Sign of the monodromy for Liouville integrable systems

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Abstract[3]

In this note we show that the monodromy of a two degree of freedom integrable Hamiltonian system has a universal sign in the case of a focus-focus singularity. We also show how to extend the monodromy index to several focus-focus fibers when the integrable system has an S^1 symmetry.

1 Introduction

The Hamiltonian monodromy of integrable systems has a surprisingly recent history dating back to Duistermaat's 1980 article [8]. Its application to quantum spectra was suggested in 1988 [5]. But it was not before 1998 – with the rigorous quantum formulation [17] and several examples [3], [7], [14], [10] (and others) – that it became a common tool for the analysis of spectra of many mathematically and physically relevant models (eg. [19]).

(Quantum) Hamiltonian monodromy is usually used to demonstrate the non-existence of global action variables (or good quantum numbers). This can be detected by a sort of "point defect" in the lattice of joint eigenvalues. The goal of our note is to sharpen this analysis by showing that this point defect can be attributed a *sign*, and in the generic case this sign is always positive (theorem 1). Moreover, as a first step in the study of systems with several isolated singularities, in theorem 3 we show how to compute the global monodromy in case of an S^1 symmetry (ie. one global action). A consequence of this sign for general systems without S^1 symmetry is that the global monodromy can cancel only for systems with complicated topology (proposition 5).

We apply our results to a simple example with two points of monodromy: the *quadratic spherical pendulum*, for which we have also numerically computed the joint spectrum.

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2 General Setup

Let M be a 4-dimensional connected symplectic manifold with symplectic form ω, let B be a 2-dimensional manifold, and let $F : M \to B$ be a smooth proper surjective Lagrangian fibration with singularities which has connected fibers. We assume that the set of critical values c_i of F is discrete and that each critical point of F in $F^{-1}(c_i)$ is a focus-focus singularity.

Recall that a point $m \in M$ with $\mathrm{d}F(m) = 0$ is called a focus-focus singularity if there exist local canonical coordinates $(x, y, \xi, \eta) \in (T^*\mathbb{R}^2, \omega = d\xi \wedge dx + d\eta \wedge dy)$ near m and a local chart of B at $F(m)$ such that the vector space spanned by the Hessians $D^2F_1(m)$ and $D^2F_2(m)$ (where (F_1, F_2) are the components of F) is generated by the standard focus-focus quadratic forms (q_1, q_2):

$$q_1 = x\xi + y\eta \qquad q_2 = x\eta - y\xi.$$

Recall that any critical point of F of Morse-Bott type (="non-degenerate" in the sense of [9]) whose critical value is isolated in B is of focus-focus type.

We are mainly interested in the case where F comes from a Liouville integrable system. Here B is a connected subset of $\mathbb{R}^2$ and $F = (H_1, H_2)$, where H_i are Poisson commuting Hamiltonians. Typically, M is a connected open subset of a symplectic manifold $\widetilde{M}$ where F may have non focus-focus critical points, see [9].

3 Monodromy

Let $B_r = B \setminus \{c_i\}$ be the set of regular values of F and denote by F_r the restriction of F to $M_r = F^{-1}(B_r)$. Then F_r is a regular Lagrangian fibration over B_r with compact connected fibers. In a local chart of B_r the fibration $F_r = (H_1, H_2)$ is a Liouville integrable system. By the Arnol'd-Liouville theorem, the fibers of F are affine 2-torii on which the flows of the Hamiltonian vector fields $\mathcal{X}_{H_1}$ and $\mathcal{X}_{H_2}$ define a linear action of $\mathbb{T}^2$. The 2-torus bundle $F_r : M_r \to B_r$ obtained this way is locally trivial. In fact it is *locally* a principal 2-torus bundle. The obstruction for it to be globally a principal bundle is the monodromy μ. More precisely, monodromy is the holonomy of a $\mathbb{Z}^2$-bundle over B_r whose fiber is the lattice of 2π-periodic vector fields, which in a local chart on B_r about c are given by linear combinations of $\mathcal{X}_{H_1}$ and $\mathcal{X}_{H_2}$ whose flow on $F^{-1}(c)$ is 2π-periodic. For more details, see [8], [4, Appendix D]. Let $\mathcal{P} \to B_r$ be this bundle of period lattices. Then the monodromy $\mu \in \mathrm{Hom}(\pi_1(B_r), \mathrm{Aut}(\mathcal{P}))$.

Given a point $c \in B_r$, a period lattice $\mathcal{P}_c$ with basis $\{X_1, X_2\}$ and a loop γ in B_r passing through c, the monodromy $\mu_c(\gamma)$ is a matrix in $\mathrm{Gl}(2,\mathbb{Z})$, whose conjugacy class in $\mathrm{Gl}(2,\mathbb{Z})$ is invariant under a change of basis. If γ encircles a single critical value $\widetilde{c}$ of F_r, then there is a basis $\mathcal{B}$ such that the monodromy is the unipotent matrix

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \tag{1}$$

see [20], [6]. Here k is a nonzero integer called the *monodromy index* of γ relative to the basis $\mathcal{B}$. The absolute value $|k|$ is invariant under conjugation by elements of $\mathrm{Gl}(2,\mathbb{Z})$ and hence is independent of the choice of basis $\mathcal{B}$. We call $|k|$ the *absolute monodromy index*. In [2], [12] and [20] it was shown that this latter index is precisely the number of focus-focus critical points in $F^{-1}(\widetilde{c})$. Moreover, $F^{-1}(\widetilde{c})$ is homeomorphic to a $|k|$-pinched 2-torus.

4 Oriented monodromy

Suppose now that B_r is oriented, which is indeed the case when B_r is an open subset of $\mathbb{R}^2$. Then there is a induced orientation on the Liouville torii and hence on the bundle of period lattices $\mathcal{P}$. This induced orientation is determined as follows. Let $\{\alpha_1, \alpha_2\}$ be a positively oriented ordered basis of $T^*_c B_r$, which is dual to a positively oriented basis of $T_c B_r$. Then the ordered basis of tangent vectors to $F^{-1}(c)$ given by the set of vector fields $\{\omega^\sharp(F^*(\alpha_1)), \omega^\sharp(F^*(\alpha_2))\}$ is said to be positively oriented. In the case of our two degree of freedom Liouville integrable system, if we use the standard orientation on $\mathbb{R}^2$, then $\{\mathcal{X}_{H_1} \restriction_{F^{-1}(c)}, \mathcal{X}_{H_2} \restriction_{F^{-1}(c)}\}$ gives the induced positive orientation for the 2-torus $F^{-1}(c)$.

We define the *oriented monodromy index* of the oriented loop γ in B_r around the focus-focus critical value $\widetilde{c}$ to be the integer k in (1) when the basis chosen to compute it is positively oriented. The number k is invariant under conjugation by orientation preserving automorphisms. When referring to the *oriented monodromy index of a focus-focus critical value* $\widetilde{c}$ we assume that γ is positively oriented.

Remark *In this article we use the convention of* (1) *to write the monodromy matrix as a lower triangular matrix (instead of an upper triangular one), which amounts to a sign convention for k.*

Theorem 1 *The oriented monodromy index of a focus-focus critical value is positive and hence is equal to the number of focus-focus critical points in the critical fiber.*

Proof. Using Eliasson's theorem [9] one can find a chart near a focus-focus critical point (which corresponds to the critical value 0) so that $F = g(q_1, q_2)$, where g is a local diffeomorphism of $\mathbb{R}^2$, and $q_1 = x\xi + y\eta$, $q_2 = x\eta - y\xi$, where (x, ξ, y, η) are coordinates for $\mathbb{R}^4$ with symplectic form $\mathrm{d}x \wedge \mathrm{d}\xi + \mathrm{d}y \wedge \mathrm{d}\eta$. Using the symplectomorphism

$$(x, \xi, y, \eta) \to (-x, -\xi, y, \eta)$$

one may change the sign of q_2, if necessary, to ensure that the ordered basis $\{\mathcal{X}_{q_1}, \mathcal{X}_{q_2}\}$ is positively oriented. In other words, we can ensure that the local diffeomorphism g is orientation preserving, that is, $\det Dg(0) > 0$. Following [18] we can choose a point c near the critical value 0 and an ordered basis $\mathcal{B}$ of the form $\{\alpha\, \mathcal{X}_{q_1} + \beta\, \mathcal{X}_{q_2}, \mathcal{X}_{q_2}\}$, where $\alpha, \beta > 0$, for which the monodromy matrix is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Since $\mathcal{B}$ has the same orientation as the ordered basis $\{\mathcal{X}_{q_1}, \mathcal{X}_{q_2}\}$ and hence as the ordered basis $\{\mathcal{X}_{H_1}, \mathcal{X}_{H_2}\}$, we see that the monodromy index is positive. □

Note that theorem 1 is purely local, since a small enough neighborhood of a focus-focus critical value is always orientable.

Making no orientability assumptions on B_r, theorem 1 can be phrased as follows.

Theorem 1 (bis) *The monodromy index k of a loop in B_r around a single focus-focus critical value is positive if and only if the loop and the basis chosen to compute k have the same orientation.*

5 Parallel transport

The fibration $F_r : M_r \to B_r$ endows B_r with an integral affine structure, whose charts are the action coordinates. This affine structure induces a parallel transport on TB_r, whose holonomy is the contragredient of the holonomy of the 2-torus bundle $\mathcal{P} \to B_r$, that is, the monodromy. For more details see [1].

Suppose that $\widetilde{c}_1$ and $\widetilde{c}_2$ are two critical values of F that can be joined by a path $\Gamma : [0, 1] \to B$ such that $\Gamma : (0, 1) \to B_r$. Assume that a neighborhood of Γ in B is orientable and fix a small loop γ_i which encircles $\widetilde{c}_i$ in the positive sense. We obtain

Corollary 2 *The monodromy index of γ_1 with respect to some basis $\mathcal{B}$ has the same sign as the monodromy index of γ_2 computed with respect to a basis obtained by parallel transport of $\mathcal{B}$.*

Proof. The holonomy of the affine manifold B being dual to the monodromy, has determinant 1. Hence parallel transport is orientation preserving. $\square$

6 Case of S^1 symmetry

Locally, a focus-focus singularity always admits an S^1 symmetry. However this symmetry does not in general extend globally, in particular when several critical fibers are present. This issue will be discussed in section 7.

We show in this section how to extend the oriented monodromy index to several focus-focus points when the fibration F has a global S^1 symmetry.

Here B is oriented an connected. Let G be the monodromy group of the regular fibration (= the image under μ of the fundamental group $\pi_1(B_r)$). For any $c \in B_r$, G acts on the lattice $H_1(F^{-1}(c), \mathbb{Z}) \simeq \mathbb{Z}^2$.

Theorem 3 *Suppose that B is oriented, connected and simply connected. Then the following properties are equivalent*

1. *each element of G has a non-trivial fixed point in $H_1(F^{-1}(c), \mathbb{Z})$;*
2. *there is a non-trivial $X \in H_1(F^{-1}(c), \mathbb{Z})$ that is fixed by G;*
3. *G is Abelian;*
4. *there is a symplectic S^1 action on (M, ω) that preserves the fibration F;*
5. *there is a Hamiltonian S^1 action on (M, ω) that preserves the fibration F;*
6. *there is a unique group homomorphism $\bar{\mu} : \pi_1(B_r) \to \mathbb{Z}$ such that for any $\gamma \in \pi_1(B_r)$ the monodromy $\mu(\gamma)$ with respect to a positively oriented basis is conjugate in $\mathrm{Sl}(2, \mathbb{Z})$ to $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ with $k = \bar{\mu}(\gamma)$.*

Proof. First note that properties 1 and 2 are of course independent of the choice of the base point c. We choose an oriented basis of $H_1(F^{-1}(c), \mathbb{Z})$, which allows us to identify G with a subgroup of $\mathrm{Sl}(2, \mathbb{Z})$ acting on $\mathbb{Z}^2$. In this proof we shall denote by $\mathcal{M}_k$ the matrix $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

The first three assertions are simple properties of $\mathrm{Sl}(2, \mathbb{Z})$.

Proof of 1⟹2. Let g_0 be a non-trivial element of G, and g be any element of G. Since g_0, g and $g_0 g$ have all 1 in their spectrum, they all have trace equal to 2. Because we can find an integral eigenvector of g_0, there is an integral basis of $\mathbb{Z}^2$ in which $g_0 = \mathcal{M}_k$ $(k \neq 0)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. But $\mathrm{Tr}(g_0 g) = a + kb + d = 2 + kb$ which implies $b = 0$. Then g must have the form $\mathcal{M}_c$. In other words the second element of our basis is necessarily a common eigenvector for all $g \in G$.

Proof of 2⟹3. Complete X into an integral basis of $\mathbb{Z}^2$. Then all $g \in G$ have the form $\mathcal{M}_{k(g)}$ in this basis. Hence they commute, by virtue of the formula

$$\mathcal{M}_k \mathcal{M}_{k'} = \mathcal{M}_{k+k'}. \tag{2}$$

Proof of 3⟹1. The fundamental group $\pi_1(B_r)$ is generated by the set $\gamma_1, \ldots, \gamma_n$, where γ_i is a small loop around a single focus-focus critical value. Since B_r is connected these loops can be deformed in B_r to pass through the point c. Hence the corresponding monodromy transformations $\mu_i = \mu(\gamma_i)$ generate G. Since they are all trigonalizable (they are conjugate to $\mathcal{M}_k$ for some k) and G is Abelian, they are simultaneously trigonalizable. Now the product law (2) implies property *1*.

Proof of 2⟹4. Recall that $H_1(F^{-1}(c), \mathbb{Z})$ is isomorphic to the period lattice $\mathcal{P}_c$: in a local chart of B_r where $F = (H_1, H_2)$, the periodic vector fields on the torus $F^{-1}(c)$ of the form $x\mathcal{X}_{H_1} + y\mathcal{X}_{H_2}$ for constant x and y are determined uniquely by the homology class of any of their orbits.

Thus we identify X with its representant in $\mathcal{P}_c$. By parallel transport it locally extends to a flat local section of $\mathcal{P}$, that is, a 2π-periodic vector field X on $F^{-1}(U)$, where U is a small neighborhood of $c \in B_r$. The 1-form $i_X\omega$ is invariant under the joint flow of F, and hence is of the form $F^*\beta$, for a 1-form β on U. By Liouville-Arnold theorem, $\mathrm{d}\beta = 0$, hence X is symplectic.

Since by hypothesis the action of the monodromy group G on X is trivial, X can be extended to a global section of the bundle of period lattices $\mathcal{P}$

over B_r. The 2π-flow of this vector field defines a symplectic S^1 action on $F^{-1}(B_r)$ preserving the fibration.

At the focus-focus singularity m, the period lattice no longer exists. However near m there is a unique 2π-periodic Hamiltonian vector field (with prescribed orientation) that is tangent to the Lagrangian foliation (see for instance [17]). Hence the above S^1 action extends uniquely to a global S^1 action on M preserving the fibration F. Note that this shows that the 1-form $i_X\omega$ is the pull-back by F of a global closed 1-form β on B.

Proof of 4$\Longrightarrow$5. Let Φ be the symplectic S^1 action and let X be the infinitesimal generator of Φ. Since Φ is symplectic, X is locally Hamiltonian. Since F is preserved by Φ, X is locally constant on the leaves in any action-angle coordinates. Hence X is actually a section of $\mathcal{P}$ above B_r. Hence we are in the situation of the proof above, and there is a closed 1-form β on B such that $i_X\omega = F^*\beta$.

Since $H^1(B) = 0$, β is exact, namely $\beta = \mathrm{d}L$. Hence $X = \mathcal{X}_{F^*L}$ is a Hamiltonian vector field on (M, ω). Thus the S^1 action Φ is Hamiltonian on (M, ω) with momentum map $L \circ F$.

Proof of 5$\Longrightarrow$6. As in the proof above, we let L be a smooth function on B such that $X = \mathcal{X}_{L\circ F}$, where X is the generator of the S^1 action. Since L is a global action, the 1-form $\mathrm{d}L$ is invariant under parallel transport on T^*B_r defined by the integral affine structure on B_r. Thus X is fixed by the monodromy group G: hence the hypotheses of assertion *2* are satisfied. Recall the choice of generators γ_i in the proof of 3$\Longrightarrow$1. Then X can be completed to an integral basis of $\mathcal{P}_c$ in which for all i, $\mu(\gamma_i) = \mathcal{M}_{k_i}$ for some $k_i \in \mathbb{Z}$. We define $\bar{\mu}$ to be the homomorphism that assigns to a loop $\gamma = \gamma_{i_1} \cdots \gamma_{i_p}$ the integer $k = k_{i_1} + \cdots + k_{i_p}$. Note that $\bar{\mu}$ realizes an isomorphism between G and $d\,\mathbb{Z}$ where d is the gcd of $(k_1, \ldots, k_n)$.

Proof of 6$\Longrightarrow$1. Obvious, since any matrix of the form $\mathcal{M}_k$ has a fixed point. $\square$

Corollary 4 *Suppose that there is a global Hamiltonian S^1 action on (M, ω) preserving F. Then the monodromy index along an embedded, positively oriented loop γ in B_r increases with the number of focus-focus critical values inside γ. In particular it can never cancel out.*

Proof. Each each focus-focus critical value adds a positive integer to the global monodromy index. $\square$

7 Vanishing of the monodromy

Some integrable systems do not have an S^1 action. For instance if B is a sphere this would contradict corollary 4, since a loop around all focus-focus critical values would be contractible. However, even without an S^1 action, it is not easy to have the monodromy cancel along an embedded loop, as shown in the following proposition.

Proposition 5 *Assume that B is oriented, connected and simply connected. Let γ be an embedded loop in B_r such that the monodromy along γ is trivial. Let n be the number of focus-focus critical values inside γ, and suppose that they are all simple: their index is* 1. *Then n is a multiple of 12.*

Proof. This is a consequence of the following lemma. See also Moishezon [13, p.179]. □

Lemma 6 *Suppose that there are matrices $A_1, A_2, \dots, A_n$ in* $\mathrm{Sl}(2,\mathbb{Z})$ *such that*

$$\prod_{i=1}^{n}(A_i F A_i^{-1}) = \mathrm{id}, \tag{3}$$

where $F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. *Then n is a multiple of* 12.

Proof. *(In order to stick to the usual conventions for the modular group, we shall use $T = {}^tF$ instead of F. The result follows by transposing* (3).*)* It is well known (see [15]) that the modular group $G = \mathrm{Sl}(2,\mathbb{Z})/\{\pm I\}$ admits the following presentation

$$G = \langle S, T; \quad S^2 = (ST)^3 = I\rangle,$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From this it easily follows that $\mathrm{Sl}(2,\mathbb{Z})$ admits the following presentation

$$\mathrm{Sl}(2,\mathbb{Z}) = \langle S, T; \quad S^4 = I, S^2 = (ST)^3\rangle.$$

Therefore the abelianization K of $\mathrm{Sl}(2,\mathbb{Z})$ is the group

$$K = \langle S, T; \quad S^4 = I, S^2 = (ST)^3, ST = TS\rangle,$$

which yields $K = \langle S, T; T^{12} = I, S = T^{-3}\rangle$. Hence $K \simeq \mathbb{Z}/12\mathbb{Z}$ and T is a generator of K. The image of the formula (3) in K gives $T^n = I$, which implies that n is a multiple of 12. □

As pointed to us by V. Matveev and O. Khomenko [11], from the data in the hypothesis of lemma 6, one can construct an integrable system with $12k$ focus-focus fibers and whose local monodromy around each critical value c_i is equal in some fixed basis to $A_i F A_i^{-1}$. Hence the monodromy around all critical values is the identity. This is done by pasting together a chain of fibrations with one focus-focus fiber, where the gluing maps between two tori are given by the A_i's. We therefore obtain a singular torus fibration over an open disc in $\mathbb{R}^2$ that cannot admit any S^1 symmetry, due to corollary 4.

To have an example of a sequence of matrices in $\mathrm{Sl}(2,\mathbb{Z})$ satisfying the hypotheses of lemma 6, take $A_{2j} = I$ and $A_{2j+1} = S$. Then the product $TSTS^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ is of order 6 in $\mathrm{Sl}(2,\mathbb{Z})$.

If one constructs a singular Lagrangian fibration over the open disc in $\mathbb{R}^2$ using these gluing matrices A_j, with 12 focus-focus critical values, we see that we can obtain as monodromy matrices of oriented loops the following ones: T (loop around one critical value), T^{-1} (because of (3)), S^{-1} (which is obtained by looping around the first three critical values, since $TSTS^{-1}T = S^{-1}$), and finally S (again because of (3)). Therefore by arbitrarily composing the corresponding loops together, we obtain any matrix of $\mathrm{Sl}(2,\mathbb{Z})$.

When B is a Riemann surface, one can show further that the number of focus-focus points (if they are all simple) is equal to $12k$, where k is the Euler characteristic of B. See [16] for more details. For example in [21] Tien Zung constructs an integrable system on a $K3$ surface which yields a singular Lagrangian fibration over S^2 with 24 simple focus-focus points.

8 Example with S^1 symmetry.

Consider the quadratic spherical pendulum. This is a Hamiltonian system on $TS^2 \subseteq T\mathbb{R}^3$ (with coordinates (x, ξ)) defined by

$$\langle x, x \rangle = 1 \text{ and } \langle x, \xi \rangle = 0,$$

where $\langle \, , \, \rangle$ is the usual Euclidean inner product. The symplectic form on TS^2 is the restriction of $\sum_{i=1}^3 dx_i \wedge d\xi_i$ to TS^2. The Hamiltonian is

$$H(x, \xi) = \tfrac{1}{2} \langle \xi, \xi \rangle + V(x_3),$$

where $V(x_3) = 2(x_3 - \alpha)^2$ with $\alpha \in (0, 1)$. H is invariant under the lift of rotation around the x_3 axis to TS^2. Hence H Poisson commutes with the angular momentum $K(x,\xi) = \langle \xi \times x, e_3 \rangle$. Thus the quadratic spherical pendulum is Liouville integrable with energy momentum mapping

$$F : TS^2 \to \mathbb{R}^2 : (x,\xi) \mapsto (H(x,\xi), K(x,\xi)),$$

that is, $F = (H, K)$. The set of critical values of F (see figure 1) is composed of two points $A = (2(1-\alpha)^2, 0)$ and $B = (2(1+\alpha)^2, 0)$ and a smooth parabola-like curve parametrized by

$$\begin{cases} h &= 2z^{-1}(\alpha - z)(1 + z\alpha - 2z^2) \\ k &= \pm 2(1-z^2)\sqrt{\alpha/z - 1} \end{cases} \qquad \text{for } z \in (0, \alpha].$$

It is straightforward to check that each point on the above curve cor-

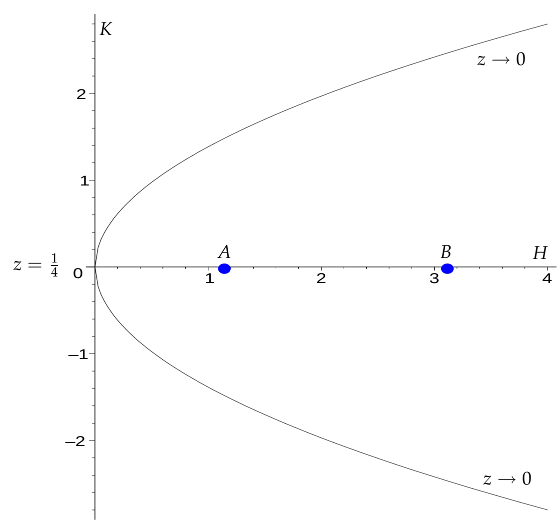

Figure 1: critical values of the momentum map F. Here $\alpha = 1/4$.

responds to a relative equilibrium of the quadratic spherical pendulum, whose image under the tangent bundle projection is a horizontal circle on S^2 with $x_3 = \pm z$. The isolated points are unstable equilibria namely, the poles of S^2, which are of focus-focus type. Since the fibers $F^{-1}(A)$ and $F^{-1}(B)$ contain each a single critical point, both A and B have oriented monodromy index 1. Hence the global index around both points is 2.

9 Semiclassical quantization

The constancy of the sign of the monodromy is easily seen on a semiclassical joint spectrum. The latter has a local lattice structure admitting a discrete parallel transport, which is an asymptotic version of the integral affine structure on B_r. For more details see [17]. This shows

Theorem 7 *Let a positively oriented basis $\mathcal{B}$ of the quantum lattice around a focus-focus point evolve in the positive sense. Then we obtain a final basis by applying to $\mathcal{B}$ a 2×2 matrix which is conjugate in $\mathrm{Sl}(2,\mathbb{Z})$ to $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ with $k \geq 0$.*

We illustrate theorem 7 with the quantum quadratic spherical pendulum. Let $\hat{H}$ and $\hat{K}$ be the self-adjoint operators acting on $L^2(S^2)$ defined as follows:

$$\begin{aligned}\hat{H} &= \tfrac{\hbar^2}{2}\Delta_{S^2} + V(x_3)\\ \hat{K} &= -\tfrac{\hbar}{i}\tfrac{\partial}{\partial\theta},\end{aligned}$$

where Δ_{S^2} is the Laplace-Beltrami operator on S^2 (with positive eigenvalues), $V = 2(x_3-\alpha)^2$ and θ is the polar angle around the vertical axis (Ox_3). $\hat{H}$ et $\hat{K}$ are $\hbar$-differential operators that commute: $[\hat{H},\hat{K}] = 0$ and hence define a quantum integrable system. Their classical limit is given by the principal symbols H and K in $C^\infty(T^*S^2)$, which are of course the Hamiltonians of section 8.

Figure 2 shows the joint spectrum of $\hat{H}$ and $\hat{K}$ for $\alpha = 1/4$ and $\hbar = 0.1$. For such "large" values of $\hbar$ the easiest way to compute the spectrum globally is to express the matrix associated to $\hat{H}$ in the basis of standard spherical harmonics (they are also eigenfunctions of $\hat{K}$). The action of the potential V is obtained from the recurrence relation of the Legendre polynomials. This matrix can be cut to a finite size without any important loss in the accuracy of the computation, due to the fact that the modes we are looking at are microlocalized in a region of bounded energy $H \leq H_{\max}$, which is compact. We have used this method of calculation to produce figure 2.

The best way to have precise results near critical values for small $\hbar$ would be to use the singular Bohr-Sommerfeld rules of [18].

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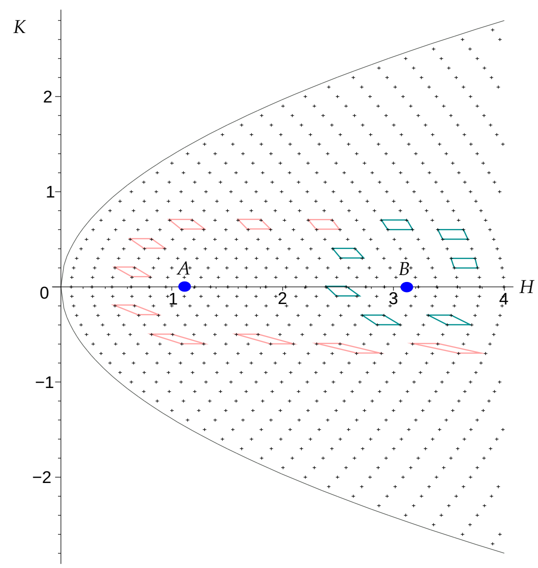

Figure 2: Joint spectrum for the quadratic spherical pendulum. The quantum monodromy is represented by the deformation of a small cell of the asymptotic lattice.

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References

[1] L. Bates, *Monodromy in the Champagne bottle*, Z. Angew. Math. Phys. **42** (1991), 837–847.

[2] A.V. Bolsinov and A.T. Fomenko, *Application of classification theory for integrable hamiltonian systems to geodesic flows on* 2*-sphere and* 2*-torus and to the description of the topological structure of momentum mappings near singular points*, J. Math. Sci. (Dynamical systems, 1) **78** (1996), 542–555.

[3] M.S. Child, *Quantum states in a Champagne bottle*, J.Phys.A. **31** (1998), 657–670.

[4] R. Cushman and L. Bates, *Global aspects of classical integrable systems*, Birkhäuser, 1997.

[5] R. Cushman and J.J. Duistermaat, *The quantum spherical pendulum*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 475–479.

[6] ______, *Non-hamiltonian monodromy*, J. Differential Equations **172** (2001), 42–58.

[7] R. Cushman and D.A. Sadovskií, *Monodromy in the hydrogen atom in crossed fields*, Phys. D **142** (2000), no. 1–2, 166–196.

[8] J.J. Duistermaat, *On global action-angle variables*, Comm. Pure Appl. Math. **33** (1980), 687–706.

[9] L.H. Eliasson, *Hamiltonian systems with Poisson commuting integrals*, Ph.D. thesis, University of Stockholm, 1984.

[10] L. Grondin, D. Sadovskii, and B. Zhilinskií, *Monodromy in systems with coupled angular momenta and rearrangement of bands in quantum spectra*, Phys. Rev. A (3) **65** (2001), no. 012105, 1–15.

[11] V. Matveev, *Private communication*, email 5 November 2001.

[12] ______, *Integrable hamiltonian systems with two degrees of freedom. topological structure of saturated neighborhoods of saddle-saddle and focus points*, Mat. Sb. **187** (1996), 29–58.

[13] B. Moishezon, *Complex surfaces and connected sums of complex projective planes*, Lecture Notes in Mathematics, no. 603, Springer-Verlag, 1977.

[14] D.A. Sadovskií and B.I. Zhilinskií, *Monodromy, diabolic points, and angular momentum coupling*, Phys. Lett. A **256** (1999), no. 4, 235–244.

[15] J.-P. Serre, *Cours d'arithmétique*, Presses Universitaires de France, 1970.

[16] M. Symington, in preparation.

[17] S. Vũ Ngọc, *Quantum monodromy in integrable systems*, Commun. Math. Phys. **203** (1999), no. 2, 465–479.

[18] ______, *Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type*, Comm. Pure Appl. Math. **53** (2000), no. 2, 143–217.

[19] H. Waalkens and H.R. Dullin, *Quantum monodromy in prolate ellipsoidal billiards*, Ann. Physics **295** (2002), 81–112.

[20] Nguyên Tiên Zung, *A note on focus-focus singularities*, Diff. Geom. Appl. **7** (1997), no. 2, 123–130.

[21] ______, *Symplectic topology of integrable hamiltonian systems, II: Topological classification*, Preprint Univ. Montpellier, `math.DG/0010181`, 2000.