Dynamics of $\text{Out}(F_n)$ on the boundary of outer space.

Vincent Guirardel

July 1998

Abstract

In this paper, we study the dynamics of the action of $\text{Out}(F_n)$ on the boundary ∂CV_n of outer space: we describe a proper closed $\text{Out}(F_n)$-invariant subset $\mathcal{F}_n$ of ∂CV_n such that $\text{Out}(F_n)$ acts properly discontinuously on the complementary open set. Moreover, we prove that there is precisely one minimal non-empty closed invariant subset $\mathcal{M}_n$ in $\mathcal{F}_n$. This set $\mathcal{M}_n$ is the closure of the $\text{Out}(F_n)$-orbit of any simplicial action lying in $\mathcal{F}_n$. We also prove that $\mathcal{M}_n$ contains every action having at most $n-1$ ergodic measures. This makes us suspect that $\mathcal{M}_n = \mathcal{F}_n$. Thus $\mathcal{F}_n$ would be the limit set of $\text{Out}(F_n)$, the complement of $\mathcal{F}_n$ being its set of discontinuity.

Outer space CV_n has been introduced by M. Culler and K. Vogtmann as an analogue of Teichmüller space for the group $\text{Out}(F_n)$ of outer automorphisms of the non-abelian free group F_n. Outer space is the set of minimal free isometric actions of F_n on simplicial $\mathbb{R}$-trees modulo equivariant homothety. It has a natural compactification $\overline{CV}_n$ in the set of minimal isometric actions of F_n on $\mathbb{R}$-trees. Both CV_n and $\overline{CV}_n$ are endowed with a natural action of $\text{Out}(F_n)$ by precomposition.

Like the Teichmüller space $\mathcal{T}_S$ of a closed surface S, CV_n is a contractible space, the action of $\text{Out}(F_n)$ on CV_n is properly discontinuous and not cocompact. The quotient being a finite disjoint union of open simplices, it may be thought of as having finite volume (see [CV]). Moreover, every outer automorphism of F_n fixes a point in $\overline{CV}_n$ (see [BH, Lus]).

Outer space has proven to be useful in the study of $\text{Out}(F_n)$. M. Culler and K. Vogtmann computed the virtual cohomological dimension of $\text{Out}(F_n)$ ([CV]) using outer space. Furthermore, M. Bestvina and M. Feighn showed that $\text{Out}(F_n)$ is $(2n-3)$-connected at infinity by using some Morse theory on a bordified version of outer space ([BF3]). However, outer space happens to be more complicated than Teichmüller space and not much is known about this space and its compactification.

Thurston theory shows that the mapping class group of a closed orientable surface S acts with dense orbits on $\partial\mathcal{T}_S$, the boundary of Thurston's compactification of $\mathcal{T}_S$ (see [FLP] for instance). When the surface is not orientable, there is an open invariant subspace of full measure in $\partial\mathcal{T}_S$ consisting of measured foliations having a regular closed one-sided leaf ([DaNo]). The action is not properly discontinuous on this set since infinite order Dehn twists fix some of its points. Surprisingly, it seems to be unknown whether the mapping class group acts with dense orbits on the complementary closed set.

In this paper, we try to understand the analogous problem in outer space: does $\mathrm{Out}(F_n)$ act with dense orbits on the boundary of outer space ? The answer to this question is no.

Definition. *Let $\mathcal{O}_n$ be the set of simplicial F_n-actions T such that*

- *T has trivial edge stabilizers*
- *T has cyclic vertex stabilizers*
- *whenever $\mathrm{Stab}\, v \neq \{1\}$, $\mathrm{Stab}\, v$ acts transitively on the set of incident orbits.*

Equivalently, T lies in $\mathcal{O}_n$ if and only if every non-trivial group in the graph of groups T/F_n is cyclic and is attached to a terminal vertex of T/F_n.

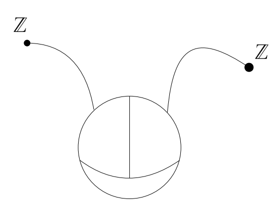


Figure 1: A typical action in $\mathcal{O}_n$ (trivial groups are omitted)

Figure 1 shows a typical action in $\mathcal{O}_n$. Because of its friendly face with antennae, I was suggested to christen the actions in $\mathcal{O}_n$ *Martian actions* (many thanks to Claire!). The set $\mathcal{O}_n$ can also be seen to be the set of simplicial actions in $\overline{CV_n}$ with finite stabilizer in $\mathrm{Out}(F_n)$. It is clearly invariant under the action of $\mathrm{Out}(F_n)$.

Theorem 1. *The set $\mathcal{O}_n$ is open in $\overline{CV_n}$ and $\mathrm{Out}(F_n)$ acts properly discontinuously on $\mathcal{O}_n$.*

Since $CV_n \subsetneq \mathcal{O}_n$, the closed set $\mathcal{F}_n = \overline{CV_n} \setminus \mathcal{O}_n$ is a proper invariant compact subset of ∂CV_n. M. Feighn pointed out that the intersection of the closure of the spine of outer space ([CV]) with ∂CV_n is a subset of $\mathcal{F}_n$.

Theorem 2. *Let $n \geq 3$. Let T be a simplicial action lying in $\mathcal{F}_n$ and let T' be a small action of F_n on an $\mathbb{R}$-tree. Then there exists a sequence α_k of elements of $\mathrm{Out}(F_n)$ such that*

$$\lim_{k\to\infty} T'.\alpha_k = T$$

This theorem has an interesting corollary about the dynamics of $\mathrm{Out}(F_n)$ on ∂CV_n:

Corollary. *For $n \geq 3$, there exists precisely one minimal non-empty closed invariant subset in $\overline{CV}_n$. This set $\mathcal{M}_n$ is the closure of the orbit of any simplicial action lying in $\mathcal{F}_n$ under the action of* $\mathrm{Out}(F_n)$.

It would be interesting to know whether $\mathcal{F}_n = \mathcal{M}_n$. If the equality held, $\mathcal{F}_n$ would be equal to the intersection of the closure of the spine of CV_n with ∂CV_n. Moreover, $\mathcal{F}_n$ could be thought of as a limit set of $\mathrm{Out}(F_n)$ and $\mathcal{O}_n$ as a domain of discontinuity like in the theory of Kleinian groups.

An argument by Bestvina and Feighn ([BF2]) shows that any action in $\overline{CV}_n$ in which there exists an arc with non-trivial stabilizer lies in $\mathcal{M}_n$. Furthermore, since any action in $\overline{CV}_n$ can be decomposed as a graph of actions with dense orbits (see [Lev2, GL]), proving $\mathcal{F}_n = \mathcal{M}_n$ reduces to showing that any action in $\overline{CV}_n$ with dense orbits belongs to $\mathcal{M}_n$. We prove that this is true under a technical condition:

Theorem 3. *Let $n \geq 3$ and let $T \in \overline{CV}_n$ be an action of F_n with dense orbits. Assume that the Lebesgue measure on T is the sum of at most $n-1$ ergodic measures. Then T lies in $\mathcal{M}_n$.*

Remark. There is a bound coming from the topological dimension of ∂CV_n for the number of ergodic measures of an action in ∂CV_n (see [BF2, GL] and section 5.1). This bound can be seen to be $3n - 4$. Therefore, we suspect that Theorem 3 still holds with no assumption on the Lebesgue measure so that $\mathcal{F}_n = \mathcal{M}_n$.

We also note than if $\alpha \in \mathrm{Out}(F_n)$ is irreducible with irreducible powers, then it has only two fixed points in $\overline{CV}_n$ ([Lus2]) which implies that they are uniquely ergodic and hence lie in $\mathcal{M}_n$.

After some definitions in section 1, we introduce in section 2 the *folding to approximate* technique to obtain approximations of a simplicial action. This technique rules out some natural candidates to be open, and leads to the definition of the set $\mathcal{O}_n$. In section 3, we prove that $\mathcal{O}_n$ is open and that the action of $\mathrm{Out}(F_n)$ on $\mathcal{O}_n$ is properly discontinuous (theorem 1). In section 4, we use the *folding to approximate* technique to study the dynamics of $\mathrm{Out}(F_n)$ on $\mathcal{F}_n = \overline{CV}_n \setminus \mathcal{O}_n$ and prove Theorem 2. In section 5, we introduce the tools of measure theory on $\mathbb{R}$-trees needed to prove Theorem 3.

This work is a part of a Ph-D thesis defended at the Université Toulouse III in January 1998. Many thanks to my advisor Gilbert Levitt who encouraged me, carefully checked my work, and suggested many improvements.

1 Preliminaries

1.1 Group actions on $\mathbb{R}$-trees

Basic facts about $\mathbb{R}$-trees may be found in [Sha, Sha2].

Definition. *An* $\mathbb{R}$-tree *is a metric space* T *such that between two points* $x, y \in T$, *there exists precisely one topological arc (denoted by* $[x, y]$*), and this arc is isometric to an interval in* $\mathbb{R}$.

In this paper, every $\mathbb{R}$-tree will be endowed with an *isometric* action of a finitely generated group. For simplicity, we will denote by the same letter T the tree and the action. We will also simply say *action* to talk about an *action on an* $\mathbb{R}$*-tree*. Most often, the group considered will be the free group F_n on n letters. If an isometry g of an $\mathbb{R}$-tree has no fixed point, then it has a translation axis isometric to $\mathbb{R}$ and we say that g is *hyperbolic*. When g has a fixed point, it is called *elliptic*. The characteristic set $\operatorname{Char} g$ of g is either its axis or the set of its fixed points, depending on whether g is elliptic or hyperbolic.

An action on an $\mathbb{R}$-tree is said to be *minimal* if it has no proper invariant subtree and if it is not reduced to one point. If an action of a finitely generated group Γ on an $\mathbb{R}$-tree T has no global fixed point, then there is a unique invariant minimal subtree of T and it is the union of the translation axes of hyperbolic elements in Γ. All the actions we consider are henceforth assumed to be minimal.

We will call *simplicial* $\mathbb{R}$*-tree* (or simply a *simplicial tree*) a connected simply-connected simplicial 1-complex together with a metric which makes it an $\mathbb{R}$-tree. For shortness' sake, we will say *simplicial action* to mean a simplicial isometric action on a simplicial $\mathbb{R}$-tree. We will always assume that a simplicial action has no inversion i. e. that no edge is flipped by any element of Γ since one can reduce to this case by performing a barycentric subdivision.

A *morphism of* $\mathbb{R}$*-trees* $f : T \to T'$ is a continuous map such that every arc in T may be subdivided into finitely many intervals which are isometrically embedded in T' by f. We will also be interested in maps *preserving alignment* i. e. such that $a \in [b, c] \Rightarrow f(a) \in [f(b), f(c)]$. Note that a morphism of $\mathbb{R}$-trees which preserves alignment is an isometry.

In an $\mathbb{R}$-tree T, a *germ* at a point $x \in T$ is the a germ of isometric applications $[0, \varepsilon] \to T$ sending 0 to x. The set of germs at a point $x \in T$ is in one to one correspondence with the connected components of $T \setminus \{x\}$. A point $x \in T$ is called a *branch point* if there are at least three germs at x. In a simplicial tree, the branch points are the vertices of valence at least three. We will sometimes use the *projection* of a point x on a closed subtree S: it is the point in S closest to x.

1.2 The topology on a set of actions on $\mathbb{R}$-trees

Two actions of a group Γ on $\mathbb{R}$-trees T and T' are identified if there exists an equivariant isometry between T and T'. Sometimes, in projectivised spaces, we will identify T and T' if there exists an equivariant homothety between them.

On any set of minimal actions of a fixed finitely generated group Γ, one can consider the *translation lengths topology*. This topology is based on the *length function* of an action (T, Γ). It is the function $l_T : \Gamma \to \mathbb{R}_+$ defined by

$$l_T(\gamma) = \inf_{x \in T} d(x, \gamma.x).$$

The translation lengths topology is the smallest topology that makes continuous the functions $T \mapsto l_T(\gamma)$ for $\gamma \in \Gamma$. An abelian action is an action whose length function is the absolute value of a morphism $\Gamma \to \mathbb{R}$. For sets of non-abelian actions of a finitely generated group, this topology is Hausdorff (see [CuMo]).

A set of minimal actions of a fixed finitely generated group Γ on $\mathbb{R}$-trees can also be equipped with the *equivariant Gromov topology*. This topology roughly says that two actions are close if they look the same metrically in restriction to a finite subtree while only considering the action of a finite subset of Γ. Here *finite subtree* means a subtree which is the convex hull of finitely many points. Let's give a definition to make this more precise:

Definition. *Consider two actions of a finitely generated group Γ on two $\mathbb{R}$-trees T and T', and take $\varepsilon > 0$, a finite subset F of Γ, and two finite subtrees $K \subset T$ and $K' \subset T'$. An F-*equivariant ε-approximation* between K and K' is a binary relation $R \subset K \times K'$ satisfying the three following conditions:*

- *for every point $x \in K$, there exists a point $x' \in K'$ such that xRx'*
- *for every point $x' \in K'$, there exists a point $x \in K$ such that xRx'*
- *if xRx' and yRy', then for all $g, h \in F$, the numbers $d_T(g.x, h.y)$ and $d_{T'}(g.x', h.y')$ are ε-close to each other.*

When xRx', we say that x' is an approximation point of x. If T is an action, for any $\varepsilon > 0$, any finite subset F of Γ, and any finite subtree $K \subset T$, consider the set $V_T(\varepsilon, F, K)$ consisting of actions (T', Γ) such that there exists a finite subtree $K' \subset T'$ with an F-equivariant ε-approximation between K and K'. By definition, the sets $V_T(\varepsilon, F, K)$ form a neighbourhood basis of T in the equivariant Gromov topology. Note that ε-approximations behave nicely with respect to the Hausdorff topology: an F-equivariant ε-approximation between K and K' such that K' is at a Hausdorff-distance η from K'_1, naturally defines an F-equivariant $(\varepsilon + 2\delta)$-approximation between K et K'_1.

The equivariant Gromov topology is always finer than the translation lengths topology and is equivalent to the equivariant Gromov topology on sets of non-abelian actions (see [Pau2]).

A group is said to be *small* if it doesn't contain any subgroup isomorphic to the free group F_2. An arc in an $\mathbb{R}$-tree is called *non-degenerate* if it contains more than one point. A small action is an action such that the stabilizer of any non-degenerate arc is small. Note that throughout this article, the stabilizer of a set is understood to be its pointwise stabilizer. We assume from now on that Γ itself is not small. Then any small action of Γ is non-abelian. Moreover, the projectivised space of small actions of Γ is compact in both topologies (see [CuMo, Pau]).

The set of actions of Γ on $\mathbb{R}$-trees is naturally endowed with a right action of $\mathrm{Aut}(\Gamma)$ by precomposition. Since two actions are identified if there exists an equivariant isometry between them, the subgroup $\mathrm{Inn}(\Gamma)$ of inner automorphisms acts trivially on this set, hence we are left with an action of the group $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ of outer automorphisms of Γ.

1.3 Outer space and very small actions

Definition. *Outer space (sometimes called* Culler-Vogtmann space*) is the set CV_n of free (minimal isometric) actions of F_n on simplicial $\mathbb{R}$-trees modulo equivariant homothety.*

CV_n is invariant under the action of $\mathrm{Out}(F_n)$. It is a disjoint union of open simplices obtained by equivariantly modifying the lengths of the edges of a tree in CV_n, and $\mathrm{Out}(F_n)$ preserves this decomposition.

CV_n is contained in the projectivised space of small actions of F_n. Its closure $\overline{CV_n}$ in this space is therefore compact. Moreover, M. Cohen, M. Lustig ([CL]) and M. Bestvina and M. Feighn ([BF2]) have proved that $\overline{CV_n}$ is exactly the space of very small actions of F_n.

Definition. *An action of F_n on an $\mathbb{R}$-tree T is said to be* very small *if*

- *it is small*
- *triod stabilizers are trivial (a triod is the convex hull of three points which are not aligned)*
- *for every $k \neq 0$ and every $g \in F_n$, $\mathrm{Fix}\, g^k = \mathrm{Fix}\, g$.*

2 Origami: folding to approximate

2.1 Definitions of folds

The goal of this section is to describe a tool which will be fundamental in this paper: we use folds to get approximations of some simplicial trees. The idea of folding is not new, J. R. Stallings already used this technique in [Sta], and many others used this notion (see [BF] and [Dun] for instance). However, we will consider not only *edge*-folding but rather *path*-folding in simplicial trees. We first need some technical conditions so that the folds behave nicely.

Definition. *Let T be a simplicial action of F_n without inversion. Let α, β be two embedded edge-paths in T starting from the same point x. We assume that α and β run through the same number of edges and we denote by $\alpha_1, \dots, \alpha_p$ and by $\beta_1, \dots, \beta_p$ the edges of α and β. We say that α and β satisfy the hypothesis (H) if*

$(H1)$ *for all $i = 1, \dots, p$, α_i and β_i have the same length*

$(H2)$ *α_1 and β_1 are distinct edges*

($H3$) *there exists an equivariant orientation of the edges of T (called the folding orientation) such that α_i and β_i are positively oriented for $i = 1, \ldots, p$. In this case, we say that α and β are well oriented.*

Clearly, ($H3$) means that there exists an orientation of the quotient graph T/F_n such that the projections of α and β are well oriented.

Definition. *Let T be a simplicial action of F_n without inversion and let α_1 and β_1 be two oriented edges satisfying (H).*

The elementary fold *between α_1 and β_1 is the quotient of T by the smallest equivariant equivalence relation in T which identifies α_1 with β_1 and also identifies their terminal vertices. The simplicial complex $T/_{\alpha_1\sim\beta_1}$ thus obtained is a tree (see for instance [BF]), it has a natural metric and an isometric action of F_n without inversion. The quotient map $f : T \to T/_{\alpha_1\sim\beta_1}$ is called the folding map.*

Definition. *Let T be a simplicial action of F_n without inversion and let α and β be two edge paths satisfying (H).*

The fold *between $\alpha = \alpha_1 \ldots \alpha_p$ and $\beta = \alpha_1 \ldots \alpha_p$ is the quotient $T/_{\alpha\sim\beta}$ of T by the smallest equivariant equivalence relation in T which identifies α_i with β_i for $i = 1 \ldots p$. It is a composition of elementary folds f_i*

$$T \xrightarrow{f_1} T_1 = T/_{\alpha_1\sim\beta_1} \xrightarrow{f_2} T_2 = T_1/_{f_1(\alpha_2)\sim f_1(\beta_2)} \xrightarrow{f_3} \cdots \xrightarrow{f_p} T_p = T/_{\alpha\sim\beta}.$$

The elementary folds f_i are called intermediate folds. We denote by $q_i = f_i \circ \cdots \circ f_1 : T \to T_i$ and by $q = f_p \circ \cdots \circ f_1 : T \to T/_{\alpha\sim\beta}$ the folding map.

This decomposition shows that $T/_{\alpha\sim\beta}$ is a simplicial tree with a natural isometric action of F_n with no inversion.

2.2 Preimage of an edge

We define the preimage of an edge e' of a simplicial tree T' under a simplicial map $f : T \to T'$ to be the set $f^{-1}(e)$ of edges which map to e under f (and not the set of points in T which are mapped to a point of the closed edge e). The main interest of the hypothesis ($H3$) is the following remark.

Lemma 2.1. *If $f : T \to T/_{\alpha_1\sim\beta_1}$ is the elementary fold between the edges α_1 and β_1 satisfying (H) then $f^{-1}(e)$ is either a single edge or a set of edges having the same origin according to the folding orientation. We say that these edges are centrifugal.*

Remark. If ($H3$) is not satisfied, then $f^{-1}(e)$ may be unbounded.

Proof. $T/_{\alpha_1\sim\beta_1}$ is the quotient of T by the equivalence relation generated by the binary relation $\sim_1$ described by $e \sim_1 e'$ if there exists $g \in F_n$ such that $\{g.e, g.e'\} = \{\alpha_1, \beta_1\}$. One needs only to notice that if $e \sim_1 e' \sim_1 e''$, then e, e', e'' are centrifugal. □

Note that lemma 2.1 implies that if α, β satisfy (H) then none of the intermediate folds can be isometries (i.e. the intermediate folds satisfy $(H2)$) because $f_1(\alpha_2) = f_1(\beta_2)$ would contradict the lemma.

The following corollary is the tool which allows us to get approximations from folds.

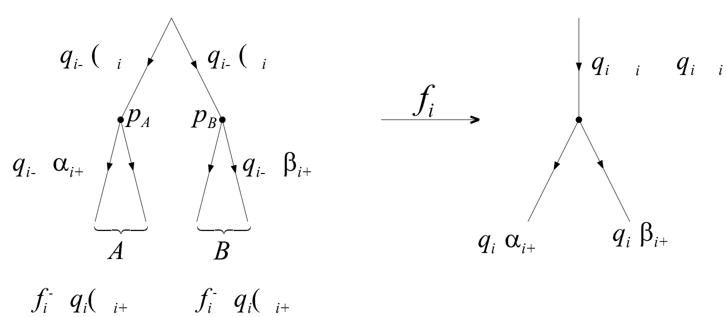
Corollary 2.2. *Take two edge paths α and β in a simplicial tree satisfying (H), and denote by $q : T \to T/_{\alpha\sim\beta}$ the folding map. Suppose that each intermediate fold f_i is a fold between two edges with trivial stabilizer.*

If e, e' are two adjacent *edges in T which are identified by q, then they are identified under the first intermediate fold f_1.*

Remark. The hypothesis on the intermediate folds f_i can be weakened but the corollary is false with no hypothesis at all on f_i.

Proof. Assume on the contrary that there exists an index $i > 0$ such that $q_i(e) \neq q_i(e')$ and $q_{i+1}(e) = q_{i+1}(e')$. This implies that $q_i(e) \neq q_i(e')$ are centrifugal.

Suppose first that $q_i(\alpha_{i+1})$ and $q_i(\beta_{i+1})$ don't lie in the same orbit of T_i. Then, the fact that $\operatorname{Stab} q_i(\alpha_{i+1}) = \operatorname{Stab} q_i(\beta_{i+1}) = \{1\}$ implies that an edge is identified with $q_i(\alpha_{i+1})$ and $q_i(\beta_{i+1})$ through f_{i+1} only if it equals $q_i(\alpha_{i+1})$ or $q_i(\beta_{i+1})$. Therefore, we can assume without loss of generality that $q_i(e) = q_i(\alpha_{i+1})$ and $q_i(e') = q_i(\beta_{i+1})$. Thanks to the previous corollary, $q_{i-1}(e)$ and $q_{i-1}(\alpha_{i+1})$ have the same origin, and similarly for $q_{i-1}(e')$ and $q_{i-1}(\beta_{i+1})$. But this prevents them from being adjacent, which is a contradiction.



Suppose now that there exists an $h \in F_n$ such that $h.q_i(\alpha_{i+1}) = q_i(\beta_{i+1})$ (this h is unique because $\operatorname{Stab} q_i(\alpha_{i+1}) = \{1\}$). In this case, the set of edges which are identified with $q_i(\alpha_{i+1})$ by f_{i+1} is exactly $h^{\mathbb{Z}}.q_i(\alpha_{i+1})$ so we can assume that $q_i(e) = q_i(\alpha_{i+1})$ and $q_i(e') = q_i(h^k.\alpha_{i+1})$ for some $k \neq 0$. Now let A and B be the preimages of $q_i(\alpha_{i+1})$ and $q_i(\beta_{i+1})$ under f_i. The previous corollary implies that A and B are two sets of centrifugal edges whose centers are p_A and p_B, the terminal points of $q_{i-1}(\alpha_i)$ and $q_{i-1}(\beta_i)$. Now since $q_{i-1}(e') \in h^k.A$, and because $q_{i-1}(e)$ and $q_{i-1}(e')$ are centrifugal, $h^k.A$ is a set of centrifugal edges with center p_A. Therefore, h^k fixes p_A and h sends p_A to p_B, so h fixes the midpoint of $[p_A, p_B]$ which is the origin of $q_{i-1}(\alpha_i)$. Hence h^k fixes $q_{i-1}(\alpha_i)$ which contradicts the assumption on the fold f_i. ☐

2.3 Folding to approximate

We are now ready to prove the *folding to approximate lemma.*

Folding to approximate lemma. *Let T be a simplicial action of F_n without inversion. Let α and β be two paths in T with origin x satisfying the (H) condition such that* $\operatorname{Stab} x$ *is infinite. Let w_k be a sequence of distinct elements in* $\operatorname{Stab} x$ *and let $T^{(k)} = T/_{\alpha \sim w_k.\beta}$. Assume that each intermediate fold is a fold between edges with trivial stabilizer.*

Then $T^{(k)}$ converges to T as $k \to \infty$.

Proof. We only need to prove that two incident edges e, e' of T are identified by only finitely many folds $q^{(k)} : T \to T^{(k)}$. As a matter of fact, this will imply that any finite subtree of T isometrically embeds in $T^{(k)}$ under $q^{(k)}$. To prove the convergence in the equivariant Gromov topology, take K to be a finite subtree of T and F a finite subset of F_n, and let K' be the convex hull of K and $F.K$. For k large enough, $q^{(k)}$ is an isometry in restriction to K' hence it gives an F-equivariant 0-approximation between K and $q^{(k)}(K)$.

Now we prove that two incident edges e, e' of T are identified by only finitely many folds $q^{(k)}$. Thanks to the previous corollary, we only need to check that they are identified by finitely many of the elementary folds $f_1^{(k)}$ between α_1 and $w_k.\beta_1$.

When α_1 and β_1 are not in the same orbit, e and e' are identified by $f_1^{(k)}$ if and only if there exists $g \in F_n$ such that $g.\{e, e'\} = \{\alpha_1, w_k.\beta_1\}$. This occurs for at most one k since $\operatorname{Stab} \alpha_1 = \{1\}$, g is unique.

When $\beta_1 = h.\alpha_1$ (h is then unique), e and e' are identified by $f_1^{(k)}$ for some index k if and only if there exists $g \in F_n$ such that $g.e = \alpha_1$ and $g.e' = (w_k.h)^{i_k}.\alpha_1$ for some $i_k \in \mathbb{Z}\backslash\{0\}$. If k_0 and k are such indices, then $g.e' = (w_{k_0}h)^{i_{k_0}}.\alpha_1 = (w_k h)^{i_k}.\alpha_1$ so $(w_{k_0}h)^{i_{k_0}} = (w_k h)^{i_k}$ and $w_k h$ lies in the finite set of roots of $(w_{k_0}h)^{i_{k_0}}$ which can hold for at most finitely many k. □

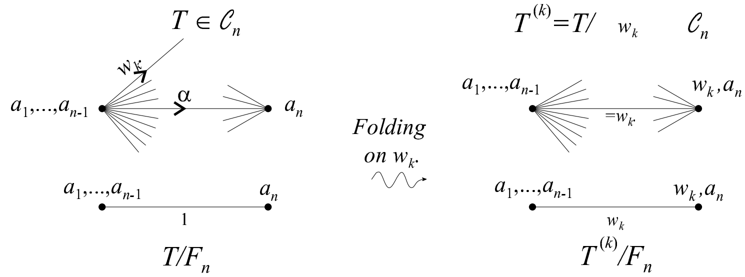
3 An open invariant subset of outer space.

3.1 Looking for an open invariant subset.

This *folding to approximate* lemma will be the cornerstone of section 4. But first, this lemma will show us that some natural candidates for open and invariant sets are in fact not open.

The first candidate for open set in $\overline{CV}_n$ is the set $\mathcal{C}_n$ of very small actions T in which there is a non-degenerate arc I containing no branch point of T and such that $\operatorname{Stab} I = \{1\}$. It is a natural candidate because the set of systems of isometries which give an action in $\mathcal{C}_n$ is precisely the set of systems of isometries whose suspension have a family of compact simply-connected leaves, and this property is stable under perturbation (see prop. IV.1 in [Lev]). Moreover, thanks to the exhaustive study of CV_2 by M. Culler and K. Vogtmann, it is easy to check that $\mathcal{C}_2$ is open in $\overline{CV}_2$. However, $\mathcal{C}_n$ is not open for $n \geq 3$: for instance if $T \in \mathcal{C}$ is the action shown on figure

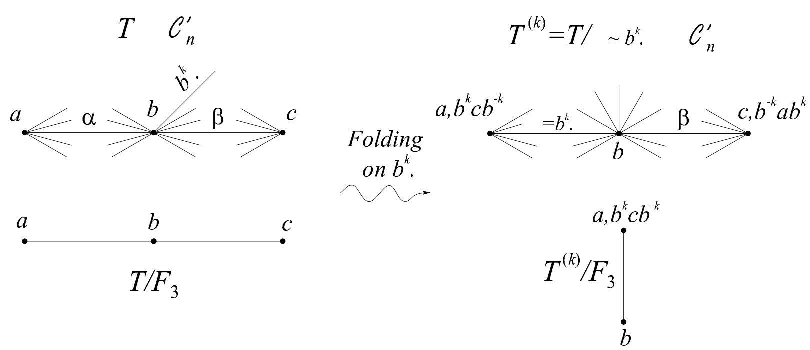
2, a folding operation allows us to approximate T by a very small simplicial action whose edge stabilizers are not trivial. Thus, this approximating action doesn't lie in $\mathcal{C}_n$.



$T \in \mathcal{C}_n$ is approximated by $T^{(k)} = T/_{\alpha \sim w_k.\alpha} \notin \mathcal{C}_n$

Figure 2: $\mathcal{C}_n$ is not open in $\overline{CV}_n$ for $n \geq 3$

In view of this example, we see that the presence of a non cyclic vertex stabilizer allows many approximations, so we may consider a second candidate for open set: the set $\mathcal{C}'_n$ of very small simplicial actions with cyclic edge and vertex stabilizers. Once again, it is natural because one can prove that the set of systems of isometries which define an action in $\mathcal{C}'_n$ is open ([Gui, th. 4.4.5]). One also checks that $\mathcal{C}'_2$ is open in $\overline{CV}_2$. But for $n \geq 3$, the *folding to approximate* lemma shows that $\mathcal{C}'_n$ is not open (see figure 3). The reason is that one can perform folds at a vertex with non-trivial stabilizer which is not terminal in the quotient graph T/F_n (a vertex is terminal if it has valence 1).



$T \in \mathcal{C}'_n$ is approximated by $T^{(k)} = T/_{\alpha \sim w_k.\beta} \notin \mathcal{C}'_n$

Figure 3: $\mathcal{C}'_n$ is not open in $\overline{CV}_n$ for $n \geq 3$.

This leads us to consider the following set $\mathcal{O}_n$:

Definition. *We define $\mathcal{O}_n$ to be the set of simplicial F_n-actions T such that*

- *T has trivial arc stabilizers*
- *T has cyclic vertex stabilizers*
- *whenever* $\operatorname{Stab} v \neq \{1\}$, $\operatorname{Stab} v$ *acts transitively on the set of incident edges.*

Equivalently, T lies in $\mathcal{O}_n$ if the edge groups of the graph of groups T/F_n are trivial, and if the only non-trivial vertex groups are cyclic and are attached to terminal vertices of T/F_n.

3.2 $\mathcal{O}_n$ is open in $\overline{CV}_n$

Theorem 1. *The set $\mathcal{O}_n$ is open in $\overline{CV}_n$ and* $\operatorname{Out}(F_n)$ *acts properly discontinuously on $\mathcal{O}_n$.*

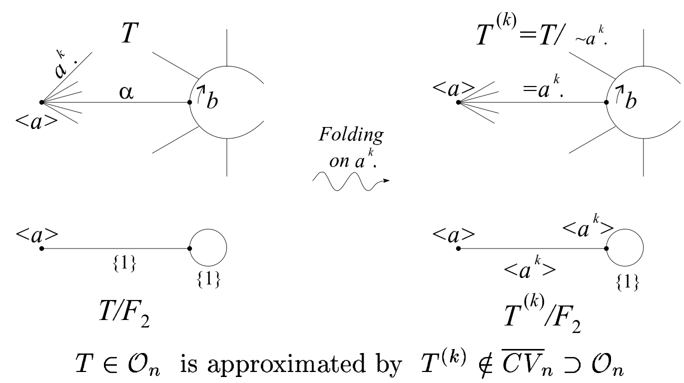


Figure 4: $\mathcal{O}_n$ is not open in the set of small actions.

Remark. Using the *folding to approximate* lemma, one proves that $\mathcal{O}_n$ is not open in the whole set of small actions as shown on figure 4.

Proof of theorem 1. With no additional work, we will prove the well known fact that CV_n is open in the set of all actions of F_n. The proof of Theorem 1 goes as follows: we start with an action $T \in \mathcal{O}_n$ and consider a fundamental domain D for this action. Given an action $T' \in \overline{CV}_n$ close enough to T in the equivariant Gromov topology, there is a finite subtree D' in T' which approximates D. The main step is to build a fundamental domain Δ for T' starting from D'.

Fundamental domain and adapted basis for an action in $\mathcal{O}_n$

Let T be an action in $\mathcal{O}_n$ and consider the quotient metric graph of groups $Q = T/F_n$. Let τ be a maximal subtree of Q and $\tilde{\tau}$ a preferred lift so that we get (using Bass-Serre theory) an identification between F_n and $\pi_1(Q, \tau)$ and an equivariant isometry between T and the universal cover of Q. Now choose an orientation for every edge in

$Q \setminus \tau$ and one generator for every non-trivial vertex group of Q. The set of elements in $\pi_1(Q,\tau)$ corresponding to the edges of $Q \setminus \tau$ with the chosen orientations and to the chosen generators of the vertex groups provides a preferred basis B of F_n.

This basis B has the following property: for every $\gamma \in B \cup B^{-1}$, either $\gamma.\tilde{\tau} \cap \tilde{\tau}$ is a single point that we denote by χ_γ (this happens when γ corresponds to a vertex group generator) or there is an edge joining $\tilde{\tau}$ to $\gamma.\tilde{\tau}$ (when γ comes from an edge in $Q \setminus \tau$) in which case we call χ_γ the midpoint of this edge. We define D to be the union of $\tilde{\tau}$ and of the segments joining χ_γ to $\tilde{\tau}$ ($\gamma \in B \cup B^{-1}$). The following properties clearly hold:

Lemma 3.1. *For every $\gamma \in B \cup B^{-1}$,*

- $\gamma.D \cap D = \{\chi_\gamma\}$
- χ_γ *is a terminal point of* D
- $\chi_\gamma = \chi_{\gamma^{-1}}$ *if and only if γ is elliptic in which case χ_γ is the only fixed point of* γ.

Observe also that D is a fundamental domain for T in the following sense:

Lemma 3.2. *D meets every orbit in T and if $x, y \in D$ are such that $y = w.x$ for some $w \in F_n \setminus \{1\}$, then either*

- $x = \chi_{\gamma^{-1}}$, $y = \chi_\gamma$, *and $w = \gamma$ for some hyperbolic generator $\gamma \in B \cup B^{-1}$,*
- *or $x = y = \chi_\gamma$ and $w = \gamma^p$ for an elliptic $\gamma \in B \cup B^{-1}$.*

Constructing Δ from D'

We now define $V_\varepsilon(T)$ a neighbourhood of T: we set $F = \{1\} \cup B \cup B^{-1}$ and set $V_\varepsilon(T) = V_T(\varepsilon, F, D)$ to be the set of actions $T' \in \overline{CV}_n$ such that there exists a finite subtree D' in T' with an F-equivariant ε-approximation between D and D'. We are going to show that if $T' \in V_\varepsilon(T)$ for some small enough ε, then $T' \in \mathcal{O}_n$.

Denote by d the length of the shortest edge in D, and assume that ε is small enough compared to d. Then any element $\gamma \in B \cup B^{-1}$ which is hyperbolic in T must be hyperbolic in T'. Note however that an elliptic element in $B \cup B^{-1}$ may be hyperbolic in T' (but its translation length must be small compared to d). We start by changing D' into D'_1 such that $\gamma.D'_1 \cap D'_1 = \emptyset$ for $\gamma \in B \cup B^{-1}$.

Definition. *Let K be a finite tree and $\delta > 0$. We call δ-interior of K the set $\mathrm{int}_\delta(K)$ of points of K which are the midpoint of a segment of K of length 2δ.*

Clearly, $\mathrm{int}_\delta(K)$ is a finite subtree of K and if K has diameter at least 2δ, $\mathrm{int}_\delta(K)$ is non-empty and K lies in the δ-neighbourhood of $\mathrm{int}_\delta(K)$.

Lemma 3.3. *Let $\delta = 3\varepsilon$, and let $D'_1 = \mathrm{int}_\delta(D')$. If ε is small enough compared to d, then*

$$\forall \gamma \in B \cup B^{-1} \qquad \gamma.D'_1 \cap D'_1 = \emptyset.$$

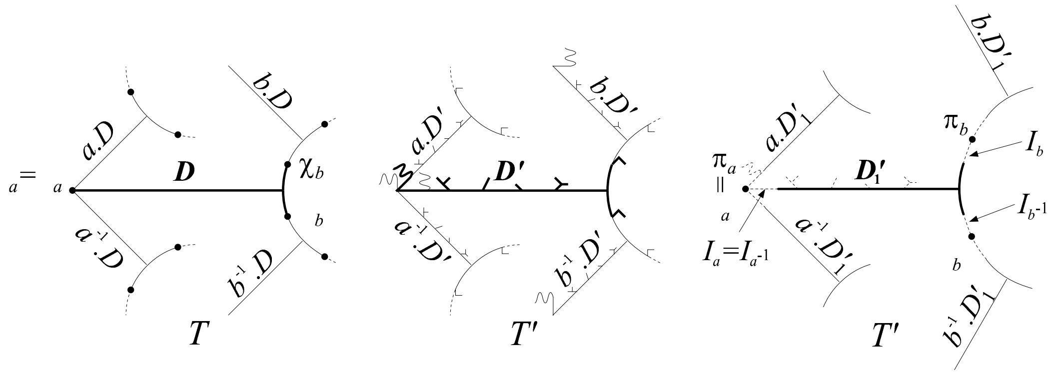


Figure 5: the fundamental domain $D \subset T$, and its approximations, D' and $D'_1 \subset T'$

Proof. Assume on the contrary that there exists $y' = \gamma.x' \in D'_1 \cap \gamma.D'_1$ for some $\gamma \in B \cup B^{-1}$ and argue towards a contradiction. By definition of the δ-interior, there are some points $x'_1, x'_2, y'_1, y'_2 \in D'$ such that x'_1, x', x'_2 (resp. y'_1, y', y'_2) are aligned in this order and δ-far from each other (we say that a, b, c are aligned in this order if $b \in [a, c]$).

Consider some approximation points x_1, x, x_2, y_1, y, y_2 in D of $x'_1, x', x'_2, y'_1, y', y'_2$. Since $y' = \gamma.x'$, $d(\gamma.x, y) \leq \varepsilon$. But $\chi_\gamma \in [\gamma.x, y]$ since $\gamma.x$ and y lie in the two subtrees $\gamma.D$ and D which intersect only in $\{\chi_\gamma\}$. Therefore, x is ε-close to χ_γ^{-1} and every branch point of D is at least $(d - \varepsilon)$-far from x.

The distance from x to its projection p on the segment $[x_1, x_2]$ is at most $3\varepsilon/2$. As a matter of fact, in an $\mathbb{R}$-tree the distance from a point a to its projection on a segment $[b, c]$ is the Gromov product

$$(b|c)_a = \frac{1}{2}\left(d(a, b) + d(a, c) - d(b, c)\right).$$

Now the fact that $d(x, x_1)$ and $d(x, x_2)$ are greater than $\delta - \varepsilon > 3\varepsilon/2$ implies that p has to be distinct from x_1 and x_2. Moreover, p cannot be a branch point of D if ε is small compared to d. This means that x must lie in $[x_1, x_2]$. But then, since $[\chi_{\gamma^{-1}}, x]$ doesn't contain any branch point of D, either $\chi_{\gamma^{-1}}, x_1, x, x_2$ or $\chi_{\gamma^{-1}}, x_2, x, x_1$ are aligned in this order. Since $d(x, x_1), d(x, x_2) \geq \delta - \varepsilon > \varepsilon$, this prevents x from being ε-close to $\chi_{\gamma^{-1}}$ which gives a contradiction. ☐

Since D' is in the δ-neighbourhood of D'_1, there is an F-equivariant ε_1-approximation between D and D'_1 for $\varepsilon_1 = \varepsilon + 2\delta = 7\varepsilon$. Hence, we forget the approximation between D and D' and we concentrate on the approximation between D and D'_1. For every $\gamma \in B \cup B^{-1}$ we choose an approximation point $\chi'_\gamma \in D'_1$ of χ_γ.

It will now be easy to construct a fundamental domain Δ for T' by adding to D'_1 the segments I_γ defined as follows (see figure 5):

Definition. *For every* $\gamma \in B \cup B^{-1}$*, we define the points* π_γ *in* T' *and the segments* $I_\gamma \subset T'$ *as follows:*

- *if* γ *is elliptic in* T'*, we call* $\pi_\gamma = \pi_{\gamma^{-1}}$ *the projection of* D'_1 *on* $\mathrm{Fix}_{T'}\,\gamma$ *and we take* $I_\gamma = I_{\gamma^{-1}}$ *to be the segment joining* D'_1 *to* π_γ*.*
- *if* γ *is hyperbolic in* T'*, we call* π_γ *the midpoint of the intersection of* $\mathrm{Axis}_{T'}(\gamma)$ *with the segment joining* D'_1 *to* $\gamma.D'_1$ *(so that* $\gamma.\pi_{\gamma^{-1}} = \pi_\gamma$*). We then take* I_γ *to be the segment joining* D'_1 *to* π_γ*.*

We then define $\Delta = D'_1 \cup \bigcup_{\gamma \in B \cup B^{-1}} I_\gamma$*.*

Remark. Note that by minimality, Δ meets every orbit of T': the fact that for all $\gamma \in B \cup B^{-1}$, $\gamma.\Delta \cap \Delta \neq \emptyset$ implies that $F_n.\Delta$ is connected and invariant, and hence must be equal to T'.

Lemma 3.4. *The arc* I_γ *is contained in the* ε_1*-neighbourhood of* χ'_γ*.*

Proof. Since I_γ is contained in the segment joining D'_1 to $\gamma.D'_1$, every segment $[p, q]$ with $p \in D'_1$ and $q \in \gamma.D'_1$ contains I_γ. Hence $I_\gamma \subset [\chi'_\gamma, \gamma.\chi'_{\gamma^{-1}}]$, but since $\chi_\gamma = \gamma.\chi_{\gamma^{-1}}$, we get $d(\chi'_\gamma, \gamma.\chi'_{\gamma^{-1}}) \leq \varepsilon_1$. □

This lemma implies that there is an F-equivariant ($\varepsilon_2 = 3\varepsilon_1$)-approximation between D and Δ for which π_γ is an approximation point of χ_γ.

Δ is a fundamental domain for T'

Lemma 3.5. *If* ε *is small enough compared to* d*, then* π_γ *is a terminal point of* Δ*, and for all* $\gamma \in B \cup B^{-1}$*,*

$$\gamma.\Delta \cup \Delta = \{\pi_\gamma\}.$$

Moreover, one has $\pi_\gamma = \pi_{\gamma'}$ *if and only if* $\gamma = \gamma'$ *or* $\gamma = \gamma'^{-1}$ *with* γ *elliptic in* T'*. Finally, if* γ *is elliptic in* T'*, then the germ of* Δ *at* π_γ *is not fixed by* γ*, and if* γ *is hyperbolic, the germ of* Δ *at* π_γ *points towards the negative half-axis of* γ*.*

Proof. By construction, it is clear that for all $\gamma \in B \cup B^{-1}$,

$$\gamma.\left(D'_1 \cup I_\gamma \cup I_{\gamma^{-1}}\right) \cap \left(D'_1 \cup I_\gamma \cup I_{\gamma^{-1}}\right) = \{\pi_\gamma\}.$$

To prove that $\gamma.\Delta \cap \Delta = \{\pi_\gamma\}$ we just have to check that

$$\gamma.I_{\gamma'} \cap \Delta = \emptyset$$

for $\gamma' \in B \cup B^{-1} \setminus \{\gamma, \gamma^{-1}\}$. But since $I_{\gamma'}$ is contained in the ε_1-neighbourhood of $\pi_{\gamma'}$, and since $\gamma.\chi_{\gamma'}$ is far apart from D (at least at a distance d), we see that $\gamma.\pi_{\gamma'}$ must be far apart from Δ.

Now by construction, π_γ is a terminal point of $D'_1 \cup I_\gamma \cup I_{\gamma^{-1}}$ and must remain terminal in Δ since the intervals added to $D'_1 \cup I_\gamma \cup I_{\gamma^{-1}}$ are far from $I_\gamma \cup I_{\gamma^{-1}}$. The last claims follow immediately. □

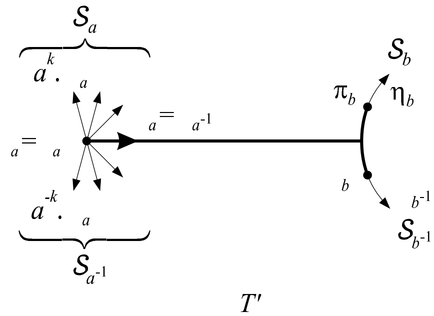


Figure 6: Δ and the germs η_γ

To prove that Δ is a fundamental domain for T', we use the same technique as [CV2]. We first introduce some notations.

Definition. *Let $\gamma \in B \cup B^{-1}$.*

- *If γ is hyperbolic in T', we call η_γ the germ of the positive axis of γ at π_γ. We also set*
$$\mathcal{S}_\gamma = \left\{ x \in T' \setminus \{\pi_\gamma\} \,\middle|\, \mathrm{germ}_{\pi_\gamma}([\pi_\gamma, x]) = \eta_\gamma \right\}.$$

- *If γ is elliptic in T', we call $\eta_\gamma = \eta_{\gamma^{-1}}$ the germ of Δ at π_γ. We also set*
$$\mathcal{S}_\gamma = \left\{ x \in (T', F_n) \setminus \{\pi_\gamma\} \,\middle|\, \exists k > 0 \text{ s.t. } \mathrm{germ}_{\pi_\gamma}([\pi_\gamma, x]) = \gamma^k.\eta_\gamma \right\}.$$

Lemma 3.6. *If T' is very small, or if T is free simplicial, and if ε is small enough, then the sets $\mathcal{S}_\gamma$ are pairwise disjoint and do not meet Δ.*

Proof. Assume that $\gamma \in B \cup B^{-1}$ is hyperbolic in T'. Since η_γ and the germ of Δ at π_γ point respectively towards the positive and negative half-axis of γ, $\mathcal{S}_\gamma$ doesn't intersect Δ.

Assume now that $\gamma \in B \cup B^{-1}$ is elliptic in T'. Then γ has to be elliptic in T (so that T is not free). Here we use the hypothesis that T' is very small: since γ doesn't fix η_γ, γ^k doesn't fix η_γ so the germs $\gamma^k\eta$ are all distinct and $\mathcal{S}_\gamma$, $\mathcal{S}_{\gamma^{-1}}$ and Δ are pairwise disjoint.

There remains only to prove that $\mathcal{S}_\gamma \cap \mathcal{S}_{\gamma'} = \emptyset$ when $\pi_\gamma \neq \pi_{\gamma'}$. But then $\overline{\mathcal{S}_\gamma} = \mathcal{S}_\gamma \cup \pi_\gamma$ is a subtree of T' which cannot intersect $\overline{\mathcal{S}_{\gamma'}}$ because otherwise their union would be connected hence would contain $[\pi_\gamma, \pi_{\gamma'}] \subset \Delta$ which is impossible. □

Remark. We note the following facts:

- $\gamma.\pi_{\gamma^{-1}} = \pi_\gamma$
- $\gamma.(\Delta \setminus \{\pi_{\gamma^{-1}}\}) \subset \mathcal{S}_\gamma$
- $\gamma.\mathcal{S}_{\gamma'} \subset \mathcal{S}_\gamma$ for all $\gamma' \in (B \cup B^{-1}) \setminus \{\gamma^{-1}\}$

Corollary 3.7. *The finite tree Δ is a fundamental domain for T' in the following sense:*

- *Δ meets every orbit in T'*
- *assume that $x, y \in \Delta$ are such that $x = w.y$ with $w \neq 1$. Then*
  - *either $x \neq y$ in which case $x = \pi_{\gamma^{-1}}$, $y = \pi_\gamma$ and $w = \gamma$ for some $\gamma \in B \cup B^{-1}$ hyperbolic in T'*
  - *or $x = y = \pi_\gamma$ and $w = \gamma^p$ for some $\gamma \in B \cup B^{-1}$ elliptic in T'.*

Proof. We have already noted that Δ meets every orbit. Now write $w = \gamma_p \dots \gamma_1$ as a reduced word with $\gamma_i \in B \cup B^{-1}$. The previous remark shows that $\gamma_i \dots \gamma_1.x \in \mathcal{S}_{\gamma_i} \cup \{\pi_{\gamma_i}\}$. Moreover, if for some index i, $\gamma_i \dots \gamma_1.x \neq \pi_{\gamma_{i+1}^{-1}}$, then $\gamma_{i+1} \dots \gamma_1.x \in \mathcal{S}_{\gamma_{i+1}}$. We thus get inductively that $\gamma_p \dots \gamma_1.x \in \mathcal{S}_{\gamma_p}$ so $\gamma_p \dots \gamma_1.x$ can't lie in Δ. Therefore, $x = \pi_{\gamma_1^{-1}}$ and if $p = 1$, we are done. Otherwise, we have $\pi_{\gamma_1} = \pi_{\gamma_2^{-1}}$ which implies $\gamma_2 = \gamma_1$ since w is reduced. We obtain recursively that $w = \gamma_1^p$, and γ_1 is elliptic in T' since $\pi_{\gamma_1} = \pi_{\gamma_1^{-1}}$. □

$\mathcal{O}_n$ is open in $\overline{CV_n}$

Proposition 3.8. *$\mathcal{O}_n$ is open in $\overline{CV_n}$ and CV_n is open in the set of all actions of F_n on $\mathbb{R}$-trees.*

Proof. Take T' close enough to an action T in $\mathcal{O}_n$ and assume moreover that T' is very small in the case when T is not free. Then the set Δ constructed above is a fundamental domain in the sense of corollary 3.7.

First of all, T' is simplicial because T' is the union of the translates of Δ and $w.\Delta$ may only meet $w_0.\Delta$ in some $w_0.\pi_\gamma$. Moreover, if T is free, corollary 3.7 shows that T' is free.

If T is not free, corollary 3.7 implies that edge stabilizers are trivial and that vertex stabilizers are cyclic. Now if x has non-trivial stabilizer, we may assume (up to the action of F_n) that $x = \pi_\gamma$ and $\operatorname{Stab} x = \gamma^{\mathbb{Z}}$ for some $\gamma \in B \cup B^{-1}$. But since Δ and the sets $\mathcal{S}_{\gamma'}$ cover T', the set of germs at π_γ is $\gamma^{\mathbb{Z}}.\eta_\gamma$. This implies that $\operatorname{Stab} x$ acts transitively on the set of edges incident to x. We thus conclude that $T' \in \mathcal{O}_n$. □

The stabilizer in $\operatorname{Out}(F_n)$ of every $T \in \mathcal{O}_n$ is finite.

Lemma 3.9. *The stabilizer in $\operatorname{Out}(F_n)$ of every $T \in \mathcal{O}_n$ is finite.*

Proof. Assume that $\alpha \in \operatorname{Aut}(F_n)$ fixes T i. e. that there is an equivariant homothety h between T and $T.\alpha$. This homothety naturally induces a homothety of the finite metric graph T/F_n, which implies that h must be an isometry. Since $\operatorname{Id} : T \to T.\alpha$ is α-equivariant (i. e. $\operatorname{Id}(g.x) = \alpha(g).\operatorname{Id}(x)$) $f = \operatorname{Id} \circ h : T \to T$ is α-equivariant.

In [Bass], it is proved that f induces an *automorphism* φ of the graph of groups $Q = T/F_n$ in the following sense:

Definition. *If Q is a graph of groups, an automorphism φ of Q consists in*

- *an automorphism φ of the underlying graph (an isometry in our case)*
- *an isomorphism $\varphi_v : \Gamma_v \to \Gamma_{\varphi(v)}$ for each vertex $v \in Q$*
- *for every oriented edge e of Q, an isomorphism $\varphi_e : \Gamma_e \to \Gamma_{\varphi(e)}$ such that $\varphi_e = \varphi_{\bar{e}}$*
- *for each vertex $v \in Q$, an element $\gamma_v \in \pi_1(Q, \varphi(v))$*
- *for every oriented edge e of Q, an element $\gamma_e \in \pi_1(Q, \varphi(o(e)))$ such that $\delta_e := \gamma_{o(e)}^{-1}\gamma_e \in \Gamma_{o(e)}$*

such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma_e & \xrightarrow{i_{\bar{e}}} & \Gamma_{o(e)} \\ \downarrow{\scriptstyle \varphi_e} & & \downarrow{\scriptstyle I_{\delta_e}^{-1} \circ \varphi_{o(e)}} \\ \Gamma_{\varphi(e)} & \xrightarrow{i_{\varphi(\bar{e})}} & \Gamma_{\varphi(o(e))} \end{array}$$

Remark. When edge stabilizers are trivial, the diagram automatically commutes.

Such a morphism φ induces α on the fundamental group of Q in the following sense: one defines $\varphi^* : \pi_1(Q, v) \to \pi_1(Q, \varphi(v))$ by setting, for every loop $(g_0, e_1, g_1, \ldots, e_n, g_n)$ in the graph of groups Q

$$\varphi^*(g_0 e_1 \ldots e_n g_n) = \left(\gamma_{v_0}\varphi_{v_0}(g_0)\gamma_{v_0}^{-1}\right)\left(\gamma_{e_1}\varphi(e_1)\gamma_{\bar{e}_1}^{-1}\right)\ldots\left(\gamma_{e_n}\varphi(e_n)\gamma_{\bar{e}_n}^{-1}\right)\left(\gamma_{v_n}\varphi_{v_n}(g_n)\gamma_{v_n}^{-1}\right).$$

Then there exists a path p_v in the graph of groups Q joining v to $\varphi(v)$ such that the induced morphism $I_{p_v} : \pi_1(Q, \varphi(v)) \to \pi_1(Q, v)$ is such that $I_{p_v} \circ \varphi^*$ induces α on $\pi_1(Q, v)$ (see [Bass]).

Denote by $\mathrm{Aut}(Q)$ the group of automorphisms of the graph of groups Q and $\mathrm{Aut}_0(Q)$ the finite index subgroup of $\mathrm{Aut}(Q)$ consisting of automorphism inducing the identity on the underlying graph of Q. We only have to prove that $\mathrm{Aut}_0(Q)$ has finite image in $\mathrm{Out}(\pi_1(Q, v))$.

Let $(g_0, e_1, g_1, \ldots, e_n, g_n)$ be a loop based at v in the graph of groups Q. If v_i is a terminal vertex of Q, $e_{i-1} = \bar{e}_i$ and since Γ_{v_i} is abelian, $\gamma_{\bar{e}_{i-1}}^{-1}\gamma_{v_i}$ commutes with $\varphi_{v_i}(g_i)$. This implies

$$(\gamma_{e_{i-1}} e_{i-1} \gamma_{\bar{e}_{i-1}}^{-1})(\gamma_{v_i}\varphi_{v_i}(g_i)\gamma_{v_i}^{-1})(\gamma_{e_i} e_i \gamma_{\bar{e}_i}^{-1}) = \gamma_{e_{i-1}} e_{i-1} \varphi_{v_i}(g_i) e_i \gamma_{e_{i-1}}^{-1}.$$

If v_i is not a terminal point in Q then $\gamma_{\bar{e}_{i-1}}^{-1}\gamma_{v_i} \in \Gamma_{v_i} = \{1\}$ so

$$(\gamma_{e_{i-1}} e_{i-1} \gamma_{\bar{e}_{i-1}}^{-1})(\gamma_{v_i}\varphi_{v_i}(g_i)\gamma_{v_i}^{-1})(\gamma_{e_i} e_i \gamma_{\bar{e}_i}^{-1}) = \gamma_{e_{i-1}} e_{i-1} \varphi_{v_i}(g_i) e_i \gamma_{\bar{e}_i}^{-1}.$$

Therefore, when $v = v_0 = v_n$ is not terminal (which we may assume without loss of generality), we derive that

$$\varphi^*(g_0 e_1 \dots e_n g_n) \quad = \quad \gamma_{v_0}\varphi_{v_0}(g_0)\varphi(e_1)\varphi_{v_1}(g_1)\dots\varphi_{v_{n-1}}(g_{n-1})\varphi(e_n)\varphi_{v_n}(g_n)\gamma_{v_0}^{-1}.$$

Therefore, the image in $\mathrm{Out}(\pi_1(Q,v))$ of $\mathrm{Aut}_0(Q)$ is a quotient of the direct product of the automorphism groups of the vertex groups Γ_{v_i}, which are finite since Γ_{v_i} are cyclic. □

The action of $\mathrm{Out}(F_n)$ on $\mathcal{O}_n$ is properly discontinuous

There is a natural decomposition of $\mathcal{O}_n$ into open simplices: if $T \in \mathcal{O}_n$, we call $\sigma(T) \subset \mathcal{O}_n$ the set of actions obtained by changing equivariantly the lengths of the edges of T (each length remaining non-zero). These open simplices form a partition of $\mathcal{O}_n$ which is preserved by $\mathrm{Out}(F_n)$. Moreover, there are finitely many orbits of simplices since such an orbit corresponds to an unmarked graph of groups (with no metric) which appears as a quotient of an action in $\mathcal{O}_n$.

We also consider the set $\bar{\sigma}(T)$ (which may not be a closed simplex) of actions in $\mathcal{O}_n$ obtained by changing equivariantly the lengths of the edges of T (here 0 is allowed but the action obtained must lie in $\mathcal{O}_n$). We then call $St(T)$ the *star* of T, i. e. the set of actions T' such that $T \in \bar{\sigma}(T')$. Equivalently, $St(T)$ is the set of simplicial actions T' such that there exists an equivariant application from T' to T which preserves alignment. Of course, $St(T)$ is a union of open simplices and if $T' \in St(T)$, $St(T) \subset St(T')$. This union is finite because if σ_n was an infinite sequence of such simplices, then up to taking a subsequence, there would exist $\alpha_n \in \mathrm{Out}(F_n)$ sending σ_n to σ_0 and stabilizing $\sigma(T)$, but this contradicts the fact that the stabilizer of the barycenter of $\sigma(T)$ is finite.

Proposition 3.10. *For all $T \in \mathcal{O}_n$, $St(T)$ is open in $\overline{CV}_n$.*

Remark. This proposition implies that an action T in $\mathcal{O}_n \setminus CV_n$ may only be approximated in CV_n by actions T' whose quotient graphs have a separating edge. Indeed, for $T' \in CV_n$ close enough to T, T' lies in $\sigma(T)$ which means that there is an equivariant map $T' \to T$ preserving alignment. The preimage of a terminal edge of T/F_n is a separating edge in T'/F_n. Since the quotient graphs of the actions contained in the spine of outer space have no separating edge, this means that the closure in $\overline{CV}_n$ of the spine of outer space is contained in $\mathcal{F}_n = \overline{CV}_n \setminus \mathcal{O}_n$.

Proof. We consider T' close enough to T and $\Delta \subset T'$ the fundamental domain of T' constructed above. We note that Δ is the convex hull of the points π_γ for $\gamma \in B \cup B^{-1}$ (because this convex hull meets every orbit of T' by minimality of T').

To prove the proposition we have to find an equivariant map from T' to T preserving alignment. The following lemma gives a map from Δ to D, linear on each edge of Δ, which preserves alignment and sends π_γ to χ_γ. Because of corollary 3.7, such an application naturally extends to an equivariant map from T' to T which preserves alignment. □

Lemma 3.11. *Let D and Δ be two finite trees together with an ε-approximation between them. Assume that for every terminal vertex π_γ of Δ there is an approximation point χ_γ which is terminal in D. If ε is small compared to the length d of the shortest edge of D, then there exists a natural application $f : \Delta \to D$ linear on the edges of Δ, preserving alignment and sending π_γ to χ_γ.*

Proof. We first define f on the terminal vertices of Δ by sending π_γ to χ_γ. To extend f to a branch point b of Δ, we consider a triod $(\pi_{\gamma_1}, \pi_{\gamma_2}, \pi_{\gamma_3})$ such that $\{b\} = [\pi_{\gamma_1}, \pi_{\gamma_2}] \cap [\pi_{\gamma_2}, \pi_{\gamma_3}] \cap [\pi_{\gamma_3}, \pi_{\gamma_1}]$ and we want to set $\{f(b)\} = [\chi_{\gamma_1}, \chi_{\gamma_2}] \cap [\chi_{\gamma_2}, \chi_{\gamma_3}] \cap [\chi_{\gamma_3}, \chi_{\gamma_1}]$. Note that $f(b)$ is either a branch point or a terminal point of D (this happens when f identifies two terminal points of Δ) hence it is a vertex of D.

The point $f(b)$ is independent of the choice of the triod because $f(b)$ is $(3\varepsilon/2)$-close of an approximation point b' of b and two vertices of D are at least at a distance d. By construction, f preserves alignment in restriction to the set of branch points and terminal vertices of D. just extend f linearly on edges to conclude. □

Proposition 3.12. *The action of* $\mathrm{Out}(F_n)$ *on* $\mathcal{O}_n$ *is properly discontinuous.*

Proof. Let K be a compact subset of $\mathcal{O}_n$. Since K is covered by a finite number of stars $St(T_i)$ each of which is a finite union of open simplices, K is covered by finitely many open simplices. Since the decomposition of $\mathcal{O}_n$ into simplices is equivariant, the proposition reduces to proving that the stabilizer of an open simplex is finite. Thus, the proposition follows from the fact that the barycenter of a simplex in $\mathcal{O}_n$ has finite stabilizer. □

This completes the proof of Theorem 1. □

4 Dynamics of $\mathrm{Out}(F_n)$ on $\mathcal{F}_n$.

In this section, we study the dynamics of $\mathrm{Out}(F_n)$ on the closed invariant subset $\mathcal{F}_n = \overline{CV}_n \setminus \mathcal{O}_n$ of the boundary of outer space.

Theorem 2. *Let T be* simplicial *in $\mathcal{F}_n$ and let T' be any small action $(n \geq 3)$. Then there exists a sequence α_k of elements of* $\mathrm{Out}(F_n)$ *such that*

$$\lim_{k\to\infty} T'.\alpha_k = T$$

The following corollary is a straightforward consequence of Theorem 2:

Corollary 4.1. *For $n \geq 3$, there exists precisely one minimal non-empty closed invariant subset of outer space. This set $\mathcal{M}_n$ is the closure of the orbit of any simplicial action lying in $\mathcal{F}_n$ under the action of* $\mathrm{Out}(F_n)$.

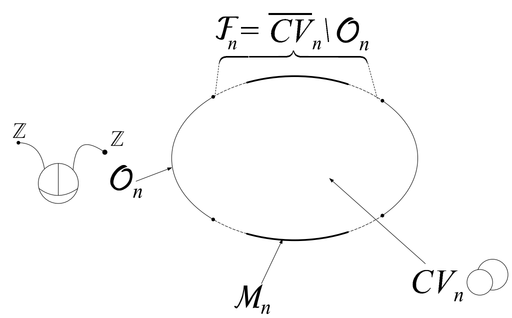


Figure 7: The boundary of outer space

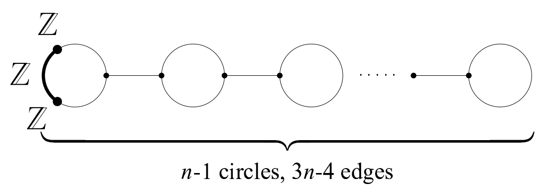


Figure 8: A $(3n-5)$-dimensional simplex in $\mathcal{M}_n$

Remark. In [GL], D. Gaboriau and G. Levitt show that the topological dimension of ∂CV_n equals $3n-5$, thus refining a theorem by M. Bestvina and M. Feighn ([BF2]). It is easy to find a simplex of simplicial actions of dimension $3n-5$ in $\mathcal{F}_n$ and hence in $\mathcal{M}_n$ (see figure 8). Therefore, the topological dimension of every open set in $\mathcal{M}_n$ equals $3n-5$.

Another easy consequence of Theorem 2 is that the set of actions in $\mathcal{M}_n$ having trivial stabilizer in $\mathrm{Out}(F_n)$ is a dense G_δ in $\mathcal{M}_n$.

The proof of Theorem 2 is analogous to the proof of the minimality of the action of the mapping class group of an orientable surface on the boundary of its Teichmüller space. The first step is a theorem by Cohen-Lustig about dynamics of Dehn twists in $\overline{CV}_n$ which is the analogue of the fact that if $i(c,\mathcal{F}) \neq 0$ for a curve c and a measured foliation $\mathcal{F}$, then iterating Dehn twists around c on $\mathcal{F}$ makes it converge to c. In a second step, we introduce a particular kind of action which we call "special curve". A special curve T has the property that given any action $T' \in \overline{CV}_n$, there exists an automorphism $\alpha_0 \in \mathrm{Out}(F_n)$ such that "$i(T, T'.\alpha_0) \neq 0$". Using the dynamics of Dehn twists, we see that the $\mathrm{Out}(F_n)$-orbit of T' accumulates on T. In a third step, using the *folding to approximate* technique, we show that any *simplicial* action $T \in \mathcal{F}_n$ may be approximated by a "special curve", which proves Theorem 3.

4.1 Dynamics of Dehn twists.

In [CL], M. Cohen and M. Lustig study the dynamics of *multiple Dehn twists.* This theorem will be the engine enabling us to show that $\mathrm{Out}(F_n)$-orbits accumulate on some actions.

Definition ([CL]). *Let Q be a graph of groups and e_0 be an oriented edge of Q. Consider an element z_0 of the center of the edge group γ_e. We still denote by e the element of the Bass group $\beta(Q)$ corresponding to an edge e. The* single Dehn twist *with twistor z_0 is the automorphism of $\pi_1(Q,v)$ induced by the automorphism D of $\beta(Q)$ defined by*

- $D(e_0) = e_0.i_{e_0}(z_0)$, $D(e_0^{-1}) = e_0^{-1}.i_{e_0^{-1}}(z_0)^{-1}$
- $D(e) = e$ *for every edge e distinct from* e_0, e_0^{-1}
- $D(r) = r$ *for every element r of a vertex group.*

The non-oriented edge corresponding to e_0 is called the twisted edge. A multiple Dehn twist *of Q is the (commuting) composition of single Dehn twists D_k on distinct edges.*

Recall that a graph of actions on $\mathbb{R}$-trees $\mathcal{Q}$ is a graph of groups Q together with the following data: for every vertex $v \in Q$ there is an action of the vertex group Γ_v on an $\mathbb{R}$-tree T_v (which may be reduced to one point), and for every oriented edge e, an attaching point $p_e \in T_{t(e)}$ fixed by $i_e(\Gamma_e)$. A graph of actions naturally defines an $\mathbb{R}$-tree $T_{\mathcal{Q}}$ endowed with an action of $\pi_1(Q)$ (see [Lev2], or [CL, combination lemma]).

If Q is a graph of groups, we denote by $\mathrm{triv}(Q)$ the set of edges of Q with trivial edge group (it can be identified with the union of the open edges with trivial group in the geometric realization of the graph underlying Q). If $\tilde{Q}$ is a subtree of T, $\mathrm{triv}(\tilde{Q})$ is the set of edges of $\tilde{Q}$ with trivial stabilizer.

Cohen and Lustig's theorem about dynamics of Dehn twists ([CL])

The data. *Let T be a very small simplicial action of F_n and let $Q = T/F_n$ be the quotient graph of groups whose fundamental group is identified with F_n. Consider a union A of connected components of $Q \setminus \mathrm{triv}(Q)$. For every component A_0 of A, we consider an $\mathbb{R}$-tree T_{A_0} endowed with a small action of $\Gamma_{A_0} = \pi_1(A_0)$ and an attaching point $p_e \in T_{A_0}$ for each edge in $Q \setminus A$ incident on A_0.*

The construction. *Let T' be the simplicial action obtained by collapsing to one point every connected component of the preimage of A in T. The graph of groups $Q' = T'/F_n$ is obtained by collapsing each connected component A_0 of A and the corresponding vertex group is $\Gamma_{A_0} = \pi_1(A_0)$. We denote by $\frac{1}{k}T_{A_0}$ the action obtained by dividing by k the metric of T_{A_0}. We denote by $\mathcal{Q}_k$ the graph of actions obtained from Q' by attaching to a vertex A_0 the tree $\frac{1}{k}T_{A_0}$, and the trivial action for any vertex of Q' which is not the image of a component of A. The attaching points are the points p_e. We denote by T_k the F_n-action corresponding to $\mathcal{Q}_k$.*

The result. *Under the hypothesis that every edge group in A_0 has no fixed point in T_{A_0}, there exists Dehn twists D_k on Q such that the sequence of actions $T'_k = T_k.D_k$ converges to T as $k \to \infty$ (in the projectivised space).*

Here is a consequence of this theorem: take T is a very small simplicial action for which none of the edge stabilizers is trivial, and assume that T' is a small action such that for every edge e of T, the stabilizer of e has no fixed point in T'. Then there exists Dehn twists D_k such that $T'.D_k \xrightarrow{k\to\infty} T$ (just take $A = T/F_n$). Therefore, we may think of an action $T \in \overline{CV}_n$ whose edge stabilizers are all non-trivial as an analogue of a (non-connected) *curve* in a surface. In this analogy, assuming that every edge stabilizer of T has no fixed point in T' corresponds to supposing that the intersection number of every connected component of a (non-connected) curve ($\sim T$) with a measured foliation ($\sim T'$) is non-zero. So, by analogy, (and to make notations lighter), we will say "$i(T, T') \neq 0$" when it is satisfied. So we may restate this particular case of the theorem as follows:

Corollary 4.2. *If $T \in \overline{CV}_n$ is a "curve" and if T' is a small action such that "$i(T, T') \neq 0$", then there exist Dehn twists D_k such that $T'.D_k$ converges to T.*

Remark. We have no satisfactory definition of what could be an intersection number $i(T, T')$ for reasonable actions $T, T' \in \overline{CV}_n$, that's why we keep quotes in the notation "$i(T, T') \neq 0$".

4.2 "Special curves": some actions on which every $\mathrm{Out}(F_n)$-orbit accumulates

Definition. *We say that an action $T \in \overline{CV}_n$ is a "special curve" if it is a "curve" (i. e. it is simplicial, very small, and every edge stabilizer is non-trivial) and if there exists a basis $\langle a_1, \ldots, a_n \rangle$ of F_n such that for all $g \in F_n$ fixing an edge in T, g or g^{-1} is a conjugate of a* positive *word in the a_i's.*

This condition is essentially technical. However, it is useful in the following proposition:

Proposition 4.3. *Let $T \in \overline{CV}_n$ be a "special curve". Then the $\mathrm{Out}(F_n)$-orbit of any small action accumulates on T.*

Proof. Using corollary 4.2 we only have to prove that there exists $\alpha \in \mathrm{Out}(F_n)$ such that "$i(T, T'.\alpha) \neq 0$". We know that there exists a basis B of F_n such that every edge stabilizer of T is generated by a conjugate of a positive word in B. Therefore, the problem reduces to showing that there exists a basis B' of F_n such that every non-trivial positive word in this basis is hyperbolic in T': one can just take α to be the automorphism sending B to B'. So we just have to prove the following lemma. □

Lemma 4.4. *For any small action T of F_n, there exists a basis in which every non-trivial positive word in this basis is hyperbolic in T.*

Proof. We first show that for any action of F_n with no global fixed point, there exist a basis of F_n containing a hyperbolic element. Start with any basis $\langle a_1, \ldots, a_n \rangle$ of F_n and assume that every a_i is elliptic. Then $a_i a_j$ is elliptic if and only if $\mathrm{Fix}\, a_i \cap \mathrm{Fix}\, a_j \neq$

$\emptyset$. If for every $i \neq j$, the basis $\langle a_1, \ldots, a_{i-1}, a_i a_j, a_{i+1}, \ldots, a_n \rangle$ is composed of elliptic elements only, then T has a global fixed point (Serre's lemma).

We now prove that any small action has a basis composed of hyperbolic elements. We first notice that if a is hyperbolic and b is elliptic, ab is hyperbolic unless $\mathrm{Fix}\, b \cap \mathrm{Axis}\, a$ contains exactly one point x. Moreover, if $b.\mathrm{Axis}\, a \neq \mathrm{Axis}\, a$, one easily checks that for k large enough, $a^k b$ is hyperbolic. But if $b.\mathrm{Axis}\, a = \mathrm{Axis}\, a$, then b^2 and ab^2a^{-1} fix this axis and don't commute which contradicts the fact that T is small. Now, starting with a basis $\langle a_1, \ldots, a_n \rangle$ of F_n such that a_1 is hyperbolic, there exist integers k_i such that the basis $\langle a_1, a_1^{k_2} a_2, \ldots, a_1^{k_n} a_n \rangle$ consists of hyperbolic elements.

From this basis $\langle b_1, \ldots, b_n \rangle$, we can deduce another basis consisting of hyperbolic elements whose axes have a common non-degenerate segment I and whose orientations coincide: take I to be any non-degenerate interval in $\mathrm{Axis}\,(b_1)$ and note that for k_i large enough, $b_1^{k_i} b_i b_1^{k_i}$ is hyperbolic and its axis contains I. So one may take a basis of the form $\langle b_1, (b_1^{k_2} b_2 b_1^{k_2})^{\pm 1}, \ldots, (b_1^{k_n} b_n b_1^{k_n})^{\pm 1} \rangle$. To prove that every positive word in this basis is hyperbolic, we just to notice that if a and b are hyperbolic isometries such that the intersection of their axes contains a non-degenerate interval I and whose orientations coincide, then ab is hyperbolic, its axis contains I and its orientation coincides with those of a and b. □

4.3 Approximation of a simplicial action in $\mathcal{F}_n$ by a "special curve".

Because of proposition 4.3, the proof of Theorem 2 reduces to the following proposition:

Proposition 4.5. *The set of "special curves" is dense in the set of simplicial actions in $\mathcal{F}_n$.*

To prove this proposition, we will proceed in three steps. In the first step, we essentially approximate a simplicial action $T \in \mathcal{F}_n$ by a simplicial action $T' \in \mathcal{F}_n$ with trivial edge stabilizers and cyclic vertex stabilizers using the dynamic of Dehn twists. In the second step, using the *folding to approximate* technique, we approximate T' by an action T'' with trivial edge stabilizers and whose quotient graph is a tree. Finally in the third step, using once again the *folding to approximate* technique, we approximate T'' by a "special curve".

First step: approximation to get rid of components of $Q \setminus \mathrm{triv}(Q)$ with non-cyclic fundamental group

Remember that $\mathrm{triv}(Q)$ denotes the set of edges of Q with trivial edge group.

Proposition 4.6. *Any simplicial action $T \in \mathcal{F}_n$ may be approximated by a simplicial action $T' \in \mathcal{F}_n$ whose quotient graph of groups $Q' = T'/F_n$ has the following properties:*

- *every component of $Q' \setminus \mathrm{triv}(Q')$ has cyclic fundamental group (as a graph of groups)*

- *At most one component of $Q' \setminus \mathrm{triv}(Q')$ is not reduced to one point.*

Proof. Consider the union A of the components of $Q \setminus \mathrm{triv}(Q)$ whose fundamental group is not cyclic (as a graph of groups). Consider a component A_0 of A, Γ_{A_0} its fundamental group and $m \geq 2$ the rank of this free group. We consider a simplicial action T'_{A_0} of A_0 whose quotient graph of groups is a $(m-1)$-rose whose edge stabilizers are trivial, and whose vertex stabilizer is infinite cyclic. We choose any attaching points for edges of $Q \setminus A$ incident on A_0.

To apply Cohen and Lustig's theorem, we need the edge groups of A_0 not to fix any point in T_{A_0}. To achieve this, we apply the following lemma to a set $\{g_1, \dots, g_k\}$ consisting of one generator of each edge group of A_0 and we change T_{A_0} to $T_{A_0}.\alpha$.

Lemma 4.7. *Let $g_1, \dots, g_k$ be a finite set of elements of Γ_{A_0}. If T_{A_0} is an action as above, there exists an automorphism α of Γ_{A_0} such that $g_1, \dots, g_k$ are hyperbolic in $T_{A_0}.\alpha$.*

Proof. There is a natural basis $\langle a_1, \dots, a_m \rangle$ of Γ_{A_0} such that $g \in \Gamma_{A_0}$ is elliptic in T_{A_0} if and only if g is conjugate to a_1. If φ is the automorphism of Γ_{A_0} fixing $a_2, \dots, a_m$ and sending a_1 to a_1a_2, then for every $g \in \Gamma_{A_0}$ there exists at most one $p \in \mathbb{Z}$ such that $\varphi^p(g)$ is conjugate to a_1, so that we can take α to be a power of φ. □

Now consider the sequence of actions T'_k constructed in the theorem about dynamics of Dehn twists. We know that T'_k converges to T. But T'_k is simplicial, very small, and lies in $\mathcal{F}_n$ since there exists a non-terminal vertex of $Q'_k = T'_k/F_n$ with non-trivial group (by choice of the actions T_{A_0}). By construction, no component of $Q'_k \setminus \mathrm{triv}(Q'_k)$ has a non-cyclic fundamental group. This proves the first part of proposition 4.6.

Now assume that T satisfies the first part of proposition 4.6 and not the second one. Then take A to be the union of all but one connected components of $Q \setminus \mathrm{triv}(Q)$ not reduced to one point. For every component A_0 of A, we consider a free action of the cyclic group $\pi_1(A_0)$ on a line T_{A_0}. Using the theorem about dynamics of Dehn twists, we get an approximation of T which is very small and which lies in $\mathcal{F}_n$ (because it has a non-trivial edge stabilizer). This conclude the proof of proposition 4.6. □

Second step: approximation by an action whose quotient graph is a tree

Proposition 4.8. *For $n \geq 3$, any simplicial action in $\mathcal{F}_n$ may be approximated by a simplicial action $T' \in \mathcal{F}_n$ with trivial edge stabilizers such that T'/F_n is a tree.*

Proof. Thanks to Proposition 4.6, we may assume that the quotient graph of groups $Q = T/F_n$ satisfies the following: every component of $Q \setminus \mathrm{triv}(Q)$ has cyclic fundamental group and $Q \setminus \mathrm{triv}(Q)$ has at most one component not reduced to one point. Therefore, either T has trivial edge stabilizers or $Q \setminus \mathrm{triv}(Q)$ contains exactly one component not reduced to one point, and this component has cyclic fundamental group (as a graph of groups). First of all, we approximate T so that the lengths of its edges are all rational. We then multiply the metric by an integer and maybe subdivide some edges so that every edge in T has length 1. We first consider the case when T has trivial edge stabilizers: the treatment of the other case is similar but is a bit more technical.

When T has trivial edge stabilizers.

Since T has trivial edge stabilizers and cyclic vertex stabilizers, the hypothesis $T \in \mathcal{F}_n$ means that there exists a non-terminal vertex $\bar{x} \in Q$ with non-trivial stabilizer. We prove the proposition by induction on the number of edges in T/F_n: the idea is to perform on T a *folding to approximate* operation from such a vertex and to choose the folding paths so that the folded action has trivial edge stabilizers and contains a vertex with non-trivial stabilizer whose projection in the quotient graph of groups is not terminal. The following lemma tells us about the length of paths required to perform such an interesting fold.

Lemma for folding sub-paths. *Let T be a simplicial action with trivial edge stabilizers and whose edges all have length 1. Let $\bar{\alpha} = \bar{\alpha}_1\bar{\alpha}_2\dots$ and $\bar{\beta} = \bar{\beta}_1\bar{\beta}_2\dots$ be two (maybe infinite) paths in $Q = T/F_n$ with same origin $\bar{x}$ and well oriented with respect to an orientation of Q. Assume that the vertex $\bar{x}$ has non-trivial group and that $\bar{\alpha}_1 \neq \bar{\beta}_1$. We also suppose that one of the following conditions is satisfied:*

1. *$\bar{\alpha}$ and $\bar{\beta}$ are infinite*
2. *$\bar{\alpha}$ is strictly longer than $\bar{\beta}$ and the terminal vertex of $\bar{\beta}$ has a non-trivial group*

Then there exists sub-paths $\bar{\alpha}', \bar{\beta}'$ of $\bar{\alpha}, \bar{\beta}$ with the same (non-zero) length such that for any lift α' and β' of $\bar{\alpha}'$ and $\bar{\beta}'$ with same initial vertex x and for every sequence of distinct elements $w_k \in \operatorname{Stab} x$, the actions $T^{(k)} = T/_{\alpha' \sim w_k.\beta'}$ converge to T, $T^{(k)}$ has trivial edge stabilizers, and its quotient graph of groups $T^{(k)}/F_n$ has a non-terminal vertex with non-trivial group (in particular, $T^{(k)} \in \mathcal{F}_n$).

Remark. The action $T^{(k)}$ still has edges of length 1 and its quotient graph $T^{(k)}/F_n$ has strictly fewer edges than T.

Proof. We denote by $T_i^{(k)}$ the action obtained after the i-th intermediate fold of α with $w_k.\beta$, $Q_i^{(k)} = T_i^{(k)}/F_n$, $q_i : T \to T_i^{(k)}$ the folding map and $\bar{q}_i^{(k)} : Q \to Q_i^{(k)}$ the induced application.

We first notice that if for $i > 1$ the edges $q_{i-1}(\alpha_i)$ and $q_{i-1}(w_k.\beta_i) \subset T_{i-1}^{(k)}$ that define the i-th intermediate fold are in the same orbit, then their common vertex x_i has non-trivial stabilizer, and its projection $\bar{x}_{i-1} \in Q_{i-1}^{(k)}$ is not a terminal vertex since it belongs to the interior of the well oriented hence immersed path $q_{i-1}(\alpha_{i-1} \cup \alpha_i)$. Moreover, the fact that those edges $q_{i-1}(\alpha_i)$ and $q_{i-1}(w_k.\beta_i)$ lie in the same orbit does not depend on w_k since the fold between two paths is the quotient by the smallest equivariant relation identifying them.

Therefore, if there exists an index i such that $q_{i-1}(\alpha_i)$ and $q_{i-1}(w_k.\beta_i)$ lie in the same orbit, we consider the smallest such i_0 ($i_0 > 1$) and take α' and β' to be the restriction of α and β to the first $i_0 - 1$ edges. Therefore, every $T_i^{(k)}$ has trivial edge stabilizers for $i < i_0$ and we can apply the *folding to approximate* lemma to conclude.

If there is no such index i, we can apply the *folding to approximate* lemma to any of the $T_i^{(k)}$, but we are looking for an index i such that $Q_i^{(k)}$ has a non-terminal vertex with non-trivial group. In this situation, every intermediate fold reduces by one the number of edges of the quotient graph which prevents α and β from being both infinite. Hence the second assumption must hold. This implies that we can take $i = |\beta|$ (i. e. $\beta' = \beta$ and α' is the restriction of α having the same length as β). Indeed, if v denotes the terminal vertex of β, $q_i^{(k)}(v)$ lies in the interior of the well oriented arc $q_i^{(k)}(\alpha)$ so its projection in $Q_i^{(k)}$ is non-terminal and has non-trivial group. $\square$

Now, going back to the proof of proposition 4.8 in the case that T has trivial edge stabilizers (and edges with length 1), we argue by induction on the number of edges in T/F_n using the *lemma for folding sub-paths*.

To apply this lemma, it is sufficient to find an infinite path $\bar{\alpha}$ in $Q = T/F_n$ which is well oriented for some orientation of the edges of $\bar{\alpha}$ in Q, starting at a vertex $\bar{x}$ with non-trivial group and such that there exists an edge $\bar{e}$ in Q with origin $\bar{x}$ not contained in $\bar{\alpha}$. As a matter of fact, we can then inductively construct a well oriented path $\bar{\beta}$ starting at $\bar{e}$, by following any edge with the right orientation (whenever it is already in $\bar{\alpha}$ or $\bar{\beta}$). The only case where this can't be done is when $\bar{\beta}$ reaches a terminal vertex of Q, but the stabilizer of this vertex has to be be non-trivial by minimality. Therefore, in this situation, we can apply the *lemma for folding sub-paths* to conclude.

We may assume that Q is not a tree because otherwise there is nothing to prove. Hence, there exists an embedded circle in Q. If there exists such a circle C not containing $\bar{x}$, we define $\bar{\alpha}$ to be the path following a simple arc joining $\bar{x}$ to C before turning around C infinitely many times. Since $\bar{x}$ is not terminal in Q, there exists an edge $\bar{e}$ with origin $\bar{x}$ which is not in $\bar{\alpha}$. A similar argument works if $\bar{x}$ has valence at least 3 and $\bar{x} \in C$.

Therefore, the only remaining case is when $\bar{x}$ has valence 2 and every embedded circle in Q contains $\bar{x}$ which may only happen when Q has the homotopy type of a circle (as a simple graph). Let C be the unique embedded circle in Q ($\bar{x} \in C$ by hypothesis). We distinguish two cases. If there exists a vertex $\bar{v} \neq \bar{x}$ in Q whose group is non-trivial and such that the length of the two simple paths $\bar{\alpha}$ and $\bar{\beta}$ joining $\bar{x}$ to $\bar{v}$ are distinct, we can apply the *lemma for folding sub-paths*.

If no such vertex exists, it means that Q can be obtained from C by gluing finitely many trees (maybe 0) on the point $\bar{u}$ which is antipodal to $\bar{x}$ in C ($\bar{u}$ may not be a vertex if $|C|$ is odd in which case $Q = C$). In this case, we take $\bar{\alpha}$ and $\bar{\beta}$ to be the two simple paths joining $\bar{x}$ to $\bar{u}$ in C (maybe in the barycentric subdivision of Q). We consider two lifts α and β of $\bar{\alpha}$ and $\bar{\beta}$ with same origin x and w_k a sequence of distinct elements in $\operatorname{Stab} x$. It is clear that every intermediate fold of $q^{(k)} : T \to T^{(k)} = T/_{\alpha \sim w_k.\beta}$ is a fold between edges with trivial stabilizers so we can apply the *folding to approximate lemma.* Moreover, $Q^{(k)} = T^{(k)}/F_n$ is a tree so we just have to check that $T^{(k)} \notin \mathcal{O}_n$. If $\bar{q}^{(k)}$ denotes the map induced by $q^{(k)}$ on the level of quotient graphs, the group of $\bar{q}^{(k)}(\bar{u})$ is always non-trivial. It is even non-cyclic when the group of $\bar{u}$ is non-trivial. Therefore, $T^{(k)} \in \mathcal{F}_n$ as soon as the group of $\bar{u}$ is non-trivial or when Q is not a circle (because $\bar{q}^{(k)}(\bar{u})$ is not terminal in $Q^{(k)}$). In the remaining situation where Q is a circle and the group of $\bar{u}$ is trivial, the only vertex with non-trivial group is $\bar{x}$. This implies that the group of $\bar{x}$ is free of rank $n-1$, and hence cannot be cyclic for $n \geq 3$. Therefore the stabilizer of $q^{(k)}(x)$ is non-cyclic and $T^{(k)} \in \mathcal{F}_n$ which concludes the proof of proposition 4.8 in the case when T has trivial edge stabilizers.

When T has non-trivial edge stabilizers.

In this case, we show how to approximate T by an action satisfying the hypotheses of the previous case, i. e. an action with trivial edge stabilizers and such that there exists a non-terminal vertex in its quotient graph with non-trivial group.

We denote by Q the quotient graph of groups T/F_n as usual. As before, we can assume that edges of T have length 1, that every component of $Q \setminus \operatorname{triv}(Q)$ has cyclic fundamental group and $Q \setminus \operatorname{triv}(Q)$ has exactly one component I which is not reduced to one point. Since the fundamental group of I as a graph of groups is cyclic, and since its edge groups are non-trivial, $\pi_1(I)$ has a global fixed point in T, so I must be a tree (an HNN extension is never trivial). Moreover, the fact that T is very small says that I is an interval and that every edge morphism is an isomorphism onto the corresponding vertex group which means that the connected components of the preimage of I in T are intervals. We are going to prove a *folding to approximate lemma for edge stabilizers* by performing a fold on an action obtained thanks to Cohen-Lustig's theorem about dynamics of Dehn twists.

In order to apply Cohen-Lustig's theorem, we set $A = I$, and take T_I to be a line with a non-trivial action of $\pi_1(I) \simeq \mathbb{Z}$ by translations. We choose a point x in T_I and take every attaching point p_e to be x. We consider T_k and $T'_k = T_k.D_k$ as in Cohen and Lustig's theorem. Note that the graphs of groups $Q_k = T_k/F_n$ and $Q'_k = T'_k/F_n$ may be obtained from $Q = T/F_n$ by collapsing I to a vertex $\bar{v}_I$ and by adding an edge e_I and gluing its endpoints to $\bar{v}_I$.

Folding to approximate lemma for edge stabilizers. *Let T be a very small simplicial action in $\mathcal{F}_n$ such that every component of $Q \setminus \operatorname{triv}(Q)$ has cyclic fundamental group and $Q \setminus \operatorname{triv}(Q)$ has exactly one component I which is not reduced to one point. Let T'_k the sequence of actions constructed in Cohen and Lustig's theorem with $A = I$ as above.*

Assume that $\bar{\alpha}$ and $\bar{\beta}$ are two paths in $Q'_k \setminus e_I$ with origin $\bar{v}_I$. We choose some lifts $\alpha^{(k)}$ and $\beta^{(k)}$ of $\bar{\alpha}$ and $\bar{\beta}$ and we assume that they satisfy condition (H) and that when folding $\alpha^{(k)}$ on $\beta^{(k)}$, every intermediate fold is a fold between edges with trivial *stabilizer.*

Under the assumption that the first edges $\bar{\alpha}_1$ and $\bar{\beta}_1$ of $\bar{\alpha}$ and $\bar{\beta}$ correspond to edges in Q with distinct initial points, the actions $T^{(k)}$ obtained from T'_k by folding $\alpha^{(k)}$ along $\beta^{(k)}$ converge to T as $k \to \infty$.

Proof. We prove convergence in the translation lengths topology. The theorem about dynamics of Dehn twists tells us that T'_k converges to T.

If $g \in F_n$ has a fixed point in T then its translation length in T'_k approaches 0 when k tends to infinity, and since a folding map decreases distances, $l_{T^{(k)}}(g) \xrightarrow{k\to\infty} 0$.

If $g \in F_n$ is hyperbolic in T, then for large enough k, it is hyperbolic in T'_k. Moreover, because $\bar{\alpha}_1$ and $\bar{\beta}_1$ correspond to edges in Q with distinct initial points, a path in Q entering I from (the edge corresponding to) $\bar{\alpha}_1{}^{-1}$ and leaving I through $\bar{\beta}_1$ will be twisted more and more so that the corresponding path in Q'_k will go through e_I more and more often between $\bar{\alpha}_1{}^{-1}$ and $\bar{\beta}_1$. A similar fact holds for a path entering I from $\bar{\beta}_1{}^{-1}$ and leaving I through $\bar{\alpha}_1$. This implies that for k large enough, the projection of the axis of g in Q'_k never successively runs through $\bar{\alpha}_1{}^{-1}$ and $\bar{\beta}_1$ or $\bar{\beta}_1{}^{-1}$ and $\bar{\alpha}_1$. Corollary 2.2 says that two adjacent edges which are not identified by the first elementary fold are not identified in $T^{(k)} = T/_{\alpha^{(k)}\sim\beta^{(k)}}$. Therefore, for large enough k, the folding map isometrically embeds the axis of g into $T^{(k)}$ so $l_{T^{(k)}(g)} = l_{T'_k}(g)$ and therefore converges to $l_T(g)$. □

The *folding to approximate lemma for edge stabilizers* allows us to prove a version of the lemma for folding sub-paths for edge stabilizers:

Lemma for folding sub-paths for edge stabilizers. *Let T be a simplicial action whose edges have length 1, such that every component of $Q \setminus \mathrm{triv}(Q)$ has cyclic fundamental group and $Q \setminus \mathrm{triv}(Q)$ has exactly one component I which is not reduced to one point. As above, consider T'_k an approximation of T provided by Cohen and Lustig's theorem.*

Let $\bar{\alpha} = \bar{\alpha}_1\bar{\alpha}_2\dots$ and $\bar{\beta} = \bar{\beta}_1\bar{\beta}_2\dots$ be two (possibly infinite) paths in $Q'_k \setminus e_I$ with the same origin $\bar{x}$, well oriented with respect to an orientation of Q and such that $\bar{\alpha}_1$ and $\bar{\beta}_1$ correspond to edges in Q with distinct initial points.

We also suppose that one of the following conditions is satisfied:

1. *$\bar{\alpha}$ and $\bar{\beta}$ are infinite*
2. *$\bar{\alpha}$ is strictly longer than $\bar{\beta}$ and the terminal vertex of $\bar{\beta}$ has non-trivial stabilizer*

Then there exists sub-paths $\bar{\alpha}', \bar{\beta}'$ of $\bar{\alpha}, \bar{\beta}$ with the same (non-zero) length such that for any lift α' and β' of $\bar{\alpha}'$ and $\bar{\beta}'$ with the same initial vertex x, the actions $T^{(k)} = T'_k/_{\alpha'\sim\beta'}$ converge to T, $T^{(k)}$ has trivial edge stabilizers, and its quotient graph of groups $T^{(k)}/F_n$ has a non-terminal vertex with non-trivial group (in particular, $T^{(k)} \in \mathcal{F}_n$).

The proof is similar to the proof in the trivial-edge stabilizer case. With this *lemma for folding sub-paths for edge stabilizers* at hand, we just have to repeat the argument used when T had trivial edge stabilizers to find some paths $\bar{\alpha}$ and $\bar{\beta}$ satisfying its hypotheses. The only additional case is when $Q'_k \setminus e_I$ is a tree. This means that Q is a tree, and by minimality, I doesn't contain any terminal point of Q. We then consider two endpoints $\bar{s}_\alpha$ and $\bar{s}_\beta$ of Q'_k lying in two distinct components of $Q'_k \setminus \bar{v}_I$ corresponding to non-adjacent components of $Q \setminus I$. We take $\bar{\alpha}$ and $\bar{\beta}$ to be the simple paths joining v_I to $\bar{s}_\alpha$ and $\bar{s}_\beta$ respectively. If their lengths are distinct, we can apply the *lemma for folding sub-paths for edge stabilizers* and we are done since we thus get an approximation of T lying in $\mathcal{F}_n$ with trivial edge stabilizer to which we can apply proposition 4.8 (already proved in the case of trivial edge stabilizers). If $\bar{\alpha}$ and $\bar{\beta}$ have the same length, we approximate T by changing some edge lengths slightly (keeping them rational) so that the lengths of $\bar{\alpha}$ and $\bar{\beta}$ become different. After subdivision, we can once again apply the *lemma for folding sub-paths* for edge stabilizers to conclude. This concludes the proof of proposition 4.8. □

Third step: approximation by a "special curve".

Proposition 4.9. *For $n \geq 3$, any action $T \in \mathcal{F}_n$ with trivial edge stabilizers and whose quotient graph is a tree may be approximated by a "special curve".*

Thanks to proposition 4.8, this proposition will conclude the proof of Proposition 4.3 and therefore of Theorem 2.

Proof. Recall that a "special curve" is a very small simplicial action such that there exists a basis $\langle a_1, \dots, a_n \rangle$ of F_n in which every edge stabilizer is non-trivial and generated by a conjugate of a positive word in $\langle a_1, \dots, a_n \rangle$.

As above, up to approximation, subdivision and rescaling, we can assume that every edge in T has length 1. We are going to perform folding operations on T to create non-trivial edge stabilizers, and we will argue by induction on the number of orbits of edges with trivial stabilizer. As above, we denote by $Q = T/F_n$ and by $\mathrm{triv}(Q)$ the set of edges in Q with trivial group. We also denote by $\overline{\mathrm{triv}(Q)}$ the union of $\mathrm{triv}(Q)$ together with the vertices of Q adjacent to an edge of $\mathrm{triv}(Q)$.

The induction hypothesis is the following: we assume that we know how to prove the proposition for every action $T' \in \mathcal{F}_n$ whose quotient graph Q' is a tree such that $\#\mathrm{triv}(Q') < \#\mathrm{triv}(Q)$ and

1. $\overline{\mathrm{triv}(Q')}$ is connected
2. $\overline{\mathrm{triv}(Q')}$ is empty, or contains a vertex whose group is not cyclic, or a vertex with non-trivial group which is not terminal in $\overline{\mathrm{triv}(Q')}$
3. there exists a free basis $\langle a_1, \dots, a_n \rangle$ of F_n, a lift $\tilde{Q}'$ of Q' and for every connected component C of $\tilde{Q}' \setminus \mathrm{triv}(\tilde{Q}')$, a possibly empty subset $B_C \subset \{a_1, \dots, a_n\}$ such that

    a. for all $a_j \in B_C$, a_j fixes a point in C

b. the stabilizer of every edge or vertex in C has a basis composed of positive words in B_C.

Note that condition 3a implies that two sets B_C and $B_{C'}$ are disjoint for $C \neq C'$, and condition 3b shows that the union of the sets B_C equals $\{a_1, \dots, a_n\}$. The action T we are starting with satisfies the induction hypothesis: choose any lift $\tilde{Q}$ of Q in T, and for every component C of $\tilde{Q} \setminus \mathrm{triv}(\tilde{Q})$ (which is a single vertex) consider a free basis B_C of $\mathrm{Stab}\, C$ and take $\langle a_1, \dots, a_n \rangle$ to be the union of the B_C. Moreover, if an action T satisfies the induction hypothesis and $\overline{\mathrm{triv}(Q)}$ is empty, then T is a "special curve" so there is nothing to prove.

First case: $\overline{\mathrm{triv}(Q)}$ contains a vertex $\bar{x}$ which is terminal in $\overline{\mathrm{triv}(Q)}$ and has non-cyclic group (see figure 9 where bold edges correspond to edges with non-trivial stabilizer). Let x be the lift of $\bar{x}$ in $\tilde{Q}$, let s_α and s_β be two terminal vertices of $\overline{\mathrm{triv}(\tilde{Q})}$ distinct from x. Note that $\mathrm{Stab}\, s_\alpha$ and $\mathrm{Stab}\, s_\beta$ must be non-trivial since either s_α is terminal in $\tilde{Q}$ or it is the endpoint of an edge of $\tilde{Q}$ with non-trivial stabilizer. Let α and β the paths joining x to s_α and s_β respectively. Since we want $\overline{\mathrm{triv}(Q)}$ to remain connected after folding, we choose s_α and s_β so that $|\alpha \cap \beta|$ is minimal, which means that α and β bifurcate as soon as they meet a branch point p of $\overline{\mathrm{triv}(Q)}$. In particular, we take $s_\alpha = s_\beta$ only if $\overline{\mathrm{triv}(Q)}$ is a segment. If α and β do not have the same length, we shorten the longest one so that this condition is satisfied.

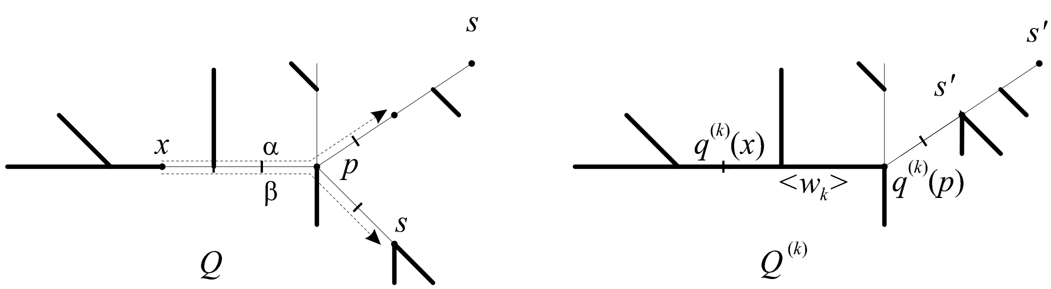


Figure 9: Folding α and β in the first case
Edges with non-trivial stabilizer are represented in bold face.

To apply the *folding to approximate* lemma (section 2.3) to α and β, we just have to choose a sequence of distinct elements $w_k \in \mathrm{Stab}\, x$. Let C be the component of $\tilde{Q} \setminus \mathrm{triv}(\tilde{Q})$ containing x and let $\{g_1, \dots, g_p\}$ be a basis of $\mathrm{Stab}\, x$ consisting of positive words in B_C. Since $\mathrm{Stab}\, x$ is not cyclic, we can choose a sequence w_k of positive words in $\{g_1, \dots, g_p\}$ which are not proper powers in F_n and which are not conjugate to elements of F_n that already fix an edge in T.

The hypotheses of the *folding to approximate* lemma are clearly satisfied, so $T^{(k)} = T/_{\alpha \sim w_k \cdot \beta}$ converge to T. Hence, we just have to prove that $T^{(k)}$ satisfy the induction hypothesis. Recall that $[x, p]$ denotes $\alpha \cap \beta$ and that $q^{(k)} : T \to T^{(k)}$ is the folding map. The stabilizer of $q^{(k)}([x,p])$ is generated by w_k, and $Q^{(k)} = T^{(k)}/F_n$ may be obtained from Q by gluing $\alpha\setminus[\bar{x},\bar{p}]$ on $\beta\setminus[\bar{x},\bar{p}]$. Therefore, $Q^{(k)}$ is a tree, and $\overline{\text{triv}(Q^{(k)})}$ is connected since $[\bar{x},\bar{p}]$ is a terminal segment of $\overline{\text{triv}(Q)}$. Moreover, $T^{(k)}$ is very small since w_k is not a proper power and $\text{Fix}_{T^{(k)}}\, w_k = q^{(k)}([x,p])$ contains no triod. Condition 2 is also satisfied by $T^{(k)}$: if $s_\alpha = s_\beta$, $\text{triv}(Q^{(k)})$ is empty; if $s_\alpha \neq s_\beta$ and if $d_T(x,s_\alpha) = d_T(x,s_\beta)$ then $q^{(k)}(s_\alpha)$ has non-cyclic stabilizer and its projection to $Q^{(k)}$ lies in $\overline{\text{triv}(Q^{(k)})}$; if $s_\alpha \neq s_\beta$ and if $d_T(x,s_\alpha) < d_T(x,s_\beta)$ (without loss of generality), then $q^{(k)}(s_\alpha)$ has non-trivial stabilizer and is not terminal in $\overline{\text{triv}(Q^{(k)})}$.

To see that condition 3 is satisfied, we consider the component S_β of $\tilde{Q} \setminus \{p\}$ containing s_β. We obtain a lift $\tilde{Q}^{(k)}$ of $Q^{(k)}$ by taking $q^{(k)}(\tilde{Q} \setminus S_\beta) \cup q^{(k)}(w_k.S_\beta)$. We change the basis $\langle a_1,\dots,a_n\rangle$ by conjugating by w_k the elements of B_C for each component C of $\tilde{Q} \setminus \text{triv}(\tilde{Q})$ contained in S_β. It is a free basis because w_k may be written in the basis $\langle a_1,\dots,a_n\rangle$ without using the letters of B_C for $C \subset S_\beta$. It is now clear that $T^{(k)}$ satisfies the induction hypothesis.

Second case: $\overline{\text{triv}(Q)}$ contains a vertex $\bar{x}$ with non-trivial stabilizer and which is not terminal in $\overline{\text{triv}(Q)}$. Let x be the lift of $\bar{x}$ in $\tilde{Q}$ and let s_α and s_β be two terminal points of $\tilde{Q}$ lying in distinct components of $\tilde{Q} \setminus \{x\}$. Note that $\text{Stab}\, s_\alpha$ and $\text{Stab}\, s_\beta$ must be non-trivial. Let α and β be the simple paths joining x to s_α and s_β respectively. If α and β do not have the same length, we shorten the longest one so that this condition is satisfied. We take any sequence w_k of distinct elements in $\text{Stab}\, x$ and we consider the folded actions $T^{(k)} = T/_{\alpha\sim w_k.\beta}$. It is an approximation of T when k is large enough thanks to the *folding to approximate* lemma.

The quotient graph $Q^{(k)} = T^{(k)}/F_n$ is obtained from Q by identifying $\bar{\alpha}$ and $\bar{\beta}$, and the stabilizers of the edges contained in $q^{(k)}(\alpha) = q^{(k)}(\beta)$ are trivial. Therefore, $\overline{\text{triv}(Q^{(k)})}$ is connected, $T^{(k)}$ is very small, and $Q^{(k)}$ is a tree. Since the stabilizers of s_α and s_β are non-trivial, $\overline{\text{triv}(\tilde{Q}^{(k)})}$ contains a non-terminal point with non-trivial stabilizer when $d(x,s_\alpha) \neq d(x,s_\beta)$ and a point with non-cyclic stabilizer when $d(x,s_\alpha) = d(x,s_\beta)$.

To see that condition 3 of the induction hypothesis is satisfied, we consider the component S_β of $\tilde{Q} \setminus \{q\}$ containing s_β. As above, we consider the lift $\tilde{Q}^{(k)}$ of $Q^{(k)}$ defined by

$$\tilde{Q}^{(k)} = q^{(k)}(\tilde{Q} \setminus S_\beta) \cup q^{(k)}(w_k.S_\beta).$$

We change the basis $\langle a_1,\dots,a_n\rangle$ by conjugating by w_k the elements of B_C for each component C of $\tilde{Q}\setminus\text{triv}(\tilde{Q})$ contained in S_β. We get that $T^{(k)}$ satisfies the induction hypothesis which ends the proof of Proposition 4.9 and hence of Theorem 2. □

5 Does $\text{Out}(F_n)$ act with dense orbits on $\mathcal{F}_n$?

We still don't know whether $\text{Out}(F_n)$ acts with dense orbits on $\mathcal{F}_n$. This question is equivalent to asking whether $\mathcal{M}_n = \mathcal{F}_n$. To prove this equality, it would be sufficient to approximate every non-simplicial action by a simplicial action lying in

$\mathcal{F}_n$. In [BF2], Bestvina and Feighn show how to approximate a very small action T by a *simplicial* very small action T'. Their argument shows that if T has a non-trivial arc stabilizer, then T' may be assumed to have a non-trivial edge stabilizer so $T' \in \mathcal{F}_n$ and $T \in \mathcal{M}_n$. They also prove that if a geometric approximation of T has an orientable surface component, then T can be approximated by a very small simplicial action with a non-trivial edge stabilizer and hence lies in $\mathcal{M}_n$.

If T is a very small action of F_n, Gaboriau and Levitt show in [GL] that T has only finitely many orbits of branch points. Therefore, we can apply [Lev2] to conclude that T may be seen as the action corresponding to a graph of actions on $\mathbb{R}$-trees whose vertex actions have dense orbits. Therefore, proving that $\mathcal{M}_n = \mathcal{F}_n$ reduces to showing that any very small action with dense orbits lies in $\mathcal{M}_n$. The following theorem partially answers this question (see section 5.12 for definitions):

Theorem 3. *Let $n \geq 3$ and let T be a very small action with dense orbits. If the Lebesgue measure on T is the sum of at most $n-1$ ergodic measures, then $T \in \mathcal{M}_n$.*

We will see in section 5.1 that because of the topological dimension of $\overline{CV}_n$, the Lebesgue measure is always a sum of at most $3n-4$ ergodic measures.

Remark. Let α be an irreducible automorphism of F_n with irreducible powers. This means that no power of α fixes a free factor of F_n up to conjugation. Then Lustig has proved that α has exactly two fixed points in $\overline{CV}_n$ and no other periodic point ([Lus2]). This implies that those fixed points are uniquely ergodic. As a matter of fact, there is a natural way to associate to an action $T \in \overline{CV}_n$ a simplex $\sigma(T) \subset \overline{CV}_n$ built on its set of invariant measures. If T is not uniquely ergodic, this simplex is not reduced to one point and some power of α fixes this simplex pointwise which is impossible. Therefore, Theorem 3 implies that the fixed points of an irreducible automorphism with irreducible powers must lie in $\mathcal{M}_n$.

5.1 Measures on $\mathbb{R}$-trees

Length measures and uniquely ergodic actions

The classical measure theory is not adapted to $\mathbb{R}$-trees because they are no locally compact. In [Pau3] is proposed an alternative called *length measure*. For shortness's sake, we will sometimes use the shortcut *measure* to mean a length measure.

Definition. *A* length measure *μ on an $\mathbb{R}$-tree T consists of a finite Borel measure μ_I for every compact interval I of T such that if $J \subset I$, $\mu_J = (\mu_I)_{|I}$.*

If T is endowed with an action of a group Γ, we say that a length measure is invariant if $\mu_{g.I} = (g_{|I})_*\mu_I$. The *Lebesgue measure* of an $\mathbb{R}$-tree is the collection of the Lebesgue measures of the intervals of T. If Γ acts by isometries on T, then the Lebesgue measure is invariant. If μ is a length measure on an $\mathbb{R}$-tree T, we write $\mu(I)$ for $\mu_I(I)$. We say that μ is non-atomic or positive if every μ_I is non-atomic or positive.

Remark. It may happen that an action with dense orbits has an invariant measure with atoms, but this is impossible if every orbit is dense in the segments.

Let $f : T \to T'$ be a map such that every segment I in T may be subdivided into finitely many intervals on which f preserves alignment (this is the case when f is a morphism of $\mathbb{R}$-trees or a map preserving alignment). Any *non-atomic* measure μ' on T' may be carried to a measure $\mu = f^*\mu'$ in the following way: let I be a segment in T and subdivide I into finitely many subsegments I_p on which f preserves alignment. Then take μ_I to be the only (non-atomic) measure on I such that for every interval J inside some I_p, $\mu_I(J) = \mu'_{f(I_i)}(f(J))$.

Measures and maps preserving alignment

From now on, we only consider positive invariant measures.

Let T be an $\mathbb{R}$-tree with an isometric action of Γ. If $q : T \to T'$ is an equivariant 1-Lipschitz map preserving alignment, by carrying to T the Lebesgue measure of T', we obtain an invariant positive measure whose density with respect to the Lebesgue measure is at most 1. Conversely, given a invariant positive measure μ on T whose density with respect to the Lebesgue measure is at most 1, we consider the pseudo-metric on T given by $d_\mu(x, y) = \mu([x, y])$. One easily checks that making this pseudo-metric Hausdorff gives an $\mathbb{R}$-tree T_μ. This tree is naturally endowed with an isometric action of Γ and the quotient map $q : T \to T_\mu$ preserves alignment. Note that by if μ is obtained by pulling back μ' under $f : T \to T'$ then T_μ is isometric to T'.

Here are some simple properties of maps preserving alignment:

Lemma 5.1. *Let T and T' be $\mathbb{R}$-trees endowed with an isometric action of a group Γ and let $q : T \to T'$ be an equivariant map preserving alignment.*

Then the preimage of a convex set is convex. For every $\gamma \in \Gamma$, $\operatorname{Char}_{T'} \gamma = q(\operatorname{Char}_T \gamma)$. Moreover, if γ is hyperbolic in T and elliptic in T' then γ has only one fixed point $a = q(\operatorname{Axis}_T \gamma)$ in T'.

Proof. Let K' be a convex set in T' and let $a, b \in K = q^{-1}(K')$. Every $x \in [a, b]$ is sent by q to a point in $[q(a), q(b)]$ so K is convex. Now, $q(\operatorname{Char}_T \gamma) \subset \operatorname{Char}_{T'} \gamma$ because a point a lies in the characteristic set of γ if and only if $a \in [\gamma^{-1}.a, \gamma.a]$. If γ is hyperbolic in T', then $q(\operatorname{Char}_T \gamma)$ is connected and γ-invariant, so it must contain the axis of γ in T'. If γ is elliptic in T', then the preimage of a fixed point of γ in T' is connected and γ-invariant. Hence it must intersect the characteristic set of γ in T. Therefore, $\operatorname{Char}_{T'} \gamma = q(\operatorname{Char}_T \gamma)$ and γ fixes at most one point in T' when it is hyperbolic in T. □

Corollary. *Let T and T' be two minimal F_n actions and $q : T \to T'$ be a map preserving alignment. If T is very small then so is T'.*

Proof. From the previous lemma, T' is small because any element fixing the non-degenerate arc $[x, y]$ fixes the arc joining the subtrees $q^{-1}(x)$ and $q^{-1}(y)$. The previous lemma allows one to deduce that $\operatorname{Fix}_{T'} \gamma = \operatorname{Fix}_{T'} \gamma^k$ from $\operatorname{Fix}_T \gamma = \operatorname{Fix}_T \gamma^k$. Finally, $\operatorname{Fix}_{T'} \gamma = q(\operatorname{Fix}_T \gamma)$ shows that γ may not fix any triod in T'. □

Ergodic measures

A homothety of a length measure μ is the multiplication of every μ_I by a same positive real number.

Definition. *We say that an $\mathbb{R}$-tree endowed with an isometric action of a group Γ is* uniquely ergodic *if the Lebesgue measure is the only non-zero positive invariant measure on T up to homothety.*

In particular, if T is uniquely ergodic, and if $q : T \to T'$ is equivariant and preserves alignment, then q is a homothety.

If T is an action of a group Γ, we denote by $M(T)$ the convex cone of invariant positive measures on T.

A subset $E \subset T$ is said to be measurable if each intersection of E with an arc of T is measurable. Thus, a function $f : T \to \mathbb{R}$ is measurable if its restriction to every interval of T is measurable. We say that a measurable subset $E \subset T$ has *μ-measure* 0 if for every arc I of T, $\mu_I(E \cap I) = 0$, and E has μ-full measure if $T \setminus E$ has measure 0. A function $f : T \to \mathbb{R}$ is constant μ-almost everywhere if there exists $c \in \mathbb{R}$ such that $f^{-1}(c)$ has full μ-measure.

Definition. *A measure $\mu \in M(T) \setminus \{0\}$ is said to be* ergodic *if the following equivalent conditions hold:*

1. *every Γ-invariant measurable function is constant μ-almost everywhere*
2. *every measure $\nu \in M(t)$ with density at most one with respect to μ is homothetic to μ*
3. *μ is extremal in $M(T)$, i. e. if $\mu = \mu_1 + \mu_2$ with $\mu_1, \mu_2 \in M(T)$, then μ_1 and μ_2 are homothetic to μ*
4. *every measurable invariant subset of T either has full or 0 measure with respect to μ.*

Proof of the equivalence of the conditions. $1 \Rightarrow 2$ because if $\nu \in M(T)$ has density at most one with respect to μ, on every arc I we may write $\nu_I = f_I \mu_I$ for some measurable functions f_I defined μ_I-almost everywhere, and the f_I are the restrictions μ-almost everywhere of an invariant measurable function $f : T \to \mathbb{R}$. $2 \Rightarrow 3 \Rightarrow 4$ are clear. If f is a Γ-invariant measurable function which is not constant almost everywhere, then there exists $M \in \mathbb{R}$ such that neither $A^+ = \{x \in T | f(x) \geq M\}$ nor $A^- = \{x \in T | f(x) < M\}$ have μ-measure 0. □

Note that if T is uniquely ergodic, then the Lebesgue measure is ergodic. We denote by $M_0(T)$ the set of non-atomic invariant positive measures on T and $M_1(T) \subset M_0(T)$ the set of invariant positive measures with density at most 1 with respect to the Lebesgue measure. Both $M_0(T)$ and $M_1(T)$ are convex.

Lemma 5.2. *A non-atomic measure μ is ergodic if and only if $M_1(T_\mu)$ has dimension* 1.

Proof. The measure μ is ergodic if and only if every non-zero measure whose density is at most 1 with respect to μ is homothetic to μ. Now there is a natural isomorphism between the set of measures on T with density at most 1 with respect to μ and the set of measures on T_μ with density at most 1 with respect to the Lebesgue measure: if $q : T \to T_\mu$ denotes the quotient map, the isomorphism is given by $\nu \in M_1(T_\mu) \mapsto q^*\nu$. $\square$

Weak topology on sets of measures

The set $M(T)$ is naturally endowed with the weak topology (see [Pau3]). For this topology, a sequence $\mu^{(k)}$ of measures converges to μ if and only if for every interval I and every continuous function $f : I \to \mathbb{R}$,

$$\int f\, d\mu_I^{(k)} \xrightarrow{k\to\infty} \int f\, d\mu_I.$$

This topology *is not* projectively compact in general. One should keep in mind the following phenomenon ([Pau3]): if I is an arc in T, if $b \in I \setminus \partial I$ is a branch point of T, and if δ_k is the Dirac measure at $x_k \notin I$ with $x_k \to b$, then δ_k *does not* converge to the Dirac measure at b.

If T is a minimal action of a finitely generated group, then there exists a finite tree $K \subset T$ such that every arc I of T may be subdivided into finitely many sub-arcs which may be sent into K by an element of Γ. Therefore, the set $M_0(T)$ of non-atomic length measures on T is naturally identified with the set of (usual) measures μ on K which are Γ-invariant i. e. such that for all $\gamma \in \Gamma$,

$$(\gamma_{|K\cap\gamma^{-1}.K})_*\mu_{|K\cap\gamma^{-1}.K} = \mu_{|K\cap\gamma.K}.$$

The topology induced on $M_0(T)$ by the weak topology coincides with the usual topology on the space of invariant measures on K. This implies that $M_1(T)$ is compact (but it contains the null measure). This identification may be extended to the set of measures for which no branch point of T has non-zero measure, but we won't need this fact. Note that on $M_0(T)$, the applications $\mu \mapsto \mu(I)$ are continuous for every arc I (because the measures in $M_0(T)$ have no atom).

Measures and simplices

Lemma 5.3. *Let T be a minimal action of a non-abelian finitely generated group Γ with dense orbits. Then the map σ_T from $M_0(T) \setminus \{0\}$ to the set of actions of Γ on $\mathbb{R}$-trees modulo equivariant isometry defined by $\sigma_T(\mu) = T_\mu$ is one-to-one.*

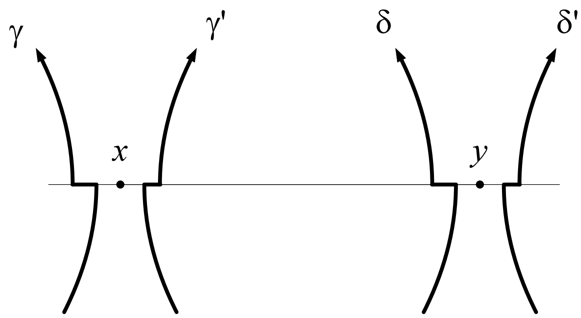
Remark. This lemma is of course false if we don't assume that T has dense orbits. This map σ_T is linear in the following sense:

$$l_{\sigma(t_1\mu_1+t_2\mu_2)} = t_1 l_{\sigma(\mu_1)} + t_2 l_{\sigma(\mu_2)} \qquad \text{for all } t_1, t_2 \geq 0.$$

The map σ_T is continuous on $M_0(T)$ because $\mu \mapsto \mu(I)$ is continuous for every interval I.

Proof. We can assume that T is not a line since we know in this case that T is uniquely ergodic. Since T is minimal, every T_μ is minimal (the preimage of an invariant subtree is an invariant subtree). Assume that $f : T_{\mu_1} \to T_{\mu_2}$ is an equivariant isometry for some $\mu_1, \mu_2 \in M_0(T)$. We denote by $q_i : T \to T_{\mu_i}$ the quotient maps.

For $\gamma, \delta \in \Gamma$, we denote by $\mathrm{bridge}_T(\gamma, \delta)$ the segment joining the characteristic sets of γ and δ (if they are disjoint) or their intersection point when they meet in exactly one point. We don't define $\mathrm{bridge}_T(\gamma, \delta)$ if the intersection of their characteristic sets contains more than one point.



Let x, y be two distinct points in T. We want to prove that $\mu_1([x,y]) = \mu_2([x,y])$. Since the orbits of Γ are dense in T, and since T is not a line, the branch points of T are dense in every segment: if I is an arc and if $x \in \mathring{I}$, we find a branch point in I close to x by projecting to I any branch point of T that is close enough to x. Now, since T is non-abelian, every segment is contained in the axis of a hyperbolic element (see [CuMo] or [Pau2, Lemma 4.3]). This implies that for every $\varepsilon > 0$, we can find elements $\gamma, \gamma', \delta, \delta' \in \Gamma$ hyperbolic in T whose axes are pairwise disjoint and

$$x \in \mathrm{bridge}_T(\gamma, \gamma') \quad \text{and} \quad y \in \mathrm{bridge}_T(\delta, \delta')$$

with

$$\mu_i\left(\mathrm{bridge}_T(\gamma, \gamma')\right), \mu_i\left(\mathrm{bridge}_T(\delta, \delta')\right) \leq \varepsilon \quad \text{for } i \in \{1, 2\}.$$

One has $q_i(\mathrm{bridge}_T(\gamma, \gamma')) = \mathrm{bridge}_{T_{\mu_i}}(\gamma, \gamma')$ and $q_i(x) \in \mathrm{bridge}_{T_{\mu_i}}(\gamma, \gamma')$ (and similar facts for y with δ, δ' instead of γ, γ'). This implies that

$$\left|\mu_i([x,y]) - d\left(\mathrm{bridge}_{T_{\mu_i}}(\gamma, \gamma'), \mathrm{bridge}_{T_{\mu_i}}(\delta, \delta')\right)\right| \leq 2\varepsilon.$$

But f sends $\mathrm{bridge}_{T_{\mu_1}}(\gamma, \gamma')$ and $\mathrm{bridge}_{T_{\mu_1}}(\delta, \delta')$ respectively to $\mathrm{bridge}_{T_{\mu_2}}(\gamma, \gamma')$ and $\mathrm{bridge}_{T_{\mu_2}}(\delta, \delta')$. We deduce that $\mu_1([x,y])$ is 4ε-close to $\mu_2([x,y])$ and this holds for every $\varepsilon > 0$ so that $\mu_1 = \mu_2$.

□

Corollary 5.4. *Let T be a very small F_n-action with dense orbits. Then $M_0(T)$ is a finite dimensional convex set and $M_0(T)$ is projectively compact. Moreover, T has at most $3n-4$ non-atomic ergodic measures (up to homothety), and every measure in $M_0(T)$ is a sum of these ergodic measures. Moreover, $M_1(T)$ is compact, and σ_T and $(\sigma_T)_{|M_1(T)\setminus\{0\}}$ define two simplices in outer space.*

Proof. First notice that if $\mu_1,\dots,\mu_p$ are ergodic measures which are mutually not homothetic, then they are linearly independent in $M(T)$. This is because there exist disjoint measurable sets $E_1,\dots,E_p$ that cover T such that E_i has full μ_i-measure.

On the other hand, the set of very small actions of F_n which are not free simplicial (i. e. the non-projective boundary of outer space) has topological dimension $3n-4$ (see [GL]). Since σ_T is linear, continuous and injective, $M_0(T)$ has dimension at most $3n-4$, and T has at most $3n-4$ non-atomic ergodic measures up to homothety.

We now prove that any measure μ is a sum of ergodic measures. The set of measures $M_\mu(T)$ with density at most 1 with respect to μ is compact since it is isomorphic to the set of invariant measures on a finite tree K with density at most 1 with respect to μ. The Krein-Millman theorem shows that μ is a convex combination of extremal points of $M_\mu(T)$ [Lang, Th. IV.1.5 p. 88]. Such an extremal point must be ergodic (if non-zero). □

5.2 Limits and maps preserving alignment

The following proposition is crucial in this section:

Proposition 5.5. *Let T be a minimal non-abelian action with dense orbits of a finitely generated group Γ, and assume that T is not a line. Assume we are given actions T_p, T'_p and T' such that $T_p \xrightarrow{t\to\infty} T$ and $T'_p \xrightarrow{t\to\infty} T'$, and assume that we have equivariant 1-Lipschitz maps preserving alignment $q_p : T_p \to T'_p$.*

Then there exists a natural equivariant 1-Lipschitz map $q : T \to T'$ preserving alignment.

Remark. This proposition can easily be checked to hold under the weaker assumption that q_p is 1-lipschitz and has a backtracking constant going to 0 as p tends to infinity.

Proof. Let K_p and K'_p be two exhaustions of T and T' by finite subtrees, F_p an exhaustion of Γ by finite subsets, and ε_p a sequence of numbers decreasing towards zero. By passing to a sub-sequence, we may assume that

- there is an F_p-equivariant ε_p-approximation R_p between $K_p \subset T$ and $H_p \subset T_p$
- there is an F_p-equivariant ε_p-approximation R'_p between $K'_p \subset T'$ and $H'_p \subset T'_p$.

Here is a method to construct $x' = q(x) \in T'$. Take $x \in T$ and assume that p is large enough so that $x \in K_p$. Let $x_p \in H_p$ be an R_p-approximation point of x and let $x'_p = q_p(x_p)$. Let y'_p be an R'_p-approximation point of the projection of x'_p on H'_p. We are going to prove that $d(x'_p, H'_p) \xrightarrow{p\to\infty} 0$ and that y'_p converges in T' to a point which we will define to be $q(x)$.

As in the proof of lemma 5.3, for every $\varepsilon > 0$, we can find hyperbolic elements $\gamma, \delta \in \Gamma$ such that

- Axis $(\gamma) \cap$ Axis $(\delta) = \emptyset$
- $x \in \mathrm{bridge}_T\,(\gamma, \delta)$
- the diameter of $\mathrm{bridge}_T\,(\gamma, \delta)$ is at most ε.

An easy argument about the Gromov topology shows that for p large enough, γ and δ are hyperbolic in T_p, their axes don't intersect, they are at most 2ε-far from each other, and x_p is ε-close to $\mathrm{bridge}_{T_p}\,(\gamma, \delta)$.

Lemma 5.1 implies that the characteristic sets of γ and δ intersect in at most one point. Moreover, $q_p[\mathrm{bridge}_{T_p}\,(\gamma, \delta)] = \mathrm{bridge}_{T'_p}\,(\gamma, \delta)$. Since q_p is 1-Lipschitz, $d(x'_p, \mathrm{bridge}_{T'_p}\,(\gamma, \delta)) \leq \varepsilon$ and the diameter of $\mathrm{bridge}_{T'_p}\,(\gamma, \delta)$ is at most 2ε. To show the first part of the claim, just notice that for p large enough, H'_p contains $\mathrm{bridge}_{T'_p}\,(\gamma, \delta)$ because it meets Char γ and Char δ (this is a simple argument about the Gromov topology).

Let x''_p be the projection of x'_p on H'_p and let y'_p be an approximation point of x''_p in T'. The condition $\#(\mathrm{Char}\,\gamma \cap \mathrm{Char}\,\delta) \leq 1$ being a closed condition in the equivariant Gromov topology (see for instance [Pau2]), $\mathrm{Char}_{T'}\,\gamma \cap \mathrm{Char}_{T'}\,\delta$ contains at most one point. Moreover, since the diameter of $\mathrm{bridge}_{T'_p}\,(\gamma, \delta)$ is at most 2ε for every p, so is the diameter of $\mathrm{bridge}_{T'}\,(\gamma, \delta)$. For sufficiently large p, y'_p is 2ε-close to $\mathrm{bridge}_{T'}\,(\gamma, \delta)$, which implies that for p, q large enough, $d(y'_p, y'_q) \leq 6\varepsilon$, so y'_p is Cauchy. Note that T may not be complete (and in this case its completion is not minimal). But the argument above shows that if γ_0, δ_0 are fixed hyperbolic elements of Γ such that $x \in \mathrm{bridge}_T\,(\gamma_0, \delta_0)$, $d(y'_p, \mathrm{bridge}_{T'}\,(\gamma_0, \delta_0))$ tends to 0 as p tends to infinity. Since $\mathrm{bridge}_{T'}\,(\gamma_0, \delta_0)$ is compact, y'_p converges to a point in this set which proves the claim.

The limit $q(x)$ of y'_p is independent of the choices made since we may apply the claim to the sequence obtained by alternating the terms of two sequences ${y'_p}^{(1)}$ and ${y'_p}^{(2)}$ corresponding to different choices. The fact that q is equivariant and 1-Lipshitz is clear. To prove that q preserves alignment, pick $a, b, c \in T$ aligned in this order, i. e. such that $(a|c)_b = 0$. Some approximation points a_p, b_p, c_p in T'_p satisfy $(a_p|c_p)_{b_p} \leq 3\varepsilon_p/2$. Since a 1-Lipschitz map preserving alignment decreases the Gromov product, $(a''_p|c''_p)_{b''_p} \leq (a'_p|c'_p)_{b'_p} \leq 3\varepsilon_p/2$ where a'_p, b'_p, c'_p are the images through q of a_p, b_p, c_p and a''_p, b''_p, c''_p are their projection on H'_p (this projection is 1-Lipschitz and preserves alignment). We deduce that $(q(a)|q(c))_{q(b)} = 0$, and q preserves alignement. □

5.3 Approximation of actions with few ergodic measures

Theorem 3. *Let $n \geq 3$ and let T be a very small action with dense orbits. If the Lebesgue measure on T is the sum of at most $n-1$ ergodic measures, then $T \in \mathcal{M}_n$.*

Remark. There exist actions for which the Lebesgue measure is non-ergodic since Keynes-Newton and Keane have shown how to build interval exchanges for which the Lebesgue measure is non-ergodic ([KeNe, Kea]). In fact, the number of ergodic measures of an orientable measured foliation on a compact orientable surface with fundamental group F_n is at most $n-1$ and equality is reached (see [Sa] for instance). The number of ergodic measures of a non-orientable measured foliation on a non-orientable surface with fundamental group F_n is at most $3n-4$. More recently, Martin proved that there exists non-ergodic systems of isometries of exotic type ([Mar]). There is a very easy way to construct non-geometric very small actions with dense orbits for which the Lebesgue measure is not ergodic: start from two non-geometric free F_3-actions T_1, T_2 with dense orbits. Given two base points $*_1$ and $*_2$ in T_1, T_2, the action $T = T_1 \underset{*_1=*_2}{*} T_2$ of $F_3 * F_3$ has dense orbits and is free. The Lebesgue measure of T is not ergodic since one may multiply by λ_1 and λ_2 the metrics on T_1 and T_2.

Proof. We have to approximate T by simplicial actions in $\mathcal{F}_n$. We first prove the theorem when the Lebesgue measure on T is ergodic, since the proof is simpler.

Take a sequence of very small (or even free) simplicial actions T_p converging to T. Given an edge $e_p \in T_p$, we consider the action T'_p obtained by collapsing to a point every edge which is not in the orbit of e_p (T'_p may be seen as $(T_p)_{\mu_p}$ where μ_p is the restriction of the Lebesgue measure on $F_n.e_p$). The collapsing map $q_p : T_p \to T'_p$ is 1-Lipschitz and preserves alignment.

We show that e_p may be chosen so that a subsequence of T'_p converges to a very small action T' (without rescaling the metric on T'_p): take $g \in F_n$ hyperbolic in T. Since T_p has at most $3n-3$ orbits of edges, there is an edge e_p of T_p whose orbit contributes at least $1/3n-3$ to the translation length of g (if I is a fundamental domain for the action of g on its axis, $|I \cap F_n.e_p| \geq l_{T_p}(e_p)/3n-3$). Compactness of $\overline{CV}_n$ and the fact that $l_{T'_p}(g)$ remains bounded away from 0, implies that up to taking a subsequence, we may assume that T'_p converges to a very small action T'.

Proposition 5.5 and ergodicity then show that T' is homothetic to T. Morever, since the quotient graph of T'_p has exactly one edge, T'_p cannot lie in $\mathcal{O}_n$. T'_p being simplicial, we get that $T'_p \in \mathcal{M}_n$. Therefore, T' (and hence T) lies in $\mathcal{M}_n$.

Now let's turn to the proof of the general case. First, Bestvina and Feighn show that T may be approximated by simplicial very small actions T_p such that there exist equivariant morphisms of $\mathbb{R}$-trees $f_p : T_p \to T$ ([BF2]). Let λ be the Lebesgue measure on T, and let $\mu_1, \dots, \mu_k$ be ergodic measures such that $\lambda = \mu_1 + \dots + \mu_k$ for some $k \leq n-1$. Denote by $\nu_i^p = f_p^* \mu_i$ the pull-back measure on T_p. The density of ν_i^p with respect to the Lebesgue measure on T_p is at most 1. Let $T_p^i = (T_p)_{\nu_i^p}$ be the corresponding simplicial action.

We show that T_p^i converges to T_{μ_i} when $p \to \infty$. For every $g \in F_n$, $l_{T_p^i}(g) \leq l_{T_p}(g)$ so if g is elliptic in T then $l_{T_p^i}(g)$ converges to $l_{T_{\mu_i}}(g) = 0$. When g is hyperbolic in T, g is hyperbolic in T_p for large p. Let I be an interval of length $l_{T_p}(g)$ in

$\text{Axis}_{T_p}(g)$ and subdivide I into sub-intervals isometrically embedded in T through f_p. This subdivision may be refined so that there exists a finite union E of the sub-intervals such that $f_p(E)$ is an interval of length $l_T(g)$ contained in $\text{Axis}_T(g)$, and such that f_p is one-to-one in restriction to $E \setminus \partial E$. This implies that f_p is isometric in restriction to each component of E, so that the Lebesgue measure of $I \setminus E$ is $l_{T_p}(g) - l_T(g)$ and thus tends to 0 as p tends to infinity. In the same way, $l_{T_p^i}(g) - l_{T_{\mu_i}} = \nu_i^p(I \setminus E)$, hence tends to 0 since $\nu_i^p(I \setminus E)$ is smaller than the Lebesgue measure of $(I \setminus E)$. This shows that T_p^i converges to T_{μ_i} when $p \to \infty$.

The argument in the ergodic case tells us that for each $i \in \{1, \dots, k\}$, up to taking a subsequence, we can collapse edges in T_p^i to obtain actions $T_p'^i$ having exactly one orbit of edges, and which converge to some action T'^i homothetic to T_{μ_i} since the Lebesgue measure of T_{μ_i} is ergodic. Let $0 < t_i \leq 1$ be such that $T'^i = t_i.T_{\mu_i}$. We denote by σ_i^p the measure on T_p corresponding to this collapse: σ_i^p is the restriction of ν_i^p to the non-collapsed edges. With these notation, $T_p'^i = (T_p)_{\sigma_i^p}$.

Now consider the measure σ on T_p defined by

$$\sigma_p = \sum_{i=1}^{k} \frac{1}{t_i} \sigma_p^i.$$

Then $(T_p)_{\sigma_p}$ tends to T as p tends to infinity. Since each σ_i^p is non-zero on at most 1 orbit of edges, $(T_p)_{\sigma_p}$ has at most $k \leq n-1$ orbits of edges. To conclude, just notice that any very small simplicial action having at most $n-1$ orbits of edges cannot lie in $\mathcal{O}_n$. $\square$

References

[Bass] H. Bass. *Covering theory for graph of groups*, Journal of Pure and Applied Algebra **89** (1993), 3-47.

[Be] M. Bestvina. *$\mathbb{R}$-trees in topology, geometry, and group theory*, Prépublication (1997).

[BF] M. Bestvina & M. Feighn. *Bounding the complexity of simplicial group actions on trees*, Invent. Math. **103** (1991) 449-469.

[BF2] M. Bestvina & M. Feighn. *Outer limits*, preprint, (Oct. 92).

[BF3] M. Bestvina & M. Feighn. *Topology at infinity of* $\text{Out}(F_n)$, preprint, (1997).

[BH] M. Bestvina, & M. Handel. *Train tracks for automorphisms of free group*, Ann. Math. **135** (1992), 1-51.

[CL] M. Cohen & M. Lustig. *Very small group actions on $\mathbb{R}$-trees and Dehn twist automorphisms,* Topology **34** (1995) 575–617.

[CuMo] M. CULLER & J.W. MORGAN. *Group actions on* $\mathbb{R}$*-trees,* Proc. London Math. Soc. **55** (1987), 571–604.

[CV] M. CULLER & K. VOGTMANN. *Moduli of graphs and outer automorphisms of free groups,* Invent. math. **84** (1986), 91–119.

[CV2] M. CULLER & K. VOGTMANN. *The boundary of outer space in rank two,* in Arboreal group theory (R. C. Alperin ed.) Springer-Verlag 1991.

[DaNo] C. DANTHONY & A. NOGUEIRA. *Measured foliations on nonorientable surfaces* Ann. Scient. Ec. Norm. Sup., **23**, 4ème serie (1990), 469–494.

[Dun] M. J. DUNWOODY *Folding sequences* preprint.

[FLP] A. FATHI, F. LAUDENBACH & V. POENARU. *Travaux de Thurston sur les surfaces,* Astérisque **66–67**, Soc. Math. France, 1979.

[GL] D. GABORIAU AND G. LEVITT. *The rank of actions on* $\mathbb{R}$*-trees,* Ann. Sc. E.N.S. **28** (1995), 549–570.

[Gui] V. GUIRARDEL. *Actions de groupes sur des arbres réels et dynamique dans la frontière de l'outre-espace,* thèse de l'université Toulouse III (1998).

[Kea] M. KEANE. *Non-ergodic interval exchange transformations*, Israel J. Math. **26** (1977), 188–196.

[KeNe] H. B. KEYNES & D. NEWTON. *A minimal non-uniquely ergodic interval exchange transformation*, Math. Z. **148** (1976), 101–105.

[Lev] G. LEVITT.*La dynamique des pseudogroupes de rotations,* Invent. Math. **113** (1992), 633–670.

[Lev2] G. LEVITT. *Graphs of actions on* $\mathbb{R}$*-trees,* Comment. Math. Helvetici **69** (1994) 28–38.

[Lang] S. LANG. *Real and Functional Analysis.* Third edition. Graduate texts in Mathematics 142. Springer Verlag, New York, 1993.

[Lus] M. LUSTIG. *Automorphisms, train track, and non-simplicial* $\mathbb{R}$*-tree actions*, to appear in Comm. in Algebra.

[Lus2] M. LUSTIG. In preparation.

[Mar] R. MARTIN. *Non-uniquely ergodic foliations of thin type* Erg. Th. & Dyn. Sys. **17** (1997), 667–674.

[Pau] F. PAULIN.*Topologie de Gromov équivariante, structures hyperboliques et arbres réels,* Inv. Math. **94** (1998) 53–80.

[Pau2] F. PAULIN. *The Gromov topology on $\mathbb{R}$-trees,* Topology and its applications **32** (1989) 197–221.

[Pau3] F. PAULIN. *De la géométrie et la dynamique des groupes discrets,* Habilitation à diriger les recherches, E.N.S. Lyon, 28 juin 95.

[Sa] E. SATAEV. *On the number of invariant measures for flows on orientable surfaces,* math. USSR Izv. 9 (1975), 813–830.

[Sha] P. SHALEN. *Dendrology of groups, an introduction* Essays in group theory, (S. M. Gersten éd.), MSRI Publ., vol. 8, Springer-Verlag, 1987, pp. 265–319.

[Sha2] P. SHALEN. *Dendrology and its applications,* Group theory from a geometrical viewpoint, (E. Ghys, A. Haefliger, & A. Verjovski éd.), World scientific, 1991.

[Sta] J. R. STALLINGS. *Topology of finite graphs,* Invent. Math. **71** (1983) 551–565.

Titre Français :
Dynamique de $\mathrm{Out}(F_n)$ sur la frontière de l'outre-espace.

Résumé :

Dans cet article, nous étudions la dynamique de l'action du groupe $\mathrm{Out}(F_n)$ sur la frontière ∂CV_n de l'outre-espace : nous décrivons un sous-ensemble fermé propre $\mathcal{F}_n$ de ∂CV_n invariant sous l'action de $\mathrm{Out}(F_n)$ et tel que $\mathrm{Out}(F_n)$ agisse proprement discontinûment sur l'ouvert complémentaire. Nous prouvons ensuite qu'il existe un unique fermé non-vide invariant non vide $\mathcal{M}_n$ dans $\mathcal{F}_n$. Cet ensemble $\mathcal{M}_n$ est l'adhérence de l'orbite de toute action simpliciale appartenant à $\mathcal{F}_n$. Nous démontrons enfin que $\mathcal{M}_n$ contient toutes les actions ayant au plus $n-1$ mesures ergodiques. Ce dernier résultat rend probable l'égalité de $\mathcal{M}_n$ et de $\mathcal{F}_n$, de sorte que $\mathcal{F}_n$ serait l'ensemble limite de $\mathrm{Out}(F_n)$, le complémentaire de $\mathcal{F}_n$ étant son domaine de discontinuité.

Present address:
Vincent Guirardel
Department of Mathematics,
University of Utah,
Salt Lake city UT 84112
USA. e-mail: guirarde@picard.ups-tlse.fr