Magnetostatic field computations based on the coupling of finite element and integral representation methods

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Abstract—**We present an original method to compute the magnetic field generated by some electromagnetic device through the coupling of an integral representation formula and a finite element method. The unbounded three dimensional magnetostatic problem is formulated in terms of the reduced scalar potential. Through an integral representation formula, an equivalent problem is set in a bounded domain and discretized using a standard finite element method. As a byproduct an integral representation formula is proposed to compute the magnetic field in any point of the space from the reduced scalar potential without numerical differentiation.**

I. Introduction

Our goal is to present an original method, and to compare it to classical ones, in order to compute the magnetic field generated by an electromagnetic device made of a weak ferromagnetic material. Nowadays a lot of work has been done in magnetostatics and numerous numerical methods are known to solve such problems. The choice of the physical unknown either the magnetic strength **H**, or the magnetic induction **B**, or the magnetic potential, the choice of the numerical approximation methods either finite difference (FD) or finite element (FEM) or boundary element (BEM) methods will lead to many different schemes. Each one has its own advantages and drawbacks, see [9], [5], for the choice of the unknowns, [1], [10] for (FEM), and [3] for (BEM).

However as the problem is set in an unbounded domain, an artificial boundary is generally introduced at a finite distance from the electromagnetic device to use (FEM) methods. Thus the behavior of the solution at infinity is handled through an approximate boundary condition set on this artificial boundary [2]. On the other hand we have the boundary element method where an integral representation formula is used to write the problem as an integral equation on the boundary of the device. As a drawback of this method we need to compute nearly singular integrals which require the use of elaborated quadrature schemes. It can also be irrelevant to use the integral representation formulae to compute the solution in a large number of points in the interior or exterior domains.

Here we present a third way to compute the magnetic unknowns through the coupling of a finite element method with an integral representation formula. Namely, an integral representation formula is used to take into account the behavior of the solution in the exterior domain while a finite element approximation is used in the interior domain. This approach is now classical and there exist different ways to write the coupled problem. The one we use originates from the work of A. Jami and M. Lenoir in the field of hydrodynamics, see [8]. The magnetostatic problem is written using the reduced scalar potential [9], [5]. The basic idea of the method is to bound the exterior domain using an artificial boundary that can be close to the device boundary but always distinct from it. The boundary condition to set on this artificial boundary is obtained using a boundary integral representation formula of the solution, the support for the integral representation being the device boundary. As the two boundaries are distinct, all the involved integrals are regular and standard quadrature schemes are used. The magnetic potential in the interior domain surrounded by the artificial boundary is computed using a standard finite element approximation. Then we use an integral representation formula, deduced from the one for the potential, to compute the magnetic field in any point of the space. This computation is achieved through the evaluation of a surface integral without numerical differentiation and therefore without any loss of accuracy.

Our method reconciles the advantages of both (FEM) and (BEM). The behavior of the solution at infinity is handled exactly by an exact boundary condition set on an artificial boundary. As the artificial boundary and the device boundary differ, all the involved integrals are regular.

Our computational approach is well suited for shape optimization where the area of interest is often well localized, the air-gap of an electromagnet for instance. The coupling boundary can be chosen close to the electromagnetic device to reduce the size of the interior domain where the finite element method is employed. The magnetic field at the node on the control surface can be computed efficiently using the integral representation formula.

The content of the paper is the following. Section 2 is devoted to the magnetostatic problem, the coupling procedure, and the approximation scheme. In section 3 the implementation is addressed and two examples are detailed, while a conclusion ends the paper.

We denote by $\mathbb{L}^2(\Omega)$, $\Omega \subset \mathbb{R}^3$, the set of square integral functions over Ω and by $\mathbb{H}^m(\Omega)$ $m \geq 1$ the set of square integral functions over Ω having their first m derivatives in $\mathbb{L}^2(\Omega)$. If ψ is a function defined over $\mathbb{R}^3$, we denote by

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$\psi\Big|_{\Omega}$ its restriction to the domain Ω and by $[\psi]$ its jump across the interface.

II. The magnetostatic problem

To compute the magnetic field induced by a magnet composed of a weakly ferromagnetic core and an inductor characterized by a time independent density current $\mathbf{j}$ in a three dimensional geometry, we start from the basic equations of linear magnetostatics. It is classical to deduce a mathematical problem for the reduced scalar magnetic potential, denoted φ here. In the sequel Ω represents the bounded set containing a weakly ferromagnetic material, μ its relative magnetic permeability. The boundary of Ω is denoted by Σ and its complement Ω^c. The magnetic field $\mathbf{H}$ is expressed as the sum of the field generated by the coils $\mathbf{H_s}$ and of the reaction of the ferromagnetic piece, of the form $\mathbf{H_m} = -\nabla\varphi$.

A. The reduced scalar potential problem

The problem reads : find φ such that

$$\left\{\begin{array}{lcll} \Delta\varphi & = & 0 & \text{in } \Omega \text{ and } \Omega^c, \\ [\mu\dfrac{\partial\varphi}{\partial n}] & = & (\mu-1)g & \text{on } \Sigma, \end{array}\right. \tag{1}$$

with $g(x) = \mathbf{H_s}(x)\cdot\mathbf{n}(x)$ for $x \in \Sigma$ and $\mathbf{n}$ the unit outward normal to the surface Σ. In the sequel g will be considered as a given function since $\mathbf{H_s}$ can be easily computed.

Problem (1) is unfit for numerical computation as it is set in the whole space. We present a way to reduce it to an equivalent problem set over a bounded domain.

B. Integral representation formulae

Let G denote the Green function associated to the three dimensional Laplacian, $G(x,y) = \frac{1}{4\pi|x-y|}$ for $x, y \in \mathbb{R}^3, x \neq y$, and G_n denote its normal derivative on Σ.

The magnetic potential φ is harmonic in the exterior domain Ω^c, so the Green representation formula, together with the second Green identity and the interface conditions, leads to

$$\begin{aligned} \varphi(y) &= (\mu-1)\int_\Sigma g(x)\ G(x,y)\ \mathrm{d}\sigma(x) \\ &\quad -(\mu-1)\int_\Sigma \varphi(x)\ G_n(x,y)\ \mathrm{d}\sigma(x). \end{aligned} \tag{2}$$

Let Γ be an *artificial* surface surrounding Ω. We define a family D^λ, $\lambda > 0$ of boundary differential operators on Γ by $D^\lambda u = \frac{\partial u}{\partial n} + \lambda u$. ¿From (2) we have for y in Γ,

$$\begin{aligned} D^\lambda\varphi(y) &= (\mu-1)\int_\Sigma g(x)\ D^\lambda G(x,y)\ \mathrm{d}\sigma(x) \qquad (3) \\ &\quad -(\mu-1)\int_\Sigma \varphi(x)\ D^\lambda G_n(x,y)\ \mathrm{d}\sigma(x). \end{aligned}$$

Relation (3) will be used as boundary condition on Γ. Problem (1) is equivalent to : find $\phi \in \mathbb{H}^1(\Omega_\Gamma)$ such that

$$\left\{\begin{array}{l} \Delta\phi = 0 \quad \text{in } \Omega, \\ \Delta\phi = 0 \quad \text{in } \Omega_{\Sigma\Gamma}, \\ \mu\dfrac{\partial\phi}{\partial n}\Big|_\Omega - \dfrac{\partial\phi}{\partial n}\Big|_{\Omega_{\Sigma\Gamma}} = g \quad \text{on } \Sigma, \\ D^\lambda\phi(y) = (\mu-1)\displaystyle\int_\Sigma g(x)\ D^\lambda G(x,y)\ \mathrm{d}\sigma(x) \\ \qquad -(\mu-1)\displaystyle\int_\Sigma \phi(x)\ D^\lambda G_n(x,y)\ \mathrm{d}\sigma(x) \quad \text{on } \Gamma; \end{array}\right. \tag{4}$$

the solutions φ to (1) and ϕ to (4) coincide in Ω_Γ, the bounded domain inside Γ, and $\Omega_\Gamma = \overline{\Omega} \cup \Omega_{\Sigma\Gamma}$. Moreover φ could be computed in any point in Ω_Γ^c using (2). In the subsection (II-E) below we will present a way to use this formula.

C. Variational formulation

To dicretize problem (4) it is convenient to write it in variational form. It is not difficult to check that its variational formulation reads : find $\phi \in \mathbb{H}^1(\Omega_\Gamma)$ such that,

$$b^\lambda(\phi,\psi) + k^\lambda(\phi,\psi) = f^\lambda(\psi) \qquad \forall\psi \in \mathbb{H}^1(\Omega_\Gamma), \tag{5}$$

where we define the bilinear forms $b^\lambda, k^\lambda : \mathbb{H}^1(\Omega_\Gamma) \times \mathbb{H}^1(\Omega_\Gamma) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} b^\lambda(\phi,\psi) &= \mu\int_\Omega \nabla\phi\cdot\nabla\psi\ \mathrm{d}\omega + \int_{\Omega_{\Sigma\Gamma}} \nabla\phi\cdot\nabla\psi\ \mathrm{d}\omega \\ &\quad +\lambda\int_\Gamma \phi\ \psi\ \mathrm{d}\gamma, \end{aligned} \tag{6}$$

$$\begin{aligned} k^\lambda(\phi,\psi) &= (\mu-1)\int_\Gamma \psi(y)\ \Big\{ \\ &\quad \int_\Sigma \phi(x)\ D^\lambda G_n(x,y)\ \mathrm{d}\sigma(x)\Big\}\ \mathrm{d}\gamma(y), \end{aligned} \tag{7}$$

and the linear form $f^\lambda : \mathbb{H}^1(\Omega_\Gamma) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} f^\lambda(\psi) &= (\mu-1)\int_\Sigma g\ \psi\ \mathrm{d}\sigma \\ +(\mu-1)\int_\Gamma \psi(y)\ &\left\{\int_\Sigma g(x)\ D^\lambda G(x,y)\ \mathrm{d}\sigma(x)\right\}\ \mathrm{d}\gamma(y). \end{aligned} \tag{8}$$

Clearly the bilinear form b^λ is continuous and elliptic on $\mathbb{H}^1(\Omega_\Gamma)$. The continuous bilinear form $b^\lambda + k^\lambda$ is not elliptic on $\mathbb{H}^1(\Omega_\Gamma)$ (due to k^λ) but it can be considered as a compact perturbation of a continuous elliptic operator on $\mathbb{H}^1(\Omega_\Gamma)$, which ensure existence and uniqueness of the solution.

In [6] a similar problem is analyzed through a different coupling of a boundary integral method and a finite element method. The authors introduce a mixte variational formulation using $\Lambda = \dfrac{\partial\phi}{\partial n}\Big|_\Gamma$ as unknown.

D. Finite element approximation

Let $\mathcal{T}_h$ be a triangulation over the domain Ω_Γ of Lagrange type k isoparametric finite elements that fits the interface

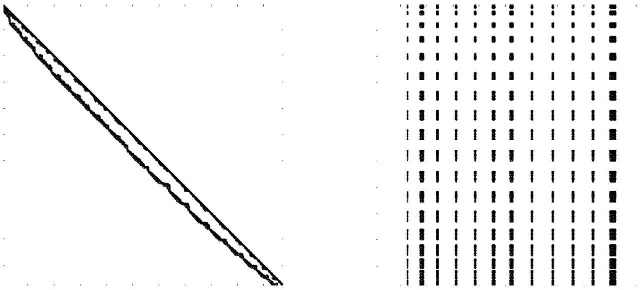

Fig. 1. Sparsity pattern for the symmetric matrix $\mathcal{B}$ (on the left) and $\mathcal{K}$ (on the right). As $\mathcal{B}$ is symmetric, only its lower part is shown.

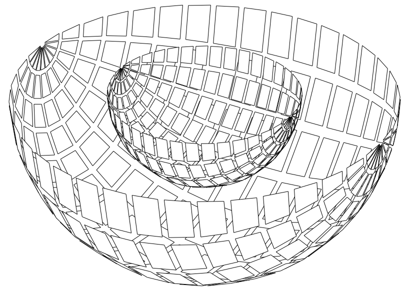

Fig. 2. Mesh of the domain Ω_Γ. The 2 boundaries Σ and Γ are represented.

Σ. Let V_h denote the Lagrange finite element approximation space over the triangulation $\mathcal{T}_h$. We define b_h^λ, k_h^λ the bilinear forms over V_h that approximate the bilinear forms b^λ, k^λ over $\mathbb{H}^1(\Omega_\Gamma)$ and f_h^λ the linear form over V_h that approximate f^λ.

The (FE) discretization of (5) reads: find $\phi_h \in V_h$ such that for all $\psi_h \in V_h$,

$$b_h^\lambda(\phi_h, \psi_h) + k_h^\lambda(\phi_h, \psi_h) = f_h^\lambda(\psi_h)$$

or find in matrix form $\Phi \in \mathbb{R}^N$ such that

$$(\mathcal{B} + \mathcal{K})\ \Phi = \mathcal{F}.$$

In figure 1, the sparsity pattern for the two matrices is shown. The matrix $\mathcal{B}$ is symmetric and sparse but the matrix $\mathcal{K}$ is not.

We have investigated the behavior of the discretization error. The study required a careful handling because of the non-standard coupling terms in (5). Using Lagrange type k isoparametric finite elements the discretization error is proven to be of order $O(h^k)$. We refer to [7] for a way to proceed.

E. Computation of the magnetic field

As a byproduct of our approach we get integral representation formulae to compute the potential or the magnetic field in any point of the space. From the expression (2) for φ, we can express $\mathbf{H_m} = -\nabla\varphi$ as

$$\begin{aligned}\mathbf{H_m}(\mathbf{y}) &= (\mu - 1)\int_\Sigma g(x)\nabla_y G(x,y)\ \mathrm{d}\sigma(x) \\ &\quad -(\mu-1)\int_\Sigma \varphi(x)\nabla_y G_n(x,y)\ \mathrm{d}\sigma(x). \qquad (9)\end{aligned}$$

Other ways to compute the magnetic field would generally imply a loss of accuracy. While attractive, formula (9) cannot be used as it is. For large values of μ ($\mu \sim 10^3$) which is the case in most applications, the two terms in the right-hand side nearly cancel. In fact we can write an asymptotic for φ,

$$\varphi = \varphi_0 + \frac{1}{\mu}\varphi_1 + \frac{1}{\mu^2}\varphi_2 + \cdots$$

where the functions $\varphi_k(x)$ can be characterized as solutions of Neumann problems stated inside Ω for $x \in \Omega$, as solutions of Dirichlet problems stated inside $\Omega_{\Sigma\Gamma}$ for $x \in \Omega_{\Sigma\Gamma}$. We can check the formula, for $y \in \Omega^c$

$$\mathbf{H_m}(\mathbf{y}) = (\mathbf{1} - \mu)\int_\Sigma (\varphi - \varphi_\mathbf{0})(\mathbf{x})\nabla_\mathbf{y}\mathbf{G_n}(\mathbf{x},\mathbf{y})\ \mathrm{d}\sigma(\mathbf{x}). \quad (10)$$

In practice we will compute φ_1, and φ_2 if necessary, and with (10) we deduce $\mathbf{H_m}$.

III. Numerical implementation

The numerical implementation of the above method is achieved using the numerical program MÉLINA developed at the *Institut de Recherche Mathématique de Rennes*, University of Rennes 1 by D. Martin (`www.maths.univ-rennes1.fr/~dmartin/melina`). It is an open collection of Fortran libraries dedicated to the solution of partial differential problem by finite element method. All the computations were done on an INTEL PIII 700Mhz biprocessor personal computer.

A. A test example

We have considered the case where the domain Ω is the ball of radius 1 cm and magnetic permeability $\mu = 10^3$. The inductor field $\mathbf{H_s}$ was assumed to be constant in intensity and direction so that an exact expression for ϕ is known. The coupling boundary Γ was the sphere of radius 2 cm. We have meshed the bounded domain Ω_Γ with 3920 elements, see figure 2.

The computation of the reduced scalar potential required 39 seconds. The computation of the coupling terms was fast (0.7 s). The major part of the time was spent in assembling the matrix of the system (19.1 s) and in solving the linear system (16.4 s). The quadratic error over the domain was about 1%. Figure 3 shows the isolines for the reduced scalar potential in the plane $x = 0$.

Using the integral representation formula (10) we have computed the total magnetic field $\mathbf{H_m}$ in 408 arbitrary points. Computing the magnetic field from the potential took 31.5 s, with 31.4 s dedicated to the resolution of the intervening problems.

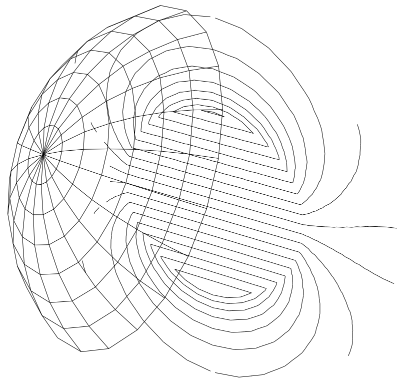

Fig. 3. Isolines for the reduced scalar potential in the plane $x = 0$.

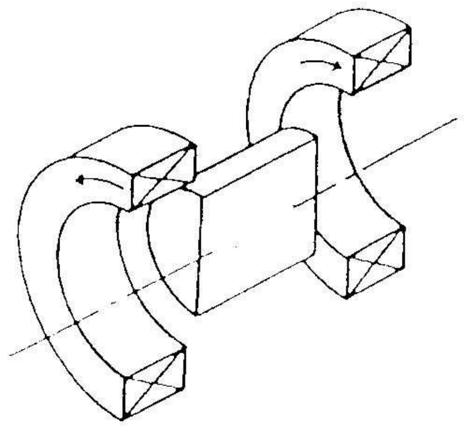

Fig. 4. Electromagnet which consists of a cylindrical core situated inside a pair of coils.

B. Example of an electromagnetic device

We consider an electromagnet which consists of a cylindrical core situated inside a pair of coils, see figure 4. The currents in the two coils are imposed in opposite direction and of constant density (1 A/mm^2). The electromagnet core has the following physical and geometric features: diameter: 1 cm, length: 3 cm, magnetic permeability $\mu = 10^3$. We have bounded the domain with a sphere of radius 4 cm and have meshed the bounded domain Ω_Γ with 5840 elements. The computation of the reduced scalar potential required 136 s while the computation of the magnetic field in 570 arbitrary points took 107 s.

IV. Conclusion

A method to compute the static magnetic field generated by an electromagnetic device has been presented. The unbounded magnetostatic problem for the reduced scalar potential is solved through the coupling of an integral representation formula with a finite element method. In our approach, the coupling boundary is chosen close to the device boundary but distinct to avoid singular kernels in the integrals. An integral representation formula is used to compute the magnetic field in any point of the space from the reduced scalar potential without need for numerical derivation.

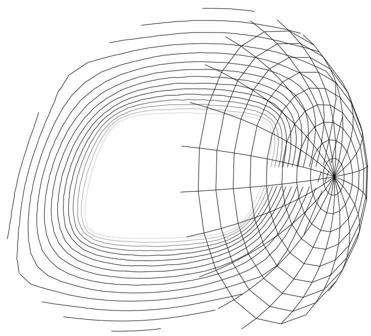

Fig. 5. Isolines for the reduced scalar potential in the plane $x = 0$.

This computational approach is well suited for electromagnetic device shape optimization where the area of interest is localized; with slight modifications it can be used to treat also axisymmetric problems. The coupling boundary can be placed close to the electromagnetic device to reduce the size of the interior domain where the finite element method is employed. The magnetic field at the node of the control surface can be computed efficiently using an integral representation formula.

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