Reading Small Actions of a One-Ended Hyperbolic Group on $\mathbb{R}$-Trees from its JSJ Splitting.

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Abstract

In this paper, we show how the canonical JSJ splitting of a one-ended hyperbolic group allows to understand all its small actions on $\mathbb{R}$-trees: they are obtained by blowing up the surface type vertices into an action corresponding to a measured foliation on the corresponding orbifold, by blowing up elementary type vertices into a finite tree with bounded complexity, and by collapsing some edges. We deduce that every small action of a one-ended hyperbolic group on an $\mathbb{R}$-tree is geometric. We also derive a strong uniqueness property of the JSJ splitting.

1 Introduction, statement of results

In [Sel2], Z. Sela introduced the JSJ splitting of a torsion free one-ended hyperbolic group. This splitting is inspired by the works of W. H. Jaco, P. B. Shalen and K. Johannson on the characteristic submanifold of an irreducible 3-manifold ([JaSh, Jo]). The JSJ splitting of a group Γ is a splitting of this group over cyclic groups with some maximality properties. Thus, every elementary splitting of Γ can be *read* from the JSJ decomposition (here, *elementary* splitting means amalgamated product or HNN extension). In these splittings, the surface groups play a particular role due to the fact that two splittings of a surface group corresponding to two intersecting closed simple curves have no common refinement. Therefore, the surface groups are considered as a whole and are not split in the JSJ splitting. Since then, the JSJ splitting has been generalized by E. Rips and Z. Sela to one-ended finitely presented groups ([RiSe]), by M. J. Dunwoody and M. E. Sageev for splittings over slender groups with a one-ended like hypothesis on Γ ([DuSa]), and by K. Fujiwara and P. Papasoglu without this hypothesis ([FuPa]).

B. Bowditch gave a topological approach of this splitting in [Bo2]. It deals with one-ended hyperbolic groups (maybe with torsion) over virtually cyclic groups. This splitting can be read from the topology of the boundary at infinity $\partial\Gamma$ of Γ and in particular from the structure of its local cut points. This theorem uses the fact that $\partial\Gamma$ is locally connected, which was recently proved by B. Bowditch and G. A. Swarup ([Bo1, Bo3, Lev5, Swa, Bo4]) using a theorem by M. Bestvina and G. Mess saying that local connexity is implied by the absence of global cut points ([BM]).

In this article, we use Bowditch's approach of the JSJ splitting because of its strong uniqueness properties. Our main theorem shows how this splitting allows to understand every small action of a one-ended hyperbolic group on $\mathbb{R}$-trees: they all can be *read* from the JSJ splitting, i. e. may be obtained by blowing up the JSJ-tree in a special way (see definition 1.4). This allows us to derive that small actions of a one-ended hyperbolic group are geometric, thus generalizing Skora's Theorem. To make those statements precise, we need to introduce more precisely the JSJ-splitting.

The JSJ splitting of a one-ended hyperbolic group. Let's first recall some definitions. In all the sequel, we will use the terminology *two-ended* for (infinite) virtually cyclic groups.

Definition 1.1. *A* fuchsian *group* Γ *is a non-elementary group (i. e. infinite and not 2-ended) together with a properly discontinuous isometric action on* $\mathbb{H}^2$ *(this action may not be faithful but its kernel is finite). Its* convex core *is the smallest non-empty closed invariant convex subset of* $\mathbb{H}^2$. Γ *is* convex cocompact *if* Γ *acts cocompactly on its convex core. Its* peripheral subgroups *are the setwize stabilizers of the boundary components of its convex core.*

Remark. The quotient of the convex core of Γ by the action of Γ is a 2-orbifold with boundary whose fundamental group is the quotient of Γ by the kernel of the action.

In the following, we blurry the difference between a splitting, a graph of groups, and the action of its fundamental group on its universal cover. Note that a small action of the one-ended hyperbolic group Γ on a simplicial tree is an action with 2-ended edge stabilizers because by Stallings' theorem, the one-ended group Γ doesn't split over a finite group.

Theorem 1.2 ([Bo2] th. 0.1, 5.28 and prop. 5.30). *Let* Γ *be a one-ended hyperbolic group. There exists a canonical action* (S,Γ) *of* Γ *on a simplicial tree* S *with 2-ended edge stabilizers called the JSJ splitting with* *the following properties. The vertices of S are of three types according to their stabilizer:*

- *the* elementary type *vertices, whose stabilizers are 2-ended. Their valence in S is finite.*
- *the* surface type *vertices, whose stabilizers are non elementary convex cocompact fuchsian groups, and which are quasi-convex in Γ. Their peripheral subgroups are precisely the stabilizers of the incident edges.*
- *the* rigid type *vertices which are not of the previous types. Their stabilizer don't have any non trivial action on any simplicial tree T with 2-ended edge stabilizers such that the stabilizers of the edges incident on v in S fix a point in T. In other words, the quotient graph of groups S/Γ cannot be refined at a rigid vertex into a splitting over 2-ended groups.*

Moreover, two adjacent vertices in S are not of the same type and the action of Γ preserves the type. Hence it has no inversion. Moreover, if v is a valence 2 *vertex in S then the two neighbours of v are of rigid type.*

This splitting is canonical in the sense that for every automorphism $\alpha \in$ $\mathrm{Aut}(\Gamma)$ *there is an equivariant automorphism between $S.\alpha$ and S (if $\rho : \Gamma \to$* $\mathrm{Isom}\,(S)$ *is the morphism defining the action of Γ on S, $S.\alpha$ is the action of Γ on S defined by $\rho' = \rho \circ \alpha$).*

The JSJ splitting is maximal in the following sense: if Γ splits over a 2-ended group E, then E is conjugate to a subgroup of a vertex stabilizer of S of surface or elementary type.

Finally, the JSJ splitting is invariant under any automorphism of Γ: for every $\alpha \in$ $\mathrm{Aut}(\Gamma)$*, there exists an automorphism $h_\alpha : S \to S$ such that $h_\alpha(g.x) = \alpha(g).h_\alpha(x)$ for every $x \in S, g \in \Gamma$.*

Remark. It may happen that S is reduced to one point. Then Γ is either a cocompact fuchsian group, or a *rigid* group in the sense that it doesn't split over any 2-ended group. We won't consider those cases in the sequel.

Reading small actions from the JSJ splitting An (isometric) group action on an $\mathbb{R}$-tree is *small* if the pointwise stabilizer of any non-degenerate arc doesn't contain a non-abelian free group. When the group considered is hyperbolic, then arc stabilizers are finite or two-ended. A *blow up* of a simplicial action S is an action obtained by replacing equivariantly every vertex v of S by an $\mathbb{R}$-tree T_v, by gluing equivariantly on T_v edges which were incident on v in S, and by giving equivariantly a (maybe 0) length to the edges of S. Equivalently, a blow up of S is the action corresponding to a graph of actions on $\mathbb{R}$-trees with underlying graph of groups S/Γ, and with a (maybe 0) length for every edge in S/Γ (see [Lev4, Gui]).

The goal of this paper is the following theorem:

Theorem 1.3 (Reading Theorem). *Every minimal small action of a one-ended hyperbolic group Γ on an $\mathbb{R}$-tree can be read from its JSJ decomposition in the sense of definition 1.4.*

Definition 1.4. *Let (S, Γ) be a small action on a simplicial tree. We classify the vertices into three types according to their stabilizers: a vertex is of elementary type if its stabilizer is 2-ended, it is of surface type if it is a convex cocompact fuchsian group and if its peripheral subgroups are precisely the stabilizers of the incident edges, and rigid otherwise.*

Assume we are given a small minimal action (T, Γ). We say that (T, Γ) can be read *from (S, Γ) if the edge stabilizers of S, the rigid groups and the elementary groups fix a point in T and if there is an equivariant isometry between (T, Γ) and the action corresponding to a graph of actions of the following form: Let $Q = S/\Gamma$ be the quotient graph of groups of the JSJ-splitting, Γ_v and Γ_e the vertex and edge groups, and i_e the edge morphisms.*

- *For each surface type vertex in Q, take a small (maybe trivial) minimal action (T_v, Γ_v) such that for each oriented edge e incident on v, $i_e(\Gamma_e)$ has a fixed point p_e in T_v.*

- *For each elementary type vertex v in Q, consider an action of Γ_v on a finite tree T_v having a global fixed point (it is not assumed to be minimal), and take for each oriented edge e incident on v a point p_e fixed by Γ_e. We require T_v to be the convex hull of the union of the (finite) orbits of the points p_e.*

- *For each rigid vertex v, we take T_v to be the trivial action of Γ_v on a point.*

- *We assign to every non-oriented edge of Q a maybe 0 length. We require that the edges incident on an elementary type vertex have 0 length.*

Remark. We don't assume in this definition that the rigid type groups are actually rigid in the sense of Theorem 1.2 but this fact has to be true if every small action can be read from (S, Γ).

By Skora's Theorem, the small actions of a surface type group Γ_v with elliptic peripheral subgroups are well known: they are obtained by taking the space of leaves of a Γ_v-invariant measured foliation on its convex core (or equivalently by taking the dual tree of an invariant measured lamination). In particular, these actions are geometric. Roughly speaking, an action is geometric if it comes similarly by taking the space of leaves from a cover of finite 2-complex with a measured foliation (see [LP] or 5.1 for a formal definition).

We then deduce from the Reading Theorem 1.3 a generalization of Skora's Theorem:

Corollary 1.5. *Every small action of a one-ended hyperbolic group is geometric.*

The Reading Theorem 1.3 also implies a strong uniqueness property of the JSJ splitting:

Corollary 1.6. *Let Γ be a one-ended hyperbolic group. The JSJ splitting of Γ is the only minimal small simplicial action (S, Γ) in which no edge connects 2 elementary type vertices such that every small action of Γ can be read from S up to equivariant homeomorphism (i. e. up to subdivision of edges and equivariant simplicial isomorphism).*

Moreover, it is unique up to equivariant simplicial isomorphism (without subdivision) if we add the requirements that no edge of S is adjacent to 2 vertices having the same type and that the two neighbours of a valence 2 vertex have the same type (these properties are satisfied by the JSJ splitting).

The proof of the theorem goes as follows. Using the construction of the JSJ-splitting, we first prove that edge stabilizers of the JSJ-splitting S fix a point in T. This implies that a rigid group $\Gamma(v)$ fix a point in T because Rips Splitting Theorem would otherwise provide a splitting of $\Gamma(v)$ contradicting its rigidity. This point is unique because the action on T is small so there is no choice in the attaching points of the incident edges. We then study the action of a surface type group $\Gamma(v)$ and show that the peripheral subgroups fix a unique point in the minimal Γ_v invariant subtree T_v. These points will be the attaching points of the corresponding edges. We then associate to an elementary vertex $v \in S$ the convex hull T_v of the attaching point of the edges originating from v. This provides a blow up of S with an equivariant morphism of $\mathbb{R}$-trees to T. In the last part we check that this morphism is an isometry.

In section 2, we recall some definitions. In section 3 we sum up Bowditch's construction of the JSJ splitting. The proof of Theorem 1.3 is given in section 4, and the corollaries are proven in section 5.

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2 Preliminaries

Generalities about $\mathbb{R}$-trees may be found in [Sha2, Sha1]. An $\mathbb{R}$-tree is an arcwise connected metric space in which any topological arc is isometric to an interval of $\mathbb{R}$. Simplicial trees endowed with a path metric provide examples of $\mathbb{R}$-trees. All the actions on $\mathbb{R}$-trees we will consider are *isometric* actions of *finitely generated* groups. We will often shortly say *action* to mean a group action on an $\mathbb{R}$-tree.

Given an isometric action (T, Γ), we say that $g \in \Gamma$ is *elliptic* if it has a fixed point in T. In this case $\operatorname{Fix} g$ is a subtree of T. Otherwise, we say that g is *hyperbolic*, in which case there is a unique g-invariant line called its *axis* on which g acts by translations. More generally, we will say that a subgroup $\Gamma_0 \subset \Gamma$ is elliptic if it fixes a point in T. The translation length $l_T(g) = \inf_{x \in T} d(x, g.x)$ is reached on the axis of g if g is hyperbolic, and on its fixed set if it is elliptic.

Two actions of Γ on $\mathbb{R}$-trees will be identified if there is an equivariant isometry between them. Unless otherwise stated, we will always assume that the actions we consider are minimal, i. e. have no proper invariant subtree. We can always reduce to this case since when a finitely generated group Γ acts on an $\mathbb{R}$-tree T with no global fixed point, there is a unique minimal non-empty Γ invariant subtree: it is the union of the translation axes of hyperbolic elements of Γ ([CuMo]).

A morphism of $\mathbb{R}$-trees f between two actions (T, Γ), (T', Γ) is an equivariant map such that every arc in T can be subdivided into finitely many sub-arcs in restriction to which f is an isometry. A map $f : T \to T'$ *preserves alignment* if $x \in [y, z]$ implies $f(x) \in [f(y), f(z)]$. A typical example of a map preserving alignment is the map induced by collapsing edges in a simplicial tree. Also note that if a morphism of $\mathbb{R}$-trees preserves alignment, then it is an isometry.

Definition. *A* graph of actions *on $\mathbb{R}$-trees Q is a graph of groups Q with vertex groups Γ_v, edge groups Γ_e and edge morphisms $i_e : \Gamma_e \to \Gamma_{t(e)}$ together with the following data:*

- *for each vertex v an action (T_v, Γ_v) of the corresponding vertex group*
- *for each oriented edge e incident to $v = t(e)$, an attaching point $p_e \in T_v$ fixed by $i_e(\Gamma_e)$*
- *a (maybe 0) length for each non-oriented edge of Q.*

We define the fundamental group of $\mathcal{Q}$ to be the fundamental group of Q. To a graph of actions $\mathcal{Q}$ naturally corresponds an action $(T_{\mathcal{Q}}, \pi_1(\mathcal{Q}))$: it is obtained from the universal cover T_Q of the graph of groups Q by replacing a vertex v of T_Q by a copy of the corresponding vertex $\mathbb{R}$-tree $T_{\bar{v}}$, by gluing equivariantly edges incident to a vertex v on T_v according to the attaching points, and by collapsing the 0-length edges. The action $(T_{\mathcal{Q}}, \Gamma)$ is called a blow-up *of $(T_Q, \pi_1(Q))$.*

Finally, we will also need a topology on the set of actions of a group Γ on $\mathbb{R}$-trees. The *translation length topology* is the weakest topology which makes the functions $T \mapsto l_T(\gamma)$ continuous for every $\gamma \in \Gamma$. In other words, (T_n, Γ) converges to (T, Γ) if and only if for every $\gamma \in \Gamma$, $l_{T_n}(\Gamma) \xrightarrow{n\to\infty} l_T(\Gamma)$. This topology is Hausdorff on the set of small actions of a finitely generated group containing a non-abelian free group ([CuMo]).

A set of minimal actions of a fixed finitely generated group Γ on $\mathbb{R}$-trees can also be equipped with the *equivariant Gromov topology*. This topology roughly says that two actions are close if they look the same up to some small ε in restriction to a finite subset of T while only considering the action of a finite subset of Γ. More precisely, a neighbourhood basis for (T, Γ) is given by the family $V_T(\varepsilon, F, \{x_1, \dots, x_n\})$ for $\varepsilon > 0$, F finite subset of Γ and $x_1, \dots, x_n \in T$ defined as follows: $T' \in V_T(\varepsilon, F, \{x_1, \dots, x_n\})$ if there exists $x'_1, \dots, x'_n \in T'$ such that for all $i, j \in \{1, \dots, n\}$ and for all $g, h \in F$

$$|d(g.x_i, h.x_j) - d(g.x'_i, h'.x_j)| < \varepsilon.$$

This topology is often more convenient than the translation length topology because it is more geometric. However, on the set of small actions of a group Γ containing a non-abelian free group, those topology coincide ([Pau2]).

We will use the following theorem which claims the density of simplicial actions in the set of small actions of a hyperbolic group.

Theorem 2.1 ([Gui]). *Let Γ be a hyperbolic group. Every minimal small action of Γ on an $\mathbb{R}$-tree can be approximated by a minimal small action of Γ on a* simplicial *tree.*

Moreover, if $\Gamma_1, \dots, \Gamma_k \subset \Gamma$ are finitely generated subgroups of Γ which fix a point in T, they may be asked to fix a point in the approximation.

3 Construction of the JSJ splitting (Bowditch).

Everything in this section can be found in [Bo2]. Bowditch's construction of the JSJ splitting both relies on the topology of the boundary $\partial\Gamma$ of Γ and on the dynamics of the action of Γ on $\partial\Gamma$. Here, we assume that Γ is a one-ended hyperbolic group, which is now known to imply that $\partial\Gamma$ is locally connected as proved by Bowditch and Swarup ([Bo1, Bo3, Lev5, Swa, Bo4]). Thus, from the topological point of view, $\partial\Gamma$ is a connected locally connected metrizable compact set (a Peano set –about Peano sets, see for instance [HoY]) and the crucial point is the study of local cut points. From the dynamical point of view, Γ acts on $\partial\Gamma$ as a *uniform convergence group*. By definition a group Γ acts as a convergence group on $\partial\Gamma$ if for every infinite sequence γ_i of distinct elements in $\partial\Gamma$, one can extract a subsequence γ_{k_i} whose dynamics is loxodromic, i. e. with two distinguished points called *repelling point* and *sink* such that for every compact K not containing the repelling point, $\gamma_{k_i}.K$ converges uniformly to the sink. This condition is equivalent to saying that Γ acts properly discontinuously on $(\partial\Gamma)^3$ minus the big diagonal. The convergence group Γ is a *uniform* convergence group if it acts cocompactly on $(\partial\Gamma)^3$ minus the big diagonal.

3.1 Simplicial actions and quasi-convexity

Let Γ be a hyperbolic group.

Definition. *A subgroup H of Γ is quasi-convex if there exists a constant C such that every geodesic in Γ joining two points in H remains within distance C from H.*

Definition. *A simplicial action is finitely supported if it has a finite subtree meeting every orbit. For instance, every minimal action of a finitely generated group is finitely supported.*

Proposition 3.1 ([Bo2]). *If (T, Γ) is a finitely supported simplicial action with quasi-convex edge stabilizers, then the vertex stabilizers are quasi-convex.*

A quasi-convex subgroup H of Γ is hyperbolic, and its boundary ∂H is naturally identified with its *limit set* $\Lambda H \subset \partial\Gamma$, i. e. the set of accumulation points of H in $\partial\Gamma$. An infinite subgroup of Γ which doesn't contain F_2 is virtually cyclic (2-ended), quasi-convex, and its limit set consists of 2 points. These points are the repelling point and the sink of the powers of any infinite order element γ in this subgroup. In this case, there is no need to take a subsequence in the convergence property, so we can denote $\gamma^+, \gamma^- \in \partial\Gamma$ the sink and the repelling point for the positive powers of γ. The previous proposition implies that the vertex stabilizers of a (minimal) small simplicial action are quasi-convex.

Let T be a simplicial tree (or an $\mathbb{R}$-tree). A ray in T is an equivalence class of isometric applications $r : [0, \infty) \to T$ where $r \sim r'$ if the intersection of their images is not compact. The set of rays of T is denoted ∂T. We also denote $\Gamma(v)$ and $\Gamma(e)$ the stabilizers of a vertex v or an edge e in an action (T, Γ).

Proposition 3.2 ([Bo2]). *Let (T, Γ) be a simplicial action with quasi-convex edge stabilizers and let $*$ be a base point. For every ray r of T, consider a sequence γ_i of elements of Γ such that $|[*, \gamma_i.*] \cap r| \to \infty$ (where $|.|$ denotes the length). Then γ_i converges to a point $j(r) \in \partial\Gamma$ depending only on r. Moreover, j induces an identification between ∂T and*

$$\partial\Gamma \setminus \left(\bigcup_{v \in T} \Lambda\Gamma(v) \right).$$

Remark. If r^+, r^- denote respectively the positive and negative semi-axis of an element γ hyperbolic in T, then $j(r^+) = \gamma^+$ and $j(r^-) = \gamma^-$.

3.2 Splittings and local cut points.

From now on, we assume that Γ is one-ended. We will use the local connexity of Γ and the fact that it has no global cut point, i. e. no point a is such that $\partial\Gamma \setminus a$ is disconnected.

Proposition 3.3 ([Bo2]). *Let (T, Γ) be a small simplicial action, and let e be an edge in T. Then $\partial\Gamma \setminus \Lambda\Gamma(e)$ is not connected. More precisely, denote T_1 and T_2 the two connected components of $T \setminus \mathring{e}$ and consider for $i \in \{1, 2\}$*

$$U_i = j(\partial T_i) \cup \left(\bigcup_{v \in T_i} \Lambda\Gamma(v) \setminus \Lambda\Gamma(e) \right).$$

Then U_1 and U_2 are two nonempty disjoint open sets of $\partial\Gamma$ whose union is $\partial\Gamma \setminus \Lambda\Gamma(e)$.

Remark. We will often use this proposition in the following form: if γ is hyperbolic in a simplicial tree T endowed with a small action of Γ and if e is an edge in the axis of γ, then $\Lambda\Gamma(e)$ disconnects γ^+ from γ^- in $\partial\Gamma$.

Proposition 3.3 implies that the two points of $\Lambda\Gamma(e)$ are local cut points:

Definition. *We say that* $x \in \partial\Gamma$ *is a* local cut point *if the connected locally compact locally connected set* $\partial\Gamma \setminus \{x\}$ *has at least two ends. We call* valence *of* x *and we denote* $\mathrm{val}(x) \in \mathbb{N} \cup \{\infty\}$ *the number of ends of* $\partial\Gamma \setminus \{x\}$.

We denote $\partial\Gamma(2)$ *the set of points of valence* 2 *in* $\partial\Gamma$ *and* $\partial\Gamma(3+)$ *the set of points of valence at least* 3.

3.3 The structure of local cut points

Since $\partial\Gamma$ has no global cut point, for all $x, y \in \partial\Gamma$, every connected component of $\partial\Gamma \setminus \{x, y\}$ accumulates both on x and y. Therefore, the number of connected components of $\partial\Gamma \setminus \{x, y\}$ is at most equal to $\mathrm{val}(x)$ and $\mathrm{val}(y)$. Moreover, one can easily check (see [Bo2]) that the local connexity implies that the number of connected components of $\partial\Gamma \setminus \{x, y\}$ is finite.

Definition. *For* $x, y \in \partial\Gamma(2)$, *we denote* $x \sim y$ *if* $x = y$ *or* $\partial\Gamma \setminus \{x, y\}$ *has* *2 connected components.*

Theorem 3.4 ([Bo2], Lemma and Propositions 3.1, 5.15, 3.7, 5.18, 5.17). *The relation* $\sim$ *is an equivalence relation on* $\partial\Gamma(2)$. *Moreover, if* σ *is an* $\sim$*-class with an isolated point, then* $\#\sigma = 2$ *and the setwise stabilizer of* σ *is 2-ended (hence infinite).*

If σ *has no isolated point, we say that its closure* $\bar{\sigma}$ *is a necklace. Then* $\bar{\sigma} \setminus \sigma \subset \partial\Gamma(3+)$. *In particular,* $\bar{\sigma} = \bar{\sigma}'$ *implies* $\sigma = \sigma'$. *Unless* Γ *is a cocompact fuchsian group,* $\bar{\sigma}$ *is a Cantor set. The quaternary relation on* $\bar{\sigma}$ *"*$\{x, z\}$ *disconnects* y *from* t *in* $\partial\Gamma$*" is a cyclic order which allows to see* $\bar{\sigma}$ *as a subset of a circle where* $\{x, z\}$ *disconnects* y *from* t *in* $\partial\Gamma$ *if and only if the same is true in the circle. In particular, for* $x, y, z, t \in \bar{\sigma}$, $\{x, z\}$ *disconnects* y *from* t *in* $\partial\Gamma$ *if and only if* $\{y, t\}$ *disconnects* x *from* z *in* $\partial\Gamma$. *Finally, the setwise stabilizer* Q *of* $\bar{\sigma}$ *is quasi-convex and* $\bar{\sigma} = \Lambda(Q)$.

Definition. *Let* $\bar{\sigma}$ *be a necklace. We say that* $\{x, y\} \subset \bar{\sigma}$ *is a jump if no pair* $\{z, t\}$ *disconnects* x *from* y.

Remark. Since a necklace has no isolated point, two distinct jumps cannot intersect.

Definition. *Let* $\bar{\sigma}$ *be a necklace and* Q *its setwise stabilizer. We call* peripheral subgroups *of* Q *the setwise stabilizers of the jumps.*

Theorem 3.5 ([Bo2], Propositions 4.9 and 5.21). *If* $\bar{\sigma}$ *is a necklace, then its setwise stabilizer* Q *is conjugate to a convex cocompact fuchsian* *group, the peripheral groups of Q corresponding to the peripheral subgroups of the fuchsian group.*

We now recall the properties of cut points of valence at least 3.

Definition. *If $x, y \in \partial\Gamma(3+)$, we denote $x \approx y$ if* $\mathrm{val}(x) = \mathrm{val}(y) = \#\pi_0(\partial\Gamma \setminus \{x, y\})$.

Theorem 3.6 ([Bo2], Lemma 3.8 and prop. 5.13). *If $x \approx y$ and $y \approx z$ then $x = z$. We then say that $\{x, y\}$ is a $\approx$-pair. Every $x \in \partial\Gamma(3+)$ belongs to a $\approx$-pair, and the setwise stabilizer of this pair is 2-ended (in particular infinite).*

Theorem 3.7 ([Bo2], Propositions 5.6 and 5.21). *If γ is an infinite order element in Γ then $\mathrm{val}(\gamma^+) = \mathrm{val}(\gamma^-) = \#\pi_0(\partial\Gamma \setminus \{\gamma^+, \gamma^-\})$. If γ^+ or γ^- is a local cut point, then $\gamma^+ \sim \gamma^-$ or $\gamma^+ \approx \gamma^-$.*

Moreover, if $J = \{a, b\}$ is a jump in a necklace $\bar{\sigma}$, then either $a \sim b \in \sigma$ or $a \approx b \in \bar{\sigma} \setminus \sigma$.

3.4 Constructing a tree from the local cut points

Consider V_e the set of $\sim$-classes with cardinal 2 and of the $\approx$-pairs, and V_s the set of the $\sim$-equivalence classes with no isolated point. V_e will be the set of elementary type vertices, and V_s the set of surface type vertices. The rigid type vertices will appear indirectly.

To construct the JSJ-tree S, B. Bowditch uses a *betweenness* relation denoted xyz on $V_e \cup V_s$. Every subset of a tree with the relation "$xyz \iff y \in (x, z)$" is an example of a betweenness relation (this kind of relation was introduced by Ward).

Definition. *The ternary relation xyz is a* betweenness *relation on a set X if the following properties hold.*

1. *If xyz then $x \neq z$*
2. *xyz and xzy are never simultaneously true.*
3. *xyz if and only if zyx*
4. *If xyz and if $w \neq y$ then xyw or wyz.*

We say that a set together with a betweenness relation is a pretree.

In a pretree, we call intervals the sets $(x,z) = \{y|xyz\}$ and $[x,z] = (x,z) \cup \{x,z\}$. A pretree is said to be *median* if for all x,y,z, $[x,y] \cap [y,z] \cap [x,z] \neq \emptyset$ (then it contains exactly one point). A pretree is *discrete* if the intervals are finite. To a discrete median pretree V naturally corresponds a simplicial tree with vertex set V by putting an edge between the points x,y such that $(x,y) = \emptyset$.

Given a discrete pretree V, one can naturally embed it in a discrete median pretree: just add to V the *stars* of V of cardinal at least 3. A star of V is a subset E of V such that no point in V is between two points in E and if $x \in V \setminus E$, there exists $x,y \in E$ such that xyz.

Now, on $V_e \cup V_s$, we define the ternary relation $\sigma_1\sigma_2\sigma_3$ if σ_2 is distinct from σ_1, σ_3, and there exists $x_1 \in \sigma_1$, $a_2, b_2 \in \sigma_2$ and $x_3 \in \sigma_3$ such that $\{a_2, b_2\}$ disconnects x_1 from x_3 in $\partial\Gamma$.

Theorem 3.8 ([Bo2], prop. 3.20). *The relation $\sigma_1\sigma_2\sigma_3$ is a betweenness relation which makes $V_e \cup V_s$ a discrete pretree. It is naturally endowed with an action of Γ (coming from the action of Γ on $\partial\Gamma$).*

Definition. *Let S be the simplicial tree corresponding to the pretree $V_e \cup V_s$ made median. Its set of vertices is $V_e \coprod V_s \coprod V_r$ where V_r is the set of stars of $V_e \cup V_s$. This tree S is endowed with an action of Γ. It is the JSJ splitting of Γ.*

This tree S is invariant by any automorphism α (and even any quasi-isometry) of Γ since α induces an homeomorphism of $\partial\Gamma$. Theorem 1.2 then follows from the properties of local cut points.

4 Small actions of Γ on $\mathbb{R}$-trees can be read from its JSJ splitting

In this section we prove Theorem 1.3. We consider (T, Γ) a fixed small action of a one-ended hyperbolic group Γ on an $\mathbb{R}$-tree. We denote by (S, Γ) the action of Γ on the JSJ-tree.

4.1 Edge stabilizers of S fix a point in T

Proposition 4.1. *Edge stabilizers of S fix a point in T.*

Proof. Since any two adjacent vertices of S don't have the same type, any edge e is at least adjacent to an elementary vertex or to a surface type vertex. If e is adjacent to a surface type vertex v, then $\Gamma(e)$ is a peripheral subgroup of $\Gamma(v)$. If e is adjacent to an elementary type vertex v, it has finite index in $\Gamma(v)$. Therefore we just have to prove that elementary groups and peripheral subgroups of surface type groups fix a point in T. This is done in Lemma 4.3 and 4.4. $\square$

We first recall a useful lemma.

Lemma 4.2 ([Bo2], Lemma 3.3 and 5.6). *Let $x, y \in \partial\Gamma$ such that $x \sim y$ or $x \approx y$, and let Γ_0 be a 2-ended subgroup for Γ. We denote $\{a, b\} = \Lambda\Gamma_0$.*

If $\{a, b\}$ disconnects x from y then $a, b, x, y \in \partial\Gamma(2)$ and $a \sim b \sim x \sim y$.

Proof. Since $\{a, b\}$ disconnects x from y, a and b are local cut points and Theorem 3.7 says that $a \approx b$ or $a \sim b$.

If $x \approx y$, then $\partial\Gamma \setminus \{x, y\}$ has at least 3 connected components (by definition of $\approx$) and each of them accumulates on both x and y (because $\partial\Gamma$ has no global cut point). Therefore, $\{a, b\}$ can't disconnect x from y which is a contradiction.

If $x \sim y$ and $a \approx b$, the symmetrical argument shows that $\{x, y\}$ can't disconnect a from b. Hence a and b lie in the same component of $\partial\Gamma \setminus \{x, y\}$ and the closure of the other component connects x to y in $\partial\Gamma \setminus \{a, b\}$, thus contradicting the hypothesis.

We deduce that $x \sim y$ and $a \sim b$. Then by Lemma 3.3 of [Bo2], the fact that $\{a, b\}$ disconnects x from y implies $a \sim b \sim x \sim y$. $\square$

Lemma 4.3. *The elementary groups of S fix a point in T.*

Proof. If v is an elementary type vertex in S, it either corresponds to a $\approx$-pair or to an $\sim$-class of cardinal 2 which we call σ. Then by Theorems 3.6 and 3.4, the setwise stabilizer of σ is infinite and 2-ended. Let γ be an infinite order element in this stabilizer so that $\sigma = \{\gamma^+, \gamma^-\}$. We only have to prove that γ is elliptic in T.

Assume on the contrary that γ is hyperbolic in T. By approximation Theorem 2.1, we can approximate (T, Γ) by a small simplicial action (T_0, Γ). If (T_0, Γ) is close enough to (T, Γ), then γ has to be hyperbolic in T_0. Now take an edge e in the axis of γ in T_0. Its stabilizer is not finite since Γ is one-ended. Let $\{a, b\} = \Lambda\Gamma(e)$. Proposition 3.3 implies that $\{a, b\}$ disconnects γ^+ from γ^-. The previous lemma (4.2) then implies $\gamma^+ \sim \gamma^- \sim a \sim b$ thus giving a contradiction with the fact that σ is either a $\approx$-pair or to an $\sim$-class of cardinal 2. $\square$

Lemma 4.4. *Let v be a surface type vertex. Then the peripheral groups of v are elliptic in T.*

Proof. Let σ be the $\sim$-class corresponding to v and $\bar{\sigma}$ the corresponding necklace. By definition, a peripheral subgroup of $\Gamma(v)$ is the setwise stabilizer of a jump $\{x, y\} \subset \bar{\sigma}$.

Theorem 3.7 says that either $x \approx y$ or $x \sim y$. In the first case, Stab $\{x, y\}$ is the stabilizer of an elementary vertex and we already know that it fixes a point in T. Now assume that $x \sim y$, and suppose that there is $\gamma \in$ Stab $\{x, y\}$ which is hyperbolic in T and argue towards a contradiction.

Now use Theorem 2.1 to approximate (T, Γ) by a small simplicial action (T_0, Γ) in which γ is hyperbolic. Consider an edge e in the axis of γ in T_0. By Proposition 3.3, $\Lambda\Gamma(e)$ disconnects γ^+ from γ^-. Since $\{x, y\} = \{\gamma^+, \gamma^-\}$, Lemma 4.2 implies $x \sim y \sim \Lambda\Gamma(e)$ so that $\Lambda\Gamma(e) \subset \sigma$. Therefore, $\{x, y\}$ cannot be a jump of $\bar{\sigma}$ since $\Lambda\Gamma(e)$ disconnects x from y. □

4.2 Rigid groups fix a point in T

Lemma 4.5. *If v is a rigid type vertex, then $\Gamma(v)$ has a fix point in T, and it is unique.*

Proof. The maximality of the JSJ splitting claims that a rigid type group $\Gamma(v)$ doesn't have any non-trivial small splitting in which the stabilizers of the incident edges in S are elliptic (Theorem 1.2). If $\Gamma(v)$ had no global fixed point in T, then Rips Relative Splitting Theorem would give such a splitting and there would be a contradiction (see [BF2]).

This fix point is unique because $\Gamma(v)$ contains a non-abelian free group hence can't fix an arc in T since (T, Γ) is small. □

4.3 Action of surface type groups

Let v be a surface type vertex. We denote by T_v the minimal $\Gamma(v)$-invariant subtree of T.

Remark. T_v is well defined since if $\Gamma(v)$ has a global fixed point, it is unique because (T, Γ) is small.

Lemma 4.6. *Let $\Gamma_0 \subset \Gamma$ be a 2-ended subgroup, let v be a surface type vertex and σ the corresponding $\sim$-class. If Γ_0 fixes a non degenerate arc in T_v then $\Lambda\Gamma_0 \subset \sigma$, $\Gamma_0 \subset \Gamma(v)$ and $\Lambda\Gamma_0$ is not a jump in $\bar{\sigma}$.*

This corollary is immediate:

Corollary 4.7. *Every finite index subgroup of a peripheral group of* $\Gamma(v)$ *has a unique fixed point in* T_v.

Proof of the lemma. If Γ_0 fixes a non degenerate arc in T_v, there is a hyperbolic element $\gamma \in \Gamma(v)$ whose axis meets $\mathrm{Fix}\,\Gamma_0$ in a non-degenerate arc. Theorem 2.1 shows that (T, Γ) may be approximated by a small simplicial action (T_0, Γ) in which Γ_0 is elliptic. Now if T_0 is close enough to T, γ has to be hyperbolic in T_0 and it is an easy Gromov topology exercise to check that for T_0 close enough to T, Γ_0 has to fix a non degenerate arc contained in the axis of γ in T_0.

Since T_0 is simplicial, Lemma 3.3 applies and shows that $\Lambda\Gamma_0$ disconnects γ^+ from γ^-. Since $\gamma \in \Gamma(v)$, and since $\Lambda\Gamma(v) = \bar{\sigma}$, γ^+ and γ^- lie in $\bar{\sigma}$. And Theorem 3.7 implies $\gamma^+ \sim \gamma^-$ or $\gamma^+ \approx \gamma^-$ since $\bar{\sigma} \subset \partial\Gamma(2) \cup \partial\Gamma(3+)$. If we set $\{a, b\} = \Lambda\Gamma_0$, Lemma 4.2 implies that $a \sim b \sim \gamma^+ \sim \gamma^- \in \sigma$ since $\bar{\sigma} \setminus \sigma \subset \partial\Gamma(3+)$. Now since Γ_0 preserves $\{a, b\} \in \sigma$, it must fix the $\sim$-class of $\{a, b\}$ which is σ. Therefore $\Gamma_0 \subset \Gamma(v) = \mathrm{Stab}\,(\sigma)$.

Now, since the relation "$\{x, z\}$ disconnects y from t" is a cyclic order relation on $\bar{\sigma}$, $\{\gamma^+, \gamma^-\}$ disconnects the two points of $\Lambda\Gamma_0$ which prevents $\Lambda\Gamma_0$ to be a jump. □

Lemma 4.8. *If* $v \neq v'$ *are surface type vertices,* T_v *and* $T_{v'}$ *intersect in at most one point.*

Proof. If the lemma is false, there exists $\gamma \in \Gamma(v)$ and $\gamma' \in \Gamma(v')$ whose axes intersect in a non-degenerate arc. Now take (T_0, Γ) a small simplicial approximation of (T, Γ) (Theorem 2.1). If T_0 is close enough to T, γ and γ' are hyperbolic in T_0 and their axes still intersect in a non-degenerate arc. Let e be an edge in this non-degenerate arc, and denote $\{a, b\} = \Lambda\Gamma(e)$. Let σ, σ' be the $\sim$-classes corresponding to v and v' so that $\gamma^+, \gamma^- \in \bar{\sigma}$ and $\gamma'^+, \gamma'^- \in \bar{\sigma}'$. Proposition 3.3 shows that $\{a, b\}$ disconnects γ^+ from γ^- and Lemma 4.2 concludes that $a \sim b \sim \gamma^+ \sim \gamma^-$. The same argument applied to γ'^+, γ'^- shows that $\gamma^+ \sim \gamma^- \sim \gamma'^+ \sim \gamma'^-$ so that $\sigma = \sigma'$ and $v = v'$. □

4.4 Assigning subtrees of T to vertices and edges of S

In order to blow up S, we now want to assign subtrees of T to vertices and edges of S in an equivariant way. The subtrees assigned to edges should be some arcs. We already defined a subtree T_v for every vertex of surface type. If v if of rigid type, it fixes a unique point in T which we call T_v. Now if e is an (oriented) edge incident on a vertex v of surface type in S, its stabilizer is a peripheral subgroup of Γ_v and thus fixes exactly one point p_e in T_v

(Lemma 4.4, 4.7). If e is incident on a rigid type vertex v, we set p_e to be the only point in T_v. Now if e is an oriented edge incident on an elementary type vertex, we set $p_e = p_{\bar{e}}$ (where $\bar{e}$ denotes the edge in S with the opposite orientation). It is well defined because two vertices of the same type in S can't be adjacent. So, for every edge e of S, we can define T_e to be the arc $[p_e p_{\bar{e}}]$. In this way, for every edge incident to an elementary type vertex, T_e is reduced to a point. This allows to give to an edge e in S the length of the corresponding arc T_e. Finally, if v is an elementary type vertex, we define T_v to be the convex hull of the finite set $\{p_e | e \text{ incident on } v\}$.

Clearly, this assignment is equivariant. Therefore, it defines a blow up $(\tilde{S}, \Gamma)$ of (S, Γ). There is a natural equivariant map $f : \tilde{S} \to T$ which sends vertex tree T_v to itself via the identity, and sends an edge e (which may of length 0) isometrically on T_e. This map f is a morphism of $\mathbb{R}$-trees since every arc in $\tilde{S}$ is a finite union of arcs contained in vertex and edge trees on which f is isometric. Therefore, to show that f is an isometry, we just have to check that f cannot identify two non-degenerate arcs I, I' contained in some T_x, T'_x for x, x' vertices or edges of S. Therefore Lemma 4.9 ends the proof of Theorem 1.3.

Lemma 4.9. *If x, x' are vertices or edges of S such that $T_x \cap T_{x'}$ contains more than one point, then $x = x'$.*

Proof. If x or x' is a rigid type vertex or an edge incident on an elementary vertex, this is trivial since the corresponding tree is reduced to one point. If both x and x' are surface type vertices, Lemma 4.8 answers the question.

If $x = v$ is a surface type vertex and x' is either an elementary vertex or an edge not adjacent to an elementary vertex, then a finite index subgroup Γ_0 of $\Gamma(x')$ fixes $T_{x'}$ pointwise. Let σ denote the $\sim$-class corresponding to v. If $T_v \cap T_{x'}$ is not reduced to one point, Lemma 4.6 implies that $\Lambda\Gamma(x') = \Lambda\Gamma_0 \subset \sigma$ and $\Lambda\Gamma(x')$ is not a jump in $\bar{\sigma}$. This is impossible when x' is an elementary vertex because $\Lambda\Gamma(x')$ is either a $\approx$-pair or an $\sim$-class of cardinal 2. When x' is an edge, it must be adjacent to a surface type vertex v', $\Lambda\Gamma(x')$ being a jump of $\bar{\sigma}'$ (with obvious notations). Since $\Lambda\Gamma(x') \subset \sigma \subset \partial\Gamma(2)$ and $\bar{\sigma}' \backslash \sigma' \subset \partial\Gamma(3+)$, we have $\Lambda\Gamma(x') \subset \sigma'$ so $\sigma \cap \sigma' \neq \emptyset$, $\sigma = \sigma'$ and $v = v'$. It is a contradiction since $\Lambda\Gamma(x')$ is a jump in $\bar{\sigma}'$ and not in $\bar{\sigma}$.

There only remains to look at the case when x and x' are either elementary vertices or edges not adjacent to elementary vertices.

The following fact follows easily from Bowditch's work.

Fact ([Bo2]). *If e is an edge of S such that $\Gamma(e)$ is commensurable to $\Gamma(v)$*

for some (elementary) vertex $v \in S$, then e is incident on v. $\square$

Assume that x and x' are either elementary vertices or edges not adjacent to elementary vertices. Since the pointwise stabilizer of $T_x \cap T_{x'}$ must be small, the groups $\Gamma(x)$ and $\Gamma(x')$ must be commensurable so that $\Lambda\Gamma(x) = \Lambda\Gamma(x')$.

This implies $x = x'$ if x and x' are elementary vertices. If x is an elementary vertex and x' is an edge not adjacent to an elementary type vertex, $\Lambda\Gamma(x) = \Lambda\Gamma(x')$ implies that the edge x' is adjacent to the elementary vertex x which is absurd.

Now assume that x and x' are elementary edges adjacent to surface type vertices v, v'. If $v = v'$, since $\Lambda\Gamma(x) = \Lambda\Gamma(x')$ are the same jump in the necklace corresponding to $v = v'$, we deduce that $x = x'$. On the other hand, if $v \neq v'$, one gets with obvious notations $\bar{\sigma} \cap \bar{\sigma}' \supset \Lambda\Gamma(x) = \Lambda\Gamma(x') \neq \emptyset$ and $\bar{\sigma} \cap \bar{\sigma}' \subset \partial\Gamma(3+)$ (otherwise $\sigma = \sigma'$). Hence $\Lambda\Gamma(x) = \Lambda\Gamma(x')$ corresponds to an elementary vertex which has to be adjacent to both x and x', which is once again a contradiction. $\square$

5 Proof of the corollaries

5.1 Every small action of a one-ended hyperbolic group is geometric

Let's first give a precise definition of a geometric action.

Definition ([LP]). *Take Σ a finite 2-complex endowed with a measured foliation $\mathcal{F}$. Let $\rho : \pi_1(\Sigma) \to \Gamma$ be a morphism onto Γ. Consider $\bar{\Sigma}$ the covering of Σ defined by this morphism. $\mathcal{F}$ lifts to a Γ-invariant measured foliation $\bar{\mathcal{F}}$ on $\bar{\Sigma}$. This induces a pseudo-metric δ on $\bar{\Sigma}$ obtained by integration of the transverse measure. Let $T(\bar{\Sigma})$ be the metric space obtained by identifying 2 points at δ-distance 0. It is naturally endowed with an isometric action of Γ.*

We say that (T, Γ) is geometric *if it is equivariantly isometric to $T(\bar{\Sigma})$ for some $(\Sigma, \mathcal{F}, \rho)$ such that every edge of $\bar{\Sigma}$ transverse to the foliation is isometrically embedded into $T(\bar{\Sigma})$.*

Given a geometric action (T, Γ) of a finitely presented group, there is a standard way to represent it as above. Let $< S || \mathcal{R} >$ be a finite presentation of Γ. Let D be a big finite subtree of T (i. e. the convex hull of a finite number of points). For each generator $g \in S$, consider the partial isometry $\varphi_g = g_{|D \cap g^{-1}(D)}$ which is the maximal restriction of g going from D to D, and denote by $X = (D, \{\varphi_g\})$ the system of isometries with domain D and generators $\{\varphi_g\}$. We suppose that D is big enough so that each relation in $\mathcal{R}$ "expresses itself" in X, i. e. if r is the relation corresponding to the word $g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n}$, then $\mathrm{dom}\,(\varphi_{g_1}^{\varepsilon_1} \dots \varphi g_n{}^{\varepsilon_n}) \neq \emptyset$ ($\mathrm{dom}\, w$ denotes the domain of definition of the partial isometry w).

We next build a 2-complex Σ by gluing on D, for each generator φ of X, a band $(\mathrm{dom}\,\varphi) \times [0,1]$ where $(x, 0)$ and $(x, 1)$ are glued with x and $\varphi(x)$ respectively. Each band is foliated by $\{*\} \times [0,1]$, and we consider the transverse measure which gives to every arc of D a measure equal to its length.

Given any base point $*$ in D, the fundamental group of Σ is canonically identified with the free group with free basis S. So there is a natural morphism $\rho : \pi_1(\Sigma, *) \to \Gamma$. Let $\bar{\Sigma}$ be the covering space of Σ corresponding to ρ endowed with the lift of the measured foliation on Σ. Let $T_{\bar{\Sigma}}$ be the space obtained by making Hausdorff the pseudo-metric on $\bar{\Sigma}$ obtained by integration of the transverse measure (it is a quotient of the space of leaves of $\bar{\Sigma}$). The edges of $\bar{\Sigma}$ transverse to the foliation embed into $T_{\bar{\Sigma}}$. The action of Γ on $\bar{\Sigma}$ gives a natural action of Γ on $T_{\bar{\Sigma}}$. Then it follows from [LP] that if D is big enough, there is a natural equivariant isometry between $T_{\bar{\Sigma}}$ and T. We say that (Σ, ρ) is a standard form for (T, Γ). If we are given finitely many finitely generated subgroups $\Gamma_i \subset \Gamma$ fixing some points p_i in T, we may choose D big enough so that the domains of the words corresponding to a finite generating set of Γ_i contain p_i. Thus, every Γ_i fixes a leaf of $\bar{\Sigma}$ which projects to p_i into T.

Now, here is a statement of the Theorem of Skora.

Theorem 5.1 ([Sko]). *Let Γ be a non-elementary convex cocompact fuchsian group. Let (T, Γ) a minimal small action such that the peripheral groups fix a point in T.*

Then (T, Γ) is geometric and can be obtained as the space of leaves of a Γ-invariant measured foliation on the convex core of Γ such that the boundary components are contained in leaves of the foliation.

This theorem was proved by Skora in the setting of actions of hyperbolic surfaces with boundary. The more general statement above follows easily. Indeed, the kernel N of the action of Γ on $\mathbb{H}^2$ being finite and normal, it must fix every point in T. Thus we are reduced to the case where Γ is the fundamental group of a hyperbolic 2-orbifold. But Selberg Theorem says that this orbifold has a finite cover which is a surface S (see for instance [Zie]). This cover may chosen to be Galois so that

$\pi_1(S) \lhd \Gamma$ and $\Gamma/\pi_1(S)$ acts on S. The Theorem of Skora applied to $(T, \pi_1(S))$ provides a measured lamination on S which must be invariant under the action of $\Gamma/\pi_1(S)$ since $(T, \pi_1(S))$ is also invariant. This lamination thus lifts to the desired Γ-invariant lamination on the universal cover.

In view of the Reading Theorem 1.3, proving Corollary 1.5 reduces to the following proposition.

Proposition 5.2. *Let Γ be a finitely presented group, and (S, Γ) a simplicial action with finitely generated edge stabilizers. If (T, Γ) is a blow up of (S, Γ) such that every vertex action is geometric, then (T, Γ) is geometric.*

This result may be compared to the Theorem of Levitt-Paulin saying that a simplicial action of a finitely generated group is geometric if and only if edge stabilizers are finitely generated ([LP]).

Remark. In [GL], the authors construct a non-geometric action as a free product of geometric actions of F_n by using attaching points which are not in the (minimal) tree but in its metric completion. This case doesn't fit the hypotheses of the proposition because the completion of a geometric action needn't be geometric.

Proof. First for each vertex $\bar{v}$ of S/Γ take a finite foliated 2-complex $\Sigma_{\bar{v}}$ with a morphism $\rho_{\bar{v}} : \pi_1(\Sigma_{\bar{v}}) \to \Gamma$ in standard form representing $(T_{\bar{v}}, \Gamma_{\bar{v}})$. Let $\bar{\Sigma}_{\bar{v}}$ denote the corresponding cover of $\Sigma_{\bar{v}}$. We also assume that edges of $\bar{\Sigma}_{\bar{v}}$ transverse to the foliation isometrically embed into $T_{\bar{v}}$. Since edge stabilizers are finitely generated, we may choose those $\bar{\Sigma}_{\bar{v}}$ so that if $\bar{e}$ is an oriented edge incident on $\bar{v}$, $i_{\bar{e}}(\Gamma(\bar{e}))$ fixes a leaf in $\bar{\Sigma}_{\bar{v}}$ which maps to $p_{\bar{e}} \in T_{\bar{v}}$. Now for each vertex $v \in S$, take a copy $\bar{\Sigma}_v$ of $\bar{\Sigma}_{\bar{v}}$ (where $\bar{v}$ is the image of v in S/Γ) so that Γ acts on the disjoint union of the $\bar{\Sigma}_v$.

Now for every non-oriented edge $\bar{e}$ in S/Γ, take a finite graph $G_{\bar{e}}$ with an onto morphism $\rho_{\bar{e}} : \pi_1(G_{\bar{e}}) \to \Gamma_{\bar{e}}$, and let $\bar{G}_{\bar{e}}$ be the corresponding cover. Take $\bar{\Sigma}_{\bar{e}}$ to be the product $[0, l(\bar{e})] \times \bar{G}_{\bar{e}}$ foliated by $\{*\} \times \bar{G}_{\bar{e}}$. For each edge e of S, take a copy $\bar{\Sigma}_e$ of $\bar{\Sigma}_{\bar{e}}$ so that Γ acts on the disjoint union of the $\bar{\Sigma}_e$. If an oriented edge e is incident on v, let L_e be a leaf in $\bar{\Sigma}_v$ fixed by $\Gamma(e)$ and projecting to p_e in T. Choose any $\Gamma(e)$-equivariant attaching map $f_e : 0 \times \bar{G}_e \subset \bar{\Sigma}_e \to L_e$, and glue $\bar{\Sigma}_e$ on $\bar{\Sigma}_v$ along f_e. This can of course be done Γ-equivariantly. Denote by $\bar{\Sigma}$ the 2-complex thus obtained.

Now the map $T(\bar{\Sigma}_v) \to T(\bar{\Sigma})$ induced by the inclusion $\bar{\Sigma}_v \subset \bar{\Sigma}$ is an isometry. Therefore, transverse edges of $\bar{\Sigma}$ must embed into $T(\bar{\Sigma})$. Since (T, Γ) is naturally isometric to $T(\bar{\Sigma})$, we get that (T, Γ) is geometric. □

5.2 Uniqueness of the JSJ splitting.

Proof of Corollary 1.6. We have to prove that every minimal small simplicial action (S', Γ) in which every small action of Γ can be read is isomorphic to the JSJ-action (S, Γ) up to subdivision. The *moreover* part of the corollary follows immediately since its hypotheses show where valence 2 vertices should be.

We first prove that for every surface type vertex v of S, $\Gamma(v)$ fixes a point in S'. Assume that this is not true and consider S'_v the minimal $\Gamma(v)$-invariant subtree of S'. Denote by Σ_v the orbifold with boundary corresponding to v and $\tilde{\Sigma}_v$ its universal cover with the action of $\Gamma(v)$. Since S' can be read from S, the peripheral subgroups of $\Gamma(v)$ fix a point in S'. Hence, thanks to the Theorem of Skora, $(S'_v, \Gamma(v))$ corresponds to a measured geodesic lamination $\mathcal{L}$ on Σ_v. Since S' is simplicial, $\mathcal{L}$ consists of finitely many disjoint simple closed curves (or 1-orbifolds without boundary) not boundary-parallel, and the transverse measure is just a positive integer weight $w(l)$ on each leaf l. We denote by $\tilde{\mathcal{L}}$ the lift of $\mathcal{L}$ to $\tilde{\Sigma}_v$. Connected components of $\tilde{\Sigma}_v \setminus \mathcal{L}$ correspond to vertices of S'_v with valence at least 3. Every leaf $\tilde{l}$ of $\tilde{\mathcal{L}}$ corresponds to a maximal open segment of $w(l)$ edges in S'_v with $w(l) - 1$ valence-2 vertices.

Take $\tilde{l}$ any leaf in $\tilde{\mathcal{L}}$ and consider an infinite order element $\gamma \in \Gamma(v)$ preserving $\tilde{l}$. Since γ fixes an edge in S', and because any small action can be read from S', γ has to be elliptic in every small action of Γ. We will prove in Lemma 5.3 that there exists a simple closed curve (or 1-orbifold without boundary) l' which is not boundary parallel and which intersects non trivially the projection l of $\tilde{l}$ in Σ_v. Thus in the action of $\Gamma(v)$ on the simplicial tree induced by the one-leaf lamination $\{l'\}$, γ is hyperbolic. This contradiction proves that $\Gamma(v)$ fixes a point in S'.

Now, using the Reading Property of S, consider a blow up $\tilde{S}$ of S such that S' is obtained from $\tilde{S}$ by collapsing its edges of length 0. Denote by $f : \tilde{S} \to S'$ and $p : \tilde{S} \to S$ the corresponding maps. For every non-elementary type vertex $v \in S$, since $\Gamma(v)$ fixes a point in S', v is not blown up in $\tilde{S}$. Therefore, $\tilde{S}$ is simplicial, and the natural map $p : \tilde{S} \to S$ induces a bijection between vertices $v \in \tilde{S}$ such that $\mathrm{Stab}\,(v)$ is not 2-ended, and the non-elementary type vertices in S. Therefore, it will be convenient to call *non-elementary* the vertices of $\tilde{S}$ whose stabilizer is not 2-ended. We denote by $NE(S), NE(\tilde{S})$ the set of non-elementary vertices of $S, \tilde{S}$.

We now argue that for $u, v, w \in NE(\tilde{S})$, u, v, w are aligned in this order if and only if $p(u), p(v), p(w)$ are. Since p preserves alignment, the *only if* part is clear. To prove the *if* part, assume that u, v, w are not aligned and consider the center c of the triod they form. Since p does not collapse edges incident to u, v, w (they come from S), $p(u)$, $p(v)$, $p(w)$ and $p(c)$ are all distinct and $p(u)$, $p(v)$, $p(w)$ can't be aligned.

Therefore, we have 3 equivariant maps p, p^{-1}, f defined on non-elementary vertices which preserve alignment. Symmetrically, we can read S from S' and get a simplicial blow up $\tilde{S}'$, and a map $p' : \tilde{S}' \to S$ which induces a bijection between $NE(\tilde{S}')$ and $NE(S')$ and $f' : \tilde{S}' \to S$ so that p', p'^{-1}, f' preserve alignment in restriction to the sets of non-elementary vertices of the corresponding tree.

Take $v \in NE(\tilde{S})$. Since all the maps considered are equivariant, we have $\operatorname{Stab} v \subset \operatorname{Stab} f(v) = \operatorname{Stab} p'^{-1} \circ f(v) \subset \operatorname{Stab} f' \circ p'^{-1} \circ f(v) = \operatorname{Stab} p^{-1} \circ f' \circ p'^{-1} \circ f(v)$. Therefore, since $\operatorname{Stab} v$ can fix at most one point in $\tilde{S}$ because $\tilde{S}$ is a small action, we get that $p^{-1} \circ f' \circ p'^{-1} \circ f$ is the identity on $NE(\tilde{S})$. This implies the following fact which will be useful in the sequel:

Fact. *3 vertices u, v, w are aligned in this order in S if and only if their images in S' under $f \circ p^{-1}$ are aligned in this order.*

Call *branch point* in a simplicial tree any vertex with valence ≥ 3. For instance, every non-elementary vertex is a branch point because it has infinite valence (edge stabilizers have infinite index in non-elementary vertex stabilizers). Call *big edge* a segment whose endpoints are branch points and containing only vertices with valence 2 in its interior. We want to extend $f \circ p^{-1}$ to a continuous map $g : S \to S'$, sending linearly big edges to big edges.

We need to recall how an elementary vertex $v \in S$ is blown up. If $\{u_i\} \subset S$ denotes the (finite) set of neighbours of v, $u_i \in NE(S)$, so $\Gamma(u_i)$ fixes a point $u'_i \in S'$ (and $u'_i = f \circ p^{-1}(u_i)$). Then the tree T_v used to blow up v is the convex hull $\operatorname{Conv}(\{u'_i\})$.

The convex hull of $\{u_i\}$ doesn't contain any non-elementary vertex different from the points u_i themselves, and u_i is a terminal vertex in this convex hull. Since the property of being in a convex hull can be expressed only in terms of alignment,[1] the convex hull of $\{u'_i\}$ doesn't contain any non-elementary vertex not in $\{u'_i\}$. For the same reason, u'_i is a terminal vertex in $\operatorname{Conv}(\{u'_i\})$. Therefore every non-terminal vertex of $\operatorname{Conv}(\{u'_i\})$ is elementary. Since no 2 elementary vertices are adjacent in S', either $\operatorname{Conv}(\{u'_i\})$ contains no elementary vertex and $\operatorname{Conv}(\{u'_i\})$ is an edge, or $\operatorname{Conv}(\{u'_i\})$ contains exactly one elementary vertex v', and every u'_i is a neighbour of v'.

[1] $x \in \operatorname{Conv}(\{u_i\})$ iff $\exists i, j$ s.t. $x \in [u_i, u_j]$

If v has valence ≥ 3, we can naturally extend $f \circ p'^{-1}$ simplicially on $\mathrm{Conv}(\{u_i\})$ by letting g send u_i to u'_i and v to v'. Where defined, g sends branch points to branch points and big edges to big edges.

If v has valence 2, then $f \circ p^{-1}$ naturally extends to the continuous map g sending the big edge $[u_1, u_2]$ linearly to $[u'_1, u'_2]$. We have that $[u'_1, u'_2]$ is a big edge since the property of being a big edge between non-elementary vertices can be expressed only in terms of alignment of non-elementary vertices: $[u'_1, u'_2]$ is a big edge if and only if for every $w' \in NE(S')$, u'_1, u'_2, w' or u'_2, u'_1, w' are aligned in this order (this uses the fact that $\mathrm{Conv}(NE(S')) = S'$ which is implied by minimality).

Now $g : S \to S'$ is defined everywhere on S except on edges joining two non-elementary vertices. If $e = [u_1, u_2]$ is such an edge, we make g send e linearly to $[f \circ p^{-1}(u_1), f \circ p^{-1}(u_2)]$. The same argument as above shows that $[f \circ p^{-1}(u_1), f \circ p^{-1}(u_2)]$ is a big edge. Therefore, $g : S \to S'$ sends linearly big edges to big edges.

Symmetrically, $f' \circ p'^{-1}$ extends to an equivariant continuous map $g' : S' \to S$ sending branch points to branch points. Since an elementary vertex with valence ≥ 3 is characterized by the set of its neighbours, we have that $g' \circ g$ restricts to the identity on the set of all branch points.

In restriction to a big edge e, $g' \circ g$ is linear and fixes the endpoints of e. Hence, $g' \circ g$ is the identity on big edges. Therefore, $g' \circ g = \mathrm{Id}_S$ and by symmetry $g \circ g' = \mathrm{Id}_{S'}$. □

Lemma 5.3. *Let Σ be a hyperbolic 2-orbifold with geodesic boundary having a simple closed geodesic $\bar{c}$ which is not boundary parallel. Then there exists a simple closed geodesic which is not boundary parallel and intersects $\bar{c}$ non trivially.*

Proof. We denote by G the fundamental group of Σ and by c a lift of $\bar{c}$ in the universal cover $\tilde{\Sigma}$ of Σ. The fact that $\bar{c}$ is not boundary parallel means that the action on the tree dual to $G.c$ is minimal. Therefore, there is an element $g \in G$ having infinite order and whose axis λ in $\tilde{\Sigma}$ intersects c transversally. Consider such a g with smallest translation length for the hyperbolic metric on $\tilde{\Sigma}$. Note that the projection $\bar{\lambda}$ of λ in Σ is a closed curve (1-orbifold without boundary) which intersects $\bar{c}$ non trivially. We want to prove that $\bar{\lambda}$ does not intersect itself (meaning that if $h \in G$ is such that $h.\lambda \cap \lambda \neq \emptyset$ then $h.\lambda = \lambda$). Note that λ cannot be boundary parallel because it intersects c non trivially.

We give a proof in $\tilde{\Sigma}$ to avoid cases with orientability and singularities. But its meaning if Σ is a surface is the following: there are 2 ways to change locally $\bar{\lambda}$ into a shorter curve which has fewer self intersection points. One of them gives a connected curve and the other one gives a curve with 2 connected components. We prove that one of the obtained connected curves still intersects $\bar{c}$ non-trivially.

So assume that there exists $h \in G$ such that $h.\lambda$ intersects λ transversally. Denote by O the point in $\lambda \cap h.\lambda$ so that $h.O \in h.\lambda$ and $h^{-1}.O \in \lambda$. Up to composing h by some power of g, we can assume that $O \in [h^{-1}.O, g.h^{-1}O]$ (remember that λ is the axis of g). For simplicity of notations, we set $A = h.O$, $B = h^{-1}.O$, $C = gh^{-1}.O$ (see figure 1 and 2).

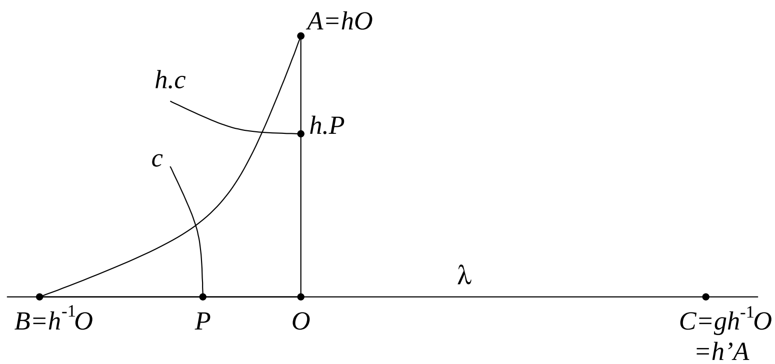

Figure 1: Case 1: when $c \cap [OA] = \emptyset$

We are going to consider the elements h and $h' = gh^{-2}$ of G. Note that $h.B = O$, $h.O = A$ and $h'.A = C$. Denote by $l(h) = \min_{x \in \tilde{\Sigma}} d(x, h.x)$ the the translation length of h. Then $l(h) < l(g)$ since $h.B = O$ and $d(B, O) < d(B, C) = l(g)$. Similarly, $l(h') < l(g)$ since $d(A, C) < d(A, O) + d(O, C) = d(B, O) + d(O, C) = l(g)$. Therefore, we just have to prove that either h or h' is loxodromic and that its axis intersects $G.c$.

We are going to use the following criterion:

Criterion. *Assume that we have $x \in \tilde{\Sigma}$, a geodesic c and an element $\gamma \in G$ such that*

- $[x, \gamma.x] \cap c \neq \emptyset$ *and* $[x, \gamma.x] \cap \gamma.c = \emptyset$
- $c \cap \gamma.c = \emptyset$ *and* $\gamma.c$ *lies in the component of* $\tilde{\Sigma} \setminus c$ *containing* $\gamma.x$
- *The projection* $\bar{c}$ *of* c *in* Σ *is compact.*

Then γ *is loxodromic and its axis intersects* c.

Proof of the criterion. The third hypothesis implies that c and $\gamma.c$ cannot meet at infinity. Consider the component C of $\tilde{\Sigma} \setminus c$ not containing x. The first hypothesis implies that c disconnects x from $\gamma.x$ and that $\gamma.c$ doesn't. Therefore, using the second hypothesis, $\gamma.c \subset C$ and $\gamma.C \subsetneqq C$. Similarly, if we denote by C' the component of $\tilde{\Sigma} \setminus c$ containing x, we have that $h^{-1}.C' \subsetneqq C'$. Therefore, γ has infinite order and γ must have a fixed point at infinity in the closure of $\gamma.C$ and a fixed point in the closure $h^{-1}.C'$. Those fixed points have to be distinct, hence γ is loxodromic and the endpoints at infinity of its axis are in the closure at infinity of $h.C$ and $h^{-1}.C'$ respectively. Thus, the axis of γ must intersect c. □

We can assume that $G.c$ intersects $[BO]$ since otherwise, it would intersect $[OC]$ and we could do a symmetric argument. We change c to another geodesic in its orbit so that c intersects $[BO]$ at a point P as close to O as possible so that $\gamma.c$ doesn't intersect the open segment (PO) for any $\gamma \in G$.

We first assume that $P \neq O$ and we consider two cases.

First case: if c doesn't intersect $[OA]$. Then $h.c$ cannot intersect $[BO]$ (see figure 1) because it cannot intersect $[PO]$ by choice of c and $h.c$ can't intersect c so it has to exit the triangle AOB via $[AB]$ (and $h.c$ can't intersect $[AB]$ twice). Therefore, the hypotheses of the criterion apply to $x = B$ and $\gamma = h$ to prove that h is loxodromic and its axis intersects c.

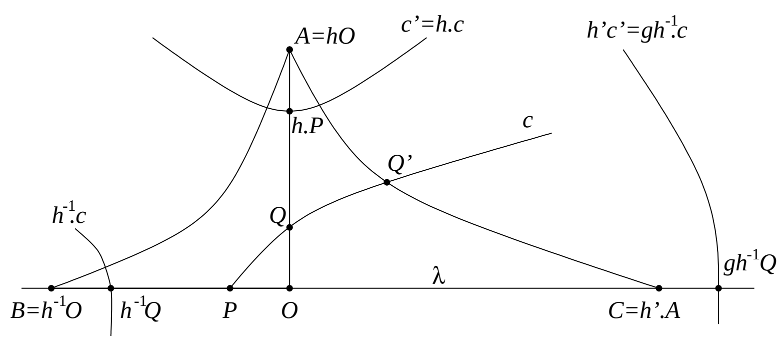

Figure 2: Case 2: when $c \cap [OA] \neq \emptyset$

Second case: if c intersects $[OA]$ at some point Q (see figure 2). The geodesic c enters the triangle AOC in Q, but c cannot intersect $[OC]$ since it already intersects λ at P. Therefore c must exit AOC through $[AC]$

and denote by Q' the point in $[AC] \cap c$. Then consider $c' = h.c$: we are going to apply the criterion to $x = A$, $\gamma = h'$ for the geodesic c'. First c' intersects $[AC]$: it enters the triangle AOC at $h.P$ and cannot intersect c. Now we show that $h'.c' = gh^{-1}.c$ doesn't intersect $[AC]$. So assume that $h'.c'$ intersects $[AC]$ at some point R. Then R cannot lie in $[Q'C]$ because $h'.c'$ already intersects λ at $gh^{-1}.Q \notin [BC]$, and $h'.c'$ would have to exit $PQ'C$ through $[PC] \subset \lambda$ which is impossible. But R cannot lie in $Q'A$ otherwise c would enter the triangle $RCgh^{-1}Q$ and could only exit through $[Cgh^{-1}Q] \subset \lambda$ thus intersecting twice λ which is impossible. We conclude that $h'.c'$ does not intersect $[AC]$. Thus we can apply the criterion to $x = A$, $\gamma = h'$ and the geodesic c' which concludes the second case.

When P=O. The treatment of this case is similar to the first case if c intersects $[AB]$: note that $h.c$ intersects AOB only at A and apply the criterion for $x \in (BO)$. If c doesn't intersect $[AB]$, the treatment is similar to the second case: $c' = h.c$ (resp. $h'c'$) intersects AOC only at A (resp. C). Moreover, h' sends the component C of $\tilde{\Sigma} \setminus c'$ containing O to the component of $\tilde{\Sigma} \setminus h'.c'$ not containing O. Therefore we can apply the criterion with x in C close to A. □

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