DYNAMICS ON MARKOV SURFACES: CLASSIFICATION OF STATIONARY MEASURES

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ABSTRACT. Consider the four punctured sphere $\mathbb{S}^2_4$. Each choice of four traces, one for each puncture, determines a relative character variety for the representations of the fundamental group of $\mathbb{S}^2_4$ in $\mathsf{SL}_2(\mathbb{C})$. We classify the stationary probability measures for the action of the mapping class group $\mathsf{Mod}(\mathbb{S}^2_4)$ on these character varieties.

RÉSUMÉ. Soit $\mathbb{S}^2_4$ la sphère privée de quatre points. À chaque choix de quatre traces, une par épointement, est associée une variété de caractères relative pour les représentations du groupe fondamental de $\mathbb{S}^2_4$ dans $\mathsf{SL}_2(\mathbb{C})$. Nous classons les mesures de probabilité stationnaires pour l'action du groupe modulaire $\mathsf{Mod}(\mathbb{S}^2_4)$ sur ces variétés.

CONTENTS

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1. INTRODUCTION

1.1. **Representations.** Let $\mathbb{S}^2_4$ denote the 2-dimensional sphere with four punctures. A presentation of its fundamental group is

$$\pi_1(\mathbb{S}^2_4) = \langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma\delta = 1 \rangle \tag{1.1}$$

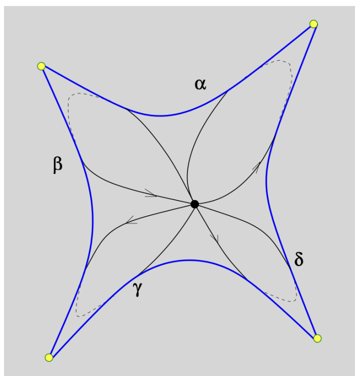

FIGURE 1

where the generators α, β, γ, δ correspond to disjoint simple loops around the punctures (see Figure 1, taken from [15]). Thus, $\pi_1(\mathbb{S}^2_4)$ is the free group on the three generators α, β, and γ.

Let $\mathsf{Rep}(\mathbb{S}^2_4)$ be the set of representations of $\pi_1(\mathbb{S}^2_4)$ in $\mathsf{SL}_2(\mathbb{C})$. Such a representation ρ is uniquely determined by the three matrices $\rho(\alpha)$, $\rho(\beta)$, $\rho(\gamma)$, so that $\mathsf{Rep}(\mathbb{S}^2_4)$ can be identified to $\mathsf{SL}_2(\mathbb{C})^3$. As in [2, 15], we associate the following traces to such a representation

$$a = \mathsf{tr}(\rho(\alpha)),\ b = \mathsf{tr}(\rho(\beta)),\ c = \mathsf{tr}(\rho(\gamma)),\ d = \mathsf{tr}(\rho(\delta)) \tag{1.2}$$

$$x = \mathsf{tr}(\rho(\alpha\beta)),\ y = \mathsf{tr}(\rho(\beta\gamma)),\ z = \mathsf{tr}(\rho(\gamma\alpha)). \tag{1.3}$$

Then, the polynomial map $\chi\colon \mathsf{Rep}(\mathbb{S}^2_4) \to \mathbb{A}^7(\mathbb{C})$ defined by

$$\chi(\rho) = (a,b,c,d,x,y,z) \tag{1.4}$$

is invariant under conjugacy, its image is the hypersurface determined by the equation

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D \tag{1.5}$$

with

$$A = ab + cd,\ B = bc + ad,\ C = ac + bd, \tag{1.6}$$

$$D = 4 - (a^2 + b^2 + c^2 + d^2) - abcd, \tag{1.7}$$

and χ is the quotient map for the action of $\mathsf{SL}_2(\mathbb{C})$ on $\mathsf{Rep}(\mathbb{S}^2_4)$ by conjugacy, in the sense of invariant theory. We shall denote the character variety $\mathsf{Rep}(\mathbb{S}^2_4)/\!/\mathsf{SL}_2(\mathbb{C})$ by $\chi(\mathbb{S}^2_4)$ (instead of $\chi(\mathsf{Rep}(\mathbb{S}^2_4))$).

For each choice of parameters A, B, C, D, we shall denote by $S_{(A,B,C,D)}$ the algebraic surface determined by the Equation (1.5); it is a cubic surface in the affine space $\mathbb{A}^3$, of degree 2 with respect to each variable x, y, or z. The family of all these surfaces will be denoted by Fam. For simplicity, we shall just write S instead of $S_{(A,B,C,D)}$ for the elements of Fam (the quadruple (A,B,C,D) is then uniquely determined by S). If the parameters A, B, C, and D are in a ring R, we denote by $S(R)$ the points of S with coordinates in R.

The compactification of S in the projective space $\mathbb{P}^3$ will be denoted by $\overline{S}$ and its boundary at infinity by

$$\partial S = \overline{S} \setminus S. \tag{1.8}$$

For every $S \in \mathsf{Fam}$, ∂S is the triangle in the hyperplane at infinity given by the equation $xyz = 0$.

1.2. **Mapping class group and Vieta involutions.** Let us view the mapping class group $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ as the subgroup of $\mathsf{Out}(\pi_1(\mathbb{S}^2_4))$ preserving the peripheral structure. Then, $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ acts on $\chi(\mathbb{S}^2_4)$ by precomposition, the conjugacy class of a representation ρ being sent to the conjugacy class of $\rho \circ \Phi^{-1}$ for any mapping class Φ. This gives a homomorphism $\Phi \mapsto f_\Phi$ into the group $\mathsf{Aut}(\chi(\mathbb{S}^2_4))$ of automorphisms of the variety $\chi(\mathbb{S}^2_4)$. As explained in [15, Sections 2.2 and 2.3], there is an isomorphism

$$\mathsf{PGL}_2(\mathbb{Z}) \ltimes H \simeq \mathsf{Mod}^{\pm}(\mathbb{S}^2_4) \tag{1.9}$$

where H is the group $(\mathbb{Z}/2\mathbb{Z})^2$, and an exact sequence

$$\mathrm{Id} \to \Gamma_2^{\pm} \to \mathsf{PGL}_2(\mathbb{Z}) \ltimes H \to \mathrm{Sym}(4) \to \mathrm{Id} \tag{1.10}$$

where $\mathrm{Sym}(4)$ corresponds to the group of permutation of the four punctures of $\mathbb{S}^2_4$ and $\Gamma_2^{\pm}$ is the congruence subgroup of $\mathsf{PGL}_2(\mathbb{Z})$ modulo 2.

(1) $\Gamma_2^{\pm}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, generated by the three involutions

$$\hat{\sigma}_x = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \ \hat{\sigma}_y = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \ \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{1.11}$$

(2) $\Gamma_2^{\pm}$ is the reflection group of an ideal triangle in the upper half plane;

(3) the action of $\Gamma_2^{\pm}$ on the character variety $\chi(\mathbb{S}^2_4)$ preserves the four coordinates a, b, c, d; in particular, it acts on each surface $S \in \mathsf{Fam}$ as a group of automorphisms of S. This action is faithfull for every S, and the image has index at most 24 in $\mathsf{Aut}(S)$ (see [15, Theorem 3.1]).

(4) the image of $\Gamma_2^{\pm}$ in $\mathsf{Aut}(S)$ is generated by the Vieta involutions

$$s_x(x,y,z) = (-x - yz + A, y, z) \tag{1.12}$$
$$s_y(x,y,z) = (x, -y - zx + B, z) \tag{1.13}$$
$$s_z(x,y,z) = (x, y, -z - xy + C). \tag{1.14}$$

In what follows, we denote by Γ the abstract group $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/2\mathbb{Z}$. Depending on the action we look at, Γ will determine a subgroup $\Gamma_2^{\pm}$ of isometries of the upper half plane, or a group of automorphisms Γ_S of S (for any S in Fam). Our goal is to describe the stochastic dynamics of this action on each of the surfaces S. For simplicity, we frequently write Γ instead of Γ_S.

1.3. **Invariant area form.** On S, the 2-form

$$\mathrm{Area} = \frac{dx\wedge dy}{2z+xy-C} = \frac{dy\wedge dz}{2x+yz-A} = \frac{dz\wedge dx}{2y+zx-B} \tag{1.15}$$

is regular and does not vanish. According to Lemma 3.5 of [15], singularities of S are quotient singularities, and Area is an area form in the sense of orbifolds (locally, in a euclidean neighborhood of the singularity, S is the quotient $\mathbb{C}^2/G$ for some finite group and Area is the quotient of a G-invariant symplectic 2-form on $\mathbb{C}^2$).

When S is defined over $\mathbb{R}$, Area is also defined over $\mathbb{R}$. To distinguish between the holomorphic 2-form on $S(\mathbb{C})$ and the real 2-form on $S(\mathbb{R})$, we use the notation $\mathrm{Area}_{\mathbb{C}}$ and $\mathrm{Area}_{\mathbb{R}}$.

The 2-form $\mathrm{Area}_{\mathbb{C}}$ (resp. $\mathrm{Area}_{\mathbb{R}}$) is multiplied by -1 under the action of each of the involutions s_x, s_y, s_z.

When S is defined over $\mathbb{R}$, $S(\mathbb{R})$ may have a compact connected component; we denote such a component by $S(\mathbb{R})_c$ (see Section 2.2 for a precise definition when S is singular). With our notation, it may happen that $S(\mathbb{R})_c$ be reduced to a point; otherwise, it is a (possibly singular) sphere and the restriction of $\mathrm{Area}_{\mathbb{R}}$ to $S(\mathbb{R})_c$ determines (a) an orientation of this sphere and (b) a probability measure $\nu_{\mathbb{R}}$ on $S(\mathbb{R})_c$, defined by

$$\nu_{\mathbb{R}}(B) = \frac{1}{\int_{S(\mathbb{R})_c}\mathrm{Area}_{\mathbb{R}}}\int_B \mathrm{Area}_{\mathbb{R}} \tag{1.16}$$

for any borel subset B of $S(\mathbb{R})_c$. We shall refer to this measure as the **symplectic measure** on $S(\mathbb{R})_c$. It is Γ_S-invariant.

1.4. **Random dynamics.** Let X be a locally compact metric space. We endow the group $\mathsf{Homeo}(X)$ with the compact-open topology. Let μ be a probability measure on $\mathsf{Homeo}(X)$. Denote by Ω the product space $\mathsf{Homeo}(X)^{\mathbb{N}}$ and endow it with the probability measure $\mu^{\mathbb{N}}$. We shall say that a property holds for a **typical** element $\omega\in\Omega$ if it holds for ω in a measurable subset Ω' with $\mu^{\mathbb{N}}(\Omega')=1$. For every $\omega=(f_0,f_1,\dots)\in\Omega$, we set

$$f_\omega^n = f_{n-1}\circ\dots\circ f_0. \tag{1.17}$$

A probability measure ν on X is μ-stationary if

$$\int_{\mathsf{Homeo}(X)} (f_*\nu)\, \mathrm{d}\mu(f) = \nu, \tag{1.18}$$

and such a measure is ergodic if it can not be written as a convex combination of two distinct stationary measures. Fix a point q in X. Then, consider the random orbit $(f^n_\omega(q))$ and the empirical measures

$$\nu_N(\omega; q) = \frac{1}{N}\sum_{j=1}^{N} \delta_{f^j_\omega(q)}. \tag{1.19}$$

A theorem of Breiman (see [3]) says that, for a subset of measure 1 in Ω, if $\nu_{n_i}(\omega; q)$ converges towards a probability measure ν as n_i goes to infinity, then ν is μ-stationary. Thus, to understand how random orbits distribute, we have to describe stationary measures. We apply this viewpoint to the dynamics of Γ_S.

Main Theorem.– *Let μ be a probability measure on $\{s_x, s_y, s_z\}$ with*

$$\mu(s_x)\mu(s_y)\mu(s_z) > 0.$$

Let S be an element of Fam. *Let ν be a probability measure on $S(\mathbb{C})$. If ν is μ-stationary and ergodic, then the support of ν is compact, ν is invariant, and either ν is given by the average on a finite orbit of Γ_S, or*

- *the parameters A, B, C, and D defining S are real and in $[-2,2]$,*
- *$S(\mathbb{R})$ has a unique bounded component $S(\mathbb{R})_c$, of dimension 2, and*
- *ν coincides with the symplectic measure $\nu_{\mathbb{R}}$ induced by* $\mathrm{Area}_{\mathbb{R}}$ *on $S(\mathbb{R})_c$.*

This extends a result of Chung for the dynamics on the compact part $S(\mathbb{R})_c$ when $(A,B,C,D) = (0,0,0,D)$ with $D \in [3.9, 4[$ (see [16, Theorem B]).

Since finite orbits have been classified, our Main Theorem gives a complete description of all μ-stationary measures; in particular,

– *on each surface $S \in$ Fam, except on $S_{(0,0,0,4)}$, the set of μ-stationary measures is a finite dimensional simplex with at most 6 vertices, the maximum 6 being realized only by $S_{(0,0,0,3)}$;*

– *if $S(\mathbb{R})$ has a compact component $S(\mathbb{R})_c$ that does not contain any finite orbit, then given any $q \in S(\mathbb{R})_c$, the empirical measures $\nu_N(\omega; q)$ almost surely converge to the symplectic measure $\nu_{\mathbb{R}}$ as N goes to $+\infty$.*

Our result certainly holds when the support of μ generates the group Γ and satisfies an exponential moment condition. We wrote the proof assuming that the support is equal to $\{s_x, s_y, s_z\}$ to simplify the exposition, as it allows us to avoid the use of the general theory of random walks on Γ.

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2. THE MARKOV SURFACES AND THEIR AUTOMORPHISMS

2.1. **Smoothness of $S(\mathbb{C})$.** In [2], Benedetto and Goldman study the topology of the surfaces $S \in \mathsf{Fam}$. Their first result says that S is singular if, and only if at least one of the parameters a, b, c, or d is equal to ± 2 or there is a reducible representation with boundary traces a, b, c, d; moreover, the latter case occurs if and only if the following discriminant vanishes:

$$\Delta = (2(a^2+b^2+c^2+d^2)-abcd-16)^2-(4-a^2)(4-b^2)(4-c^2)(4-d^2).$$

Example 2.1. The **Cayley cubic** $S_{\mathrm{Ca}} = S_{(0,0,0,4)}$ is defined by the equation $x^2+y^2+z^2+xyz=4$. It has four singularities, the maximum for an irreducible cubic surface. It is the quotient of the multiplicative group $\mathbb{G}_m \times \mathbb{G}_m$ by $\eta(u,v) = (1/u, 1/v)$, the quotient map being

$$(u,v) \mapsto (-u-1/u, -v-1/v, -uv-1/(uv)). \tag{2.1}$$

2.2. **Topology of $S(\mathbb{R})$.** Let n be the number of boundary traces a, b, c, d in the interval $]-2,2[$. According to [2, Theorem 1.2], if $S(\mathbb{R})$ is smooth then its Euler characteristic is equal to $2n-2$ and $S(\mathbb{R})$ is homeomorphic to

(1) a quadruply punctured sphere if $n=0$ and $abcd<0$;
(2) a disjoint union of a triply punctured torus and a disk if $n=0$ and $abcd>0$;
(3) a disjoint union of a triply punctured sphere and a disk if $n=1$;
(4) a disjoint union of an annulus and two disks if $n=2$;
(5) a disjoint union of four disks if $n=3$;
(6) a disjoint union of four disks and a sphere if $n=4$.

In particular, if $S(\mathbb{R})$ has a compact connected component, then this component is unique and is homeomorphic to a sphere.

When S is allowed to have singularities, we define the **components** of $S(\mathbb{R})$ as follows. Firstly, if $S(\mathbb{R})$ has an isolated point, this point will be one of the components of $S(\mathbb{R})$; note that there is at most one isolated point, it must be one of the singularities of $S(\mathbb{R})$, and a well chosen perturbation of the coefficients (a,b,c,d) will turn this point into a small sphere (resp. will make this point disappear). Secondly, consider the smooth part $S(\mathbb{R}) \setminus \mathsf{Sing}(S)$, and split it as a disjoint union of connected components $S(\mathbb{R})^0_i$; then, replace each

$S_i^0(\mathbb{R})$ by its closure (equivalently, add to $S(\mathbb{R})_i^0$ the singularities of S which are contained in the closure of $S(\mathbb{R})_i^0$); the result

$$S(\mathbb{R})_i := \overline{S(\mathbb{R})_i^0} \tag{2.2}$$

will be one of the components of $S(\mathbb{R})$. For instance, in the case of the Cayley cubic, $S(\mathbb{R})$ has a compact component: it contains four singularities, each of which is contained in exactly one of the unbounded components.

With this definition at hand, the classification of Benedetto and Goldman remains correct, except that the components are not necessarily disjoint (they may touch at singularities) and the sphere in Case (6) can be reduced to a singular point. In particular, one gets the following result: *if $S(\mathbb{R})$ has a compact component, this component is unique, and it is homeomorphic to a sphere or a singleton.* If such a component exists, it will be denoted by $S(\mathbb{R})_c$; when $S(\mathbb{R})$ has an isolated point, then $S(\mathbb{R})_c$ is equal to this point.

2.3. **Representations in the compact component.** The group $\mathsf{SL}_2(\mathbb{C})$ has two real forms, one is $\mathsf{SL}_2(\mathbb{R})$, the other is SU_2. Since SU_2 is compact, representations in SU_2 give points in $S(\mathbb{R})_c$; but points of $S(\mathbb{R})_c$ may also correspond to representations in $\mathsf{SL}_2(\mathbb{R})$ (see [2, Proposition 1.4]). In fact, according to [15, Lemma 2.7], the map

$$\Pi\colon (a,b,c,d) \in \mathbb{C}^4 \mapsto (A,B,C,D) \in \mathbb{C}^4 \tag{2.3}$$

from Equations (1.6) and (1.7) is a ramified cover of degree 24 and $\mathrm{Jac}(\Pi) = -\frac{1}{2}\Delta$, where Δ is the discriminant from Section 2.1. The following automorphisms of $\mathbb{C}^4$ generate a group $Q \subset \mathsf{Aut}(\mathbb{C}^4)$ of order 8:

(a) the simultaneous sign change of the parameters a, b, c, and d
(b) the permutations of a, b, c, and d which are a composition of two transpositions with disjoint supports.

The ramified cover Π is invariant under the action of Q. But Π is not Galois, and to understand the structure of the 24 points in the fibers of Π, one has to introduce the Okamoto correspondences: see Section 3 of [15] for this.

If $S(\mathbb{R})$ has a compact component, then (A,B,C,D) is contained in $[-2,2]^4$ and $\Pi^{-1}\{(A,B,C,D)\}$ is also entirely contained in $[-2,2]^4$. Moreover, different choices of (a,b,c,d) in $\Pi^{-1}\{(A,B,C,D)\}$ lead to different types of representations, as summarized in the following result (see [15, Theorem B and Proposition 3.13]).

Theorem 2.2. *Let (A,B,C,D) be real parameters. If the smooth part of $S(\mathbb{R})$ has a bounded component, then all parameters (a,b,c,d) in $\Pi^{-1}\{(A,B,C,D)\}$ are real. Moreover, the points of $S(\mathbb{R})_c$ are conjugacy classes of SU_2 and $\mathsf{SL}_2(\mathbb{R})$-representations, depending on the choice of (a,b,c,d): modulo the action of the group Q, each point of $S(\mathbb{R})_c$ corresponds to two SU_2-representations and one $\mathsf{SL}_2(\mathbb{R})$-representation.*

3. Invariant compact subsets and invariant measures on them

In this section, we summarize some known results concerning Γ_S-invariant compact subsets of $S(\mathbb{C})$ and derive from them a classification of Γ_S-invariant probability measures on $S(\mathbb{C})$.

3.1. **Finite orbits.**

Boalch, and Lisovyy and Tykhyy obtained a classification of all finite orbits of Γ in the surfaces $S \in \mathsf{Fam}$. We summarize their results.

3.1.1. *Short orbits.* Finite orbits of Γ with at most 4 elements are classified in [22, Lemma 39]. These finite orbits come in families, depending on 3, 2, or 1 parameters, up to permutation of the coordinates. Fixed points coincide with singularities of S, and form a 3-parameter family, the parameters (a,b,c,d) being on the locus $(4-a^2)(4-b^2)(4-c^2)(4-d^2)\Delta = 0$. Orbits of length 2 depend on 2 parameters, for instance $\{(x,0,0),(x',0,0)\}$ is such an orbit if $A = x+x'$, $B = 0$, $C = 0$, and $D = 4+x+x'$. Orbits of length 3 or 4 depend on 1 parameter; for a generic choice of the parameter, the points in the orbit correspond to representations of $\pi_1(\mathbb{S}^2_4)$ with an infinite image.

3.1.2. *Finite groups.* Let F be a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$. Then, each representation of $\pi_1(\mathbb{S}^2_4)$ into F gives rise to a finite orbit of Γ in the character variety $\chi(\mathbb{S}^2_4)$. Changing F into a conjugate subgroup of $\mathsf{SL}_2(\mathbb{C})$ does not change the corresponding orbit. The finite orbits obtained with this method have been described by Boalch in [4, 5, 6]. Note that changing the parameters (a,b,c,d) in $\Pi^{-1}(A,B,C,D)$ may turn a representation inside a finite group into a representation with infinite image.

Example 3.1. Consider the Cayley cubic $S_{\mathrm{Ca}} = S_{(0,0,0,4)}$ described in Example 2.1. The group $\mathsf{GL}_2(\mathbb{Z})$ acts by monomial transformations on the multiplicative group $\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$ and each point of type $(e^{2i\pi p/q}, e^{2i\pi p'/q'})$ has a finite orbit for this action. On the other hand, this action commutes to the involution η and induces a subgroup of $\mathsf{Aut}(S)$ that contains $\Gamma_{S_{\mathrm{Ca}}}$ as a finite index subgroup (see [15]). Thus, their projections

$$(-2\cos(2\mathrm{i}\pi p/q), -2\cos(2\mathrm{i}\pi p'/q'), -2\cos(2\mathrm{i}\pi(p/q+p'/q'))) \tag{3.1}$$

give points on the Cayley cubic with finite $\Gamma_{S_{\mathrm{Ca}}}$-orbits. The Cayley cubic is the unique example of a surface $S \in \mathsf{Fam}$ containing infinitely many finite orbits.

3.1.3. *Boalch-Klein orbit.* In [5], Boalch constructs a finite orbit that is not given by any representation into a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$ (it comes from a representation into a finite subgroup of $\mathsf{SL}_3(\mathbb{C})$, though). The surface is defined by the equation

$$x^2+y^2+z^2+xyz = x+y+z \tag{3.2}$$

and the orbit is made of the seven points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$. There are 24 choices of parameters (a,b,c,d) giving rise to $(A,B,C,D) = (1,1,1,0)$. One of them is

$$a = b = c = 2\cos(2\pi/7), \quad d = 2\cos(4\pi/7). \tag{3.3}$$

For such a choice, the image of the representation $\pi_1(\mathbb{S}^2_4) \to \mathsf{PSL}_2(\mathbb{C})$ is conjugate to the triangle group $T(2,3,7)$. The Galois conjugates of this representations provide two distinct representations in SU_2 (corresponding to conjugates of a, b, c, d). Modulo Okamoto symmetries, all orbits of length 7 are given by these three representations.

3.1.4. *Classification.* Lisovyy and Tykhyy proved that *every finite orbit of* Γ *is given by one of the above examples.* The parameters for finite orbits are listed in Lemma 39 page 147, Theorem 1 page 149 and Table 4 page 150 of [22]. Each point with a finite orbit of length ≥ 5 is determined by a representation of $\pi_1(\mathbb{S}^2_4)$ in a finite subgroup of SU_2, except for the Boalch-Klein orbit.

3.2. **Invariant compact subsets and probability measures.** From Lemma 4.3 and Theorems B and C of [15], one gets the following result.

Theorem 3.2. *Let S be an element of* Fam. *If Γ_S preserves a compact subset K of $S(\mathbb{C})$ with at least five elements then*

(1) *the parameters a, b, c, and d are in* $[-2,2]$*;*
(2) *$S(\mathbb{R})$ has a unique compact component $S(\mathbb{R})_c$;*
(3) *K is either finite or equal to $S(\mathbb{R})_c$.*

In [20] and [23], Goldman, Pickrell and Xia prove that the symplectic measure $\nu_{\mathbb{R}}$ is ergodic for the action of Γ_S on $S(\mathbb{R})_c$. Together with Theorem 3.2, this gives a classification of invariant probability measures with compact support (and, as we shall see in Corollary 6.3, a classification of all invariant probability measures):

Corollary 3.3. *Let ν be a Γ_S-invariant probability measure on $S(\mathbb{C})$ with compact support. If ν is ergodic, then either ν is the counting measure on a finite orbit of Γ_S, or the parameters a, b, c, and d are in $[-2,2]$, $S(\mathbb{R})$ has a unique compact component and ν is the symplectic measure $\nu_{\mathbb{R}}$ introduced in Section 1.3.*

Since the proof is a simple variation on the arguments of [20], [15], and [13], we only sketch it.

Proof. Denote by K the support of ν. If ν has an atom at a point $q \in S(\mathbb{C})$ then the orbit of q contains at most $\nu(\{q\})^{-1}$ points because ν is a probability measure. Then, by ergodicity, ν coincides with the counting measure on $\Gamma_S(q)$.

Now, suppose that ν does not have any atom. If there is an irreducible algebraic curve $C \subset S(\mathbb{C})$ with $\nu(C) > 0$, then $\nu(f(C) \cap C) = 0$ for every $f \in \Gamma_S$ that does not preserve C. Thus, the orbit of C under Γ_S is a union of at most $\nu(C)^{-1}$ irreducible curves and we obtain a contradiction because Γ_S does not preserve any curve (see [15, Theorem D]). Thus, we may assume that K is infinite and $\nu(Z) = 0$ for every proper algebraic subset of $S(\mathbb{C})$.

From Theorem 3.2, the parameters A, B, C, and D are real and K coincides with $S(\mathbb{R})_c$. Consider the projection $\pi \colon S \to \mathbb{R}$ onto the first axis; its image is contained in $[-2,2]$. The composition of the second and third involutions acts on S by

$$g(x,y,z) = (x, -xz - y + B, x^2 z - z + C - Bx). \tag{3.4}$$

If the fiber $S(\mathbb{R})_{x_0} := \pi^{-1}\{x_0\}$ is non-empty and smooth, then it is an ellipse, it is diffeomorphic to a circle, and g acts on it as a rotation, the angle of which is given by $2cos(\pi\theta) = x_0$ (see [19, §5] and [15]). Thus, if x_0 is not in $2\cos(2\pi\mathbb{Q})$, g preserves a unique probability measure λ_{x_0} on $S(\mathbb{R})_{x_0}$: this measure is smooth, it corresponds to the Lebesgue measure on the circle.

Consider the projection $\nu_x := \pi_*(\nu)$. This measure does not have any atom. Thus, if we desintegrate ν with respect to the fibration π, the conditional measures coincide with the measures λ_{x_0} because they are g-invariant. Now, projecting on the second axis and applying the same argument, we obtain a smooth measure ν_y on the base, with smooth conditional measure. This shows that ν is absolutely continuous with respect to the symplectic measure $\nu_{\mathbb{R}}$, and by the ergodicity theorem of [20, 23], ν is equal to $\nu_{\mathbb{R}}$. □

4. Dynamics at infinity

We extend Γ_S as a group of birational transformations of $\overline{S}$, the completion of S in $\mathbb{P}^3$, and study its dynamics near the hyperplane at infinity.

Remark 4.1. We present all computations and estimates for the complex surface $S(\mathbb{C})$. On the other hand, the same computations work equally well on $S(\mathbb{K})$ if $\mathbb{K}$ is a local field, for some non-trivial absolute value $|\cdot|$. In that case, $\|\cdot\|$ must be a norm on $\mathbb{K}^3$ which is compatible with $|\cdot|$. Good examples to keep in mind are $\mathbb{K}=\mathbb{Q}_p$, or $\mathbb{K}=\mathbb{F}_p[t]$, p a prime.

4.1. **Taylor expansion at infinity.** We add a variable w to get homogeneous coordinates $[x:y:z:w]$ on $\mathbb{P}^3$; then, the hyperplane at infinity is $\{w=0\}$, and its intersection with $\overline{S}$ is the triangle $\partial S=\overline{S}\setminus S$ given by the equation $xyz=0$. The vertices of this triangle are $p_1=[1:0:0:0]$, $p_2=[0:1:0:0]$, and $p_3=[0:0:1:0]$. Near p_3, we can use the local coordinates x,y,w, with $z=1$. The equation of S becomes, locally,

$$(1+x^2+y^2)w+xy=(Ax+By+C)w^2+Dw^3. \tag{4.1}$$

Thus, S is smooth near p_3, its tangent plane at p_3 is the "horizontal" plane $w=0$, and S is locally the graph of a function $\varphi_3\colon (x,y)\mapsto w=\varphi_3(x,y)$. Since $\overline{S}$ intersects the plane $\{w=0\}$ on the coordinate axis, φ_3 is divisible by xy; then, its Taylor expansion starts by

$$\begin{aligned}\frac{\varphi_3(x,y)}{-xy} &= 1-(x^2+Cxy+y^2)-(Ax+By)xy\\ &\quad +(x^4+3Cx^3y+(2-D+2C^2)x^2y^2+3Cxy^3+y^4)+\dots\end{aligned}$$

Hence

$$\frac{\varphi_3(x,y)}{-xy}=1-(x^2+Cxy+y^2)-(Ax+By)xy+O(\|(x,y)\|^4), \tag{4.2}$$

where the norm is any (fixed) norm in the local (x,y)-coordinates. The coefficients in this expansion are polynomial functions of (A,B,C,D) with integer coefficients.

Similarly, we can write locally $\overline{S}$ as a graph $(y,z)\mapsto w=\varphi_1(y,z)$ near p_1, and as a graph $(z,x)\mapsto w=\varphi_2(z,x)$ near p_2 (note the cyclic permutation of the local coordinates); we shall use the notation

- $(u_1, v_1, \varphi_1(u_1, v_1)) = (y, z, \varphi_1(y, z))$ near the point p_1,
- $(u_2, v_2, \varphi_2(u_2, v_2)) = (z, x, \varphi_2(z, x))$ near the point p_2, and
- $(u_3, v_3, \varphi_3(u_3, v_3)) = (x, y, \varphi_3(x, y))$ near the point p_3.

More precisely, (u_i, v_i, w) are local coordinates near p_i, the projection from $\overline{S}$ to the (u_i, v_i)-plane being a local diffeomorphism.

We can now compute the Taylor expansion of the involutions s_x, s_y, and s_z near the points p_i where they are well defined. For instance, at p_3 both s_x and s_y are well defined, while s_z has an indeterminacy. From Equation (1.12),

$$s_x[x : y : z : w] = [-xw - yz + Aw^2 : yw : zw : w^2] \tag{4.3}$$

Since s_x maps $p_3 = [0 : 0 : 1 : 0]$ to $p_1 = [1 : 0 : 0 : 0]$, we use the local coordinates $(u_3, v_3) = (x, y)$ and $(u_1, v_1) = (y, z)$. Combining Equations (4.2) and (4.3) with $w = \varphi_3(x, y)$, we can write $s_x(x, y) = (y', z')$ with

$$y' = xy[1 - Cxy - y^2 - Bxy^2] + \cdots \tag{4.4}$$

$$z' = x[1 - Cxy - y^2 - Bxy^2 - x^2(x^2 + 2Cxy + 2y^2)] + \cdots \tag{4.5}$$

where the dots correspond to terms of degree at least 6. Thus if we start with the point of coordinates (u_3, v_3), that is with the point $[u_3 : v_3 : 1 : \varphi_3(u_3, v_3)] \in S$, the coordinates (u_1', v_1') of its image by s_x satisfy

$$u_1' = u_3 v_3 [1 + O(\| (u_3, v_3) \|^2)] \tag{4.6}$$

$$v_1' = u_3 [1 + O(\| (u_3, v_3) \|^2)]. \tag{4.7}$$

This is not a surprise, indeed locally s_x is the blow down of the axis $\{u_3 = 0\}$. A similar computation can be done near each of the vertices of the triangle ∂S. To summarize it, we introduce the following two matrices.

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \ B := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.8}$$

We denote by $(u, v) \mapsto (u, v)^M$ the monomial action of a matrix $M \in \mathsf{Mat}_2(\mathbb{Z})$; for instance, $(u, v)^A = (v, uv)$ and $(u, v)^B = (uv, u)$.

Proposition 4.2. *In the local coordinates (u_1, v_1) near the point $p_1 \in \overline{S}$, (u_2, v_2) near p_2, and (u_3, v_3) near p_3, the involution s_x acts by*

$$s_x(u_2, v_2) = (v_2(1 + O(\| (u_2, v_2) \|^2)), u_2 v_2 (1 + O(\| (u_2, v_2) \|^2))$$

$$s_x(u_3, v_3) = (u_3 v_3 (1 + O(\| (u_3, v_3) \|^2), v_3(1 + O(\| (u_3, v_3) \|^2)).$$

Thus, s_x acts as the monomial map $(u_2, v_2)^A$ near p_2 and as $(u_3, v_3)^B$ near p_3, up to multiplication by functions of type $1 + O(\| (u_i, v_i) \|^2)$.

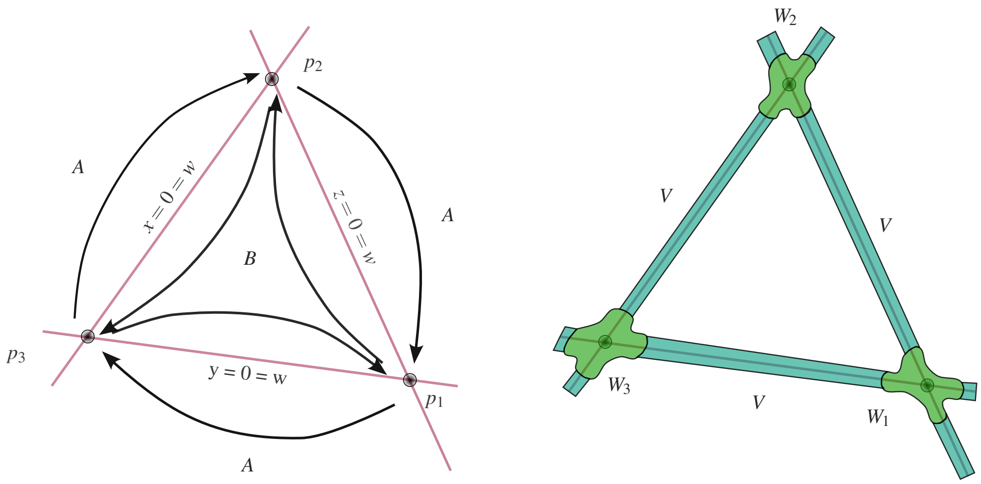

FIGURE 2. **On the left**, one sees the triangle at infinity ∂S with the three vertices corresponding to the indeterminacy points of the involutions. For instance, s_x maps the line $\{w = 0 = x\}$ to the point $[1:0:0:0]$, and locally near $[0:1:0:0]$ it behaves like the monomial map $(u_2, v_2)^A$. **On the right**, one sees the two types of open sets used to confine the dynamics near infinity: small neighborhoods W_i near the vertices, in green, small neighborhoods V around the edges; the neighborhoods of type V depend on the words.

Similar formulas hold for s_y and s_z, where the choice of A or B in the monomial expansion is given on Figure 2.

Note that if we set $\alpha_i = -\log|u_i|$ and $\beta_i = -\log|v_i|$, then the action of s_x near p_2 transforms (α_2, β_2) in (α_1', β_1') with

$$\alpha_1' = \beta_2 + O(\exp(-2 \parallel (\alpha_2, \beta_2) \parallel)) \tag{4.9}$$

$$\beta_1' = \alpha_2 + \beta_2 + O(\exp(-2 \parallel (\alpha_2, \beta_2) \parallel)). \tag{4.10}$$

Here, as (u_2, v_2) approaches ∂S in S, then (α_2, β_2) goes to ∞ in $\mathbb{R}^2_+$.

4.2. **Dynamics of the semi-group $\langle A, B\rangle_{s.g.}$ on $\mathbb{R}^2_+$.** The last paragraph shows that, to describe the action of Γ_S near ∂S, we must look at the linear dynamics of the semi-group generated by A and B on $\mathbb{R}^2_+$. This semi-group is the free semi-group on two generators $\langle A, B\rangle_{s.g.}$; its elements are uniquely represented by words in A, and B, such as $w(A, B) = AABABBAAAB$ (we do not distinguish a word in A and B from the corresponding element in $\langle A, B\rangle_{s.g.}$); the length of such a word is just its number of letters; a sequence of words (w_n) is said to be increasing if $w_{n+1} = v_n \cdot w_n$ is the concatenation of w_n with a word v_n of positive length. In the next lemma, $\| \cdot \|_1$ is the ℓ_1-norm on $\mathbb{R}^2$.

Lemma 4.3. *Let (α, β) be a point of $\mathbb{R}^2_+$. Let (w_n) be an increasing sequence of words in the semi-group $\langle A, B \rangle_{s.g.}$. Then, given any point $(\alpha, \beta) \neq (0,0)$ in $\mathbb{R}^2_+$,*

- *either $\| w_n(\alpha, \beta) \|_1$ goes to $+\infty$ as n goes to $+\infty$,*
- *or $\alpha = 0$ and $w_n = B^{\varepsilon(n)}(AB)^{\ell(n)}$ for some increasing sequence $\ell(n) \in \mathbb{Z}_+$ and some sequence $\varepsilon(n) \in \{0,1\}$,*
- *or $\beta = 0$ and $w_n = A^{\varepsilon(n)}(BA)^{\ell(n)}$ for some increasing sequence $\ell(n) \in \mathbb{Z}_+$ and some sequence $\varepsilon(n) \in \{0,1\}$.*

If $\alpha\beta > 0$, the orbit of $(w_n(\alpha, \beta))$ of (α, β) in $\mathbb{R}^2_+$ is discrete and contains at most $\min(\alpha, \beta)^{-1} R$ points at distance $\leq R$ from the origin.

Proof. For any $(\alpha, \beta) \in \mathbb{R}^2_+$ and for $U \in \{A, B\}$ we have $\| U(\alpha, \beta) \|_1 \geq \| (\alpha, \beta) \|_1 + \min(\alpha, \beta)$. Thus, if $w_m(\alpha, \beta)$ is in the interior of $\mathbb{R}^2_+$ for some $m \geq 1$, then $\| w_n(\alpha, \beta) \|_1 \geq a(n-m)$ for some $a > 0$ and all n. Now, $w_n(\alpha, \beta)$ stays permanently on the boundary of $\mathbb{R}^2_+$ if and only if $\alpha = 0$ (resp. $\beta = 0$) and w_n is an alternating sequence of A and B, as stated in the lemma. $\square$

Lemma 4.4. *Let (α, β) be a point of $\mathbb{R}^2_+$. Let (w_n) be an increasing sequence of words in the semi-group $\langle A, B \rangle_{s.g.}$. Let (α, β) be a point in $\mathbb{R}^2_+$ such that $\alpha\beta \neq 0$. Denote by (α_n, β_n) the coordinates of $w_n(\alpha, \beta)$.*

- *either $\min(\alpha_n, \beta_n)$ goes to $+\infty$ as n goes to $+\infty$,*
- *or there is an index n_0 such that, for all $n \geq n_0$,*
$$\min(\alpha_n, \beta_n) = \alpha_{n_0} \ \text{ and } \ w_n = B^{\varepsilon(n)}(AB)^{\ell(n)} w_{n_0}$$
for an increasing sequence $\ell(n) \in \mathbb{Z}_+$ and some sequence $\varepsilon(n) \in \{0,1\}$.

The proof is the same as for the previous lemma. The reason why there is only one exceptional case is that we can change a sequence of type $(BA)^{\ell(n)}$ into a sequence of type $B(AB)^{\ell(n)-1}v$ with $v = A$ and concatenate v with w_{n_0}.

Now, we consider the following process, acting again on $\mathbb{R}^2$. An increasing sequence of words $w_n = U_{L(n)} U_{L(n-1)} \cdots U_2 U_1$ is given, but now the U_k are not in $\{A, B\}$; each U_k is a small (non-linear) perturbation of A or B, of type

$$U_k(\alpha, \beta) = V_k(\alpha, \beta) + P_k(\alpha, \beta) \tag{4.11}$$

where $V_k \in \{A, B\}$ and $\| P_k(\alpha, \beta) \|_1 \leq C \exp(-2 \| (\alpha, \beta) \|_1)$ for some fixed constant C (that does not depend on (α, β) or (w_n)).

Lemma 4.5. *Let R be a positive number such that $R \geq 2C\exp(-2R)$. If (α,β) satisfies $\alpha \geq R$ and $\beta \geq R$, then for any sequence of words as above, we have*

$$\| w_n(\alpha,\beta) \|_1 \geq \| (\alpha,\beta) \|_1 + \frac{R}{2}n.$$

In particular, $w_n(\alpha,\beta)$ goes to ∞ with n. Moreover, each coordinate of $w_n(\alpha,\beta)$ goes to $+\infty$, except if there is an integer n_0 such that, for all $n \geq n_0$,

$$w_n = B^{\varepsilon(n)}(AB)^{\ell(n)} w_{n_0}$$

for some increasing sequence $\ell(n) \in \mathbb{Z}_+$, some sequence $\varepsilon(n) \in \{0,1\}$.

Indeed, for $V \in \{A,B\}$ and any map $P\colon \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $\| P(\alpha,\beta) \|_1 \leq C\exp(-2 \| (\alpha,\beta) \|_1)$, we obtain

$$\begin{aligned} \| V(\alpha,\beta) + P(\alpha,\beta) \|_1 &\geq \| (\alpha,\beta) \|_1 + R - C\exp(-2 \| (\alpha,\beta) \|_1) & (4.12)\\ &\geq \| (\alpha,\beta) \|_1 + \frac{R}{2}. & (4.13)\end{aligned}$$

The proof is now the same as for Lemma 4.3 and Lemma 4.4.

4.3. **Application: dynamics at infinity.** Sections 4.2 and 4.1 can be combined as follows. Fix some open neighborhoods W_1, W_2, W_3 of p_1, p_2, p_3 in S with local coordinates (u_i,v_i) as in § 4.1 (neighborhoods for which Proposition 4.2 holds). Set $(\alpha_i,\beta_i) = (-\log|u_i|, -\log|v_i|)$; there is a constant R_0 such that $\{(u_i,v_i)\,;\ \alpha_i \geq R_0 \text{ and } \beta_i \geq R_0\}$ is contained in W_i for each index i. Then, Equations (4.9) and (4.10) and their siblings for other choices of involutions and local coordinates say the following. There is a constant C such that if (u_2,v_2) is in W_2, then $s_x(u_2,v_2)$ is in W_1, and if we write its coordinates in W_1 as $(u_1',v_1') = s_x(u_2,v_2)$ then their logarithms $\alpha_1' = -\log|u_1'|$ and $\beta_1' = -\log|v_1'|$ satisfy

$$(\alpha_1',\beta_1') = A(\alpha_2,\beta_2) + P_{1,2}(\alpha_2,\beta_2) \qquad (4.14)$$

where $P_{1,2}$ is defined on $[R_0,+\infty[\times[R_0,+\infty[$ and

$$\| P_{1,2}(\alpha,\beta) \|_1 \leq C\exp(-2 \| (\alpha,\beta) \|_1). \qquad (4.15)$$

Similar formulas hold for the other involutions in the open sets W_i where they are well defined; the matrices A or B are to be chosen as on Figure 2, and the perturbations $P_{i,j}$ depend on the open sets; the constant C can be chosen uniformly, i.e. independently of the indices.

We say that a word w in s_x, s_y, s_z is **reduced** if $s_{i_{k+1}} \neq s_{i_k}$ for all pairs of successive letters in w. A reduced word $w_n = s_{i_{\ell(n)}} \cdots s_{i_1}$ gives rise to a sequence s_{i_1}, $s_{i_2} \circ s_{i_1}$, $\cdots$, $s_{i_{\ell(n)}} \circ \cdots \circ s_{i_1}$ of birational transformations of $\overline{S}$, for any $S \in \mathsf{Fam}$, such that $s_{i_k} \circ \cdots \circ s_{i_1}$ contracts the generic point of the boundary ∂S onto a vertex p_{i_k} that does not coincide with the indeterminacy point of $s_{i_{k+1}}$. We use the same notation w_n for the reduced word, the element of Γ determined by it, and the associated automorphism of S, for any S in Fam. On the other hand, we prefer to use $s \circ s'$ to denote the composition in $\Gamma_S \subset \mathsf{Aut}(S)$ and ss' for products in $\Gamma \simeq \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$.

Lemma 4.5 gives the following.

Proposition 4.6. *Let S be an element of* Fam. *There are disjoint open neighborhoods W_i of the points p_i in S such that given any point q in one of the W_i, and any increasing sequence of reduced words $w_n = s_{i_{\ell(n)}} \cdots s_{i_1}$ in s_x, s_y, s_z such that the indeterminacy point of s_{i_1} is not in W_i, the sequence $w_n(q)$ stays in $W_1 \cup W_2 \cup W_3$ and goes exponentially fast to infinity as n goes to ∞.*

If the three letters s_x, s_y, s_z appear infinitely often in the words w_n as n goes to $+\infty$, then the accumulation points of $(w_n(q))$ in $\overline{S}$ are contained in the three vertices of the triangle ∂S.

This proposition describes the dynamics of reduced words when q is in $W_1 \cup W_2 \cup W_3$. Now, our goal is to describe what happens when q is near ∂S but is not in $W_1 \cup W_2 \cup W_3$.

Remark 4.7. Set $L = \{z = 0 = w\} \subset \partial S$ and denote by L^* the open subset $L \setminus \{p_1, p_2\}$. The involutions s_x and s_y are regular on L^*, and both of them acts by $[x : y : 0 : 0] \mapsto [y : x : 0 : 0]$ on L^*. On the other hand, there are points arbitrary close to $[1 : -1 : 0 : 0]$ (for instance), for which the orbit under $s_x \circ s_y$ does not remain close to L.

Consider an increasing sequence of reduced words $w_n = s_{i_{\ell(n)}} \cdots s_{i_1}$ in which each of s_x, s_y, and s_z ultimately appears. Let k be the smallest integer such that the three involutions appear among the first k letters s_{i_j}, $1 \leq j \leq k$. There are integers $\ell \geq 1$ and $\varepsilon \in \{0, 1\}$ such that

- for $n \geq k$, w_n starts with a sequence of type $s_{i_3} s_{i_1}^{\varepsilon} (s_{i_2} s_{i_1})^{\ell}$, and
- $\{i_1, i_2, i_3\} = \{x, y, z\}$.

In particular, $k = 2\ell + \varepsilon + 1$. Then, for all $n \geq k$, the birational transformation of $\overline{S}$ induced by w_n contracts each edge of ∂S onto some vertex of ∂S. Thus, if $B(p_{i_1})$ is some small neighborhood of p_{i_1} in $\overline{S}$, there exists a neighborhood V of $\partial S \setminus B(p_{i_1})$ in $\overline{S}$ such that $w_n(V)$ is contained in $W_1 \cup W_2 \cup W_3$ for all $n \geq k$. This neighborhood depends on $B(p_{i_1})$ and on k. Together with Proposition 4.6, this proves the following corollary.

Corollary 4.8. *Let $w_n = s_{i_{\ell(n)}} \cdots s_{i_1}$ be an increasing sequence of reduced words in s_x, s_y, s_z. Suppose that w_n involves each of these three letters if n is larger than some n_0. Let B be a neighborhood of the indeterminacy point p_{i_1}. There exists a neighborhood V of $\partial S \setminus B$ in $\overline{S}$ such that, for $n \geq n_0$,*

(1) w_n contracts $\partial S \setminus \{p_{i_1}\}$ onto one of the vertices of ∂S,
(2) w_n maps V into $W_1 \cup W_2 \cup W_3$,
(3) if q is any point of $V \setminus \partial S$, its orbit $(w_n(q))$ converges towards ∂S in $\overline{S}$.

Moreover, if each of the three letters s_x, s_y, s_z appears infinitely often in the sequence $(s_{i_j})_{j \geq 1}$, the set of accumulation points of $(w_n(q))$ is contained in $\{p_1, p_2, p_3\}$.

Remark 4.9. Once we know that $(w_n(q))$ is trapped in $W_1 \cup W_2 \cup W_3$, we get the following: *if $s_{i_{\ell(n)}}$ is equal to s_x, then $w_n(q)$ is in W_1; thus, if each of the involutions appears infinitely often in the sequence of letters $s_{i_{\ell(n)}}$, the accumulation point of $(w_n(q))$ coincides with $\{p_1, p_2, p_3\}$.*

5. From random paths to reduced words

5.1. **Cayley graph (see** [24]**).** Recall that the group $\Gamma = \langle s_x, s_y, s_z \rangle$ is a free product $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$. Let $\mathcal{G}_\Gamma$ be the Cayley graph of Γ for the system of generators (s_x, s_y, s_z): two vertices g, $h \in \Gamma$ are connected by an edge labelled s, for $s \in \{s_x, s_y, s_z\}$, if and only if $gs = h$. This is a trivalent tree. The points of the boundary $\partial \mathcal{G}_\Gamma$ are in one to one correspondance with infinite geodesic rays starting at the neutral element 1_Γ.

Let μ be a probability measure on $\{s_x, s_y, s_z\}$ such that $\mu(s_x)\mu(s_y)\mu(s_z) > 0$. We endow the set
$$\Omega = \{s_x, s_y, s_z\}^{\mathbb{N}} \tag{5.1}$$
with the probability measure $\mu^{\mathbb{N}}$.

To an element $\omega = (f_0, f_1, \ldots)$ of Ω, we associate a path in $\mathcal{G}_\Gamma$: the path starts at 1_Γ and visits successively the vertices f_0, $f_0 f_1$, $\ldots$, $f_0 \cdots f_n$, $\ldots$. This path is typically not a geodesic (almost surely, some of the words $f_0 \cdots f_n$ are not reduced). On the other hand, with probability 1 with respect to $\mu^{\mathbb{N}}$, this path goes to infinity in $\mathcal{G}_\Gamma$ (at a linear speed), and converges towards a unique point $\theta^+(\omega)$ of $\partial \mathcal{G}_\Gamma$. Starting at 1_Γ and following the geodesic ray corresponding to $\theta^+(\omega)$, one creates a sequence of vertices (w_n) with $\mathrm{dist}(w_n, 1_\Gamma) = n$. It is an increasing sequence of reduced words $w_n = s_{i_1} \cdots s_{i_n}$. We shall say that (w_n) is the **reduced sequence attached to** ω.

5.2. **Random dynamics at infinity.** Let us now come back to the dynamics of Γ near ∂S in $\overline{S}$, for $S \in \mathsf{Fam}$. As in Section 5.1, we consider a typical element ω of Ω with respect to $\mu^{\mathbb{N}}$; then, the reduced words

$$w_n = s_{i_1} \cdots s_{i_n} \tag{5.2}$$

converge. The inverse of w_n is

$$w_n^{-1} = s_{i_n} \cdots s_{i_1}. \tag{5.3}$$

We can extract a subsequence (n_j) to assure that $(w_{n_j}^{-1})$ also converges to some infinite reduced word w_∞^{-1}. To do that, consider a subsequence such that the first letter of w_n^{-1} is constant, then extract from it a subsequence such that the second letter is also constant, and so on. Finally, apply a diagonal process.

We shall now apply Corollary 4.8 to the sequence (w_{n_j}) (note that the indices are indexed from left to right for w_n in Equation (5.2)). We denote by m an element from the sequence n_j. Write

$$w_\infty^{-1} = s_{j_1} s_{j_2} \cdots s_{j_n} \cdots \tag{5.4}$$

Note that we use indices i_k for w_n and j_k for w_n^{-1}, and that the reduced word w_∞^{-1} provides a boundary point (and a geometric ray), denoted $\theta^-(\omega, (n_j))$.

Let m_0 be the first integer such that $s_{j_1} s_{j_2} \cdots s_{j_{m_0}}$ involves the three letters s_x, s_y, and s_z. Then, $w_m = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_k} \cdots s_{j_{m_0}} s_{j_{m_0 - 1}} \cdots s_{j_2} s_{j_1}$ for $m \geq m_0$. Recall that p_{j_1} denotes the indeterminacy point of s_{j_1}, viewed as a birational transformation of $\overline{S}$. Let B be a neighborhood of p_{j_1}. Then, Corollary 4.8 and Remark 4.9 provide a neighborhood V of $\partial S \setminus B$ in $\overline{S}$ (which depends on B and m_0) such that, for $m \geq m_0$ in the sequence (n_j),

(1) w_m contracts $\partial S \setminus \{p\}$ onto the vertex p_{i_1},
(2) w_m maps V into W_{i_1},
(3) if q is any point of $V \setminus \partial S$, its orbit $(w_{n_j}(q))$ converges towards p_{i_1} in $\overline{S}$.

Remark 5.1. Changing (n_j), the point p_{j_1} can be taken to be any of the vertices of ∂S. On the other hand, p_{i_1} is uniquely determined by ω.

Theorem 5.2. *There is a subset Ω' of Ω of full $\mu^{\mathbb{N}}$-measure such that for any $\omega = (f_0, \ldots, f_n, \ldots)$ in Ω', there is a vertex $p(\omega)$ of ∂S with the following property. Given any vertex q of ∂S, one can find a subsequence (n_j) such that, for every neighborhood B of q, there is a neighborhood V of $\partial S \setminus B$ in $\overline{S}$ that satisfies:*

(1) $f_0 \circ \cdots \circ f_{n_j}$ contracts $\partial S \setminus \{q\}$ onto $p(\omega)$;

(2) for all $v \in V$, the sequence $f_0 \circ \cdots \circ f_{n_j}(v)$ converges towards $p(\omega)$.

Remark 5.3. The sequence (n_j) depends on ω and q, but not on B. We can also impose that w_∞^{-1} starts with three distinct letters. In that case, V depends only on (the size of) B.

Proof. As above, we consider the point $\theta^+(\omega) \in \partial \mathcal{G}_\Gamma$ and the parametrization (w_n) of the corresponding geodesic ray. The sequence $(f_0, f_1, \ldots)$ is typically not reduced (consecutive letters can be equal), but we can extract a subsequence (k_j) to impose that (a) each of the products $f_0 \cdots f_{k_j} \in \Gamma$ is equal to one of the reduced words w_{m_j} and (b) $f_0 \cdots f_n \neq w_{m_j}$ for all $n > k_j$. Then, we extract a further subsequence (n_j) from (m_j) (or equivalently from (k_j)) such that $w_{n_j}^{-1}$ converges to an infinite reduced word that starts with the letter s_i, where i is the index such that $q = p_i = \mathrm{Ind}(s_i)$. This done, the conclusion follows from our previous discussion. □

To conclude this paragraph, we add a definition that will be used in Section 6.4. As above, starting with a typical sequence $\omega \in \Omega$, we associate a boundary point $\theta^+(\omega)$ and a geodesic ray which starts at 1_Γ. The first vertex that this ray visits corresponds to the first letter s_{i_1} of w_n (see Equation 5.2). We shall say that s_{i_1} is the **initial letter** of $\theta^+(\omega)$; it will be denoted by $\mathrm{In}(\theta^+(\omega))$.

6. Stationary measures

In this section, we prove our Main Theorem (see Section 1.4). We fix a probability measure μ on $\{s_x, s_y, s_z\}$ such that $\mu(s_x)\mu(s_y)\mu(s_z) > 0$ and, as in Section 5.1, we endow the set $\Omega = \{s_x, s_y, s_z\}^{\mathbb{N}}$ with the probability measure $\mu^{\mathbb{N}}$. For $\omega = (f_0, f_1, \ldots) \in \Omega$ and $n \in \mathbb{N}$, we set

$$f_\omega^n = f_{n-1} \circ \ldots \circ f_0 \in \mathsf{Aut}(S). \tag{6.1}$$

6.1. **Stationary probability measures.** Let ν be a μ-stationary probability measure on $S(\mathbb{C})$; this means that $\nu = \mu(s_x)(s_x)_*\nu + \mu(s_y)(s_y)_*\nu + \mu(s_z)(s_z)_*\nu$ (see Equation (1.18)). Let ω be a typical element of Ω. In Sections 5.1 and 5.2, we introduced a boundary point $\theta^+(\omega) \in \partial \mathcal{G}_\Gamma$ and, for appropriate subsequences, boundary points $\theta^-(\omega, (n_j)) \in \partial \mathcal{G}_\Gamma$. We shall use these constructions and Theorem 5.2 to describe the support of ν and the accumulation points of stable manifolds at infinity.

6.2. **Recurrence implies compact support.** Let us start with an example. Endow $\mathsf{GL}_2(\mathbb{Z})$ with a probability measure μ, the support of which is finite and generates $\mathsf{GL}_2(\mathbb{Z})$. The group $\mathsf{GL}_2(\mathbb{Z})$ acts linearly on $\mathbb{R}^2/\mathbb{Z}^2$, fixes the origin $o = (0,0)$, and preserves the Lebesgue measure $\mathrm{dvol} = \mathrm{d}x \wedge \mathrm{d}y$. Moreover, by [9], if $q \notin \mathbb{Q}^2/\mathbb{Z}^2$, then $\mu^{\mathbb{N}}$-almost every random trajectory $(f_\omega^n(q))$ equidistributes towards dvol. Now, let us restrict the action to the punctured torus $X = \mathbb{R}^2/\mathbb{Z}^2 \setminus \{o\}$. If one considers the puncture as a point "at infinity" in X, then (a) there is a stationary measure dvol with unbounded support, (b) $(\mu^{\mathbb{N}} \otimes \mathrm{dvol})$-almost every random trajectory $(f_\omega^n(q))$ is unbounded, and (c) every bounded orbit of $\mathsf{GL}_2(\mathbb{Z})$ is finite. The following proposition excludes this type of behavior for the dynamics of Γ on $S(\mathbb{C})$, for any $S \in \mathsf{Fam}$.

Proposition 6.1. *Let ν be a μ-stationary measure on $S(\mathbb{C})$. Then ν has compact support.*

Remark 6.2. If $\mathbb{C}$ is replaced by a local field $\mathbb{K}$, then the same proposition holds for μ-stationary measures on $S(\mathbb{K})$ (see Remark 4.1).

Proof. **Step 1.–** By classical results due to Furstenberg and Guivarc'h-Raugi, see [7, Lemma 2.1, p.19], there exists a borel subset $\Omega' \subset \Omega$ such that (a) $\mu^{\mathbb{N}}(\Omega') = 1$, (b) for every $\omega \in \Omega'$, the sequence $(f_0 \circ \ldots \circ f_{n-1})_*\nu$ converges towards a probability measure ν_ω on $S(\mathbb{C})$, and (c) the family of measures $(\nu_\omega)_{\omega\in\Omega'}$ satisfies

$$\nu = \int_\Omega \nu_\omega \, d\mu^{\mathbb{N}}(\omega). \tag{6.2}$$

Step 2.– Now we follow the argument used by Bougerol and Picard to prove [8, Lemma 3.3]. For every $\omega \in \Omega'$ and every increasing sequence of integers (n_j), set

$$H(\omega,(n_j)) = \{x \in S(\mathbb{C}) \; ; \; f_0 \circ \ldots \circ f_{n_j}(x) \text{ does not tend to } \partial S\}. \tag{6.3}$$

Let $\varphi : S(\mathbb{C}) \to \mathbb{R}^+$ be a smooth function with compact support. We have

$$\lim_{j\to+\infty} \int_{S(\mathbb{C})} \varphi(f_0 \circ \ldots \circ f_{n_j})\, d\nu = \int_{S(\mathbb{C})} \varphi \, d\nu_\omega. \tag{6.4}$$

Since φ has compact support, we get $\varphi(f_0 \circ \ldots \circ f_{n_j}(x)) = 0$ for every $x \in S(\mathbb{C}) \setminus H(\omega,(n_j))$ and j large enough (depending on x). By the dominated convergence theorem, we get

$$\lim_{j\to+\infty} \int_{S(\mathbb{C})\setminus H(\omega,(n_j))} \varphi(f_0 \circ \ldots \circ f_{n_j})\, d\nu = 0. \tag{6.5}$$

Equation (6.4) then implies

$$\lim_{j\to+\infty}\int_{H(\omega,(n_j))}\varphi(f_0\circ\ldots\circ f_{n_j})\,d\nu=\int_{S(\mathbb{C})}\varphi\,d\nu_\omega, \tag{6.6}$$

which yields

$$\int_{S(\mathbb{C})}\varphi\,d\nu_\omega\leq\parallel\varphi\parallel_\infty\nu(H(\omega,(n_j))). \tag{6.7}$$

Let $1_M:\mathbb{C}^3\to[0,1]$ be a smooth function equal to 1 on $B(0,M)$ and equal to zero on $\mathbb{C}^3\setminus B(0,M+1)$. Replacing φ in Equation (6.7) by the restriction of 1_M to $S(\mathbb{C})$ and taking the limit when M tends to infinity, we obtain

$$\nu(H(\omega,(n_j))=1$$

for every $\omega\in\Omega'$ and every subsequence. In particular, the support of ν is contained in the closure of any such $H(\omega,(n_j))$.

Step 3.– Fix $\omega\in\Omega'$ that satisfies also the conclusion of Theorem 5.2 and apply this theorem for two vertices $q\neq q'$ of ∂S. This provides two sequences (n_j) and (n'_j) satisfying the following properties. For every neighborhood B of q (respectively B' of q'), there exists a neighborhood V of $\partial S\setminus B$ (respectively V' of $\partial S\setminus B'$) such that for every $x\in V$ (respectively $x\in V'$), $f_0\circ\ldots\circ f_{n_j}(x)$ (respectively $f_0\circ\ldots\circ f_{n'_j}(x)$) tends to p when j tends to infinity. Since $p\in\partial S$, we get $\overline{H(\omega,(n_j))}\cap\partial S\subset B$ and $\overline{H(\omega,(n'_j))}\cap\partial S\subset B'$; thus,

$$\overline{H(\omega,(n_j))}\cap\partial S\subset\{q\}\quad\text{and}\quad\overline{H(\omega,(n'_j))}\cap\partial S\subset\{q'\}. \tag{6.8}$$

Since $q\neq q'$, $\overline{H(\omega,(n_j))}\cap\overline{H(\omega,(n'_j))}$ is a compact subset of $S(\mathbb{C})$, and by Step 2, this implies that the support of ν is compact. $\square$

6.3. **Invariant probability measures.** We can now strengthen Corollary 3.3.

Corollary 6.3. *Let ν be a Γ_S-invariant, ergodic, probability measure on $S(\mathbb{C})$. Then either ν is the counting measure on a finite orbit of Γ_S, or the parameters a, b, c, and d are in $[-2,2]$, $S(\mathbb{R})$ has a unique compact component and ν is the symplectic measure $\nu_\mathbb{R}$ on $S(\mathbb{R})_c$.*

Indeed, if ν is invariant it is μ-stationary, hence its support is compact, and the conclusion follows from Corollary 3.3.

6.4. **Stable manifolds.** Now that we know that the support of any stationary measure ν is compact, we go on to conclude the proof of our Main Theorem.

Let σ denote the one sided left shift on Ω. We introduce the skew product $F\colon \Omega\times S(\mathbb{C})\to\Omega\times S(\mathbb{C})$, defined by

$$F(\omega,q)=(\sigma(\omega),f_0(q)) \tag{6.9}$$

for $\omega=(f_i)_{i\geq 0}\in\Omega$ and $q\in S(\mathbb{C})$. The μ-stationarity of ν is equivalent to the F-invariance of the product measure $\mu^{\mathbb{N}}\times\nu$ on $\Omega\times S(\mathbb{C})$.

Let us assume that ν is ergodic, and that its support $\mathrm{Supp}(\nu)$ is infinite. According to Proposition 6.1 and Theorem 3.2, S is defined over $\mathbb{R}$, $S(\mathbb{R})$ has a (unique) bounded component, and

$$\mathrm{Supp}(\nu)=S(\mathbb{R})_c. \tag{6.10}$$

Since $\mathrm{Supp}(\nu)$ is compact and $\mathrm{Supp}(\mu)$ is finite, we can apply Kingman's sub-additive ergodic theorem. This gives two real numbers $\lambda^+\geq\lambda^-$ such that

$$\lambda^+=\lim_{n\to+\infty}\frac{1}{n}\log\parallel D_qf_\omega^n\parallel \quad\text{and}\quad \lambda^-=\lim_{n\to+\infty}\frac{1}{n}\log\parallel (D_qf_\omega^n)^{-1}\parallel^{-1} \tag{6.11}$$

for ν-almost every $q\in S(\mathbb{R})_c$ and $\mu^{\mathbb{N}}$-almost every $\omega\in\Omega$. These real numbers are the **Lyapunov exponents** of ν; they satisfy

$$\lambda^++\lambda^-=0 \tag{6.12}$$

because the smooth part of $S(\mathbb{C})$ is endowed with the 2-form Area and $f^*\mathrm{Area}=\pm\mathrm{Area}$ for every $f\in\Gamma_S$ (see Section 1.3).

Suppose that $\lambda^+=\lambda^-=0$. By Ledrappier's invariance principle (see [17], and [21, 1] for other contexts), ν is invariant under the action of μ-almost every element of Γ_S, hence by Γ_S since the support of μ generates Γ_S. Thus, our Main Theorem follows from Corollary 3.3 in that case.

From now on, we assume that ν is **hyperbolic**, that is

$$\lambda^+>0>\lambda^-. \tag{6.13}$$

For ν-almost every $q\in S(\mathbb{C})$, we define the stable manifold

$$W^s_\omega(q):=\{q'\in S(\mathbb{C})\ ;\ \limsup_{n\to+\infty}\frac{1}{n}\log\mathsf{dist}(f^n_\omega(q),f^n_\omega(q'))<0\}. \tag{6.14}$$

Remark 6.4. By [14, Proposition 7.8], $W^s_\omega(q)$ is parametrized by an injective entire curve $\psi^s_{\omega,q}:\mathbb{C}\to S(\mathbb{C})$. Let $\overline{W}^s_\omega(q)$ be the closure of $W^s_\omega(q)$ in $\overline{S}$. By Liouville's theorem, $\overline{W}^s_\omega(q)\cap\partial S$ is not empty.

We also define

$$K(\omega) = \{q \in S(\mathbb{C}) \; ; \; (f_\omega^n(q))_{n\geq 0} \text{ is bounded in } S(\mathbb{C})\}. \tag{6.15}$$

Observe that K differs from the subset H given in Section 6.2, since the composition of the f_i's is reversed. We let $\overline{K}(\omega)$ be the closure of $K(\omega)$ in $\overline{S}$.

Lemma 6.5. *For ν-almost every q and $\mu^{\mathbb{N}}$-almost every ω, the stable manifold $W^s_\omega(q)$ is contained in $K(\omega)$. In particular, $\overline{W}^s_\omega(q) \cap \partial S \subset \overline{K}(\omega) \cap \partial S$.*

Proof. The orbit $\mathrm{Orb}_\omega(q) = \{f_\omega^n(q) \; ; \; n \geq 0\}$ is contained in the compact set $\mathrm{Supp}(\nu)$. Hence $\mathrm{Orb}_\omega(y)$ is bounded for every $y \in W^s_\omega(q)$. □

Proposition 6.6. *For $(\mu^{\mathbb{N}} \otimes \nu)$-almost every (ω, q),*

(1) the sets $\overline{K}(\omega) \cap \partial S$ and $\overline{W}^s_\omega(q) \cap \partial S$ are equal to the indeterminacy point of the initial letter $\mathrm{In}(\theta^+(\omega))$;
(2) $K(\omega)$ depends on ω: it is equal to one of the three vertices of ∂S, each of them being realized with positive probability.

Proof. Write the reduced words associated to $\omega = (f_0, \ldots, f_n, \ldots)$ as $w_n = s_{i_1} \cdots s_{i_n}$, as done in Section 5.1. Choose an increasing sequence (n_j) such that the unreduced word $f_0 \circ \cdots \circ f_{n_j}$ is equal, in the group Γ, to one of the reduced words w_{m_j} and $f_0 \circ \cdots \circ f_n \neq w_{m_j}$ for all $n > n_j$. Let B be a small neighborhood of the point $p = p_{i_1} = \mathrm{Ind}(s_{i_1})$, and apply Corollary 4.8: there is a neighborhood V of $\partial S \setminus B$ such that the orbit

$$f_\omega^{n_j}(q) = (s_{i_{m_j}} \circ \cdots \circ s_{i_2} \circ s_{i_1})(q) \tag{6.16}$$

of every point $q \in V$ goes to ∞ in S as n_j goes to $+\infty$. Thus, $\overline{K}(\omega) \cap \partial S$ is contained in B. Shrinking B, $\overline{K}(\omega) \cap \partial S \subset \{p_{i_1}\}$. The first assertion follows from Remark 6.4 and Lemma 6.5. The second assertion is a consequence of the first one and Remark 5.1. □

The stable direction $E^s_\omega(q)$ is the tangent to $W^s_\omega(q)$ at the point q. We shall say that the stable directions define an invariant line field if they do not depend on ω (at least over some subset of total measure in Ω). Otherwise, we say that they **genuinely depend** on ω. In fact, there are two versions of $E^s_\omega(q)$, the complex one in $T_qS(\mathbb{C}) \simeq \mathbb{C}^2$, and the real one in $T_qS(\mathbb{R})_c \simeq \mathbb{R}^2$; since the complex one is the complexification of the real one, this definition does not depend on the version one chooses.

Proposition 6.7. *If the ergodic, hyperbolic, stationary measure ν is not invariant, the stable directions $E^s_\omega(q)$ depend genuinely on the random trajectory $\omega = (f_0, \ldots, f_n, \ldots)$.*

Indeed, we only need to apply the proof of Theorem 9.1 in [14]. This theorem assumes that the dynamical system is defined on a compact Kähler surface for only two reasons. Firstly, to use Hodge theory and Nevanlinna currents in order to show that $W^s_\omega(q)$ depends genuinely on ω; in our context, this follows from Proposition 6.6. Secondly, the compactness of the manifold is used for certain moment conditions to be satisfied; here, the same conditions are available because, according to Proposition 6.1, the study can be done on $S(\mathbb{R})_c$.

6.5. **Classification of stationary measures: proof of the Main Theorem.** Let S be an element of Fam. Let ν be an ergodic, μ-stationary, probability measure on $S(\mathbb{C})$. As explained in Section 6.4, we can assume that ν is not invariant, since otherwise our Main Theorem follows from Proposition 6.1 and Corollary 3.3. Under this assumption, the support of ν is infinite, hence equal to $S(\mathbb{R})_c$, as above. And by Ledrappier's invariance principle we can assume that ν is hyperbolic. Thus, we can restrict the dynamics, i.e. Γ_S and ν, to $S(\mathbb{R})_c$ and apply the following theorem of Brown and Rodriguez-Hertz:

Theorem 6.8 (Theorem 3.4 of [10])**.** *Let M be a closed surface, endowed with a probability measure* dvol *given by a smooth area form. Let μ be a probability measure on* $\mathsf{Diff}^2(M;\mathrm{dvol})$ *with finite support; let Γ_μ be the subgroup of* $\mathsf{Diff}^2(M;\mathrm{dvol})$ *generated by the support of μ. Let ν be a hyperbolic, ergodic, and μ-stationary probability measure on M. If for ν-almost every $q \in M$ the stable direction $E^s_\omega(q)$ depends genuinely on ω, then ν is invariant by Γ_μ.*

By Proposition 6.7, the stable directions $E^s_\omega(q)$ depend genuinely on ω for ν-almost every q. By this Theorem, ν should be invariant, contradiction.

Remark 6.9. The surface $S(\mathbb{R})_c$, in our case, may have singularities. But they are quotient singularities and the group Γ_S preserves the orbifold structure of $S(\mathbb{R})$, thus [10] can be applied without any change, even if $S(\mathbb{R})$ is singular.

7. Margulis functions and applications: an example

In this final section, we study the example from Section 3.1.3, the goal being to describe further properties satisfied by the dynamics of Γ.

7.1. **The surface.** The parameters are $(A,B,C,D) = (1,1,1,0)$. We write S for $S_{(1,1,1,0)}$ and Γ for Γ_S. The surface S is smooth and the compact component $S(\mathbb{R})_c$ is homeomorphic to a sphere. According to [22], every orbit of Γ in $S(\mathbb{C})$ is infinite, except the orbit of the origin $o = (0,0,0)$. Thus, our main theorem shows that the space of stationary measures is an interval, the endpoints of which are

- the symplectic measure $\nu_{\mathbb{R}}$, supported by $S(\mathbb{R})_c$, and
- $\delta_{\Gamma(o)}$, the counting measure on the finite orbit.

We will be interested in the following problem. Fix a point q of $S(\mathbb{C}) \setminus \Gamma(o)$. Given $\omega = (f_0, \dots, f_n, \dots) \in \Omega$, consider the empirical measures $\nu_N(\omega; q)$ defined in Equation (1.19). As already said in Section 1.4, for a typical ω any cluster value ν of $(\nu_N(\omega; q))$ in the space of measures on $S(\mathbb{C})$ is a stationary measure (though its total mass may be < 1). The question is to determine the decomposition of such a measure ν as a convex combination $\alpha\delta_{\Gamma(o)} + \beta\nu_{\mathbb{R}}$.

7.2. **Expansion along $\Gamma(o)$.** The stabilizer of o is a subgroup of Γ of index 7 that contains $f = (s_y \circ s_x)^2$, $g = (s_x \circ s_z)^2$, $h = (s_z \circ s_y)^2$. The tangent space at the origin is given by the equation $u + v + w$ and can be parametrized by $(u, v, -u - v)$. In these coordinates, the differentials of f, g, and h at the origin act on T_oS by

$$Df_o(u,v) = (2u + v, -u), \tag{7.1}$$
$$Dg_o(u,v) = (u + v, v), \tag{7.2}$$
$$Dh_o(u,v) = (u, -u + v). \tag{7.3}$$

Thus, Df_o, Dg_o, and Dh_o generate the group $\mathsf{SL}_2(\mathbb{Z}) \subset \mathsf{GL}(T_oS)$.

According to [12, Theorem 8.16], this implies that the finite orbit $\Gamma(o)$ is expanding, in the sense of [12, Section 1.3] and [16]; in other words, in average, the dynamics of Γ is infinitesimally repelling along the orbit $\Gamma(o)$. From [12, Theorem 4.3][1]), we get:

if q is a point of $S(\mathbb{R})_c \setminus \Gamma(o)$, then for a typical ω in Ω, the empirical measures $\nu_N(\omega; q)$ converge towards the symplectic measure $\nu_{\mathbb{R}}$.

In other words, as soon as $q \in S(\mathbb{R})_c$ is not on the finite orbit, then its typical random trajectories do not charge $\Gamma(o)$.

[1]In this statement and in Theorem 4.4 of [12], the ambient complex manifold is assumed to be compact, but this is not really used in the proof.

This result is an instance of the following general phenomenon.

Proposition 7.1. *Assume S is defined over $\mathbb{R}$ and contains a compact component $S(\mathbb{R})_c$. Let q be a smooth point of $S(\mathbb{R})_c$. If the orbit $\Gamma(q)$ is finite, then it is automatically expanding.*

Proof. Let Γ_q be the stabilizer of q in Γ, and let k be its index. We have to prove that the subgroup of $\mathsf{GL}\,(T_qS)$ given by the differentials D_qg, for $g \in \Gamma_q$, is non-elementary. Permuting the role of the three coordinates, we can assume that the fibers of the projections $\pi_1(x,y,z) = x$ and $\pi_2(x,y,z) = y$ containing q are smooth and are transverse at q.

Now, consider the element $h = s_y \circ s_z$ of Γ. It acts on S by $(x,y,z) \mapsto (x, H_x(y,z))$ where H_x is the affine transformation of the (y,z)-plane defined by $H_x(y,z) = L_x(y,z) + T_x$ with

$$L_x = \begin{pmatrix} x^2-1 & -x \\ x & -1 \end{pmatrix} \quad \text{and} \quad T_x = \begin{pmatrix} -B-Cx \\ -C \end{pmatrix}. \tag{7.4}$$

The trace of L_x is x^2-2, its determinant is 1. This affine map H_x preserves the conic C_x given by the equation of S, with x fixed, in the (y,z)-plane.

Now, write $q = (x_0, y_0, z_0)$, with x_0, y_0, z_0 in $[-2,2]$. Our hypotheses imply that $-2 < x_0,\ y_0 < +2$. Let $I \subset]-2,2[$ be an interval containing x_0 above which π_1 is a submersion. For $x \in I$, the real part $C_x(\mathbb{R})$ of the conic C_x is a smooth ellipse. It can be identified to $\mathbb{P}^1(\mathbb{R})$ by a homography; then, H_x induces a homography H'_x of $\mathbb{P}^1(\mathbb{R}) \simeq C_x(\mathbb{R})$ given by a matrix in $\mathsf{PSL}_2(\mathbb{R})$, the trace of which satisfies $\mathsf{Tr}(H'_x)^2 = \mathsf{Tr}(L_x) + 2 = x^2$. Writing $x = 2\cos(\theta)$, we can identify $\mathbb{P}^1(\mathbb{R})$ to the circle $\mathbb{R}/2\pi\mathbb{Z}$ and (H'_x) to the rotation of angle $\theta(x) = \arccos(x/2)$. This gives a local diffeomorphism to $I \times \mathbb{R}/2\pi\mathbb{Z}$ that conjugates h to the map $(x,\varphi) \mapsto (x, \varphi + \theta(x)) = (x, \varphi + \arccos(x/2))$. In these coordinates, the differential of the n-th iterate is

$$\begin{pmatrix} 1 & -n/\sqrt{4-x^2} \\ 0 & 1 \end{pmatrix}. \tag{7.5}$$

It is a non-trivial unipotent matrix if $n \neq 0$. Coming back to S, we obtain the following: the differential of h^k at the fixed point q is a unipotent matrix $D_qh^k \neq \mathrm{Id}$ in $\mathsf{GL}\,(T_qS)$ that fixes the direction tangent to the fiber of π_1.

Now, the same is true for $g = s_z \circ s_x$ with respect to the fibration π_2. Since the two fibrations π_1 and π_2 are transverse at q, the differentials of g^k and h^k at q generate a non-elementary subgroup of $\mathsf{GL}\,(T_qS)$, as desired. $\square$

7.3. **Expansion along $S(\mathbb{R})_c$.** The transformation f from the previous paragraph act on S in the following way. It preserves the fibration $\pi_3\colon S(\mathbb{C}) \to \mathbb{C}$, $\pi_3(x,y,z) = z$. It acts by homography on each fiber $x^2 + y^2 + z_0^2 + xyz_0 = x + y + z_0$. Along $S(\mathbb{R})_c$, π_3 is a fibration in (topological) circles (except for the singular fibers) along which f acts as a rotation by an angle that depends analytically on z_0. The behavior of g and h is similar, but with respect to the other fibrations π_2 and π_1.

Thus, the situation is slightly different from [12], in which parabolic automorphisms preserve fibrations into curves of genus 1, but the same arguments apply. From [12, Theorem 1.5], we deduce that the dynamics is expanding along $S(\mathbb{R})_c$. Then, [12, Theorem 4.5] shows that, using the complex structure, the expansion can be transfered transversally to $S(\mathbb{R})_c$, and this shows that

if q is a point of $S(\mathbb{C}) \setminus S(\mathbb{R})_c$ and ω is typical, then the only cluster value of $(\nu_N(\omega; q))$ is the zero measure: in average, the orbit of q goes to ∂S.

This argument also shows the following.

Suppose that $S \in \mathsf{Fam}$ is defined over $\mathbb{R}$, $S(\mathbb{R})$ has a smooth compact component $S(\mathbb{R})_c$, and $S(\mathbb{C})$ does not contain any finite orbit. Let q be a point from $S(\mathbb{C}) \setminus S(\mathbb{R})$. Then, for a typical random sequence ω, the only cluster value of $(\nu_N(\omega; q))$ is the zero measure.

8. Appendix

We extend the study of Section 5 to relate it to hyperbolic geometry and Furstenberg theory and to improve Proposition 6.1.

8.1. **Isometries of $\mathbb{H}$.** The group $\Gamma \simeq \Gamma_2^{\pm}$ can also be viewed as a group of isometries of the upper half plane $\mathbb{H}$. Explicitly, the generators s_x, s_y, s_z are mapped to the involutive isometries (see Section 1.2)

$$\sigma_x(z) = -\bar{z} + 2, \ \ \sigma_y(z) = \frac{\bar{z}}{2\bar{z} - 1}, \ \ \sigma_z(z) = -\bar{z}. \tag{8.1}$$

Each involution is the reflection with respect to one side of the ideal triangle with vertices 0, 1, ∞ in $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. This triangle T is a fundamental domain of the action of $\Gamma_2^{\pm}$, its images tesselate $\mathbb{H}$, and the dual graph of this tesselation can be identified to $\mathcal{G}_\Gamma$.

Remark 8.1. The Cayley map $(z - \mathsf{i})/(z + \mathsf{i})$ maps the upper half plane to the unit disk $\mathbb{D}$ and the triangle to the ideal triangle $\mathsf{T}_{\mathbb{D}} \subset \mathbb{D}$ with vertices 1, -1 and $-i$.

As in Section 5.1, let ω be a typical element of Ω and let (w_n) be the sequence of reduced words derived from ω, with $w_n = s_{i_1} \cdots s_{i_n}$. The isometries $\sigma_{i_1} \circ \cdots \circ \sigma_{i_n}$ map

T to a sequence of adjacent triangles

$$\mathsf{T}_0 = \mathsf{T},\ \ \mathsf{T}_1 = \sigma_{i_1}(T),\ \ \mathsf{T}_2 = \sigma_{i_1}(\sigma_{i_2}(T)),\ \ \ldots. \tag{8.2}$$

the dual Then, the sequence of triangles (T_n) converges in the Hausdorff topology of $\overline{\mathbb{H}}$ towards a unique boundary point $\theta^{+}_{\mathbb{H}}(\omega)$.

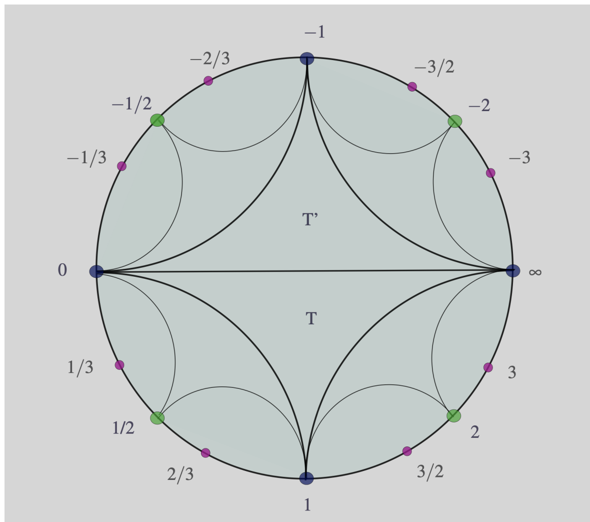

FIGURE 3. We draw a picture in the unit disk, using the change of variable $z \mapsto (z-\mathrm{i})/(z+\mathrm{i})$, but we label boundary points according to the natural parametrization $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. The triangle T and its images under the action of $\Gamma_2^{\pm}$ tesselate the hyperbolic disk. For instance, the triangle T' corresponds to $\sigma_z(\mathsf{T})$, and the magenta points correspond to curves of depth 2 in $\overline{S}_2$.

8.2. **Matrices and Furstenberg theory (see** [7]**).** Fix a norm $\|\cdot\|$ on $\mathsf{Mat}_2(\mathbb{R})$, and we recall that the boundary $\partial\mathbb{H}$ is naturally identified to the projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{P}(\mathbb{R}^2)$. The isometries σ_x, σ_y, σ_z correspond to elements $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ of $\mathsf{PGL}_2(\mathbb{R})$ (see Equation (1.11)). We lift these elements to matrices in $\mathsf{GL}_2(\mathbb{R})$, using the same notation. Typically, the norm of the product $\hat{\sigma}_{i_1}\cdots\hat{\sigma}_{i_n}$ goes exponentially fast to $+\infty$. If we normalize $\hat{\sigma}_{i_1}\cdots\hat{\sigma}_{i_n}$ by dividing by its norm, we obtain a sequence of matrices

$$[\hat{\sigma}_{i_1}\cdots\hat{\sigma}_{i_n}] = \frac{\hat{\sigma}_{i_1}\cdots\hat{\sigma}_{i_n}}{\|\hat{\sigma}_{i_1}\cdots\hat{\sigma}_{i_n}\|} \in \mathsf{Mat}_2(\mathbb{R}), \tag{8.3}$$

each of which has norm 1. Extracting a subsequence (n_j), we may assume that this sequence converges towards an element $\hat{\sigma}_\infty$ of $\mathsf{Mat}_2(\mathbb{R})$; then, with probability 1 with respect to $\mu^{\mathbb{N}}$, we have

(1) $\hat{\sigma}_\infty$ has rank 1 and its image coincides with the line corresponding to $\theta^{+}_{\mathbb{H}}(\omega)$ in the identification $\partial\mathbb{H} = \mathbb{P}^1(\mathbb{R})$; in particular, the image does not depend on the subsequence.

On the other hand the kernel does depend on (n_j). More precisely, consider the inverse of $s_{i_1}\cdots s_{i_n}$, which is the same thing as the reversed word $s_{i_n}\cdots s_{i_1}$. Then, as in Section 5.2, extract a subsequence such that each of $s_{i_{n_j}}$, $s_{i_{n_j-1}}$, etc, converges towards an element of $\{s_x, s_y, s_z\}$; in other words, extract a subsequence to make the path $s_{i_{n_j}}$, $s_{i_{n_j}}s_{i_{n_j}-1}$, etc, converge into $\mathcal{G}_\Gamma$ to a geodesic ray. This ray corresponds to a point of $\partial\mathcal{G}_\Gamma$, to a sequence of adjacent triangles in $\mathbb{H}$, and to a boundary point $\theta^-_{\mathbb{H}}(\omega,(n_j)) \in \partial H = \mathbb{P}^1(\mathbb{R})$. With such a choice,

(2) the kernel of $\hat{\sigma}_\infty$ is the line in $\mathbb{R}^2$ corresponding to the point

$$\theta^-_{\mathbb{H}}(\omega,(n_j)) \in \mathbb{P}^1(\mathbb{R}).$$

For a typical ω, each point of $\mathbb{P}^1(\mathbb{R})$ is equal to $\theta^-_{\mathbb{H}}(\omega,(n_j))$ for some subsequence (n_j). This follows from two facts: (a) the limit set $\mathrm{Lim}(\Gamma) \subset \partial\mathbb{H}$ is equal to $\mathbb{H}$ and (b) the support of the Furstenberg measure is equal to $\mathrm{Lim}(\Gamma)$.

Remark 8.2. Assume that (n_j) is chosen in such a way that $\theta^+_{\mathbb{H}}(\omega) \neq \theta^-_{\mathbb{H}}(\omega,(n_j))$ Let $I \subset \partial\mathbb{H}$ be an open neighborhood of $\theta^-_{\mathbb{H}}(\omega,(n_j))$. If I is small enough, its complement $I^c = \partial\mathbb{H}\setminus I$ is a compact neighborhood of $\theta^+_{\mathbb{H}}(\omega)$, and in the Hausdorff topology of $\partial\mathbb{H}$, the sequence $\sigma_{i_1}\circ\cdots\circ\sigma_{i_n}(I^c)$ converges towards $\theta^+_{\mathbb{H}}(\omega)$.

8.3. **Compactifications of S.**

8.3.1. Let m be a positive integer. Blow-up $\overline{S}$ at the vertices of ∂S to get a compactification by 6 rational curves organized in a hexagon, and repeat this process to get a compactification $\overline{S}_m$ of S by $N := 3\cdot 2^m$ rational curves organized in a cycle. We denote by ∂S_m its boundary.

8.3.2. The dual graph $\mathcal{C}_m$ of ∂S_m has one vertex per rational curve, and one edge between two vertices if the corresponding curves have a point in common. Topologically, $\mathcal{C}_m$ is a circle with N points on it. The graph $\mathcal{C}_m$ is obtained from $\mathcal{C}_{m-1}$ by adding a new vertex in the middle of each edge, i.e. by doing a barycentric subdivision. We say that a vertex of $\mathcal{C}_m$ has depth $k \leq m$ if it appears first in the k-th barycentric subdivision $\mathcal{C}_k$. The three lines of the triangle $\partial S_0 = \partial S$ correspond to vertices of depth 0. One way to parametrize $\mathcal{C}_m$ is the following. In $\partial\mathbb{H}$, consider the three end points of T and their images under the action of all reduced words in $\{\sigma_x,\sigma_y,\sigma_z\}$ of length $\leq m$; we obtain a circle $\partial\mathbb{H}$ with N marked points, hence a graph $\mathcal{H}_m$; the edges of $\mathcal{H}_m$ are intervals of $\partial\mathbb{H}$. There is an equivariant map sending $\partial\mathbb{H}$ to $\mathcal{C}_m$: it maps the vertices of T to the vertices of $\mathcal{C}_m$ of depth 0, then the (new) vertices in $\sigma_x(\mathsf{T})$, $\sigma_y(\mathsf{T})$, $\sigma_z(\mathsf{T})$, etc.

8.3.3. Now, consider an element f in $\Gamma \simeq \Gamma_2^{\pm}$. Suppose that the 3 vertices of $f^{-1}(\mathsf{T})$ are contained in the interior of an edge I of $\mathcal{H}_m$ and that the three vertices of $f^{-1}(\mathsf{T})$ are contained in the interior of an edge J. These edges correspond to edges I' and J' of $\mathcal{C}_m$, hence to vertices p and q of the cycle of rational curves ∂S_m. Then the birational map induced by f on $\overline{S}_m$ contracts $\partial S_m\setminus\{q\}$ onto p. Moreover, one can construct neighborhoods W_j of the vertices of ∂S_m satisfying properties which are analogous to the ones listed in Corollary 4.8 and then extend Theorem 5.2. We do not prove these properties. But they can be derived from [18] (or [11, Chapter 8]) and the computations done in Sections 4 and 5. Nguyen Bac Dang studies the dynamics of Γ on the space of valuations centered at infinity in S.

8.4. **Application.** One can now strengthen Proposition 6.1 as follows.

Theorem 8.3. *Let S_1, S_2, …, S_m be elements of* Fam. *If ν is a stationary measure for the diagonal action of Γ on $S_1(\mathbb{C})\times\cdots\times S_m(\mathbb{C})$, the support of ν is compact.*

Sketch of Proof. Set $M = S_1\times\cdots\times S_m$ and denote by $\overline{M}$ its closure in $(\mathbb{P}^3)^m$. The boundary $\partial M = \overline{M}\setminus M$ is the union of the sets

$$\partial_i M = \overline{S}_1\times\cdots\times\partial S_i\times\cdots\times\overline{S}_m. \tag{8.4}$$

Define $H(\omega,(n_i))$ as in the proof of Proposition 6.1, but for the diagonal dynamics of Γ on M.

First, assume $m=2$. The closure of $H(\omega,(n_i))$ in $\overline{M}$ intersects its boundary on a set that is contained in $\overline{S_1}\times\{q\}\cup\{q\}\times\overline{S_2}$; varying the choice of the subsequence (n_i), the point q can be chosen to be any of the vertices at infinity. Now, if $q\neq q'$,

$$(\overline{S_1}\times\{q\}\cup\{q\}\times\overline{S_2})\cap(\overline{S_1}\times\{q'\}\cup\{q'\}\times\overline{S_2}) = \{(q,q'),(q',q)\}. \tag{8.5}$$

Thus, the accumulation points of the support of ν in ∂M are contained in $\{(q,q'),(q',q)\}$ for any pair of vertices (q,q') at infinity. Since

$$\{(q,q'),(q',q)\}\cap\{(q,q''),(q'',q)\} = \emptyset \tag{8.6}$$

when q, q', q'' are pairwise distinct, this shows that the support of ν is compact.

Now, let us prove the result for any $m\geq 1$. For this we use Section 8.3. Blow-up each $\overline{S_i}$ at the vertices of ∂S_i to get a compactification by 6 rational curves organized in a hexagon, and repeat this process m times to get a compactification by $N := 3\cdot 2^m$ rational curves organized in a cycle; let q_i be the vertices of this cycle, with $1\leq i\leq N$. Then, choosing correctly (n_i), one sees that the accumulation points of $\mathrm{Supp}(\nu)$ in ∂M are contained in

$$B(I) = \{(p_1,\ldots,p_m)\ ;\ p_i\in\{q_{i_1},\ldots,q_{i_m}\}\quad\text{for each}\quad i\leq m\} \tag{8.7}$$

for each multi-index $I=(i_1,\ldots,i_m)\in\{1,\ldots,N\}^m$. The intersection of the $B(I)$ being empty, $\mathrm{Supp}(\nu)$ is compact. □

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