o}$. This means that $\mathfrak{s}_m \wedge \mathfrak{s}_k \neq \mathfrak{o}$, but then $\mathfrak{s}_m < \mathfrak{s}_k$ or $\mathfrak{s}_k < \mathfrak{s}_m$, and only the first one is possible.
II. If a term $\mathfrak{k}_n = \bigwedge_{j=1}^{\varphi(n)} \varepsilon_j \mathfrak{s}_j$ is defined, where each ε_j is 1 or -1 and the function $\varphi : \mathbf{N} \to \mathbf{N}$ is such that $1 \leqq \varphi(1) < \varphi(2) < \ldots$, then $\{\mathfrak{k}_n\}$ is an infinite decreasing sequence. Now if $\mathfrak{a} \in \mathfrak{A}$ is smaller than any element $\mathfrak{k}_n$ of the sequence, then $\mathfrak{a} = \mathfrak{o}$; for if $\mathfrak{a} \neq \mathfrak{o}$, then $\mathfrak{a} \in \mathfrak{S}$ (if $\mathfrak{a}$ is not an element of $\mathfrak{S}$ it could be adjoint to it), so $\mathfrak{a} = \mathfrak{s}_k$, but according to what has been proved in I, there is an element $\mathfrak{s}_m < \mathfrak{a} = \mathfrak{s}_k$, which is a contradiction since it has been assumed that $\mathfrak{a} < \varepsilon_m \mathfrak{s}_m$. It follows from this, and the fact that $\mathfrak{S}$ is countable, that $\bigwedge_{n=1}^{\infty} \mathfrak{k}_n = \mathfrak{o}$.
III. If, on the other hand, $k_i = \{\varepsilon_1^i, \varepsilon_2^i, \ldots, \varepsilon_n^i, \ldots\}$ is an infinite sequence whose terms are $+1$ or -1, it is clear that two such sequences k_i, k_j are different if at least for two terms ε_n^i, ε_n^j, one of them is 1 and the other is -1. A double sequence $\{\mathfrak{k}_{ni}\}$ is defined in $\mathfrak{A}$ as follows:

$$\mathfrak{k}_{ni} = \bigwedge_{j=1}^{\varphi(n)} \left(\varepsilon_j^i \mathfrak{s}_j\right) \tag{4.XII}$$

where again each ε_j^i is 1 or -1 and the function $\varphi : \mathbf{N} \to \mathbf{N}$ satisfies $1 \leqq \varphi(1) < \varphi(2) < \cdots$. Since $\mathfrak{s}_1 = \mathfrak{l}$, then it follows from conditions (*ma.i*)–(*ma.iii*) that for the elements of the double sequence $\{\mathfrak{k}_{ni}\}$ 4.X and 4.XI hold.

A Boolean algebra $\mathfrak{B}$ is said to be $\aleph_0$-distributive if for a double sequence $\{\mathfrak{r}_{ni}\}$, the distributive law is satisfied:

$$\bigwedge_n \bigvee_i \mathfrak{r}_{ni} = \bigvee_{j\in\mathbf{N}^{\mathbf{N}}} \bigwedge_n \mathfrak{r}_{nj(n)} \ .$$

A non-atomic, $\aleph_0$-distributive and complete Boolean algebra $\mathfrak{B}$ satisfying the *ccc* condition is called a *Souslin algebra*. Since 4.X and 4.XI state that (any) non-atomic, complete σ -algebra $\mathfrak{A}$ which satisfies the *ccc* condition is not $\aleph_0$-distributive, this theorem states the equivalence of Souslin's hypothesis with the non-existence of a Souslin algebra. A modern version of this theorem by Maharam is stated in the following terms:[128]

Theorem ***(J.1).*** *There exists a Souslin tree if and only if there exists a Souslin algebra.*

This Boolean algebraic version of Souslin's hypothesis will prove to be particularly useful to provide a Boolean-valued model for set-theory for which this hypothesis is true.

5. The independence of Souslin's hypothesis

5.1. The place of the ramification hypothesis

We have remarked that the approaches of Kurepa, Miller and Sierpiński gave an equivalent condition, but not a proof for Souslin's hypothesis. Nevertheless, besides these equivalences no systematic study was made concerning the relation of Souslin's problem to other unsolved set-theoretic propositions such as the continuum hypothesis or the axiom of choice. We have pointed out that Kurepa opened his memoir of 1935 by stating what he considered the three most important problems in set theory. At the end of this memoir he made some remarks concerning a possible relation between them. The first one is that any of the 9 propositions P_i $(i = 1, \ldots, 9)$ of his theorem *EOR.7* implies the following proposition:[129]

Proposition. ***(Q).*** *For any infinite ramified table T, the power of the family of all the degenerate subsets of T is greater than the power of T itself.*

Just by taking proposition P_1 it is possible to state that $2^{bT} = 2^{|T|}$, which is, after Cantor's theorem, greater than $|T|$.

After this statement he stated the conditions under which the 9 propositions P_i could be obtained:[130]

Proposition. *All the propositions P_i $(i = 1, \ldots, 9)$ can be obtained from proposition (Q) and the Generalized Continuum Hypothesis.*

This can be easily seen from the following three facts:

$$2^{bT} > |T| \quad , \tag{Q}$$

$$2^{bT} = (bT)^{+} \quad , \tag{GCH}$$

[128] *Cf.* Jech 1978, p. 220. Devlin K, Johnsbraten H 1975, p. 82.
[129] Kurepa 1935, p. 133.
[130] *Ibid.*

$$bT \leqq |T| \quad . \tag{3.XVII}$$

From these three propositions it follows that

$$bT \leqq |T| < 2^{bT} = (bT)^+$$

and, finally, $bT = |T|$.

After this proof Kurepa enunciated as a problem their possible relation:

A-t-on (GCH) → (Q)? et, par consequent (GCH) → P_i?[131]

A positive answer to this question would imply that Souslin's hypothesis (the non-existence of a *Souslin line*) is a consequence of the generalized continuum hypothesis.

At the end of *Ensembles Ordonnés et Ramifiés* Kurepa concluded by giving his idea about the relation between Souslin's hypothesis and Cantor's continuum hypothesis, as well as his conviction concerning their logical role in axiomatic set theory:

> En terminant nous exprimons la conviction que l'hypothèse [du continu] de Cantor, et l'hypothèse de ramification (P_1) ne sont pas abordables par des méthodes et principes connus de la Théorie des ensembles. Sont-elles logiquement équivalentes entre elles? ou, sont-elles deux cas particuliers (et très intéressants par leur structure logique) d'un même principe, irréductible aux axiomes et principes usuels?[132]

In the note delivered at the *Académie des Sciences de Paris* on January 20, 1936,[133] Kurepa claimed once more that the *ramification hypothesis* should have some relation with Cantor's continuum hypothesis, but also that apparently neither could be deduced from the current axioms of set theory. Besides this relation between the ramification hypothesis (and so, Souslin's hypothesis) and the continuum hypothesis, Kurepa claimed that the ramification hypothesis was not compatible with another set-theoretic conjecture related to a problem on analytic and projective sets: the hypothesis, formulated in 1935 by M. Luzin, that any subset $X \subset \mathbf{R}$ having the power $\aleph_1$ should be the complement of an *analytic set*.[134] The importance of this hypothesis is that an immediate consequence of it is the following cardinal equality:

$$2^{\aleph_0} = 2^{\aleph_1} \quad .$$

Concerning the relations between the continuum hypothesis, Luzin's hypothesis and Kurepa's ramification hypothesis, he said

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[131] "Is it that (GCH) → (Q) and consequently (GCH) → P_i?" *Ibid.*

[132] "Finally we state our conviction that Cantor's [continuum] hypothesis and the ramification hypothesis (P_1) are not solvable through the known methods of set theory. Are they logically equivalent? or are they two particular cases (and very interesting because of their logical structure) of the same principle, which is not reducible to the axioms and usual principles?". *Id.*, p. 134.

[133] Kurepa 1936.

[134] Luzin 1935, p. 129.

> Remarquons que *l'hypothèse de ramification* est dans une certaine correspondance avec l'hypothèse [du continu] de Cantor, et il semble qu'aucune d'elles n'est réductible aux axiomes courants de la théorie des ensembles. En particulier, il nous semble que *l'hypothèse de M. Luzin* es incompatible avec l'hypothèse de ramification de même qu'elle est incompatible avec l'hypothèse de Cantor.[135]

But as we will see, Cantor's continuum hypothesis is independent of Souslin's hypothesis and the latter, as shown by Martin and Solovay, it is completely "compatible" with Luzin's conjecture.

5.2. The non-provability of Souslin's hypothesis

Other set theoretic and topological propositions related to Souslin's hypothesis were found in the fifties. A consequence which becaem well-known due to its importance in point set topology was given by M. E. Rudin in 1955 [136] concerning the question of whether every ($\mathbf{T}_2$) space T is *countable paracompact*.[137] This question can be answered negatively if a negative answer to Souslin's hypothesis is proved. In other words, the existence of a Souslin line implies the existence of a $\mathbf{T}_2$ space which is not countable paracompact. Another important consequence implied in a remark made by Kurepa is that a Souslin line is an example of a continuous ordered set which is not metrizable.

But a completely new direction for investigations of Souslin's problem was given when, in the language of model theory, the question was raised whether Souslin's hypothesis could be proved within the frame of set theoretic axioms. Two important and completely independent studies on this question were stated by T. Jech and by S. Tennenbaum.

In a series of notes published in the BULLETIN DE L'ACADEMIE DES SCIENCES POLONAISE in 1965, P. Vopěnka announced a series of important proofs on the independence of some set-theoretic propositions. These independent proofs were based on a method, introduced in the first note, for constructing models for set theory. As part of this series of studies, Jech published in 1967 a short paper[138] where he showed the non-provability of Souslin's hypothesis. In fact, what Jech proved was the existence of a model for set theory in which the existence of a Souslin tree was easily proved.

Jech opened his paper with a brief description of the theorems of Miller (*M.1*) and Sierpiński (*S.1*), called by him the "equivalent theorems" for Souslin's problem. In order to introduce his own version of these theorems, he defined a relation $\mathfrak{r}$ on the set $\omega_0 \times \omega_1$. First he defined, for an ordinal $\alpha < \omega_1$, the α*-th* row of $\omega_0 \times \omega_1$ as the set $h_\alpha = \omega_0 \times \{\alpha\}$. Now the relation $\mathfrak{r}$ is a *ramified graph* when it satisfies the following conditions:

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135 "We should remark that the *ramification hypothesis* is in some correspondence with Cantor's [continuum] hypothesis, and it seems that none of them is reducible to the current set theoretic axioms. Particularly, it seems that *Mr. Luzin's hypothesis* is incompatible with the ramification hypothesis, as well as with Cantor's hypothesis". Kurepa *Op. cit*.

136 Rudin 1955.

137 A space T is *countable paracompact* if for every countable covering of T, any point $x \in T$ is in some open set which intersects only a finite number of sets of the covering.

138 Jech 1967.

i) $\mathfrak{r}$ is reflexive and transitive,
ii) if $\langle x, y\rangle \in \mathfrak{r}$, and $x \in h_\alpha$ and $y \in h_\beta$, then $\alpha \leqq \beta$,
iii) if $\alpha < \beta$ and $y \in h_\beta$ then there is $x \in h_\alpha$ with $\langle x, y\rangle \in \mathfrak{r}$,
iv) if $x \neq y$ belong to the same row, there is no z with $\langle x, z\rangle \in \mathfrak{r}$ and $\langle y, z\rangle \in \mathfrak{r}$.

The obvious definition that two elements x and y are $\mathfrak{r}$-comparable if $\langle x, y\rangle \in \mathfrak{r}$ or if $\langle y, x\rangle \in \mathfrak{r}$, makes it possible to define a "chain" of the ramified graph as a relation $\mathfrak{s} \subset \mathfrak{r}$ which satisfies conditions (*i*)–(*iv*), and which is a "linear order"; this means that if $\langle x, y\rangle \in \mathfrak{s}$ and $\langle x, z\rangle \in \mathfrak{s}$ then $\langle y, z\rangle \in \mathfrak{s}$ or $\langle z, y\rangle \in \mathfrak{s}$. A subset of the domain $\mathfrak{D}(\mathfrak{r})$ of $\mathfrak{r}$ is an "anti-chain" when it contains only elements which are pairwise incomparable.

With this new definition Jech gave his own version for an "equivalence theorem",[139] obtained "after a simple modification of the Miller-Sierpiński theorem".

Theorem ***(J.2).*** *The necessary and sufficient condition for the existence of Souslin's continuum is the existence of an uncountable ramified graph, which has no uncountable chains or anti-chains.*

This equivalent theorem led Jech to the non-provability of Souslin's hypothesis. To be more precise, he constructed a model of set theory in which Souslin's hypothesis proved to be false. Taking into account the theorems of Miller and Sierpiński, Jech showed a model (a ∇-model constructed with the method introduced by P. Vopěnka) in which an uncountable ramified graph (an uncountable tree) with no uncountable chains or anti-chains exists. This uncountable graph appears to be the limit, and this is the key to Vopěnka's method, of a set of countable ramified graphs, ordered by inclusion, satisfying the condition of being *regular*, *i.e.* such that each point has at least two successor points in any upper level. Jech's conclusion is resumed in the main theorem of his paper:[140]

Theorem ***(J.3).*** *Souslin's hypothesis is not provable in set theory.*

This theorem followed immediately from the construction of a model in which a *Souslin tree* exists, since with this construction the "negation" of Souslin's hypothesis was proved to be consistent with the current set-theoretic axioms. For Jech his theorem also showed the non-provability of another set theoretic hypothesis:

> For the sake of completeness it is to be mentioned that in 1936 Kurepa formulated his ramification hypothesis, which implies Souslin's hypothesis and is therefore not provable.[141]

The construction of a model in which a Souslin tree exists was also carried out by S. Tennenbaum in 1967[142] as an application of the forcing methods introduced by P. J. Cohen for the proof of the independence of the continuum hypothesis. Tennenbaum started his work in 1963 and the next year, in a series of lectures at Harvard University, he presented the proof for the consistency of the negation of Souslin's hypothesis. Although

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139 *Ibid.*, p. 295.
140 *Ibid.*, p. 292.
141 *Ibid.*, p. 294.
142 Tennenbaum 1968.

the method employed for the construction of his model is different from the one followed by Jech, the first conclusion obtained by Tennenbaum is the same as the one obtained by Jech: Souslin's hypothesis is not provable in set theory. However, with Cohen's method Tennenbaum obtained another important fact: he was able to construct two models of set theory in which a Souslin tree exists, but for one of these models the continuum hypothesis holds, and for the other one this hypothesis fails. So Tennenbaum gave, besides the non-provability of Souslin's hypothesis, an answer to a problem posed for the first time by Kurepa concerning its relation with the continuum hypothesis: the continuum hypothesis is independent of Souslin's hypothesis. The following two theorems include these results:[143]

Theorem ***(T.1).*** *There exists a model $\mathscr{N}$ of set theory such that $\mathscr{N}$ contains a Souslin tree and satisfies* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Theorem ***(T.2).*** *There exists a model $\mathscr{R}$ of set theory such that $\mathscr{R}$ contains a Souslin tree but* $2^{\aleph_0} \neq \aleph_1$.

The existence of the two models is proved in the same way: for theorem *T.1*. Tennenbaum took first a countable transitive model $\mathscr{M}$ for set theory where the generalized continuum hypothesis holds –a model satisfying $V = L$ and so $2^{\aleph_\alpha} = \aleph_{\alpha+1}$– and then he obtained from $\mathscr{M}$ a model $\mathscr{N}$ by adjoining a generic tree $T \subset \aleph_1 \times \aleph_1$. For the theorem *T.2*. the same procedure is used by taking a model $\mathscr{M}$ where $2^{\aleph_0} \neq \aleph_1$.

Another proof of the consistency of the negation of Souslin's hypothesis was given by R. B. Jensen in 1968[144] when he showed that in the "constructible universe" $V = L$ there exists a *Souslin tree*; his proof employed his combinatorial principle known as the $\diamond$–principle.

5.3. Souslin's hypothesis and Martin's axiom

After the theorems by Jech and Tennenbaum, the independence of Souslin's hypothesis had to be proved by constructing another model for set theory in which this hypothesis holds. This task was accomplished mainly in two articles by D. Martin, R. Solovay and S. Tennenbaum[145] which we will analyze briefly because their purpose, the construction of a model where no Souslin tree exists, seems to us closely related to Kurepa's program to give a positive answer to Souslin's problem: to show that no *abnormal* ramified table could exist. The most remarkable fact is that the means with which this task was completed has a peculiar relation with the means proposed by Kurepa to accomplish his task.

The paper of Solovay and Tennenbaum was considered by their authors as a continuation of Tennenbaum's paper of 1968, where he announced that another paper would provide the proof for the consistency of Souslin's hypothesis:

143 *Ibid*.
144 Jensen 1968.
145 Solovay, Tennenbaum 1971 and Martin, Solovay 1970.

> In a later paper, written with R. Solovay, Cohen's method is extended to define models in which the answer [to Souslin's problem] is *affirmative*.[146]

This proof by Solovay and Tennenbaum was given in two steps: first a new axiom A is introduced and proved to be consistent with ZFC axioms, then Souslin's hypothesis is deduced from $A+(2^{\aleph_0} > \aleph_1)$.

Martin and Solovay in their turn considered their paper as a continuation of the paper by Solovay and Tennenbaum, even though it had been published earlier, in the sense that some other consequences of the axiom A are obtained, especially some consequences that were previously deduced from the continuum hypothesis. This fact allowed them to state that even if "the axiom arose from the consistency problem for Souslin's hypothesis", it could be possible to ask about the possibility of considering the axiom A as a *substitute* for the continuum hypothesis.

In order to introduce the new axiom A, which is due to D. A. Martin, Solovay and Tennenbaum presented the following definition: for a partial ordered set P a subset $X \subset P$ is *dense* if it satisfies:

1. For any $p \in X$, if $p \leqq q$ then $q \in X$.
2. For any $p \in P$ there exists always $q \in X$ such that $p \leqq q$.

For a family $\mathscr{F}$ of subsets of P a subset $G \subset P$ is said to be an $\mathscr{F}$-generic filter on P if:

1. For any $p \in G$ if $q \leqq p$ then $q \in G$,
2. If p_1 and $p_2 \in G$ then there is a $p_3 \in G$ such that $p_1 \leqq p_3$ and $p_2 \leqq p_3$,
3. If a dense subset X of P belongs to the family $\mathscr{F}$, then $X \cap G \neq \emptyset$.

Two elements p_1 and p_2 of P are *compatible* if there exists a third element $p_3 \in P$ such that $p_1 \leqq p_3$ and $p_2 \leqq p_3$. Otherwise p_1 and p_2 are said to be incompatible. A subset $Y \subset P$ whose elements are pairwise incompatible is an *anti-chain* and P satisfies the *countable anti-chain condition* if every anti-chain of P is at most countable.

They considered next the following statement $A_\aleph$:

($A_\aleph$) If P is a partially ordered set satisfying the *countable anti-chain condition* and $\mathscr{F}$ is a family of dense subsets of P such that $|\mathscr{F}| \leqq \aleph$, then there is an $\mathscr{F}$-generic filter on P.

The axiom A formulated by Martin is the following statement:

Axiom (**A**). *The proposition* $A_\aleph$ *holds if* $\aleph < 2^{\aleph_0}$.

If the family $\mathscr{F}$ of dense subsets of P is countable, then it is easy to show the existence of an $\mathscr{F}$-generic filter G: for if $\mathscr{F} = \{D_n\}$ is the countable family of dense subsets then it is possible to take $g_1 \in D_1$ and then to define by induction g_{n+1} as an element of D_{n+1} such that $g_{n+1} \geqq g_n \in D_n$; $G = \{g_n\}$ is an $\mathscr{F}$-generic filter. This shows that $A_{\aleph_0}$ is a theorem or else that the continuum hypothesis $2^{\aleph_0} = \aleph_1$ implies A, so that A is consistent with the Zermelo-Fraenkel axioms for set theory.

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[146] Tennenbaum *Op. cit.*

In their paper, Martin and Solovay proved that if $\aleph \geqq 2^{\aleph_0}$ then $A_\aleph$ fails.[147] So the interesting case for $A_\aleph$ is when $\aleph_1 \leqq \aleph < 2^{\aleph_0}$. Clearly in this case $A_\aleph$ should be accompanied with the negation of the continuum hypothesis.

This remark is clear in the theorem where Solovay and Tennenbaum proved that Souslin's hypothesis can be deduced from axiom $A_{\aleph_1}$:[148]

Theorem ***(S-T.1).*** *Assume Martin's axiom A. Assume further that* $2^{\aleph_0} > \aleph_1$, *then Souslin's hypothesis holds.*

If the conclusion were not true then there would exist a Souslin tree T, *i.e*. a tree of height ω_1 with only countable chains and anti-chains. Clearly for this tree, which is a partially ordered set, two elements are incomparable if they are incompatible, so the tree T satisfies the countable anti-chain condition. For every $\alpha < \omega_1$ the set D_α is defined as the union of all the levels above α, which is then a dense subset of T. By taking the family

$$\mathscr{F} = \{D_\alpha; \alpha < \omega_1\} \quad ,$$

with the assumption that $2^{\aleph_0} > \aleph_1$, then from the axiom A it is possible to deduce the existence of an $\mathscr{F}$-generic filter G on T. G is at the same time a chain of T, but in order to be an $\mathscr{F}$-generic filter, G must intersect every D_α, $\alpha < \omega_1$, so that G, as a chain, has length ω_1. This fact contradicts the initial hypothesis that T is a Souslin tree.

Martin and Solovay declared in their paper that Martin's axiom arose from the consistency problem of Souslin's hypothesis, although it had consequences in many other fields. Some of its consequences, besides Souslin's hypothesis, are the following:[149]

1. If κ is an infinite cardinal, $\kappa < 2^{\aleph_0}$, then $2^\kappa = 2^{\aleph_0}$. So if $2^{\aleph_0} > \aleph_1$, $2^{\aleph_1} = 2^{\aleph_0}$,
2. The union of less than $2^{\aleph_0}$ sets of reals of Lebesgue measure zero (of the first category) is of Lebesgue measure zero (of the first category),
3. If $2^{\aleph_0} > \aleph_1$, every projective set of reals $\left(\sum_2^1\right)$ is Lebesgue measurable and has the property of Baire,
4. $2^{\aleph_0}$ is not a real valued measurable cardinal.

In relation to Souslin's problem, by assuming Martin's axiom and the negation of the continuum hypothesis, it is possible to "kill" any Souslin tree, since the generic filter, whose existence is guaranteed by the axiom, becomes a chain of height ω_1. This means that with this axiom it is possible to prove immediately propositions P_1 (ramification hypothesis) and P_2 (reduction principle) of Kurepa's theorem *EOR. 7*.

However, for the application of Martin's axiom, two conditions play an important role:

1. First of all the condition that the partially ordered set should satisfy the countable anti-chain condition[150]. This condition was clearly explained by Martin who showed

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[147] Martin, Solovay *Op. cit*., p. 149.
[148] Solovay, Tennenbaum *Op. cit*., p. 234.
[149] Martin, Solovay *Op. cit*., p. 144.
[150] In relation to Kurepa's research, this countable antichain condition states that the "degenerate subset" having the same power of the set should be a chain and not an anti-chain.

that for a partially ordered set P which is not compelled to satisfy the countable anti-chain condition, the proposition asserting the existence of an $\mathscr{F}$-generic filter, when $\mathscr{F}$ is a family of dense subsets of P, gives a proof for the collapsing of cardinal numbers.

2. A second important condition is the negation of the continuum hypothesis. Martin saw clearly the role played by the condition

$$2^{\aleph_0} > \aleph_1 \ ,$$

together with the axiom A. Souslin's hypothesis is derived from the axiom A, which guarantees the existence of an ordered subset of power $\aleph_1$ in every tree of height ω_1 satisfying the countable anti-chain condition. This means that for the proof of Souslin's hypothesis the proposition $A_{\aleph_1}$ suffices, but in order to assure this proposition, the negation of the continuum hypothesis is necessary.

Martin and Solovay also introduced in their paper a Boolean version for axiom A:

Axiom $(\mathfrak{B}_{\aleph})$. *If* $\mathfrak{B}$ *is a (complete) Boolean algebra which satisfies the* ccc *condition, and* $\{\mathfrak{b}_{i\alpha}\}$ *is a double sequence of element of* $\mathfrak{B}$ *(*$i < \omega, \alpha < \aleph$*); there exists a homomorphism* $h : \mathfrak{B} \to \{\mathfrak{o}, \mathfrak{l}\}$*, (the two-element Boolean algebra) such that for every* $\alpha < \aleph$*,*

$$h\left(\bigvee_i \mathfrak{b}_{i\alpha}\right) = \bigvee_i h\left(\mathfrak{b}_{i\alpha}\right) \ .$$

The equivalence between $A_{\aleph}$ and $\mathfrak{B}_{\aleph}$ is shown by defining first a partially ordered set $\mathscr{P} = \mathfrak{B} - \{\mathfrak{o}\}$ (reversing the order relation of $\mathfrak{B}$) which satisfies the countable antichain condition if $\mathfrak{B}$ satisfies the *ccc* condition. This new version for Martin's axiom is introduced to underline the general correspondence between forcing and Boolean algebras, but it clearly makes it possible to give a proof for Souslin's hypothesis after the equivalence given in theorem *MA.4*.

6. Conclusion

The question raised by Souslin in 1920 involves two basic properties of the linear continuum whose relation, at least in terms of logical implication, is easily proved in one direction. But even if this question might seem quite simple to formulate, it was not originally stated by Cantor or by any of the other "creators" of set theory. As we have shown in the first part of our paper, these two conditions, the countable chain condition and the separability condition, were conceived and characterized independently. This was due to the fact that while the first one was considered as a sort of *intrinsic geometrical property* of Euclidean n-dimensional space, the second one was considered as a property of some (continuous and linear) sets of points, and also for some (abstract) ordered set.

When Sierpiński first noticed the importance and the difficulty of Souslin's problem, he linked it to the theory of order types and ordered sets. In this interpretation, the problem asked for another (equivalent) characterization for the order type λ[151] of

[151] Also for the order type θ.

the linear continuum. But in some sense the problem of a possible relation between the separability and the countable chain condition could hardly be formulated within the original framework of the theory of ordered sets created by Cantor. The idea of a family of non-overlapping intervals, and the power which corresponds to it, seemed to Cantor an external notion having no connection with the idea of an order relation.

The changes of style we presented, the balance between different approaches, and the partial results that we described, are all consequences of the fact that this problem was never studied thoroughly in an extended treatise. The only exception is Kurepa's work. Specific and isolated equivalences and different alternative formulations were given as consequences of indirect interest in the problem; therefore a global approach was difficult to provide.

Along with the different attempts to give a positive answer to Souslin's problem, it became clear that a proposition which, at first sight, seemed quite obvious in the context of continuous ordered sets, lost its self-evidence once it was interpreted in the realm of partially ordered sets. For a continuous ordered set E, the equality $p_1E = p_2E$ seemed quite convincing to Kurepa when he began his study on Souslin's problem in 1934, and certainly he had no argument other than a direct survey of continuous ordered sets, *separable* or not. When he translated this condition on linear ordered sets to the condition that every ramified table of height ω_1 should be *normal*, he realized that it became more difficult to support his conviction on some kind of survey. Nevertheless, Kurepa still believed in the truth of this condition, as he stated in his theorem *CR.IV.1*, but he soon became aware of the difficulties involved in proving this last statement.

Even if we say that Kurepa's first approach to Souslin's problem should now be considered as mathematically wrong – in the sense that contrary to what he claimed, he provided no proof of Souslin's hypothesis – this approach led him to the important shift from the theory of order types to the theory of *ramified tables*. This is the main reason why we explained Kurepa's work in detail, even though it has now been more or less forgotten. By following the context of the reformulation of the problem, we can see how Kurepa made the most important contribution to our actual understanding of the nature and the meaning of the question raised by Souslin.

Besides the obvious fact that Souslin's problem is, as the continuum hypothesis, a proposition dealing with a property of the linear continuum, and despite the fact that these two propositions are independent, it should be underlined that they both share the particular feature of being *undecidable* propositions (within the frame of Z-F axioms). Knowing the existence of set theoretic models where Souslin's hypothesis holds as well as models where it fails, it might seem, as happened with the continuum hypothesis, that no definite answer exists. But, besides the fact that we are dealing with an undecidable proposition of set theory, Souslin's hypothesis is a proposition involving two of the most basic properties of the linear continuum. And this certainly shows that if no answer is possible, our knowledge of the set theoretic properties of the continuum is still incomplete. We are certain that, like Cantor when he defined some of the basic notions of set theory, Souslin, Kurepa and Sierpiński were all convinced that the problems with which they were dealing would find a definite answer one day.

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