A SINGULAR POINCARÉ LEMMA

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ABSTRACT. We prove a Poincaré lemma for a set of r smooth functions on a $2n$-dimensional smooth manifold satisfying a commutation relation determined by r singular vector fields associated to a Cartan subalgebra of $\mathfrak{sp}(2r, \mathbb{R})$. This result has a natural interpretation in terms of the cohomology associated to the infinitesimal deformation of a completely integrable system.

1. INTRODUCTION

The classical Poincaré lemma asserts that a closed 1-form on a smooth manifold is locally exact. In other words, given m-functions g_i on an m-dimensional manifold for which $\frac{\partial}{\partial x_i}(g_j) = \frac{\partial}{\partial x_j}(g_i)$ there exists a smooth F in a neighbourhood of each point such that $g_i = \frac{\partial}{\partial x_i}(F)$.

Now assume that we have a set of r functions g_i and a set of r vector fields X_i with a singularity at a point p and fulfilling a commutation relation of type $X_i(g_j) = X_j(g_i)$. We want to know if a similar expression for g_i exists in a neighbourhood of p.

In the case g_i are n functions on the symplectic manifold $(\mathbb{R}^{2n}, \sum_i dx_i \wedge dy_i)$ and X_i form a basis of a Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{R})$ a Poincaré-like lemma exists. This result was stated by Eliasson in [4]. In [5] Eliasson provided a proof of this statement in the completely elliptic case. As far as the non-elliptic cases are concerned, no complete proof of this result is known to the authors of this note.

The analytical counterpart of this result dates back to the seventies and was proved by Vey [12]. The transition from the analytical case to the smooth case in cases other than elliptic entails a non-trivial work with flat

functions along certain submanifolds and, in our opinion, cannot be ne-
glected.

The aim of this note is to prove a more general singular Poincaré lemma;
the one that would correspond to a set of r functions on a $2n$-dimensional
manifold with $r \leq n$ fulfilling similar commutation relations determined by
a basis of a Cartan subalgebra of $\mathfrak{sp}(2r, \mathbb{R})$. In particular, in this way we
obtain a complete proof also when $r = n$ in the non-completely elliptic cases
which was missing in the literature. This result has a natural interpretation
in terms of the cohomology associated to the infinitesimal deformation of
completely integrable foliations (see section 6).

This result has applications in establishing normal forms for completely
integrable systems. The statement for $r = n$ was used by Eliasson in [4]
and [5] to give a symplectic normal form for non-degenerate singularities
of completely integrable systems. The more general result we prove here
could be useful to establish normal forms for more general singularities of
completely integrable systems.

2. The result

All the objects considered in this note will be $\mathcal{C}^\infty$. We are interested in
germ-like objects attached to a point p of a smooth manifold M^{2n}.

We denote by $(x_1, y_1, \ldots, x_n, y_n)$ a set of coordinates centered at the ori-
gin. Consider the standard symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ in a neigh-
bourhood of the origin. Take $r \leq n$ and consider the embedding $i_r \colon \mathbb{R}^{2r} \longrightarrow$
$\mathbb{R}^{2n}$ defined by $i_r(x_1, y_1, \ldots, x_r, y_r) = (x_1, y_1, \ldots, x_r, y_r, 0, \ldots, 0)$. Consider
$\omega_r = \sum_{i=1}^r dx_i \wedge dy_i$ then $i_r^*(\omega) = \omega_r$, in other words, this embedding induces
an inclusion of Lie groups $Sp(2r, \mathbb{R}) \subset Sp(2n, \mathbb{R})$. In this way $\mathfrak{sp}(2r, \mathbb{R})$ is
realized as a subalgebra of $\mathfrak{sp}(2n, \mathbb{R})$. This particular choice of subalgebra
is implicit throughout the note.

In this note we consider singular vector fields which constitute a basis of
a Cartan subalgebra of the Lie algebra $\mathfrak{sp}(2r, \mathbb{R})$ with $r \leq n$. Recall that
$\mathfrak{sp}(2m, \mathbb{R})$ is isomorphic to the algebra of quadratic forms in $2m$ variables,
$Q(2m, \mathbb{R})$, via symplectic duality. Thus the above chosen immersion induces,
in turn, an inclusion of subalgebras $Q(2r, \mathbb{R}) \subset Q(2n, \mathbb{R})$.

Cartan subalgebras of $Q(2r, \mathbb{R})$ were classified by Williamson in [17].

Theorem 2.1. (Williamson) *For any Cartan subalgebra $\mathcal{C}$ of $Q(2r, \mathbb{R})$*
there is a symplectic system of coordinates $(x_1, y_1, \ldots, x_r, y_r)$ in $\mathbb{R}^{2r}$ and a

basis $q_1, \ldots, q_r$ of $\mathcal{C}$ such that each q_i is one of the following:
(2.1)

$$q_i = x_i^2 + y_i^2 \qquad \text{for } 1 \le i \le k_e \,, \qquad \text{(elliptic)}$$

$$q_i = x_i y_i \qquad \text{for } k_e + 1 \le i \le k_e + k_h \,, \qquad \text{(hyperbolic)}$$

$$\begin{cases} q_i = x_i y_i + x_{i+1} y_{i+1}, & \text{for } i = k_e + k_h + 2j - 1, \\ q_{i+1} = x_i y_{i+1} - x_{i+1} y_i & 1 \le j \le k_f \end{cases} \quad \text{(focus-focus pair)}$$

Observe that the number of elliptic components k_e, hyperbolic components k_h and focus-focus components k_h is therefore an invariant of the algebra $\mathcal{C}$. The triple (k_e, k_h, k_f) is called the Williamson type of $\mathcal{C}$. Observe that $r = k_e + k_h + 2k_f$. Let $q_1, \ldots, q_r$ be a Williamson basis of this Cartan subalgebra. We denote by X_i the Hamiltonian vector field of q_i with respect to ω. Those vector fields are a basis of the corresponding Cartan subalgebra of $\mathfrak{sp}(2r, \mathbb{R})$. We say that a vector field X_i is hyperbolic (resp. elliptic) if the corresponding function q_i is so. We say that a pair of vector fields X_i, X_{i+1} is a focus-focus pair if X_i and X_{i+1} are the Hamiltonian vector fields associated to functions q_i and q_{i+1} in a focus-focus pair.

In the local coordinates specified above, the vector fields X_i take the following form:

- X_i is an elliptic vector field,

$$X_i = 2(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}).$$

- X_i is a hyperbolic vector field,

$$X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}.$$

- X_i, X_{i+1} is a focus-focus pair,

$$X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_{i+1}}$$

and

$$X_{i+1} = -x_i \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_i} + x_{i+1} \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_{i+1}}.$$

With all this notation at hand we can now state the result proven in this note.

Theorem 2.2. *Let $g_1, \ldots g_r$, be a set of germs of smooth functions on $(\mathbb{R}^{2n}, 0)$ with $r \le n$ fulfilling the following commutation relations*

$$X_i(g_j) = X_j(g_i), \quad \forall i, j \in \{1, \ldots, r\}$$

where the X_i's are the vector fields defined above. Then there exists a germ of smooth function G and r germs of smooth functions f_i such that,

(1) $X_j(f_i) = 0$, $\forall i, j \in \{1, \ldots, r\}$.
(2) $g_i = f_i + X_i(G)$ $\forall i \in \{1, \ldots, r\}$.

3. Preliminaries

In this section we recall some basic facts which are proved elsewhere and which will be used in the proof. Here and in the rest of the article the symbol X_i always refers to the hamiltonian vector field associated to the quadratic function q_i, as precised above.

3.1. A special decomposition for elliptic vector fields.

Assume X_i is an elliptic vector field. That is, it is the vector field associated to an elliptic $q_i = x_i^2 + y_i^2$. The following result was proved by Eliasson in [4] when $n = 1$.

Proposition 3.1. *Let g be a smooth function then there exist differentiable functions g_1 and g_2 such that:*

$$g = g_1(x_1, y_1, \ldots, x_i^2 + y_i^2, \ldots, x_n, y_n) + X_i(g_2).$$

Moreover,

(1) *g_1 is uniquely defined and satisfies $X_j(g_1) = 0$ whenever $X_j(g) = 0$;*
(2) *one can choose g_2 such that $X_j(g_2) = 0$ whenever $X_j(g) = 0$.*

Remark: There are explicit formulas for the functions g_1 and g_2 claimed above. Let ϕ_t be the flow of the vector field X_i we define,

$$g_1(x_1, y_1, \ldots, x_n, y_n) = \frac{1}{\pi} \int_0^\pi g(\phi_t(x_1, y_1, \ldots, x_n, y_n)) dt$$

and

$$g_2(x_1, y_1, \ldots, x_n, y_n) = \frac{1}{\pi} \int_0^\pi (tg(\phi_t(x_1, y_1 \ldots, x_n, y_n)) - g_1(x_1, y_1 \ldots, x_n, y_n)) dt.$$

3.2. A special decomposition for hyperbolic vector fields.

In this section we assume the vector field X_i corresponds to a hyperbolic function $q_i = x_i y_i$. As a matter of notation, S_i stands for the set $S_i = \{x_i = 0, y_i = 0\}$. When we refer to an (x_i, y_i)-flat function f along S_i we mean that

$$\frac{\partial^{k+l} f}{\partial x_i^k \partial y_i^l}_{|S_i} = 0.$$

The first result is a decomposition result for smooth functions.

Proposition 3.2. *Given a smooth function g there exist smooth functions g_1 and g_2 such that*

$$g = g_1(x_1, y_1, \ldots, x_i.y_i, \ldots, x_n, y_n) + X_i(g_2).$$

Moreover one can choose g_1 and g_2 such that $X_j(g_1) = X_j(g_2) = 0$ whenever $X_j(g) = 0$ for some $j \neq i$.

This proposition was proven by the first author of this note in [9] (Proposition 2.2.2).

The main strategy of the proof is first to find a decomposition of this type in terms of (x_i, y_i)-jets and then solve the similar problem for (x_i, y_i)-flat functions along S_i. A main ingredient in the proof of the proposition above are the following lemmas which we will be also used in the proof of the theorem in this note. The proof of the following two lemmas is also contained in [9] (lemmas 2.2.1 and 2.2.2 respectively).

Lemma 3.3. *Let g be a smooth function, the equation $X_i(f) = g$ admits a formal solution along the subspace S_i if and only if*

$$\frac{\partial^{2k} g}{\partial x_i^k \partial y_i^k}\bigg|_{S_i} = 0.$$

Lemma 3.4. *Let g be a (x_i, y_i)-flat function along the subspace S_i then there exists a smooth function f for which $X_i(f) = g$.*

Remarks:

(1) Let us point out that when $n = 1$ the decomposition claimed in Proposition 3.2 had been formerly given by Guillemin and Schaeffer [8], by Colin de Verdière and Vey in [2] and Eliasson in [4].

(2) The recipe for solving the equation specified in the lemma above in the case $n = 1$ was given by Eliasson in [4]. The recipe for the general case follows the same guidelines. It is given by the following formula.

(3.1) $\quad f(x_1, y_1, \ldots, x_n, y_n) = -\int_0^{T_i(x_1, y_1, \ldots, x_n, y_n)} g(\phi_t(x_1, y_1, \ldots, x_n, y_n))dt.$

where T_i is the function,

$$T_i(x_1, y_1, \ldots, x_n, y_n) = \begin{cases} \frac{1}{2}\ln\frac{x_i}{y_i} & x_i y_i > 0 \\ \frac{1}{2}\ln\frac{-x_i}{y_i} & x_i y_i < 0 \end{cases}$$

and $\phi_t(x_1, y_1, \ldots, x_n, y_n)$ the flow of the vector field X_i. Observe that f is defined outside the set $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are the sets: $\Omega_1 = \{(x_1, y_1, \ldots, x_n, y_n), x_i = 0\}$ and $\Omega_2 = \{(x_1, y_1 \ldots, x_n, y_n), y_i = 0\}$. In [9] it is proven that f admits a smooth continuation in the whole neighbourhood considered and that it is a solution of the equation $X_i(f) = g$.

(3) From the formula specified above one deduces that if $X_j(g) = 0$ for $j \neq i$ then $X_j(f) = 0$.

(4) In contrast to the uniqueness of the function g_1 in the decomposition obtained in proposition 3.1 for elliptic vector fields, the function g_1 specified in the decomposition is not unique. In fact, if g_1 and h_1 are two functions fitting in the decomposition their difference is an (x_i, y_i) flat function along S_i. In order to check this, observe $g_1 - h_1 = X_i(h_2 - g_2)$ where h_2 is a function such that $g = h_1 + X_i(h_2)$. Now, on the one hand the Taylor expand of $g_1 - h_1$ in the x_i, y_i variables has the form $\sum_j c_j(\check{z}_i)(x_i \cdot y_i)^j$ but, on the other hand, the Taylor expand of $X_i(h_2 - g_2)$ has the form $\sum_{jk} c_{jk}(\check{z}_i) x_i^j y_i^k$ with $j \neq k$ and since the equality $g_1 - h_1 = X_i(h_2 - g_2)$ holds we deduce that $g_1 - h_1$ is an (x_i, y_i)-flat function along S_i.

(5) Let us show the last point of the proposition. The first step in the proof of the proposition was to take care of the formal Taylor series in (x_i, y_i). Then it is easy to see that one can always choose Borel resummations of these formal expansions which are annihilated by X_j ($j \neq i$) whenever g is.

Finally we integrate the flat function using the formula (3.1), on which one can check directly that f is invariant by the flow of X_j ($j \neq i$) whenever g is, at least in a neighbourhood of any point where the formula is well defined. In other words $X_j(g) = 0$ implies $X_j(f) = 0$ at these points, and hence everywhere by continuity.

4. A SPECIAL DECOMPOSITION FOR FOCUS-FOCUS VECTOR FIELDS

The aim of this section is to prove the analogue to propositions 3.1 and 3.2 for a focus-focus pair.

But before stating and proving this result we need some preliminary material concerning the integration of equations of type $X(f) = g$ in a neighbourhood of a hyperbolic zero (in the sense of Sternberg) of the vector field

X. As we will see, the resolution of this equations is closely related to the problem of finding the desired decomposition for focus-focus pairs.

4.1. **Digression: Two theorems of Guillemin and Schaeffer.** A point is called a hyperbolic zero of a vector field X if the vector field vanishes at this point and all the eigenvalues of the matrix associated to the linear part of X have non-zero real part.

According to Sternberg's linearization theorem a vector field can be linearized in a neighbourhood of a hyperbolic zero.

The following two theorems are concerned with the integration of equations of type $X(f) = g$ in a neighbourhood of a hyperbolic zero. These theorems A and B correspond to theorems 2 and 4 in section 4 of [8].

Theorem 4.1. (Theorem A)[8]
Let V be a linear vector field on $\mathbb{R}^n$ with a hyperbolic zero at the origin and let c be a fixed constant. Then given a smooth function g flat a the origin, there exists a smooth function defined in a neighbourhood of the origin which is flat at the origin and such that:

$$V(f) + cf = g.$$

The theorem that follows is used in the proof of Theorem A. We recall it here because we will need it in order to show the smoothness of some constructions used in the next subsection. This theorem uses a trick previously used by Nelson [10] in his proof of the Sternberg's linearization theorem.

Theorem 4.2. (Theorem B)[8] *Let $U(t)$ be a group of linear transformations acting on $\mathbb{R}^n$. Let N be a subspace of $\mathbb{R}^n$ invariant under $U(t)$ and let E be the subspace of $\mathbb{R}^n$ consisting of all x in $\mathbb{R}^n$ such that*

$$\lim_{t \to \infty} ||U(t)(x) - N|| = 0.$$

Let g be a compactly supported function on $\mathbb{R}^n$ which is flat along N. Set

$$f(x,s) = -\int_0^s e^{ct} g(U(t)(x)) dt.$$

Then for all multi-indices α, $\lim_{s \to \infty} D^\alpha f(x,s)$ converges absolutely for all $x \in E$ and is a smooth function of x. Moreover this limit is flat along N.

Observe that the vector field X_i in a focus-focus pair X_i, X_{i+1} has a hyperbolic zero à la Sternberg on the set $\{x_j = c_j, y_j = d_j, j \neq i, j \neq i+1\}$ for fixed constants c_j and d_j.

4.2. **Our proposition for focus-focus pairs.** When i is the index of a focus-focus component, we denote by S_i the set $S_i = \{x_i = 0, y_i = 0, x_{i+1} = 0, y_{i+1} = 0\}$. Let us state and prove the decomposition result for focus-focus pairs.

Proposition 4.3. *Let q_i, q_{i+1} be a focus-focus pair,*

$$q_i = x_i y_i + x_{i+1} y_{i+1}$$
$$q_{i+1} = x_i y_{i+1} - x_{i+1} y_i$$

and let g_1 and g_2 be two functions satisfying the commutation relation:

$$X_i(g_2) = X_{i+1}(g_1)$$

Then there exists smooth functions f_1, f_2 and F such that

(4.1) $$X_j(f_k) = 0 \quad j \in \{i, i+1\} \quad k \in \{1, 2\}$$

such that

$$g_1 = f_1 + X_i(F)$$
$$g_2 = f_2 + X_{i+1}(F)$$

Moreover

   (1) *f_2 is uniquely defined and satisfies $X_j(f_2) = 0$ whenever $X_j(g_2) = 0$ for some j;*
   (2) *f_1 is uniquely modulo functions that are z_j-flat along S_j and satisfy (4.1);*
   (3) *one can choose F and f_1 such that $X_j(F) = X_j(f_1) = 0$ whenever $X_j(g_1) = X_j(g_2) = 0$ for some $j \neq i$.*

Remark: In the case $n = 2$ the proposition above was proven by Eliasson [4].

Proof. Here again the proof if a mild extension of Eliasson's. Without loss of generality, one can assume that $i = 1$. The flow of X_2 defines an S^1-action which will be used in the proof. We can visualise this S^1-action easily using complex coordinates $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$, so that $q_1 + iq_2 = \bar{z}_1 z_2$. The flow of q_2 is the S^1 action given by $(z_1, z_2) \mapsto e^{-it}(z_1, z_2)$ whereas the flow of q_1 is the hyperbolic dynamics given by $(z_1, z_2) \mapsto (e^{-t}z_1, e^t z_2)$ (both flows act trivially on the remaining coordinates). When we say that a function H is S^1-invariant for this action we mean that $X_2(H) = 0$.

As in the proof of Eliasson, we will first integrate along this S^1 action and then along the hyperbolic flow in an S^1-invariant way. Instead of using

the formula of Eliasson (which consists in integrating from a transversal hyperplane through the origin), we will embed everything in $\mathbb{R}^{2n}$ in order to apply the parametric versions of Theorems A and B.

The proof consists of three steps:

1. Integrating along the S^1-action. Let $\varphi_{2,t}$ be the flow of q_2. As in the elliptic case (Proposition 3.1) we define

$$F_2 = \frac{1}{2\pi} \int_0^{2\pi} (\theta - 1) g_2 \circ \varphi_{2,\theta} d\theta$$

and one obtains easily, by differentiating $F_2 \circ \varphi_{2,t}$ at $t = 0$, that

(4.2) $$X_2(F_2) = g_2 - f_2,$$

where

(4.3) $$f_2 = \frac{1}{2\pi} \int_0^{2\pi} g_2(\varphi_{2,\theta}) d\theta,$$

which is obviously S^1 invariant. Notice that if f_2 is any S^1 invariant function satisfying equation (4.2) then by integrating along the S^1 flow f_2 is necessarily of the form given by (4.3). Hence such an f_2 is indeed unique.

If we check that $X_1(f_2) = 0$, then we can write $g_2 = f_2 + X_2(F_2)$, with f_2 satisfying $X_1(f_2) = 0$ and $X_2(f_2) = 0$. That is to say, these functions g_2 and f_2 solve the second equation stated in the proposition.

One can check this directly on formula (4.3), using the commutation relation $X_1(g_2) = X_2(g_1)$ and the fact that the flows of X_1 and X_2 commute; one can also from equation (4.2) write

$$0 = X_1(f_2) + X_2(X_1(F_2) - g_1),$$

where $X_2(X_1(f_2)) = 0$. This equation can be seen as a decomposition for the zero function. Using the uniqueness of the S^1-invariant function in this decomposition we obtain

$$X_1(f_2) = 0, \quad X_2(X_1(F_2) - g_1) = 0,$$

in particular this also yields that the function $\tilde{g}_1 = g_1 - X_1(F_2)$ is S^1-invariant.

2. Formal resolution of the system. In order to solve the initial system we need to find a smooth function f_1 such that $X_1(f_1) = 0$ and $X_2(f_1) = 0$ and a smooth function F_1 solving the system

(4.4) $$\begin{aligned} X_1(F_1) &= \tilde{g}_1 - f_1 \\ X_2(F_1) &= 0, \end{aligned}$$

Once this system has been solved the desired function F solving the initial system can be written as $F = F_1 + F_2$.

In order to solve this system we will first find a formal solution using formal power series and in a further step we will take care of the remaining flat functions along S_1.

We first solve the system in formal power series in (z_1, z_2), which is fairly easy. It amounts to solving the first equation assuming that all terms in the series commute with q_2 (we can do this because $X_2(\tilde{g_1}) = 0$). As in the hyperbolic case, the formal series for f_1 is unique and is of the form $\sum c_{k,\ell}(\check{z}) q_1^k q_2^\ell$, where $\check{z} = (x_3, y_3, \ldots, x_n, y_n)$. Now we can use a Borel resummation in the variables (q_1, q_2) for f_1 and an S^1-invariant Borel resummation for F_1, which ensures that the system is reduced to the situation where the right hand-side of the first equation of (4.4) is a function g_1 which is S^1 invariant and flat at $\{z_1 = z_2 = 0\}$. These Borel resummations can be chosen uniform in the $\check{z}$ variables.

3. Solving the equation $X_1(F_1) = g_1$ for an S^1-invariant function which is flat along S_1. We could finish the proof by invoking a similar formula as for the hyperbolic case (Proposition 3.4). But checking the smoothness in all variables is not so obvious; we present here a small variant which uses Theorem A and B stated in the preceding subsection and which are contained in [8].

The strategy is exactly the same as in [8], with the additional requirement of keeping track of the S^1 symmetry. We give below the arguments for the sake of completeness.

First of all, using an S^1-invariant cut-off function in $\mathbb{R}^{2n}$, one can assume that g_1 is compactly supported while still commuting with X_2. Again, let us call this new function by g_1. It is clear that if one solves the corresponding system (4.4) in $\mathbb{R}^{2n}$, the associated germs for F_1 and f_1 will solve the initial local problem. Let $\varphi_{1,t}$ be the flow of q_1. The matrix associated to the linear vector fields X_1 has two positive and two negative eigenvalues.

We first apply Theorem B with parameters $x_j, y_j, j \neq 1$ and $j \neq 2$ with $N = S_1$, $E = E^+ = \{z_1 = 0\}$ and $U(t) = \varphi_{1,-t}$. As explained in the proof of Theorem A in [8] this allows to solve the equation to infinite order on the $2n - 2$ dimensional invariant subspace $E^+ = \{z_1 = 0\}$. Observe that the formula provided in the statement of Theorem B shows that if the function g depends smoothly on the parameters x_j and y_j for $j \neq 1$ and $j \neq 2$ then

the function f does also depend smoothly on this parameters because $\varphi_{1,-t}$ leaves the set S_1 fixed.

Therefore using an S^1-invariant Borel resummation, we are then reduced to the case where g_1 is flat on E^+ and S^1-invariant, and we terminate by a second application of Theorem B with parameters x_j, y_j, $j \neq 1$ and $j \neq 2$ with $N = E^+$ and $E = \mathbb{R}^{2n}$. That is the function F_1 is given by the formula

$$F_1 = - \int_0^\infty g_1 \circ \varphi_{1,t} dt.$$

Again this function F_1 is smooth in all the variables since g_1 is smooth in all the variables. Using this formula we see that $X_2(F_1) = 0$ because $\varphi_{1,t}$ and $\varphi_{2,\theta}$ commute.

The justification of the last claim of the proposition goes as before, by examining the explicit formulae and the Borel resummations. The claimed uniqueness of f_1 modulo z_j-flat functions along S_j is a direct consequence of the uniqueness of the formal solution in the z_j variables. Of course, one can also check it by an *a posteriori* argument as we did in the remark after lemma 3.3. $\qquad\qquad\square$

5. The proof of Theorem 2.2

Consider $s = k_e + k_h + k_f$. As we observed in section 2. we have $r = k_e + k_h + 2k_f$. Observe also that $r = s$ if there are no focus-focus components. We prove the theorem using induction on s for a fixed n.

In order to simplify the statements involving focus-focus pairs, we introduce some more notation. Let the vector fields $Y_1, Y_2, \ldots, Y_s$ be such that $Y_j = X_j$ for elliptic or hyperbolic cases (*ie.* for $j \leq k_e + k_h$) while $Y_j = X_{\sigma(j)} + \sqrt{-1} X_{\sigma(j)+1}$ for focus-focus pairs (*ie.* $j > k_e + k_h$ and $\sigma(j) := 2j - k_e - k_h - 1$). Similarly we define γ_j to be g_j for elliptic or hyperbolic indices, and $\gamma_j = g_{\sigma(j)} + \sqrt{-1} g_{\sigma(j)+1}$ for focus-focus indices.

For any $j \leq s$ let $\mathcal{C}_j$ be the space of all germs of complex functions $f \in \mathcal{C}^\infty(\mathbb{R}^{2n}, 0)$ such that $Y_j(f) = \overline{Y_j}(f) = 0$, and $\mathcal{F}_s = \cap_{j \leq s} \mathcal{C}_j$.

With these notations, the system we wish to solve has the form $\gamma_j = f_j + Y_j(G)$ ($\forall j \in \{1, \ldots, s\}$) for germs of smooth functions G and f_j, where $f_j \in \mathcal{F}_s$ and G and f_j, $j \leq k_h + k_e$ are real valued. The commutation relations are $\overline{Y_i}(\gamma_j) = Y_j(\overline{\gamma_i})$ and $Y_i(\gamma_j) = Y_j(\gamma_i)$ (of course the second one is redundant except when both Y_i and Y_j are complex).

Suppose throughout the rest of the proof that $r < n$. For any subindex i corresponding to an elliptic or hyperbolic vector field Y_i , we denote by

$z_i = (x_i, y_i)$ and $\check{z}_i = (z_1, \ldots, \check{z}_i, \ldots, z_n)$. For any subindex j corresponding to a focus-focus pair Y_j, we denote by $z_j = (x_i, y_i, x_{i+1}, y_{i+1})$ and $\check{z}_j = (z_1, \ldots, \check{z}_j, \ldots, z_n)$ (with $i = \sigma(j)$). We denote by S_j the set $S_j = \{z_j = 0\}$.

This being said, one notices that there is no more need to keep the vector fields Y_j in a particular order, which is of course most convenient for the induction process.

Sublemma 5.1. *Let Z be a (real or complex) vector field on $\mathbb{R}^{2n}$ acting trivially on the variables $(z_1, \ldots, z_s)$. Let $j \leq s$. Let f be a smooth real valued function on $\mathbb{R}^{2n}$ such that:*

(1) $f \in \mathcal{F}_s$
(2) $Z(f)$ is flat along S_j.

Then there exists a smooth real valued function $\tilde{f} \in \mathcal{F}_r$ such that

(1) $Z(\tilde{f}) = 0$
(2) $f - \tilde{f}$ is flat along S_j.

Proof. Consider the Taylor expansion of f in z_j. Because $Y_j(f) = 0$ this expansion is a formal series in q_j (in case of an elliptic or hyperbolic Y_j) or in q_i, q_{i+1} (in case of a focus-focus Y_j, with $i = \sigma(j)$). Moreover the coefficients of this expansion are functions of $\check{z}_j$ that are annihilated by X_j, $j \leq r, j \neq i$, and Z. Hence using a suitable Borel resummation one can come up with a smooth $\tilde{f}$ satisfying the requirements of our statement. $\square$

5.1. **Case $s = 1$.**

(1) The Cartan subalgebra has Williamson type $(1, 0, 0)$ or $(0, 1, 0)$. In this case there is only one function. Propositions 3.1 (in the case X_i is elliptic) and 3.2 (in the case X_i is hyperbolic) guarantee that the theorem holds.

(2) The Cartan subalgebra has Williamson type $(0, 0, 1)$. In this case there are two functions g_1 and g_2 fulfilling the conditions specified in Proposition 4.3, and the proposition guarantees that the theorem holds.

5.2. **Passing from s to $s + 1$.** By hypothesis we can construct G and $f_1, \ldots, f_s$ such that

$$\forall j \leq s, \qquad \gamma_j = f_j + Y_j(G),$$

with $f_j \in \mathcal{F}_r, \forall j \leq r$. Observe that when we pass from s to $s + 1$ we are adding a real vector field if the Williamson type changes from (k_e, k_h, k_f) to

$(k_e + 1, k_h, k_f)$ or from (k_e, k_h, k_f) to $(k_e, k_h + 1, k_f)$. In the case we increase in one the number of focus-focus components we are adding a complex vector field. The proof will go in two steps. First we modify the existing f_j and G in such a way that the new f_j's, $j \leq s$ are in $\mathcal{F}_{s+1}$. The final step is to look for a new G of the form $\tilde{G} = G + K$ which leads to the system

$$Y_1(K) = \ldots Y_s(K) = 0, \quad \tilde{\gamma}_{s+1} = f_{s+1} + Y_{s+1}(K),$$

with $Y_j(\tilde{\gamma}_{s+1}) = \overline{Y_j}(\tilde{\gamma}_{s+1}) = 0, \forall j \leq s$.

1. Let us consider the commutation relations

$$\overline{Y_{s+1}}(\gamma_j) = Y_j(\overline{\gamma_{s+1}}) \quad \text{and} \quad Y_{s+1}(\gamma_j) = Y_j(\gamma_{s+1}).$$

We distinguish three subcases:

(1) The vector field Y_j is elliptic: From the uniqueness of the function g_1 of the decomposition in Proposition 3.1 (possibly applied to the real and imaginary parts of Y_{s+1}) this condition tells us that $Y_{s+1}(f_j) = 0$. Therefore in this case no modification of f_j is required and $f_j \in \mathcal{F}_{s+1}$.

(2) The vector field Y_j is hyperbolic: By applying lemma 3.3 we deduce that the z_j-jet of $Y_{s+1}(f_j)$ is zero. We can write $Y_{s+1}(f_j) = \alpha_j$ where α_j is a z_j-flat function along S_j. We can now apply sublemma 5.1 to obtain the following decomposition $f_j = \tilde{f}_j + \phi_j$ where $\tilde{f}_j \in \mathcal{F}_{s+1}$ and $\phi_j \in \mathcal{F}_s$ is a z_j-flat function.

We may apply lemma 3.4 to the function ϕ_j to find a function φ_j satisfying $Y_j(\varphi_j) = \phi_j$. According to Proposition 3.2, this function φ_j can be chosen such that $Y_j(\varphi_j) = 0$ for $j \neq i$ and $j \leq s$. Hence for this γ_j we can write

$$\gamma_j = \tilde{f}_j + Y_j(\varphi_j + G).$$

(3) The vector field Y_j is a focus-focus complex vector field. The commutation conditions also read:

$$Y_{s+1}(\Re\gamma_j) = \Re(Y_j)(\gamma_{s+1}).$$

$$Y_{s+1}(\Im\gamma_j) = \Im(Y_j)(\gamma_{s+1}).$$

From the second equation and the uniqueness of the function f_2 obtained in Proposition 4.3 we obtain $Y_{s+1}(\Im f_j) = 0$ so we only need to modify $\Re f_j$.

Now since $\Im(Y_j)(\Re f_j) = 0$ and $\Re(Y_j)(\Re f_j) = 0$ we can invoke the uniqueness up to a flat function of the function f_1 in the decomposition claimed in proposition 4.3 applied to the first equality to deduce that $Y_{s+1}(\Re f_j)$ is z_j-flat along S_j. Hence by sublemma 5.1 applied to $Z = Y_{s+1}$ we can write $\Re f_j = h_j + \phi_j$ where h_j is a real function in $\mathcal{F}_{s+1}$ and $\phi_j \in \mathcal{F}_s$ is a real z_j-flat function along S_j; therefore as in the proof of Proposition 4.3 we can integrate ϕ_j to a function φ_j satisfying $\Re Y_j(\varphi_j) = \phi_j$. Hence

$$\gamma_j = \tilde{f}_j + Y_j(G + \varphi_j),$$

with $\tilde{f}_j = f_j - \phi_j \in \mathcal{F}_{s+1}$.

2. After considering all these cases we may write

$$g_j = \tilde{f}_j + Y_j(\varphi_j + G) \quad , \forall j \leq s$$

where $\varphi_j \in \mathcal{F}_s$ is a real function equal to the zero function for subindices corresponding to elliptic Y_j. Now define $\widetilde{G} = \sum_i \varphi_i + G$. This function satisfies

$$Y_j(\widetilde{G}) = Y_j(\varphi_j + G) \quad , \forall j \leq s.$$

To finally prove the theorem, it suffices to find a real function K and $f_{s+1} \in \mathcal{F}_{s+1}$ such that

$$\begin{cases} \gamma_j = \tilde{f}_j + Y_j(\tilde{G} + K), & \text{for } j \leq s \\ \gamma_{s+1} = f_{s+1} + Y_{s+1}(\tilde{G} + K). \end{cases}$$

But consider $\tilde{\gamma}_{s+1} := \gamma_{s+1} - Y_{s+1}(\widetilde{G})$. The commutation relations yield

$$(5.1) \qquad\qquad Y_j(\tilde{\gamma}_{s+1}) = \overline{Y_j}(\tilde{\gamma}_{s+1}) = 0$$

for $j \leq s$, and we still have (in case $s + 1$ is a focus-focus index)

$$(5.2) \qquad\qquad Y_{s+1}(\overline{\tilde{\gamma}_{s+1}}) = \overline{Y_{s+1}}(\tilde{\gamma}_{s+1}).$$

Thus our system becomes

$$\begin{cases} 0 = Y_j(K), & \text{for } j \leq s \\ \tilde{\gamma}_{s+1} = f_{s+1} + Y_{s+1}(K), \end{cases}$$

and since $\tilde{\gamma}_{s+1} \in \mathcal{F}_s$ (equation (5.1)), it is solved by an application of proposition 3.1, 3.2 or 4.3, depending on the type of Y_{s+1} (notice that the relation (5.2) is precisely the commutation relation required in the focus-focus case). This ends the proof of the theorem.

6. DEFORMATIONS OF COMPLETELY INTEGRABLE SYSTEMS

Theorem 2.2 has a natural interpretation in terms of infinitesimal deformations of integrable systems near non-degenerate singularities. This was stated without proof in [16]. Let us recall briefly the appropriate setting.

A completely integrable system on a symplectic manifold M of dimension $2n$ is the data of n functions $f_1, \ldots, f_n$ which pairwise commute for the symplectic Poisson bracket: $\{f_i, f_j\} = 0$ and whose differentials are almost everywhere linearly independent.

When we are interested in geometric properties of such systems, the main object under consideration is the (singular) lagrangian foliation given by the level sets of the momentum map $f = (f_1, \ldots, f_n)$. We introduce the notation $\mathbf{f}$ for the linear span (over $\mathbb{R}$) of $f_1, \ldots, f_n$. It is an n-dimensional vector space. It is also an abelian Poisson subalgebra of the Poisson algebra $X = (\mathcal{C}^\infty, \{,\})$. Let $\mathcal{C}_{\mathbf{f}} = \{h \in X, \{\mathbf{f}, h\} = 0\}$ be the set of functions that commute with all f_i. By Jacobi identity $\mathcal{C}_{\mathbf{f}}$ is a Lie subalgebra of X. The fact that $df_1 \wedge \cdots \wedge df_n \neq 0$ almost everywhere implies that $\mathcal{C}_{\mathbf{f}}$ is actually abelian. From now on, we are given a point $m \in M$ and everything is localised at m; in particular X is the algebra of germs of smooth functions at m.

Definition 6.1. Two completely integrable systems $\mathbf{f} = \langle f_1, \ldots, f_n \rangle$ and $\mathbf{g} = \langle g_1, \ldots, g_n \rangle$ are *equivalent* (near m) if and only if

$$\mathcal{C}_{\mathbf{f}} = \mathcal{C}_{\mathbf{g}}$$

Geometrically speaking, $\mathbf{f}$ is equivalent to $\mathbf{g}$ if and only if the functions f_i are constant along the leaves of the $\mathbf{g}$-foliation (or vice-versa).

We wish to describe infinitesimal deformations of integrable systems modulo this equivalence relation. For this we fix an integrable system $\mathbf{f}$ and introduce a deformation complex as follows. Let $L_0 \simeq \mathbb{R}^n$ be the typical commutative Lie algebra of dimension n. L_0 acts on X by the adjoint representation:

$$L_0 \times X \ni (\ell, g) \mapsto \{\mathbf{f}(\ell), g\} \in X.$$

Hence X is an L_0-module, in the Lie algebra sense, and we can introduce the corresponding Chevalley-Eilenberg complex [1]: for $q \in \mathbb{N}$, $C^q(L_0, X) = \mathrm{Hom}(L_0^{\wedge q}, X)$ is the space of alternating q-linear maps from L_0 to X (regarded merely as real vector spaces), with the convention $C^0(L_0, X) = X$. The associated differential is denoted by $d_{\mathbf{f}}$. Following [1] for a 0-cochain

$g \in X$, the 1-cochain $d_{\mathbf{f}}(g)$ is $d_f(g)(l) = \{\mathbf{f}(l), g\}, l \in L$ and for a k-cochain ϕ the $k+1$ cochain $d_{\mathbf{f}}(\phi)$ is

$$d_{\mathbf{f}}(\phi)(l_1, \ldots, l_{k+1}) = \frac{1}{k+1} \sum_{i=1}^{k+1} (-1)^{i+1} \{\mathbf{f}(l_i), \phi(\check{l}_i)\}, l_i \in L,$$

where $\check{l}_i = (l_1, \ldots, \check{l}_i, \ldots, l_{k+1})$.

Now since L_0 acts trivially on $\mathcal{C}_{\mathbf{f}}$, the quotient Lie algebra $X/\mathcal{C}_{\mathbf{f}}$ is a L_0-module, and we can define the corresponding Chevalley-Eilenberg complex: for $q \in \mathbb{N}$, $C^q(L_0, X/\mathcal{C}_{\mathbf{f}}) = \mathrm{Hom}(L_0^{\wedge q}, X/\mathcal{C}_{\mathbf{f}})$, with differential denoted by $\bar{d}_{\mathbf{f}}$.

Finally we define the *deformation complex* $C^\bullet(\mathbf{f})$ as follows:

$$0 \longrightarrow X/\mathcal{C}_{\mathbf{f}} \xrightarrow{\bar{d}_{\mathbf{f}}} C^1(L_0, X/\mathcal{C}_{\mathbf{f}}) \xrightarrow{\partial_{\mathbf{f}}} C^2(L_0, X) \xrightarrow{d_{\mathbf{f}}} C^3(L_0, X) \xrightarrow{d_{\mathbf{f}}} \cdots$$

where $\partial_{\mathbf{f}}$ is defined by the following diagram, where all small triangles are commutative ($C^k(L_0, \mathcal{C}_{\mathbf{f}})$ is always in the kernel of $d_{\mathbf{f}}$) :

For all cochain complexes, cocycles and coboundaries are denoted the standard way: $Z^q(\cdot)$ and $B^q(\cdot)$. In the analytic category a similar deformation complex was introduced recently by Van Straten and Garay ([7] and [6]) and (for the first degrees) by Stolovitch [11]. The equivalence used in the analytic category is much easier to handle due to the absence of flat functions.

Definition 6.2. $Z^1(\mathbf{f})$ is the space of infinitesimal deformations of $\mathbf{f}$ modulo equivalence.

If we fix a basis $(e_1, \ldots, e_n)$ of L_0, a cocycle $\alpha \in Z^1(\mathbf{f})$ is just a set of functions $g_1 = \alpha(e_1), \ldots, g_n = \alpha(e_n)$ (defined modulo $\mathcal{C}_{\mathbf{f}}$) such that

(6.1) $\forall i, j \qquad \{g_i, f_j\} = \{g_j, f_i\}.$

It is an infinitesimal deformation of $\mathbf{f}$ in the sense that, modulo ϵ^2,

$$\{f_i + \epsilon g_i, f_j + \epsilon g_j\} \equiv 0.$$

A special type of infinitesimal deformations of $\mathbf{f}$ is obtained by the infinitesimal action of the group G of local symplectomorphisms: given a function

$h \in X$ one can define the deformation cocycle $\alpha \in Z^1(\mathbf{f})$ by

$$(6.2) \qquad L_0 \ni \ell \mapsto \alpha(\ell) = \{h, \mathbf{f}(\ell)\} \mod \mathcal{C}_{\mathbf{f}}.$$

In other words, the set of all such cocyles, with h varying in X, is the orbit of $\mathbf{f}$ under the adjoint action on $Z^1(\mathbf{f})$ of the Lie algebra of G. From equation (6.2) one immediately sees that this orbit is exactly $B^1(\mathbf{f})$.

In the particular case that ω is the Darboux symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ and $\mathbf{f} = (q_1, \ldots, q_n)$ is a Williamson basis as specified in theorem 2.1, we can reformulate the statement of theorem 2.2 in cohomological terms.

Namely, in this case since $\{f_i, f\} = X_i(f)$, we can write $C_q = \{f \in X, X_i(f) = 0, \forall i\}$. Let α be a 1-cocycle, the cocycle condition specified in formula 6.1 reads as $X_j(g_i) = X_i(g_j)$ where $g_i = \alpha(e_i)$. But this is nothing but the commutation hypothesis of theorem 2.2 therefore there exists a function G such that $g_i = f_i + X_i(G)$. Using formula 6.2 and the definition of g_i this shows that α is a coboundary. In other words, what theorem 2.2 shows in cohomological terms is that any $\alpha \in Z^1(\mathbf{f})$ is indeed a coboundary. And this proves the following reformulation of theorem 2.2:

Theorem 6.3. *Let $q_1, \ldots, q_n$ be a standard basis (in the sense of Williamson) of a Cartan subalgebra of $\mathcal{Q}(2n, \mathbb{R})$. Then the corresponding completely integrable system $\mathbf{q}$ in $\mathbb{R}^{2n}$ is $\mathcal{C}^\infty$-infinitesimally stable at $m = 0$: that is,*

$$H^1(\mathbf{q}) = 0.$$

Remark 6.4. Our proof actually shows that the result is also true when we include a smooth dependence on parameters in the definition of the deformation complex.

This theorem should have important applications in semi-classical analysis, where we consider pseudodifferential operators with $\mathcal{C}^\infty$ symbols depending on a small parameter $\hbar$. One can define a similar deformation complex for pseudodifferential operators, where the deformation is understood with respect to the parameter $\hbar$. Then in many situations the vanishing of the classical H^1 implies the vanishing of the pseudodifferential H^1. See [16] for general remarks, and [14, 3] for applications in simple cases where the vanishing of the pseudodifferential H^1 was checked explicitly.

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