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Quantum monodromy in Integrable Systems

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Abstract. Let $P_1(h), \ldots, P_n(h)$ be a set of commuting self-adjoint h-pseudo-differential operators on an n-dimensional manifold. If the joint principal symbol p is proper, it is known from the work of Colin de Verdière [6] and Charbonnel [3] that in a neighbourhood of any regular value of p, the joint spectrum locally has the structure of an affine integral lattice. This leads to the construction of a natural invariant of the spectrum, called the quantum monodromy. We present this construction here, and show that this invariant is given by the classical monodromy of the underlying Liouville integrable system, as introduced by Duistermaat [9]. The most striking application of this result is that all two degree of freedom quantum integrable systems with a *focus-focus* singularity have the same non-trivial quantum monodromy. For instance, this proves a conjecture of Cushman and Duistermaat [7] concerning the quantum spherical pendulum.

1. Introduction

Obstructions to the existence of global action-angle coordinates for completely integrable systems are well known since Duistermaat's article [9]. It was then natural to raise the question about the impact of these obstructions on *quantum* integrable systems, at least for the (semi)-classical pseudo-differential quantisation on cotangent bundles. The first attempts in this direction were [7] and [11], both of them concerning the monodromy invariant for the example of the spherical pendulum. This system is indeed one of the simplest (along with the Champagne bottle [1]) that exhibit a non-trivial monodromy. The first of these

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articles [7] proposed a particularly interesting way of detecting the monodromy by observing a shift in the lattice structure of the joint spectrum. It is the purpose of this article to state, prove and explain this idea.

Surprisingly enough, this idea of quantum monodromy has been sleeping for ten years, before new interest resulted in its experimental discovery in the spectrum of excited water molecules [4, 5].

Back to mathematics, it turns out that, in the framework of semi-classical microlocal analysis (developed for integrable systems in [3]), there is a natural way of defining an invariant of the joint spectrum away from singularities of the principal symbols, that precisely describes the obstruction to the existence of a *global* lattice structure for the spectrum. The organisation of this article is as follows : we first extract the relevant properties of joint spectra, and define the *quantum monodromy* invariant for any set that shares these properties (section 2). Then we prove in section 3 that, for spectra, the quantum monodromy is precisely given by the classical monodromy of the underlying classical Hamiltonian system. The result is applied in section 4 to the particularly interesting case of systems admitting a *focus-focus* singularity. The last section 5 finally shows how to read off the monodromy from a picture of the spectrum. As an example, we use the spectrum of the Champagne bottle computed by Child [4].

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2. Construction of the quantum monodromy

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$, let H be a set of positive real numbers accumulating at 0, and for any h in H let $\Sigma(h)$ be a discrete subset of $\mathcal{U}$.

If B is an open subset of $\mathcal{U}$, a family $(f(h))_{h\in H}$ of smooth functions on B with values in $\mathbb{R}^n$ is called a *symbol* (of order zero) if it admits an asymptotic expansion of the form

$$f(h) = f_0 + hf_1 + h^2 f_2 + \cdots$$

for smooth functions $f_i : B \to \mathbb{R}^n$. More precisely we require that for any $\ell \geq 0$, for any $N \geq 0$, and for any compact $K \subset B$, there is a constant $C_{\ell,N,K}$ such that for all $h \in H$,

$$\left\| f(h) - \sum_{k=0}^{N} h^k f_k \right\|_\ell \leq C_{\ell,N,K} h^{N+1},$$

where $\|.\|_\ell$ denotes the C^ℓ norm in K. The symbol $f(h)$ is *elliptic* if its principal part f_0 is a local diffeomorphism of B into $\mathbb{R}^n$. The value of $f(h)$ at a point $c \in B$ will be denoted by $f(h; c)$.

A family $(r(h))_{h\in H}$ of elements of a finite dimensional vector space is said to be $O(h^\infty)$ if for any $N \geq 0$ there is a constant $C > 0$ such that $\|r(h)\| \leq Ch^N$, uniformly for all $h \in H$. If $S(h)$ is any family of sets depending on h, then the notation $f(h) \in S(h) + O(h^\infty)$ means that the function $\mathrm{dist}(f(h), S(h))$ is $O(h^\infty)$.

We will say that $\Sigma(h)$ has the structure of an "asymptotic affine lattice" whenever it can be described with a locally finite set of "asymptotic affine integral charts", in the following sense :

Definition 1. *$(\Sigma(h), \mathcal{U})$ is an "asymptotic affine lattice" if for any $c \in \mathcal{U}$, there exists a small open ball $B \subset \mathcal{U}$ around c, and an elliptic symbol $f(h) : B \to \mathbb{R}^n$ of order zero such that, for any family $\lambda(h) \in B$:*

- $\lambda(h) \in \Sigma(h) \cap B + O(h^\infty) \Longleftrightarrow f(h; \lambda(h)) \in h\mathbb{Z}^n + O(h^\infty)$
- *if $\lambda(h)$ and $\lambda'(h)$ are in $\Sigma(h) \cap B$, then $\lambda'(h) - \lambda(h) = O(h^\infty)$ if and only if for small h, $\lambda'(h) = \lambda(h)$.*

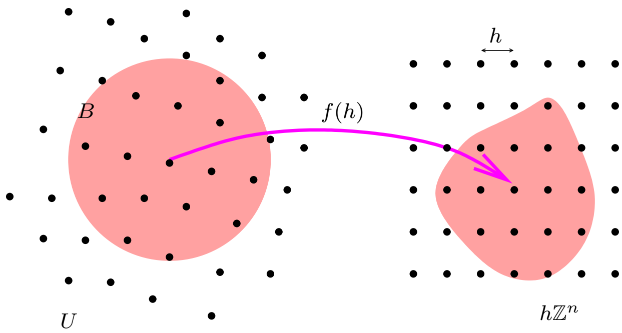

Fig. 1. An asymptotic affine lattice

Intuitively this means that zooming by a factor of $\frac{1}{h}$ inside B makes $\Sigma(h) \cap B$ converge to the standard lattice as h tends to zero. The issue here is to see what prevents $\Sigma(h)$ from *globally* converging to a lattice. Of course, the reason for this definition is that, under suitable hypothesis, the joint spectrum of a set of n commuting h-pseudo-differential operators on an n-dimensional manifold is indeed an "affine asymptotic lattice" (see the next section).

For short, a symbol $f(h)$ satisfying definition 1 will be referred to as an "affine chart" of $\Sigma(h)$.

The main point is that the transition functions associated to these charts are elements of the affine group $GA(n, \mathbb{Z})$ (following Berger [2], we denote by

$GA(n,\mathbb{R})$ the group of invertible affine transformations of $\mathbb{R}^n$, which is the semi-direct product of the linear group $GL(n,\mathbb{R})$ by the normal subgroup of translations. Some authors use the notation $\mathrm{Aff}_n(\mathbb{R})$ instead. The subgroup $GA(n,\mathbb{Z})$ consists then of elements $A \in GA(n,\mathbb{R})$ such that A and A^{-1} leave $\mathbb{Z}^n$ globally invariant).

Proposition 1. *Let $f(h)$ and $g(h)$ be two affine charts of $\Sigma(h)$, both defined on a ball B. Then there is a unique $A \in GA(n,\mathbb{Z}) \subset GA(n,\mathbb{R})$ such that*

$$\left(\frac{g(h)}{h}\right) \circ \left(\frac{f(h)}{h}\right)^{-1} = A_{\upharpoonright f(h)(B)/h} + O(h^\infty).$$

Suppose now that $\mathcal{U}$ is covered by a locally finite union of balls B_α on each of which is defined an affine chart $f_\alpha(h)$ of $\Sigma(h)$. Proposition 1 yields a family of affine linear maps $A_{\alpha\beta}$ such that on non-empty intersections $B_\alpha \cap B_\beta$

$$\frac{1}{h} f_\alpha(h) = A_{\alpha\beta}\left(\frac{1}{h} f_\beta(h)\right).$$

This in turn defines a 1-cocycle $\mathcal{M}$ in the Čech cohomology of $\mathcal{U}$ with values in the non-Abelian group $GA(n,\mathbb{Z})$.

Definition 2. *The class $[\mathcal{M}] \in \check{H}^1(\mathcal{U}, GA(n,\mathbb{Z}))$ of the cocycle defined by $A_{\alpha\beta}$ is called the* quantum monodromy *of $(\Sigma(h),\mathcal{U})$.*

Let L be the canonical homomorhism, whose kernel is the group of translations :

$$L : GA(n,\mathbb{R}) \to GL(n,\mathbb{R}).$$

Let ι be the inclusion of $GL(n,\mathbb{R})$ into $GA(n,\mathbb{R})$ such that for any $M \in GL(n,\mathbb{R})$, $\iota(M)$ leaves the origin $0 \in \mathbb{R}^n$ invariant. Then ι is an injective homomorphism that depends on the choice of the origin 0, satisfying $L \circ \iota = Id$. Any $A \in GA(n,\mathbb{R})$ can be written in a unique way

$$A = \tau(k) \circ \iota(M),$$

(which is usually written $A = M + k$), where $M = L(A) \in GL(n,\mathbb{Z})$ and $\tau(k)$ is translation by the vector $k \in \mathbb{Z}^n$.

The exact sequence of group homomorphisms

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\tau} GA(n,\mathbb{Z}) \xrightarrow{L} GL(n,\mathbb{Z}) \longrightarrow 1$$

gives rise to the following sequence of maps (which are not homomorphisms, since cohomology sets with values in a non-abelian group have no natural group structure – see [12, p.38])

$$\check{H}^1(\mathcal{U},\mathbb{Z}^n) \xrightarrow{\tau_*} \check{H}^1(\mathcal{U}, GA(n,\mathbb{Z})) \xrightarrow{L_*} \check{H}^1(\mathcal{U}, GL(n,\mathbb{Z})) \longrightarrow 1.$$

This sequence is "exact" in the sense that L_* is surjective, and if $L_*([\mathcal{M}]) = 1$, then there is an integer cocycle $[\omega] \in \check{H}^1(\mathcal{U},\mathbb{Z}^n)$ such that $[\mathcal{M}] = \tau_*([\omega])$. The surjectivity of L_* is due to the existence of the cross section ι, which gives rise to the map

$$\check{H}^1(\mathcal{U}, GA(n,\mathbb{Z})) \xleftarrow{\iota_*} \check{H}^1(\mathcal{U}, GL(n,\mathbb{Z}))$$

such that $L_*\iota_* = Id$. For the second point, we remark that if the cocycle $L(A_{\alpha\beta})$ is a coboundary, then it can be written $M_\alpha M_\beta^{-1}$. Therefore the cocycle $\iota(M_\alpha^{-1})A_{\alpha\beta}\iota(M_\beta)$ (which is equivalent to $A_{\alpha\beta}$) has a linear part equal to the identity, hence is a translation.

Remark 1. The lack of injectivity for τ_* is measured by $\check{H}^0(\mathcal{U}, GL(n,\mathbb{Z}))$: one can check that two cocycles $[k]$ and $[k']$ in $\check{H}^1(\mathcal{U}, \mathbb{Z}^n)$ yield the same element of $\check{H}^1(\mathcal{U}, GA(n,\mathbb{Z}))$ if and only if there is an $M \in \check{H}^0(\mathcal{U}, GL(n,\mathbb{Z}))$ such that $[k'] = [M \cdot k]$.

Let us now give various interpretations of the quantum monodromy $\mathcal{M}$.

The action of $GA(n,\mathbb{Z})$ on $\mathbb{Z}^n$ being effective, it is a standard fact that the cohomology set $\check{H}^1(\mathcal{U}, GA(n,\mathbb{Z}))$ classifies the isomorphism classes of fibre bundles over $\mathcal{U}$ with structure group $GA(n,\mathbb{Z})$ and fibre $\mathbb{Z}^n$ (see for instance [12, p.40–41]). Let $\mathcal{L}$ be such a lattice bundle associated to $\mathcal{M}$. The elements $A_{\alpha\beta}$ just define the transition functions between two adjacent trivialisations of $\mathcal{L}$.

Since these trivialisation functions are locally constant, there is a naturally defined parallel transport $\gamma.p$ of a point $p \in \mathcal{L}_c$ along a path γ in the base $\mathcal{U}$. This defines the holonomy of $\mathcal{L}$, as a map from $\pi_1(\mathcal{U}, c)$ into $GA(\mathcal{L}_c)$. We will always identify the latter with $GA(n,\mathbb{Z})$ by choosing an affine basis of $\mathcal{L}_c$.

The choice of such a basis is equivalent to that of a trivialisation f of $\mathcal{L}$ above c that sends this basis to the canonical basis of $\mathbb{Z}^n$; the holonomy $\boldsymbol{\mu}_f$ is then defined by :

$$f(\gamma.p) = \boldsymbol{\mu}_f(\gamma) f(p). \tag{1}$$

Finally, this is also equivalent to the choice of an affine chart $f(h)$ of $\Sigma(h)$ around c. If $\mathcal{M}$ is any cocycle associated to this trivialisation, then

$$\boldsymbol{\mu}_f(\gamma) = A_{1,\ell} \circ \cdots \circ A_{3,2} \circ A_{2,1}, \tag{2}$$

where $A_{i,j}$ denotes the transition element corresponding to a pair of intersecting open balls (B_i, B_j), and $B_1, \ldots, B_\ell$ enumerate elements of a cover of $\mathcal{U}$ encountered by $\gamma(t)$ when t runs from 0 to 1.

We shall always assume that $\mathcal{U}$ is connected, so that $\boldsymbol{\mu}_f$ does not depend on the base point c. Note that since $(\gamma'\gamma).p = \gamma.(\gamma'.p)$, we have

$$\boldsymbol{\mu}_f(\gamma'\gamma) = \boldsymbol{\mu}_f(\gamma)\boldsymbol{\mu}_f(\gamma').$$

It should be noticed that the bundles considered here have discrete fibres, so that we could reduce the discussion to the theory of coverings. The fibre bundle formulation seems however to be more natural when it comes to comparing them with objects arising in Hamiltonian systems. Nevertheless, the covering approach will be used in section 5.

Other geometric interpretations of $\mathcal{M}$ will also be discussed in section 5. For the moment just notice that the non-triviality of $[\mathcal{M}]$ is equivalent to the non-triviality of the lattice bundle $\mathcal{L}$ and to the fact that there is no globally defined symbol $f(h)$ on $\mathcal{U}$ sending $\Sigma(h)$ to the straight lattice $h\mathbb{Z}^n$.

Proof ((of proposition 1)). There are no surprises in this quite elementary proof. Let $c \in \mathcal{U}$, and $f(h)$, $g(h)$ be two affine charts of Σ defined on a ball B around c. Because of definition 1, any open ball around c contains, for h small enough, at least one element of $\Sigma(h)$. Therefore, there exists a family $\lambda(h) \in \Sigma(h) \cap B$ such that

$$\lim_{h\to 0} \lambda(h) = c.$$

Let $k \in \mathbb{Z}^n$ and let $\lambda'(h)$ be a family of elements of $\Sigma(h) \cap B$ such that

$$f(h;\lambda(h)) = f(h;\lambda'(h)) + hk + O(h^\infty).$$

Then, as h tends to zero, $\frac{\lambda'(h)-\lambda(h)}{h}$ tends towards a limit $v \in \mathbb{R}^n$ satisfying

$$k = df_0(c)v$$

(recall that f_0 denotes the principal part of $f(h)$).

Since $\lambda(h)$ and $\lambda'(h)$ are in $\Sigma(h)$, there is a family $k'(h) \in \mathbb{Z}^n$ such that

$$\left(\frac{g(h;\lambda'(h)) - g(h;\lambda(h))}{h}\right) = k'(h) + O(h^\infty).$$

The left-hand side of the above equation has limit $dg_0(c)v$ as $h \to 0$. Therefore $k'(h)$ is equal to a constant integer k' for small h, and we have

$$k' = dg_0(c)(df_0(c))^{-1}k,$$

which implies that $dg_0(c)(df_0(c))^{-1} \in GL(n,\mathbb{Z})$. Since $GL(n,\mathbb{Z})$ is discrete, there is a constant matrix $M \in GL(n,\mathbb{Z})$ such that for all $c \in B$, $dg_0(c) = M \cdot (df_0(c))$; this in turn implies the existence of a constant $k \in \mathbb{Z}^n$ such that, on B,

$$g_0 = M \cdot f_0 + k.$$

But k is necessarily zero : indeed, applying the above equality to $\lambda(h)$ gives a sequence $k'(h) \in \mathbb{Z}^n$ such that

$$hk'(h) \stackrel{\text{def}}{=} g(h;\lambda(h)) - M \cdot f(h;\lambda(h)) = k + O(h).$$

Therefore $k'(h)$ must tend to zero, and hence must equal zero for small h, implying that $k = 0$.

We have proved the existence of a smooth symbol $F(h)$ such that

$$M \cdot f(h) - g(h) = hF(h).$$

Because $F(h;\lambda(h)) \in \mathbb{Z}^n + O(h^\infty)$ and $\lim_{h\to 0} F(h;\lambda(h)) = F_0(c)$, we must have $F_0(c) \in \mathbb{Z}^n$. So

$$F_0 = const \in \mathbb{Z}^n \text{ in } B.$$

This easily implies that all lower order terms in $F(h)$ must vanish on B, so we are left with

$$F(h) = k + O(h^\infty), \text{ for a } k \in \mathbb{Z}^n.$$

This gives $g(h) = M \cdot f(h) - hk + O(h^\infty)$, which reads

$$\frac{1}{h} g(h) = A(\frac{1}{h} f(h)) + O(h^\infty),$$

with $A \in GA(n, \mathbb{Z})$ defined by $A(p) = M \cdot p - k$, $p \in \mathbb{Z}^n$. □

Remark 2. Because of the discreteness of $GA(n, \mathbb{Z})$, proposition 1 implies that there is an $h_0 > 0$ such that the transition element A is uniquely defined by $(g(h_0)/h_0)(f(h_0)/h_0)^{-1}$ acting on a finite subset of $\mathbb{Z}^n$. Therefore, when restricted to any open subset of $\mathcal{U}$ with compact closure in $\mathcal{U}$, the cocycle $[\mathcal{M}]$ is really a *quantum* object, in the sense that "you don't need to let h tend to zero" to define it.

3. Link with the classical monodromy

Let $P_1(h), \ldots, P_n(h)$ be a set of commuting self-adjoint h-pseudo-differential operators on an n-dimensional manifold X. They will be assumed to be classical and of order zero, in the sense that in any coordinate chart their Weyl symbols $p_j(h)$ have an asymptotic expansion of the form

$$p_j(h; x, \xi) = p_0^j(x, \xi) + h p_1^j(x, \xi) + h^2 p_2^j(x, \xi) + \cdots$$

Because the principal symbols $p_0^1, \ldots, p_0^n$ commute with respect to the symplectic Poisson bracket on T^*X, the map

$$T^*X \ni (x, \xi) \xrightarrow{p} (p_0^1(x, \xi), \ldots, p_0^n(x, \xi)) \in \mathbb{R}^n$$

is a momentum map for the local Hamiltonian action of $\mathbb{R}^n$ on T^*X defined by the Hamiltonian flows of the p_0^j. We will always assume that p is *proper*, so that the level sets

$$\Lambda_c = p^{-1}(c)$$

are compact. Moreover, we ask that these level sets be *connected.* Conclusions for non-connected Λ_c can be obtained by separately studying the different connected components.

Let U_r be the open subset of regular values of the momentum map p, and let $\mathcal{U}$ be an open subset of U_r with compact closure.

It follows from the Arnold-Liouville theorem that $p_{\restriction \mathcal{U}}$ is a smooth fibration whose fibres are Lagrangian tori. The structure of this fibration is semi-globally (*i.e.* in a neighbourhood of a fibre) described with the help of action-angle coordinates. However, the flat fibre bundle $H_1(\Lambda_c, \mathbb{Z}) \to c \in \mathcal{U}$ (with fibre $\mathbb{Z}^n$)

may have non-trivial monodromy, preventing the construction of *global* action variables on $p^{-1}(\mathcal{U})$ (see Duistermaat [9]). We will denote by $[\mathcal{M}_{cl}]$ (classical monodromy) the cocycle in $\check{H}^1(\mathcal{U}, GL(n,\mathbb{Z}))$ associated to this lattice bundle.

On the other hand, let $\Sigma(h)$ be the intersection with $\mathcal{U}$ of the joint spectrum of the operators $P_1(h), \ldots, P_n(h)$. It is known from [3] that this spectrum is discrete and for small h is composed of simple eigenvalues. Moreover, the following result holds :

Proposition 2 ([3]). *$\Sigma(h)$ is an asymptotic affine lattice on $\mathcal{U}$.*

We denote by $[\mathcal{M}_{qu}] \in \check{H}^1(\mathcal{U}, GA(n,\mathbb{Z}))$ the quantum monodromy of the spectrum on $\mathcal{U}$, given by definition 2.

Recall that ι denotes the inclusion of $GL(n,\mathbb{R})$ into $GA(n,\mathbb{R})$ such that for any $M \in GL(n,\mathbb{R})$, $\iota(M)$ leaves the origin $0 \in \mathbb{R}^n$ invariant.

The relation between $[\mathcal{M}_{qu}]$ and the classical monodromy $[\mathcal{M}_{cl}]$ is then given by the following theorem :

Theorem 1. *The quantum monodromy is "dual" to the classical monodromy in the following sense :*

$$[\mathcal{M}_{qu}] = \iota_*({}^t[\mathcal{M}_{cl}]^{-1}).$$

In other words, for any $c \in \mathcal{U}$ there exists a choice of basis of $H_1(\Lambda_c, \mathbb{Z})$ and of an affine chart of $\Sigma(h)$ such that the monodromy representations

$$\boldsymbol{\mu}^{cl} : \pi_1(\mathcal{U}, c) \to GL(n,\mathbb{Z})$$

and

$$\boldsymbol{\mu}^{qu} : \pi_1(\mathcal{U}, c) \to GA(n,\mathbb{Z})$$

defined by $[\mathcal{M}_{cl}]$ and $[\mathcal{M}_{qu}]$ satisfy :

$$\boldsymbol{\mu}^{qu} = \iota \circ ({}^t\boldsymbol{\mu}^{cl})^{-1}.$$

Proof. Let α be the Liouville 1-form on T^*X. Let $c_0 \in \mathcal{U}$ and for c near c_0 let $(\gamma_1(c), \ldots, \gamma_n(c))$ be a smooth family of loops on Λ_c whose homology classes form a basis of $H_1(\Lambda_c, \mathbb{Z})$. It is known from [3, 6] (see also [14] for a viewpoint closer to this article) that one can find an affine chart $f(h)$ for $\Sigma(h)$ around c such that the principal part f_0 is equal to the action integral associated to $\gamma_1, \ldots, \gamma_n$:

$$f_0(c) = \left(\frac{1}{2\pi}\int_{\gamma_1(c)} \alpha, \ldots, \frac{1}{2\pi}\int_{\gamma_n(c)} \alpha\right).$$

Because of proposition 1, any other affine chart around c having the same principal part must equal $f(h)$ (modulo $O(h^\infty)$). In this way, the choice of a local smooth basis of $H_1(\Lambda_c, \mathbb{Z})$ determines an affine chart of $\Sigma(h)$. If $(\gamma'_1(c), \ldots, \gamma'_n(c))$ is another basis of $H_1(\Lambda_c, \mathbb{Z})$ such that

$$(\gamma'(c)) = M(c) \cdot (\gamma(c)), \tag{3}$$

for a matrix $M(c) \in GL(n,\mathbb{Z})$ depending smoothly on c, then the corresponding affine charts $f(h)$ and $f'(h)$ of $\Sigma(h)$ satisfy :

$$f'(h;c) = M(c) \cdot f(h;c) + O(h^\infty).$$

Recall that the notation "$M\cdot$" here means matrix multiplication by M, which is of course the same as affine composition by $\iota(M)$.

But formula (3) says that if k and k' are trivialisation functions of the bundle $H_1(\Lambda_c,\mathbb{Z}) \to c$ associated to the basis γ and γ', then $k' = {}^tM^{-1}k$. Therefore, if ${}^tM_{\alpha\beta}^{-1}$ are transition elements for the lattice bundle $H_1(\Lambda_c,\mathbb{Z}) \to c$, then $\iota(M_{\alpha\beta})$ define a monodromy cocycle for $\Sigma(h)$. $\square$

Remark 3. The fact that the *affine* nature of quantum monodromy is here naturally reduced to an action of the *linear* group $GL(n,\mathbb{Z})$ is due the the global existence of a primitive of the symplectic form on T^*X, namely the Liouville 1-form α.

4. Monodromy of a *focus-focus* singularity

It is probably not worth discussing monodromy in arbitrary degrees of freedom, for it is a typical phenomenon of 4-dimensional symplectic manifolds (see [13]).

More precisely, let X be a 2-dimensional manifold, and let $P_1(h)$, $P_2(h)$ be two commuting self-adjoint h-pseudo-differential operators on X. As before, suppose that the momentum map $p = (p_0^1, p_0^2)$ defined by the principal symbols is proper with connected level sets.

We shall make the following hypothesis. There exists a critical point $m \in T^*X$ of p of maximal corank (*i.e.* both p_0^1 and p_0^2 are critical at m) such that, in some local symplectic coordinates (x,y,ξ,η), the Hessians $(p_0^1)''(m)$ and $(p_0^2)''(m)$ (thereafter denoted by $\mathcal{H}(p_0^1)$ and $\mathcal{H}(p_0^2)$) generate a 2-dimensional subalgebra of the algebra $\mathcal{Q}(4)$ of quadratic forms in (x,y,ξ,η) under Poisson bracket that admits the following basis (q_1,q_2) :

$$q_1 = x\xi + y\eta,$$

$$q_2 = x\eta - y\xi.$$

Such a singularity m is called a *focus-focus* singularity. The point m is then isolated amongst critical points of p. Therefore, we can choose $\mathcal{U} \subset U_r$ to be a small punctured disc around $o = p(m)$. Finally, we shall always assume that m is the only critical point of the critical level set $\Lambda_0 = p^{-1}(o)$.

It is known (probably since [15]; see for instance [14] or [8] for discussions and more references on this topic) that the fibration $p_{\restriction\mathcal{U}}$ has non-trivial monodromy, and can be described in the following way :

Near m, we know from [10] that the integrable Hamiltonian system (p_0^1, p_0^2) can be brought into a normal form given by (q_1,q_2). In other words there exists a local diffeomorphism $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, o)$ such that

$$(p_0^1, p_0^2) = F(q_1, q_2).$$

This allows one to define transversal vector fields $\mathcal{X}_1$ and $\mathcal{X}_2$ tangent to the fibres Λ_c that are equal to the Hamiltonian vector fields $\mathcal{X}_{q_1}$ and $\mathcal{X}_{q_2}$ near m. Note that $\mathcal{X}_2$ is periodic of period 2π.

Around each $c \in \mathcal{U}$, we can now define the following smooth basis $(\gamma_1(c), \gamma_2(c))$ of $H_1(\Lambda_c, \mathbb{Z}) \simeq \pi_1(\Lambda_c)$:

- $\gamma_2(c)$ is a simple integral loop of $\mathcal{X}_2$.
- Take a point on $\gamma_2(c)$; let it evolve under the flow of $\mathcal{X}_1$. After a finite time, it goes back on $\gamma_2(c)$. Close it up on $\gamma_2(c)$. This defines $\gamma_1(c)$.

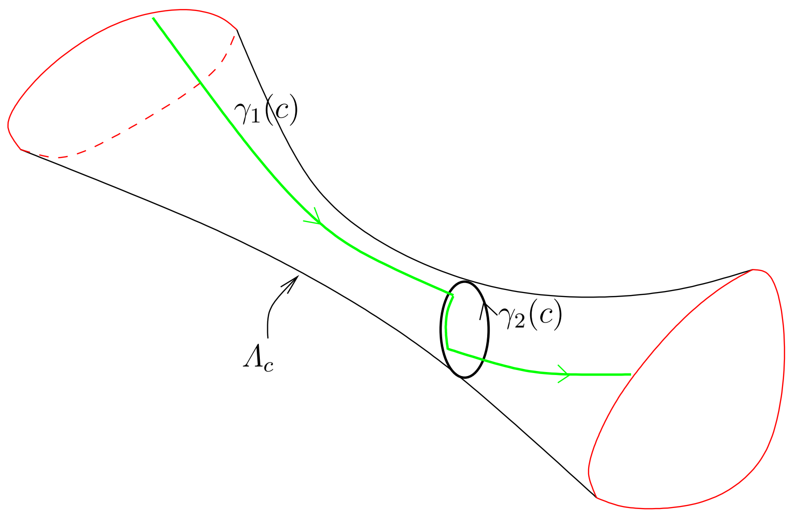

Fig. 2. the basis $(\gamma_1(c), \gamma_2(c))$

Proposition 3 ([15]). *Let $c \in \mathcal{U}$. With respect to the basis $(\gamma_1(c), \gamma_2(c))$, the action of the classical monodromy map $\boldsymbol{\mu}^{cl}$ on a simple loop $\delta \in \pi_1(\mathcal{U}, c)$ enclosing o is given by the matrix*

$$\boldsymbol{\mu}^{cl}(\delta) = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}.$$

Here ϵ is the sign of $\det M$*, where $M \in GL(2, \mathbb{R})$ is the unique matrix such that :*

$$(\mathcal{H}(p_0^1), \mathcal{H}(p_0^2)) = M \cdot (\mathcal{H}(q_1), \mathcal{H}(q_2)).$$

Note also that $M = dF(0)$.

This, together with theorem 1, proves the following result :

Theorem 2. *Let $P_1(h), P_2(h)$ be a quantum integrable system with a* focus-focus *singularity. Then there exists a small punctured neighbourhood $\mathcal{U}$ of the critical value o such that for any $c \in \mathcal{U}$, if $f(h)$ is an affine chart of the joint spectrum $\Sigma(h)$ around c having principal part*

$$\left(\frac{1}{2\pi}\int_{\gamma_1(c)} \alpha, \frac{1}{2\pi}\int_{\gamma_2(c)} \alpha\right),$$

then the value of the quantum monodromy map $\boldsymbol{\mu}_f^{qu} \in GA(2,\mathbb{Z})$ at a simple loop $\delta \in \pi_1(\mathcal{U}, c)$ enclosing o is given by the matrix

$$\boldsymbol{\mu}_f^{qu}(\delta) = \iota \begin{pmatrix} 1 & -\epsilon \\ 0 & 1 \end{pmatrix}.$$

Here ϵ is the sign of $\det M$*, where $M \in GL(2,\mathbb{R})$ is the unique matrix such that :*

$$(\mathcal{H}(p_0^1), \mathcal{H}(p_0^2)) = M \cdot (\mathcal{H}(q_1), \mathcal{H}(q_2)).$$

5. How to detect quantum monodromy

5.1. Introduction

Theorem 1 wouldn't be of much interest if one could not "read off" the quantum monodromy from a picture of the joint spectrum.

This is actually easy to do, at least in a heuristic way. The rigorous mathematical formulation may however look slightly awkward.

The first idea is the following. Given a straight lattice $\mathbb{Z}^n$, and any two points A and B in $\mathbb{Z}^n$, there is a natural parallel translation from A to B acting on $\mathbb{Z}^n$, namely the translation by the integral vector $\overrightarrow{AB}$.

Now, the joint spectrum $\Sigma(h)$ locally around any point $c \in \mathcal{U}$ looks like a lattice. If the points A and B in $\Sigma(h)$ are close enough to c and h is small enough, one can still define a parallel translation from A to B, taking points of $\Sigma(h)$ near A to points in $\Sigma(h)$ near B. This allows us to pass from one chart to another, and hence to define the notion of parallel transport along any loop through c. This yields a map from $\pi_1(\mathcal{U}, c)$ to $GL(n,\mathbb{Z})$ which is precisely the linear part of the quantum monodromy $\boldsymbol{\mu}^{qu}$. This idea is made precise in section 5.2.

The problem can also be viewed the other way round. Roughly speaking, $(\Sigma(h), \mathcal{U})$ is an affine manifold, and hence can be defined by the data of a local diffeomorphism $f(h)$ from the universal cover $\tilde{\mathcal{U}}$ of $\mathcal{U}$ to $h\mathbb{R}^n$ sending $\Sigma(h)$ to $h\mathbb{Z}^n$, and of the holonomy ν associated to it :

$$f(h;\gamma.\tilde{c}) = \nu_{\tilde{c}}(\gamma) f(h;\tilde{c}), \quad \forall \gamma \in \pi_1(\mathcal{U}), \forall \tilde{c} \in \tilde{U}.$$

Of course, ν should be related to the quantum monodromy $\boldsymbol{\mu}_f$. The diffeomorphism $f(h)$ can be seen as an "unwinding" of $\Sigma(h)$ onto $\mathbb{R}^n$. This viewpoint is developed in section 5.3.

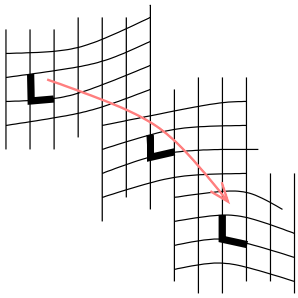

Fig. 3. parallel transport on $\Sigma(h)$

5.2. Parallel transport on $\Sigma(h)$

Let $(\Sigma(h),\mathcal{U})$ be an asymptotic affine lattice.

1. First suppose that there exists an affine chart $f(h)$ of $\Sigma(h)$ defined globally on $\mathcal{U}$. Since $f(h)$ is elliptic and sends elements of $\Sigma(h)$ into $h\mathbb{Z}^n + O(h^\infty)$, there is an $h_0 > 0$ such that for any $h < h_0$, there is an injective map $\tilde{f}(h)$ sending elements of $\Sigma(h)$ exactly into $h\mathbb{Z}^n$ and such that $\tilde{f}(h) - f(h) = O(h^\infty)$.

Because $f(h)$ is of order zero, there is a fixed open ball $\tilde{B}' \subset f(h;\mathcal{U})$ such that $\tilde{B}' \cap (h\mathbb{Z}^n)$ is contained in $\tilde{f}(h;\Sigma(h))$.

Then, one can find a smaller ball $\tilde{B} \subset \tilde{B}'$ such that for any two points $\tilde{P}$, $\tilde{Q}$ in $\tilde{B} \cap (h\mathbb{Z}^n)$, the translation by the vector $\overrightarrow{\tilde{P}\tilde{Q}}$ takes any point of $\tilde{B} \cap (h\mathbb{Z}^n)$ into $\tilde{B}' \cap (h\mathbb{Z}^n)$ (figure 4). Let us denote by B an open ball in $\mathbb{R}^n$ such that $f(h;B) \subset \tilde{B}$. Pulling back by $\tilde{f}(h)$, one thus defines the "parallel transport" $\tau_{\overrightarrow{PQ}}(A)$ of a point $A \in \Sigma(h) \cap B$ along the direction given by two points P and Q in $\Sigma(h) \cap B$. When the composition is defined, we have

$$\tau_{\overrightarrow{QR}} \circ \tau_{\overrightarrow{PQ}} = \tau_{\overrightarrow{PR}}. \tag{4}$$

Moreover, because translation in $\mathbb{Z}^n$ is an isometry, there exists a constant $C > 0$, independent of h, such that for any $A \in \Sigma(h) \cap B$

$$||\overrightarrow{Q\tau_{\overrightarrow{PQ}}(A)}|| < C||\overrightarrow{PA}||. \tag{5}$$

Because of proposition 1, any other choice of affine chart $f(h)$ gives the same parallel transport.

2. Now, let $(\Sigma(h),\mathcal{U})$ be a general asymptotic affine lattice. If γ is any path in $\mathcal{U}$, one can cover its image by open balls B_i on which parallel transport is well defined for h less than some $h_i > 0$. If $\overline{\mathcal{U}}$ is compact, as we shall always assume,

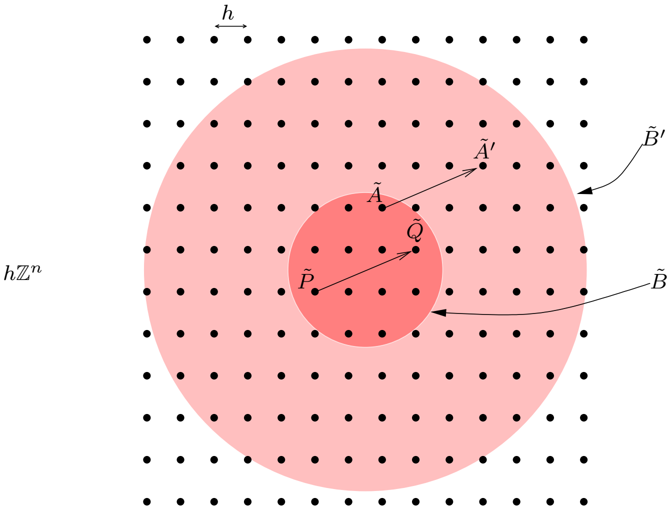

Fig. 4. parallel translation

this can be done with a finite number of such balls $B_1, \ldots, B_\ell$, ordered in a way that for each $1 \le i < \ell$, $B_i \cap B_{i+1} \neq \emptyset$.

In the following, take h to be less than $\min_i h_i$. Let $P \in \Sigma(h) \cap B_0$ and $Q \in \Sigma(h) \cap B_\ell$. For each $i = 1, \ldots, \ell - 1$, pick up a point $P_i \in \Sigma(h) \cap (B_i \cap B_{i+1})$. For h small enough, this set is not empty. Because of the estimate (5), the mapping

$$\tau_{\gamma,P,Q} \stackrel{\text{def}}{=} \tau_{\overrightarrow{P_{\ell-1}Q}} \circ \cdots \circ \tau_{\overrightarrow{P_1 P_2}} \circ \tau_{\overrightarrow{PP_1}}$$

is well-defined when restricted to a sufficiently small ball B_0 around P (here again, $\Sigma(h) \cap B_0$ won't be empty if h is small enough). Equation (4) shows that this map does not depend on the choice of the intermediate points P_i. Therefore it depends only on P, Q, and on the homotopy class of γ (as a path from a point in B_1 to a point in B_ℓ).

If $Q = P$, and γ is a loop ($B_\ell \cap B_1 \neq \emptyset$ and $B_0 \subset B_1$) then $\tau_{\gamma,P,P}$ is a map from $\Sigma(h) \cap B_0$ to $\Sigma(h) \cap B_1$ leaving P invariant. If $f(h)$ is an affine chart for $\Sigma(h)$ on B_1, then $\tilde{f}(h) \circ \tau_{\gamma,P,P} \circ \tilde{f}(h)^{-1}$ is a locally defined map $\tilde{\tau}_{\gamma,f(h),P}$ from $h\mathbb{Z}^n$ to itself leaving $\tilde{f}(h;P)$ invariant.

We know from section 2 (formula (1)) that the choice of such an affine chart allows the quantum monodromy map $\boldsymbol{\mu}_f$ to take its values in $GA(n,\mathbb{Z})$. Remember that L denotes the natural homomorphism from $GA(n,\mathbb{R})$ to $GL(n,\mathbb{R})$.

Proposition 4. *The map $\tilde{\tau}_{\gamma,f(h),P}$ is equal to the linearisation at $\tilde{P} = \tilde{f}(h;P)$ of the quantum monodromy along γ :*

$$\forall \tilde{R} \in h\mathbb{Z}^n, \quad \overrightarrow{\tilde{P}\tilde{\tau}_{\gamma,f(h),P}(\tilde{R})} = L(\boldsymbol{\mu}_f(\gamma))\overrightarrow{\tilde{P}\tilde{R}},$$

whenever the left-hand side of the above is defined.

Proof. If we choose affine charts $f_i(h)$ for $\Sigma(h)$ on each of the B_i's with $f_1 = f$, and let $A_{i,i+1}$ be the transition elements of the monodromy cocycle

$$f_i(h)/h = A_{i,i+1}(f_{i+1}(h)/h) + O(h^\infty) \quad (\text{convention } \ell + 1 \equiv 1),$$

then it is easy to check that

$$\overrightarrow{\tilde{P}\tilde{\tau}_{\gamma,f(h),P}(\tilde{R})} = L(A_{1,\ell}) \cdots L(A_{3,2})L(A_{2,1}) \cdot \overrightarrow{\tilde{P}\tilde{R}},$$

whenever the composition is defined. Using (2) finishes the proof. □

As an application, one can easily "read off" from the spectrum of the quantum Champagne bottle (figure 5) that the linear part of the quantum monodromy is conjugate to the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

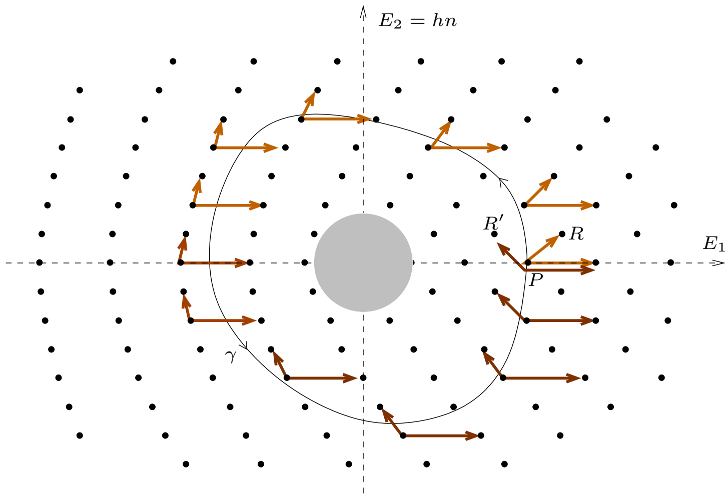

Fig. 5. Spectrum of the Champagne bottle. The gray disc encloses the *focus-focus* critical value. $R' = \tau_{\gamma,P,P}(R)$.

5.3. Unwinding the spectrum

We keep here the notation of the previous paragraph. In particular, $\Sigma(h)$ is any asymptotic affine lattice on $\mathcal{U}$, γ is a path in $\mathcal{U}$ whose image is covered by balls B_i on which local parallel translation is defined. We choose points $P \in B_1 \cap \Sigma(h)$, $Q \in B_\ell \cap \Sigma(h)$ and $P_1, P_2, \ldots, P_{\ell-1}, P_\ell = Q$ such that for $i = 1, \ldots, \ell - 1$, $P_i \in B_i \cap B_{i+1} \cap \Sigma(h)$.

Given an affine chart $f(h)$ on B_1, for h small there is a unique $k_1 \in \mathbb{Z}^n$ such that the map $\tilde{f}(h) \circ \tau_{\overrightarrow{PP_1}} \circ \tilde{f}(h)^{-1}$ is just translation by hk_1. If $B_1, \ldots, B_\ell$ are endowed with affine charts $f_1(h) = f(h), f_2(h), \ldots, f_\ell(h)$, in the same way we define $k_i \in \mathbb{Z}^n$ such that

$$\tilde{f}_i(h) \circ \tau_{\overrightarrow{P_{i-1}P_i}} \circ \tilde{f}_i(h)^{-1}$$

is translation by the vector hk_i. We unwind the points $P, P_1, \ldots, P_\ell$ onto $h\mathbb{Z}^n$ using the following procedure (see figure 6) :

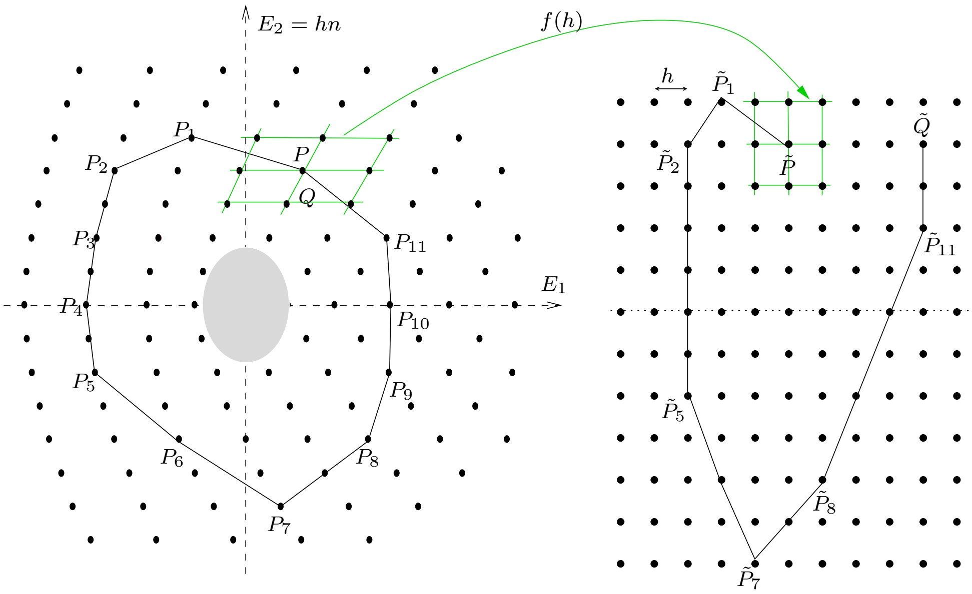

Fig. 6. Unwinding of the points P_i. We deduce that $y_{\tilde{P}} = 4$, which allows us to locate the horizontal line through the origin $0 \in h\mathbb{Z}^2$ (the dotted one).

- $\tilde{P} = \tilde{f}(h; P)$;
- $\tilde{P}_1 = \tilde{P} + hk_1 = \tilde{f}(h, P_1)$;
- $\tilde{P}_2 = \tilde{P}_1 + hL(A_{2,1}) \cdot k_2$;
- ...
- $\tilde{Q} = \tilde{P}_\ell = \tilde{P}_{\ell-1} + hL(A_{\ell,\ell-1}) \cdots L(A_{2,1}) \cdot k_\ell$.

Then one easily checks that

$$\tilde{P}_i = hA_{1,2} \circ A_{2,3} \circ \cdots \circ A_{i-1,i}(\tilde{f}_i(h; P_i)/h).$$

In particular, applying this procedure to a loop γ $(P = Q)$ proves the following :

Proposition 5. *For h small enough, the quantum monodromy $\boldsymbol{\mu}_f$ gives the end point $\tilde{Q}$ of the unwinding of any loop γ on $\mathcal{U}$ through a point $P \in \Sigma(h)$ around which we are given an affine chart $f(h)$ by the following formula :*

$$\tilde{Q} = h(\boldsymbol{\mu}_f(\gamma))^{-1}(\tilde{f}(h;P)/h).$$

Remark 4. There is a unique symbol $g(h)$ defined on the universal cover $\tilde{\mathcal{U}}$ of $\mathcal{U}$ that is an affine chart for $\Sigma(h)$ and that coincides with $f(h)$ above B_0. Then Q can be seen as the lift $\gamma.P \in \tilde{\mathcal{U}}$. The point is now that

$$g(h;Q) = \tilde{Q} + O(h^\infty).$$

For any $P \in \tilde{\mathcal{U}}$, and for any $\gamma \in \pi_1(\mathcal{U})$, there is a unique $\nu_P(\gamma) \in GA(n,\mathbb{Z})$ such that

$$g(h;\gamma.P)/h = \nu_P(\gamma)(g(h;P)/h) + O(h^\infty).$$

By definition, we have $\nu_P(\gamma\gamma') = \nu_{\gamma.P}(\gamma')\nu_P(\gamma)$. But one can show that for any loop γ such that $\gamma.P = Q$, then

$$\nu_Q(\gamma') = \nu_P(\gamma)\nu_P(\gamma')\nu_P(\gamma)^{-1}.$$

Therefore, ν_P is actually a homomorphism. Proposition 5 just says that

$$\nu_P = \boldsymbol{\mu}_f^{-1}.$$

Applying this proposition together with theorem 2 to a *focus-focus* singularity, we see that if the principal part of $f(h)$ is given by the action integrals $\frac{1}{2\pi}\int_{\gamma_1}\alpha$ and $\frac{1}{2\pi}\int_{\gamma_2}\alpha$ then, for a small loop δ enclosing the critical value o,

$$\nu(\delta) = \iota \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}.$$

In particular, the whole horizontal line through the origin consists of fixed points. Of course, locating the origin on a diagram like figure 6 may require the computation of the action at one point. However, given $\tilde{P}$ and its image $\tilde{Q}$, it is easy to find the horizontal line through the origin, for

$$\epsilon y_{\tilde{P}} = x_{\tilde{Q}} - x_{\tilde{P}}.$$

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