# A GENERAL CONSTRUCTION OF JSJ DECOMPOSITIONS

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This is an extended and updated version of a talk given by Gilbert Levitt in Barcelona on July 2, 2005 about joint work with Vincent Guirardel, currently being written. It is based on notes TEX'd by Peter Kropholler. I thank Peter for making his notes available to me. Of course, he shouldn't be held responsible for the content. Thanks also to the organizers of the Barcelona Conference on Geometric Group Theory, and to the referee for helpful suggestions.

## 1. Introduction

JSJ theory has its roots in the work of Jaco-Shalen and Johannson on 3-manifolds. In the context of group theory, JSJ decompositions were constructed by Kropholler for duality groups, and in general by Rips-Sela [RS], Dunwoody-Sageev [DS], Fujiwara-Papasoglu [FP], Scott-Swarup [SS]. Roughly speaking, the main purpose of a JSJ decomposition is to describe all splittings of a given group $G$ over a certain class $\mathcal{A}$ of subgroups.

Here are a few ideas that will be developed in this talk:

• Rather than viewing the JSJ decomposition of $G$ over $\mathcal{A}$ as a $G$-tree (a splitting) satisfying certain properties (which in general don't define it uniquely), one should *view the JSJ decomposition as a well-defined deformation space* $\mathcal{D}_{JSJ}$ satisfying a universal property.

A deformation space [Fo 1] is a collection of $G$-trees (and $\mathcal{D}_{JSJ}$ contains the trees constructed in [RS], [DS], [FP]). It is a contractible complex [GL]. A basic example: when $G = F_n$, the JSJ deformation space (over any $\mathcal{A}$) is Culler-Vogtmann's Outer space, consisting of all free $F_n$-trees (see [Vo]); it has no preferred element.

• *A deformation space* $\mathcal{D}$ *carries a lot of information.* Trees in a given deformation space have a lot in common, so many invariants may be extracted from $\mathcal{D}$ (in particular, the set of vertex stabilizers not in $\mathcal{A}$). One may also say that two trees in the same deformation space are always related by certain types of moves (expansions and collapses, sometimes slides), so the JSJ splitting of $G$ over $\mathcal{A}$ is well-defined up to these moves.

• *If* $G$ *is finitely presented, the JSJ deformation space* $\mathcal{D}_{JSJ}$ *always exists.* No assumption on $\mathcal{A}$ is necessary. Vertex stabilizers of trees in $\mathcal{D}_{JSJ}$ yield a canonical finite set of conjugacy classes of subgroups of $G$.

• The JSJ deformation space $\mathcal{D}_{JSJ}$ usually contains no preferred splitting. In order to get a canonical tree, we construct (for $G$ finitely presented) a second deformation space, the compatibility JSJ deformation space of $G$ over $\mathcal{A}$. Unlike $\mathcal{D}_{JSJ}$, it always has a preferred element, yielding an Aut$(G)$-*invariant splitting which we call the compatibility JSJ tree.* For instance, the Bass-Serre tree of the HNN extension defining $BS(m,n) = \langle a,t \mid ta^m t^{-1} = a^n \rangle$ is the cyclic compatibility tree of $BS(m,n)$ when none of $m,n$ divides the other.

## 2. Assumptions and notations

We fix a finitely generated group $G$. Let $\mathcal{A}$ be a family of subgroups which is subgroup closed and closed under conjugation ($\mathcal{A}$ is usually defined by restricting the isomorphism type: trivial, finite, cyclic, abelian, slender, small...).

We always consider simplicial $G$-trees $T$ whose edge stabilizers belong to $\mathcal{A}$. We make the standard assumptions on $T$: there are no inversions, and $T$ is minimal (it contains no proper $G$-invariant subtree). The trivial tree ($T =$ point) is permitted (the JSJ decomposition is sometimes trivial...).

An element of $G$, or a subgroup, is elliptic in $T$ if it fixes a point.

We denote by $\Gamma$ the quotient graph of groups $T/G$; this is a marked graph of groups.

All maps between trees will be $G$-equivariant, but otherwise arbitrary. Of course, any map may be redefined on edges so as to be linear or constant on every edge.

Typical examples of maps $f$ are collapses and folds. *Collapses* will be important for us. In a collapse, an edge $e$ of $\Gamma$ is collapsed to a single vertex (Figure 1). In $T$, each edge in a given $G$-orbit is collapsed to a point.

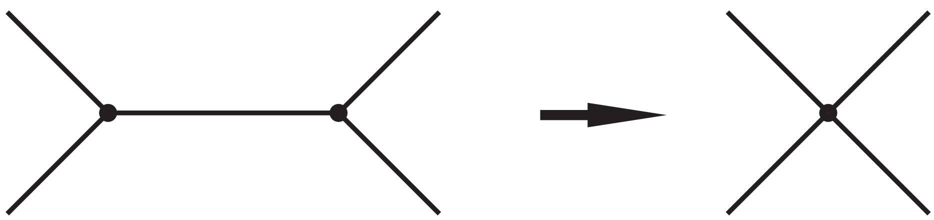

Figure 1. Collapsing.

## 3. Deformation spaces

Deformation spaces were introduced by Forester [Fo 1]. See [GL] for a detailed study.

**Definition 3.1 (domination).** $T$ *dominates* $T'$ if there is a map $f : T \to T'$. By abstract nonsense, this is equivalent to saying that, if a subgroup $H$ is elliptic in $T$, then it is elliptic in $T'$ (recall that all maps are equivariant).

**Definition 3.2 (deformation space).** $T$ and $T'$ are in the same *deformation space* if they dominate each other. Equivalently, they have the same elliptic subgroups. For experts: this is slightly stronger than saying that they have the same elliptic elements.

In this talk we will think of a deformation space as a set of (non-metric) simplicial trees. It is actually a contractible complex.

**Definition 3.3 (expansion, admissible).** An *expansion* is the opposite of a collapse. A collapse or an expansion is *admissible* if it does not change the deformation space. More on admissibility later (Remark 3.5).

**Theorem 3.4.** *Given $T, T'$, the following are equivalent:*

(1) *$T, T'$ are in the same deformation space $\mathcal{D}$.*
(2) *They are equivariantly quasi-isometric.*
(3) *The length functions are bi-Lipschitz equivalent.*
(4) *$T, T'$ are related by a finite sequence of admissible collapses and expansions.*

(4) means that $T$ may be "deformed" to $T'$ *within* $\mathcal{D}$. Most of this theorem, especially the hard part (1) $\Rightarrow$ (4), is due to Forester [Fo 1]. For experts: condition (3) is equivalent to the others only when the trees are irreducible.

Domination induces a *partial ordering* on the set of deformation spaces of $G$. The trivial tree is the smallest element. In general, there is no highest element.

**Example 1 (free group).** $G$ is the free group $F_n$, and $\mathcal{A}$ consists of the trivial group. There is a highest $\mathcal{D}$, the set of free $G$-trees. As a complex this is Culler–Vogtmann's Outer space. In this case, the minimality assumption says that the quotient graph has no valence one vertices.

Let $F$ be a finite subgroup of $\mathrm{Aut}(F_n)$, and $G_F = F_n \rtimes F$. The set of free $F_n$-trees which are $F$-invariant may be identified with a deformation space of $G_F$-trees.

**Example 2 (surface group).** $G$ is the fundamental group of a closed orientable surface, $\mathcal{A}$ is the class of cyclic subgroups. Any tree is dual to a family $\mathcal{C}$ of disjoint, non-parallel, essential simple closed curves. Deformation spaces (viewed as sets) are singletons, one for each $\mathcal{C}$. A deformation space is maximal for domination if and only if the corresponding $\mathcal{C}$ is a pair of pants decomposition.

**Example 3 (GBS groups).** Suppose $G$ acts on $T$ with all edge and vertex stabilizers infinite cyclic (we say that $G$ is a generalized Baumslag-Solitar group, or GBS group). If $G$ is not $\mathbf{Z}^2$ or a Klein bottle group, the set of all such trees is a deformation space [Fo 1] (exercise: there are infinitely many spaces if $G = \mathbf{Z}^2$, two if $G$ is a Klein bottle group).

**Remark 3.5: More on admissibility.** What does it mean to be admissible in the context of Outer space? It means that you can collapse a forest in the quotient graph but not a loop, since collapsing a loop creates a non-free tree. Theorem 3.4 is basically the statement that two marked graphs in Outer space are related by forest collapses and expansions.

In a general deformation space, collapsing an edge of $\Gamma$ is admissible if and only if the edge has distinct end points and the edge group maps onto at least one of the vertex groups: admissible collapses are just consequences of the isomorphism

$A *_B B \simeq A$. On the other hand, collapsing an edge carrying a non-trivial amalgam $A *_C B$ creates new elliptic elements (e.g. $ab$ with $a \in A \setminus C$ and $b \in B \setminus C$).

**Definition 3.6 (reduced).** $T$ is *reduced* if no collapse is admissible. For experts: This terminology comes from [Fo 1]. Scott-Wall use the name "minimal". Reduced in this sense is stronger than reduced in the sense of Bestvina-Feighn's accessibility paper.

As reduced trees are the simplest elements in a deformation space, one may try to connect them in a more direct way than by expansions and collapses.

In Outer space, reduced trees are just roses (the quotient graph has only one vertex). Two roses differ by an automorphism of $F_n$, and any generating set of $\mathrm{Aut}(F_n)$ gives a way to connect roses.

In particular, Nielsen automorphisms correspond to slides. For instance, the automorphism $a \mapsto a,\ b \mapsto ba$ in the free group on $a, b$ corresponds to sliding the edge of $\Gamma$ labelled $b$ over that labelled $a$.

**Definition 3.7 (slide).** In general, slides are based on the isomorphism $A *_E B *_F C \simeq A *_E C *_F B$, valid if $E \subset F$. A slide looks in $\Gamma$ like one end of an edge $e$ is sliding along a different edge $f$ satisfying $G_e \subset G_f$ (one or both edges may be loops), see Figure 2. Note that slide moves do not change edge and vertex groups (they are always admissible).

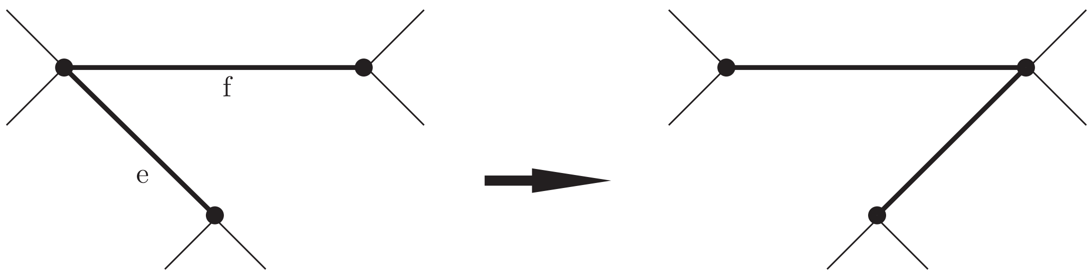


FIGURE 2. Sliding.

In Outer space, two reduced trees are connected by slides (because Nielsen automorphisms (almost) generate $\mathrm{Aut}(F_n)$). But this is not always true.

Here is an example where slide moves are not sufficient to achieve everything.

**Example 4.** Start with the standard HNN presentation $\langle a, t \mid ta^2t^{-1} = a^4 \rangle$ of the Baumslag–Solitar group $BS(2, 4)$. After an admissible expansion (Figure 3), one gets a non-reduced graph of groups with two edges, corresponding to the presentation $\langle a, t, b \mid tbt^{-1} = b^2, b = a^2 \rangle$. Sliding around the loop gives a reduced graph of groups with two edges, corresponding to $\langle a, t, \tilde{b} \mid t\tilde{b}t^{-1} = \tilde{b}^2, \tilde{b}^2 = a^2 \rangle$ (with $\tilde{b} = t^{-1}bt$). The associated Bass-Serre tree cannot be connected to the original one by slides, as the quotient graphs do not have the same number of edges (other examples in [Fo 2]).

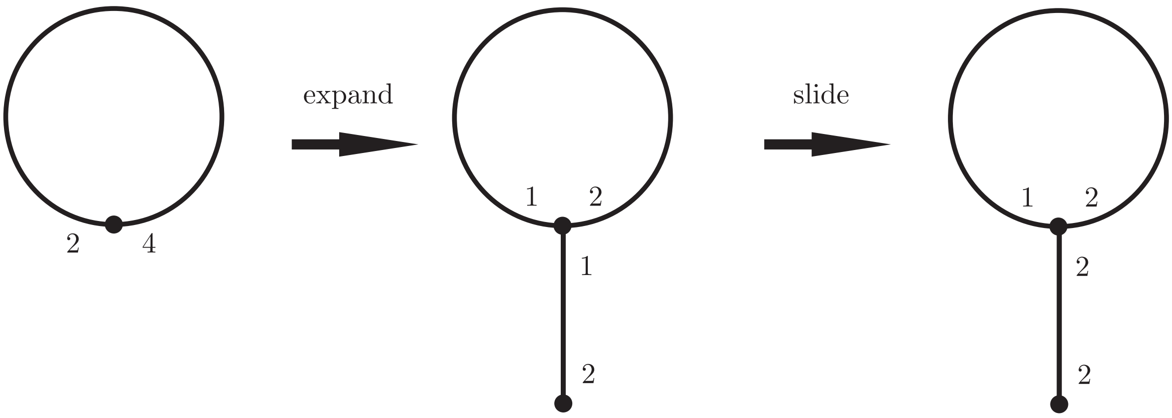


FIGURE 3

**Theorem 3.8.** *Let $T, T'$ be reduced trees in the same deformation space $\mathcal{D}$. If no group in $\mathcal{A}$ properly contains a conjugate of itself, then $T, T'$ are connected by slides.*

This was proved by Forester [Fo 3] for the deformation spaces of Example 3 (GBS groups). His proof works in general.

**Corollary 3.9.** *If $T, T', \mathcal{A}$ are as above, then $T, T'$ have the same edge stabilizers and vertex stabilizers.*

In fact the corollary remains true without assumptions on $\mathcal{A}$, but one has to consider generalized edge stabilizers (groups $H$ such that $G_e \subset H \subset G_{e'}$ for edges $e, e'$) and restrict to "big" vertex stabilizers (those not in $\mathcal{A}$).

**Theorem 3.10.** *Two reduced trees belonging to the same deformation space $\mathcal{D}$ have the same generalized edge stabilizers and the same "big" vertex stabilizers.*

## 4. The JSJ deformation space

The JSJ decomposition of $G$ over $\mathcal{A}$ is supposed to allow a description of all splittings of $G$ over groups in $\mathcal{A}$. In what sense?

$$\begin{array}{ccc} & \widehat{S} & \\ {}^{\text{collapse}}\swarrow & & \searrow^{\text{collapse}} \\ S & & T \end{array}$$

FIGURE 4

First consider an optimal situation. Suppose $G$ is a one-ended hyperbolic group and $\mathcal{A}$ is the set of virtually cyclic subgroups. Then there is a canonical JSJ tree $S$, constructed purely in terms of the topology of $\partial G$ (Bowditch [Bo]). In this case, the answer to the question asked above is the following [Gu] (Figure 4): every $T$

(with virtually cyclic edge groups) may be obtained from $S$ by first expansions and secondly collapses, but no zig-zag (later on, we will say that $S$ and $T$ are compatible). Note that we are now using non-admissible moves: we want to obtain all trees, in all deformation spaces.

In general, requiring that any tree may be obtained from $S$ as on Figure 4 may force $S$ to be the trivial tree (we will come back to this when we discuss compatibility). To obtain something more interesting, one settles for a bit less (Figure 5): *every $T$ (with edge groups in $\mathcal{A}$) should be obtainable from $S$ by expanding to some $\widehat{S}$* (which depends on $T$) *and mapping $\widehat{S}$ to $T$* (but the map $\widehat{S} \to T$ may fold, rather than just collapse).

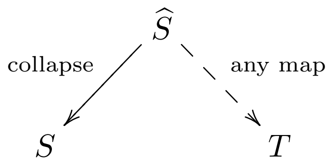


FIGURE 5

Note that, if $S$ has this property, its edge stabilizers are elliptic in every $T$ (because, unlike vertex stabilizers, edge stabilizers of $S$ are always elliptic in $\widehat{S}$). The converse is also true: if edge stabilizers of $S$ are elliptic in $T$, there is an $\widehat{S}$ as on Figure 5.

**Definition 4.1 (universally elliptic).** We say that $S$ is *universally elliptic* if its edge stabilizers are elliptic in every $T$. We say that $\mathcal{D}$ is *universally elliptic* if some (hence every) reduced tree in $\mathcal{D}$ is universally elliptic.

It follows from this discussion that the JSJ decomposition should be universally elliptic. It should also be as large as possible (note that the trivial tree is universally elliptic).

**Theorem 4.2.** *If $G$ is finitely presented, then there is a unique highest universally elliptic deformation space $\mathcal{D}$.*

**Definition 4.3 (JSJ deformation space).** We call it the *JSJ deformation space of $G$ over $\mathcal{A}$, denoted $\mathcal{D}_{JSJ}$.* It follows from the above discussion that, given any reduced $S \in \mathcal{D}_{JSJ}$ and any $T$ with edge groups in $\mathcal{A}$, there exists $\widehat{S}$ as in Figure 5.

The proof of the theorem is not hard and uses a form of Dunwoody accessibility: given a sequence of collapses $\cdots \to T_k \to \cdots \to T_1 \to T_0$, there exists $T$ dominating every $T_k$.

**Examples.** • If $G$ is free (and $\mathcal{A}$ is arbitrary), then $\mathcal{D}_{JSJ}$ is Outer space.

• If $G$ is a surface group, and $\mathcal{A}$ is the class of cyclic subgroups, then $\mathcal{D}_{JSJ}$ is trivial.

• If $G$ is accessible and $\mathcal{A}$ is the class of finite subgroups, then $T \in \mathcal{D}_{JSJ}$ if and only if vertex groups have at most one end. Note that all reduced trees in $\mathcal{D}_{JSJ}$ have the same edge and vertex groups by Corollary 3.9.

• If $G$ is a GBS group, and $\mathcal{A}$ is the class of cyclic subgroups, then $\mathcal{D}_{JSJ}$ is the deformation space of Example 3 (trees with cyclic vertex groups).

• The JSJ splittings of [RS], [DS], [FP], [Bo] belong to $\mathcal{D}_{JSJ}$.

## 5. The compatibility JSJ tree

Now we look at finding preferred (in particular, Aut($G$)-invariant) trees, in $\mathcal{D}_{JSJ}$ when possible, or in a lower $\mathcal{D}$. This is similar to the approach of Scott-Swarup [SS], who insist on invariance under automorphisms.

When (as is usually the case) $\mathcal{A}$ is invariant under automorphisms of $G$, the group Out($G$) acts on $\mathcal{D}_{JSJ}$. Deformation spaces are contractible and this gives information about Out($G$), but having an (interesting) invariant tree is much stronger.

Sometimes you have to go very low to find a tree invariant under automorphisms: with the free group only the trivial tree is invariant. At the other extreme we have the Bowditch tree $S$ of a one-ended hyperbolic group: being constructed in a canonical way from the boundary, $S$ is Aut($G$)-invariant. Having this invariant tree makes it possible to determine Out($G$) (Sela, see [Le]).

Recall the expand-then-collapse picture (Figure 4). We shall call $S$ and $T$ compatible, because they have a common expansion $\widehat{S}$ (as in [SS], we also use the term common refinement which seems more natural in this context). For instance, trees dual to disjoint curves on a surface are compatible. But two curves which meet in an essential way give rise to non-compatible splittings.

**Definition 5.1 (compatible).** Two trees are *compatible* if they have a common refinement. For experts: irreducible trees are compatible if and only if the sum of their length functions is a length function. A tree $T$ is *universally compatible* if it is compatible with every $T'$. It is *$\mathcal{D}$-compatible* if it is compatible with every $T'$ in $\mathcal{D}$.

**Facts:**

• If $T_1, \dots, T_k$ are pairwise compatible, then they have a *least common refinement (l.c.r.)*. This was also proved by Scott and Swarup (when edge groups are finitely generated).

• A deformation space $\mathcal{D}$ may contain only finitely many reduced $\mathcal{D}$-compatible trees (this is an easy form of accessibility).

Consequence of these facts: If $\mathcal{D}$ is a deformation space which contains a $\mathcal{D}$-compatible tree, then it has a preferred element $T_{\mathcal{D}}$: the l.c.r. of the reduced $\mathcal{D}$-compatible trees. In general, $T_{\mathcal{D}}$ is not reduced.

**Theorem 5.2.** *If $G$ is a finitely presented group, then the set of deformation spaces containing a universally compatible tree has a unique highest element $\mathcal{D}_c$.*

This is quite harder to prove than Theorem 4.2.

**Definition 5.3 (compatibility JSJ tree).** The preferred element $T_c$ of $\mathcal{D}_c$ is called the *compatibility JSJ tree* of $G$ over $\mathcal{A}$. It is invariant under automorphisms if $\mathcal{A}$ is invariant. Any tree $T$ (with edge groups in $\mathcal{A}$) may be obtained from $T_c$ by expanding and collapsing as on Figure 4.

**Examples ($\mathcal{A}$ as above).**

• Free group, surface group: $T_c$ is trivial.

• One-ended hyperbolic group: $T_c$ is the Bowditch tree.

• $G = BS(2,4)$. Now $T_c$ is trivial. But for $BS(2,3)$ then $T_c$ is the Bass–Serre tree of the HNN extension. Note that the canonical decomposition of [SS] is trivial in this case.

• $G$ a one-ended limit group (fully residually free group), $A = \{\text{cyclic groups}\}$, then $T_c$ (almost) belongs to $\mathcal{D}_{JSJ}$ (almost, because $\mathbf{Z}^2$ subgroups are elliptic in $T_c$ but not necessarily in $\mathcal{D}_{JSJ}$).

In the last example, $T_c$ may be constructed directly as the *tree of cylinders* of any $T$ in $\mathcal{D}_{JSJ}$.

Take an edge $e$ of $T$, and look at all edges whose stabilizers are commensurable with $G_e$. The *cylinder* of $e$ is the union of all these edges. Cylinders are subtrees meeting in at most one point. Given a cylinder $C$, let $\partial C$ be the set of vertices of $C$ also belonging to another cylinder. Then $T_c$ is obtained from $T$ by replacing every $C$ by the cone over $\partial C$.

This type of construction works for cyclic or abelian splittings of CSA groups (for instance, relatively hyperbolic groups with abelian parabolic subgroups).

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