Analytic Bergman operators in the semiclassical limit

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Abstract

Using a new quantization scheme, we construct approximate semiclassical Bergman projections on weighted L^2 spaces with analytic weights, and show that their kernel functions admit an asymptotic expansion in the class of analytic symbols. As a corollary, we obtain new estimates for asymptotic expansions of the Bergman kernel on $\mathbb{C}^n$ and for high powers of ample holomorphic line bundles over compact complex manifolds.

1 Introduction

Let $L \to X$ be a holomorphic line bundle over a closed complex manifold X, and assume that L is equipped with a positive Hermitian metric. The corresponding Chern form induces a Riemannian metric on X, and the integrated scalar product on L gives a natural Hilbert space structure on sections of L. In this work we will be interested in the so-called *semiclassical limit* L^k of high tensor powers of the line bundle L. The line bundle L^k is naturally equipped with the product Hermitian metric, and we may consider the Hilbert space $L^2(X; L^k)$ of square-integrable sections of L^k. The orthogonal projection onto holomorphic sections:

$$\Pi_k : L^2(X; L^k) \to H^0(X; L^k)$$

is called the Bergman projection. A central question in complex geometry is to understand the asymptotic behavior, as $k \to +\infty$, of the distributional

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kernel $K(x,y;k)$ of Π_k. The same problem arises in the sister theory of the Szegö projection, for which X is replaced by domains of $\mathbb{C}^n$. After the pioneer works of Fefferman [17], Boutet de Monvel–Sjöstrand [8] and Kashiwara [24] on the Szegö projection, their techniques have been transposed over to compact complex manifolds. In particular, thanks to the works by Bouche [5], Tian [30], and then Catlin [10], Zelditch [31], a complete asymptotic expansion of the Bergman function (the norm of the Bergman kernel on the diagonal) was given:

$$|K(x,x;k)|_{L_x^k} \sim k^n \left(b_0(x) + \frac{b_1(x)}{k} + \frac{b_2(x)}{k^2} + \cdots \right)$$

where the coefficients b_j are analytic functions on X.

Since then, there has been an intense activity on getting a better understanding of this expansion: extending it away from the diagonal in $X \times X$, estimating the growth of the coefficients, extending to $\mathscr{C}^\infty$-smooth or $\mathscr{C}^k$ metrics on X, etc. See for instance the expository works [4], [25], and the references therein. Here, we wish to consider the issue of the relationships between the analyticity of the metric on L and optimal estimates for b_k on or off the diagonal. The question has raised recent interest, see for instance the articles [13], [12], [19]. In this last paper, the authors prove that, if the metric is analytic, the estimate $\left|\tilde{b}_k(x,y)\right| \leq C^k k!^2$ holds locally uniformly near the diagonal in $X \times X$, where $\tilde{b}_k$ is the holomorphic extension of b_k. But they also conjecture[1] that a stronger, more natural estimate

$$\left|\tilde{b}_k(x,y)\right| \leq C^k k! \tag{1.1}$$

could hold, and relate various debates on this issue. One of our main results, Theorem 6.1 below, settles the question in a positive way, showing that the Bergman kernel admits an asymptotic expansion in the topology of analytic symbols, which implies that the more natural version (1.1) is correct. After this article was written, we realized that the estimate (1.1) (but not the asymptotic expansion) has been obtained independently by Hezari and Xu [20].[2]

[1] This question was also asked to us by S. Zelditch and L. Charles. According [20], this conjecture is attributed to Zelditch in 2014.

[2] Recently, we have received a preliminary text by A. Deleporte which gives quite a different proof of Theorem 6.1. See Remark 6.6 below.

But this result was not our unique goal. In fact, our initial motivation for undertaking this research has its roots in the spectral theory of Berezin-Toeplitz operators. Starting from a prequantizable Kähler manifold X, one can construct a Hermitian line bundle as above, and define an algebra of operators extending the usual geometric quantization scheme, see [2, 6], and also [11] and references therein. Such operators are defined by a 'symbol' f, which is a function on X, through the formula

$$T_f : u \mapsto \Pi_k(fu) : H^0(X; L^k) \to H^0(X; L^k)\,.$$

In [27] it was conjectured that, for Berezin-Toeplitz operators with analytic symbols on a Riemann surface, one has a very accurate asymptotic description, in the semiclassical limit $k \to +\infty$, of all individual eigenvalues, provided that the operator is 'nearly selfadjoint'. The conjecture was supported by the proof of this result in the case of analytic pseudo-differential operators acting on $L^2(\mathbb{R})$ or $L^2(S^1)$ [27], and, under some additional geometric assumption (related to complete integrability), on $L^2(\mathbb{R}^2)$ [26, 21, 22].

Although the idea of transposing these results to the Berezin-Toeplitz case is natural, analytic microlocal analysis was never applied to general Berezin-Toeplitz operators, and there was a fundamental obstacle to this. Namely, one should prove that the Bergman projection Π_k, viewed as a complex Fourier integral operator, has an analytic symbol with a suitable asymptotic expansion. Building the necessary theory and finally proving this result constitutes the core of this article, see Theorems 3.1, 5.5 and 6.1.

To conclude this introduction, we would like to give an informal overview of the method. As mentioned above, it had been realized for a long time that the asymptotic study of the Bergman kernel is tightly related to semiclassical analysis, see [8, 31]. More recently, other approaches have been proposed that derive asymptotic expansions in a more 'elementary' (but still semiclassical) way, see for instance [3]. The techniques that we use in this article also go back to the initial ideas, but in a more systematic and natural way: we combine a fully microlocal approach with L^2 estimates, in order to obtain a transparent 'local-to-global' principle. The first step is to construct an approximate Bergman projection by means of analytic microlocal analysis, via a new quantization scheme that we call Bargmann-Bergman (or Brg for short) quantization. By its local nature, this step does not require any geometric assumption on the underlying phase space. The second step uses an L^2-analysis of these operators (combined with the usual Hörmander $\overline{\partial}$

estimates) to show that, up to an exponentially small error in terms of the semiclassical parameter, the exact Bergman projection coincides with the microlocally constructed one. Once this is established, the estimates for the asymptotics of the Bergman kernel are a consequence of the pseudo-differential calculus in analytic classes of symbols developed in [28]. We have applied this second step to two cases, namely $\mathbb{C}^n$ (Section 5) and compact complex manifolds (Section 6). It would be interesting to apply the same program to more general non-compact geometries.

Organization of the article

In section 2 we introduce the 'Brg' quantization on weighted spaces of germs of holomorphic functions H_{Φ,x_0}, where $x_0 \in \mathbb{C}^n$ and Φ is an analytic, strictly plurisubharmonic function defined near x_0 (Definition 2.20). This particular form of quantization is directly inspired by the well-known exact formula for the Bergman projection in the setting of weighted L^2 spaces on $\mathbb{C}^n$, when the weight is quadratic (Proposition 2.4).

Section 3 contains the main microlocal result of the paper, namely: using an analytic Fourier integral operator, one obtains a microlocal equivalence between the Brg quantization and the 'usual' (but complex) Weyl quantization (Theorem 3.1). The proof consists in expressing this equivalence as a product of analytic Fourier integral operators, and proving transverse intersection of the underlying canonical relations.

In Section 4, we cast the microlocal result in terms of approximate weighted L^2 spaces on nested domains. This allows the construction of approximate Bergman projections (Proposition 4.9), which are unique modulo an exponentially small error (Proposition 4.15).

Sections 5 and 6 contain the main applications to the asymptotics of the Bergman kernel. In Section 5 we treat the case of weighted L^2 spaces on $\mathbb{C}^n$, while Section 6 deals with the Bergman projection associated with a holomorphic Hermitian line bundle.

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Contents

2 Brg quantization

2.1 FBI-Bargmann transforms and Bergman kernels

For the sake of completeness, and in order to introduce the relevant notation, we discuss here the FBI-Bargmann transform, which is in fact a generalization of the original Segal-Bargmann and FBI transforms from [1] and [9] to the case of a general strictly plurisubharmonic quadratic form $\Phi : \mathbb{C}^n \to \mathbb{R}$

(see Definition 2.2). The original case investigated by Bargmann corresponds to $\Phi(z) = \frac{1}{2}|z|^2$; the corresponding transform has been used in various settings under different names: Bargmann-Segal, Gabor, or wavepacket transforms. The general case was studied by several authors; one can find a good account of the theory in the book [32, Chapter 13]. In [28], [29] these transformations are treated as Fourier integral operators and integrated into microlocal (semiclassical) analysis.

We present here the semiclassical version. Let $0 < \hbar \leq 1$ be the semiclassical parameter. Without explicit notice, all constants in this text are implicitly independent of $\hbar$.

Definition 2.1. *Let $\phi(z, x)$ be a holomorphic quadratic function on $\mathbb{C}^n \times \mathbb{C}^n$ such that:*

i) $\Im\left(\frac{\partial^2\phi}{\partial x^2}\right)$ *is a positive definite matrix;*

ii) $\det\left(\frac{\partial^2\phi}{\partial x \partial z}\right) \neq 0.$

*The **FBI-Bargmann transform** associated with the function ϕ is the operator, denoted by T_ϕ, defined on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ by:*

$$T_\phi u(z) = c_\phi \hbar^{-3n/4} \int_{\mathbb{R}^n} e^{(i/\hbar)\phi(z,x)} u(x) \mathrm{d}x, \quad z \in \mathbb{C}$$

where:

$$c_\phi = \frac{1}{2^{n/2}\pi^{3n/4}} \frac{|\det \partial_x \partial_z \phi|}{(\det \Im \partial_x^2 \phi)^{1/4}}.$$

The canonical transformation associated with T_ϕ is given by

$$\begin{aligned} \kappa_\phi : \mathbb{C}^n \times \mathbb{C}^n &\longrightarrow \mathbb{C}^n \times \mathbb{C}^n \\ (x, -\partial_x \phi(z, x)) &\longmapsto (z, \partial_z \phi(z, x)). \end{aligned}$$

Definition 2.2. *A function $\Phi \in \mathscr{C}^2(\mathbb{C}^n; \mathbb{R})$ is called **plurisubharmonic** (respectively **strictly plurisubharmonic**) if, for all $x \in \mathbb{C}^n$, the matrix $(\partial^2_{x_j, \bar{x}_k}\Phi)^n_{j,k=1}$ is positive semidefinite (respectively positive definite).*

We shall often identify the matrix $(\partial^2_{x_j,\bar{x}_k}\Phi)^n_{j,k=1}$ (where j is the line index and j the column index) with the $(1,1)$-form $\partial_{\bar{x}}\partial_x \Phi = \sum_{j,k} \partial^2_{x_j,\bar{x}_k}\Phi \,\mathrm{d}\bar{x}_k \wedge \mathrm{d}z_j$.

Proposition 2.3 ([29]). *With the notation of Definition 2.1, define for $z \in \mathbb{C}^n$:*

$$\Phi(z) := \max_{x\in\mathbb{R}^n} -\Im\phi(z,x). \tag{2.1}$$

Then Φ is a strictly plurisubharmonic quadratic function and the canonical transformation κ_ϕ is a bijection from $\mathbb{R}^{2n}$ to

$$\Lambda_\Phi = \left\{ \left(z, \frac{2}{i}\frac{\partial \Phi}{\partial z}(z) \right) ; z \in \mathbb{C}^n \right\}. \tag{2.2}$$

Throughout the paper, we shall use the following notation.

- $L(\mathrm{d}z)$ is the Lebesgue measure on $\mathbb{C}^n$, *i.e.*

$$L(\mathrm{d}z) = \prod_{j=1}^{n} \left(\frac{i}{2} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j \right) =: \left(\frac{i}{2}\right)^n \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{2.3}$$

- $L^2_\Phi(\mathbb{C}^n) := L^2(\mathbb{C}^n, e^{-2\Phi(z)/\hbar} L(\mathrm{d}z))$ is the set of measurable functions $f : \mathbb{C}^n \to \mathbb{C}$ such that:

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-2\Phi(z)/\hbar} L(\mathrm{d}z) < +\infty.$$

- $H_\Phi(\mathbb{C}^n) := \mathrm{Hol}(\mathbb{C}^n) \cap L^2_\Phi(\mathbb{C}^n)$ is the closed subspace of holomorphic functions in $L^2_\Phi(\mathbb{C}^n)$.

- If $z, w \in \mathbb{C}^n$, $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$, then we denote by $z \cdot w$ the 'complex scalar product', *i.e.*

$$z \cdot w := \sum_{j=1}^{n} z_j w_j \ .$$

Proposition 2.4 ([29, Formula (1.12)]). *Let Φ be the strictly plurisubharmonic quadratic function defined by Equation (2.1), and let ψ be the unique holomorphic quadratic form on $\mathbb{C}^n \times \mathbb{C}^n$ such that, for all $z \in \mathbb{C}^n$:*

$$\psi(z, \bar{z}) = \Phi(z)\,.$$

The following properties hold.

i) The orthogonal projection $\Pi_\Phi : L^2_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n)$ is given by:

$$\Pi_\Phi u(z) = \frac{2^n \det(\partial^2_{z\bar{z}}\Phi)}{(\pi\hbar)^n} \int_{\mathbb{C}^n} e^{\frac{2}{\hbar}(\psi(z,\bar{w})-\Phi(w))} u(w) L(\mathrm{d}w). \tag{2.4}$$

ii) $T_\phi : L^2(\mathbb{R}^n) \to H_\Phi(\mathbb{C}^n)$ is a unitary transformation and if $T^*_\phi : L^2_\Phi(\mathbb{C}^n) \to L^2(\mathbb{R}^n)$ is the adjoint of T_ϕ, then $\Pi_\Phi = T_\phi T^*_\phi$.

The operator Π_Φ is called the Bergman projection onto H_Φ.

The FBI transform allows to obtain a correspondence between Weyl operators acting on $L^2(\mathbb{R}^n)$ and Weyl operators acting on $H_\Phi(\mathbb{C}^n)$, introduced in [29]. Let $S(\mathbb{R}^{2n})$ denote the following symbol class:

$$S(\mathbb{R}^{2n}) = \{a \in \mathscr{C}^\infty(\mathbb{R}^{2n}); \quad \forall \alpha \in \mathbb{N}^{2n}, \exists C_\alpha > 0, |\partial^\alpha a| \leq C_\alpha\}.$$

Using the parametrization (2.2) of $\Lambda_\Phi \simeq \mathbb{C}^n$, we will also use the class of symbols $S(\Lambda_\Phi)$ that we identify with $S(\mathbb{C}^n) \simeq S(\mathbb{R}^{2n})$.

Definition 2.5. *Let $a_\hbar \in S(\mathbb{R}^{2n})$. Define the **Weyl quantization** of $a_\hbar$, denoted by $\mathrm{Op}^{\mathrm{w}}_\hbar(a_\hbar)$, by the following formula, for $u \in \mathscr{S}(\mathbb{R}^n)$:*

$$\left[\mathrm{Op}^{\mathrm{w}}_\hbar(a_\hbar)u\right](x) = \frac{1}{(2\pi\hbar)^n} \iint_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(x-y)\cdot\xi} a_\hbar\left(\tfrac{x+y}{2}, \xi\right) u(y)\mathrm{d}y\mathrm{d}\xi.$$

Then $a_\hbar$ is called the (Weyl) symbol of the pseudo-differential operator $\mathrm{Op}^{\mathrm{w}}_\hbar(a_\hbar)$.

By the Calderon-Vaillancourt theorem, such an operator $\mathrm{Op}^{\mathrm{w}}_\hbar(a_\hbar)$ extends to a bounded operator on $L^2(\mathbb{R}^n)$ whose operator norm is bounded by a constant independent of $\hbar$.

Definition 2.6. *Let $b_\hbar \in S(\Lambda_\Phi)$. The **complex Weyl quantization** of the symbol $b_\hbar$ is the operator given by the contour integral:*

$$[\mathrm{Op}^{\mathrm{w}}_\Phi(b_\hbar)u](z) = \frac{1}{(2\pi\hbar)^n} \iint_{\Gamma(z)} e^{(i/\hbar)(z-w)\cdot\zeta} b_\hbar\left(\tfrac{z+w}{2}, \zeta\right) u(w)\mathrm{d}w\mathrm{d}\zeta,$$

where $\Gamma(z) = \left\{(w,\zeta) \in \mathbb{C}^{2n}; \zeta = \dfrac{2}{i}\dfrac{\partial\Phi}{\partial z}\left(\tfrac{z+w}{2}\right)\right\}$.

The following Egorov type theorem is, formally, an application of the invariance of Weyl quantization under the metaplectic representation (with complex quadratic phase).

Proposition 2.7 ([21]). *Let $a_\hbar \in S(\mathbb{R}^{2n})$. We have*

$$T_\phi \mathrm{Op}_\hbar^{\mathrm{w}}(a_\hbar) T_\phi^* = \mathrm{Op}_\Phi^{\mathrm{w}}(b_\hbar)$$

where the symbol $b_\hbar$ is given by $b_\hbar = a_\hbar \circ \kappa_\phi^{-1}$, thus $b_\hbar \in S(\Lambda_\Phi)$.

In particular, $\mathrm{Op}_\Phi^{\mathrm{w}}(b_\hbar) : H_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n)$ is uniformly bounded with respect to $\hbar$.

There exists also a connection between the Berezin-Toeplitz quantization and the complex Weyl quantization $\mathrm{Op}_\Phi^{\mathrm{w}}$ of Definition 2.6 above. Let us first recall the definition of the Berezin-Toeplitz quantization of $\mathbb{C}^n$.

Definition 2.8. *Let $f_\hbar \in S(\mathbb{C}^n)$. Define the* ***Berezin-Toeplitz quantization*** *of $f_\hbar$ as:*

$$T_{f_\hbar} := \Pi_\Phi M_{f_\hbar} \Pi_\Phi,$$

where $M_{f_\hbar} : L^2_\Phi(\mathbb{C}^n) \to L^2_\Phi(\mathbb{C}^n)$ is the operator of multiplication by the function $f_\hbar$. We call $f_\hbar$ the symbol of the Berezin-Toeplitz operator $T_{f_\hbar}$.

Remark 2.9. *In [29], Berezin-Toeplitz operators on $\mathbb{C}^n$ were denoted by $\widetilde{\mathrm{Op}}_{\hbar,0}$.*

The relation between the Berezin-Toeplitz and the complex Weyl quantizations of $\mathbb{C}^n$ is given in the next proposition, where we identify Λ_Φ with $\mathbb{C}^n$.

Proposition 2.10 ([29, (1.23)], [32, Theorem 13.10]).

i) Let $f_\hbar \in S(\mathbb{C}^n)$ admit an asymptotic expansion in powers of $\hbar$. Let $T_{f_\hbar}$ be the Berezin-Toeplitz operator of symbol $f_\hbar$. Then, we have:

$$T_{f_\hbar} = \mathrm{Op}_\Phi^{\mathrm{w}}(b_\hbar) \quad \text{in} \quad \mathcal{L}(H_\Phi(\mathbb{C}^n)),$$

where $b_\hbar \in S(\Lambda_\Phi)$ admits an asymptotic expansion in powers of $\hbar$ given, for all $z \in \Lambda_\Phi \simeq \mathbb{C}^n$, by

$$b_\hbar(z) = \exp\left(\frac{\hbar}{4}\left\langle \left(\partial^2_{z\bar{z}}\Phi\right)^{-1} \partial_z, \partial_{\bar{z}} \right\rangle\right)(f_\hbar(z)) \quad \text{in} \quad S(\Lambda_\Phi). \tag{2.5}$$

ii) Let $b_\hbar \in S(\Lambda_\Phi)$ admit an asymptotic expansion in powers of $\hbar$. Then, there exists a function $f_\hbar \in S(\mathbb{C}^n)$ such that:

$$\mathrm{Op}_\Phi^{\mathrm{w}}(b_\hbar) = T_{f_\hbar} + \mathcal{O}(\hbar^\infty) \quad \text{in} \quad \mathcal{L}(H_\Phi(\mathbb{C}^n)),$$

where $T_{f_\hbar}$ is the Berezin-Toeplitz operator of symbol $f_\hbar$, and $f_\hbar$ admits the following asymptotic expansion in powers of $\hbar$

$$f_\hbar(z) \sim \exp\left(\frac{-\hbar}{4}\left\langle\left(\partial^2_{z\bar{z}}\Phi\right)^{-1}\partial_z, \partial_{\bar{z}}\right\rangle\right)(b_\hbar(z)) \quad in \quad S(\mathbb{C}^n). \tag{2.6}$$

Remark 2.11. *In Item i), we actually don't need $f_\hbar$ to admit an asymptotic expansion in powers of $\hbar$. Then $b_\hbar$ is given by (2.5), which is an exact formula, corresponding to solving a heat equation in positive time. On the other hand, the reverse formula (2.6) is only formal.*

Remark 2.12. *If $f_\hbar = 1$, then $T_{f_\hbar} = \Pi_\Phi$. Hence Proposition 2.10 implies that the Bergman projection Π_Φ can be written as a complex pseudo-differential operator.*

2.2 Analytic symbols

Our goal will be to extend the representation formula (2.4) for the Bergman projection to the case of a general phase function Φ, which is not necessarily a quadratic form. In order to do this, we first need to discuss the microlocal classes of analytic symbols, as introduced in [28], following [7].

Definition 2.13 (Space H^{loc}_Φ)**.** *Let Ω be an open subset of $\mathbb{C}^n$. Let $\Phi \in \mathscr{C}^0(\Omega;\mathbb{R})$. Let $u_\hbar$ be a function defined on Ω. We say that $u_\hbar$ belongs to the space $H^{\text{loc}}_\Phi(\Omega)$ if:*

1. *$u_\hbar \in \mathrm{Hol}(\Omega)$;*
2. *$\forall K \Subset \Omega$, $\forall \epsilon > 0$, $\exists C > 0$, such that $|u_\hbar(z)| \leq Ce^{(\Phi(z)+\epsilon)/\hbar}$ for all $z \in K$.*

*If $u_\hbar \in H^{\text{loc}}_0(\Omega)$ (meaning that $\Phi = 0$), we say that $u_\hbar$ is an **analytic symbol**.*

Notice that analytic symbols may have a sub-exponential growth as $\hbar \to 0$: for instance the constants $\hbar^{-m}$, for any $m \geq 0$, are analytic symbols. In this work it will be important to control the polynomial growth in $\hbar^{-1}$, and hence we introduce a finer definition, as follows.

Definition 2.14. *Let $m \in \mathbb{R}$. We say that $a_\hbar \in \mathrm{Hol}(\Omega)$ is an **analytic symbol of finite order** m if $a_\hbar = \mathcal{O}(\hbar^{-m})$ locally uniformly in Ω, i.e. $\forall K \Subset \Omega$, $\exists C > 0$ such that for all $z \in K$:*

$$|a_\hbar(z)| \leq C\hbar^{-m}.$$

Naturally, analytic symbols of finite order are also analytic symbols in the sense of Definition 2.13. Let $S^0(\Omega)$ be the space of analytic symbols of order zero in Ω.

Definition 2.15. *A **formal classical analytic symbol** $\hat{a}_\hbar$ in Ω is a formal series $\hat{a}_\hbar = \sum_{j=0}^{\infty} a_j \hbar^j$, where $a_j \in \mathrm{Hol}(\Omega)$ satisfies:*

$$\forall K \Subset \Omega, \exists C > 0, \forall j \geq 0, \quad \sup_K |a_j| \leq C^{j+1} j^j.$$

We denote by $\hat{S}^0(\Omega)$ the space of such power series.

The next definition is similar to the one used in [7, Definition 2.1].

Definition 2.16. *We say that $a_\hbar \in S^0(\Omega)$ is a **classical analytic symbol** if there exists $\hat{a}_\hbar \in \hat{S}^0(\Omega)$ such that $a_\hbar$ admits the asymptotic expansion $a_\hbar \sim \hat{a}_\hbar = \sum_{j=0}^{\infty} a_j \hbar^j$, in the following sense:*

$$\forall K \Subset \Omega, \exists C > 0, \forall N \geq 0, \qquad \sup_K \left| a_\hbar - \sum_{j=0}^{N-1} a_j \hbar^j \right| \leq \hbar^N C^{N+1} N^N. \tag{2.7}$$

It is useful to introduce spaces of germs at a given point $x_0 \in \mathbb{C}^n$; we let H_{Φ,x_0} be the space of germs of the presheaf H_Φ^{loc} at x_0, *i.e.* H_{Φ,x_0} is the inductive limit:

$$H_{\Phi,x_0} := \varinjlim_{\Omega \ni x_0} H_\Phi^{\mathrm{loc}}(\Omega),$$

where Ω varies in the set $\mathcal{V}(x_0)$ of open neighbourhoods of x_0. The local space $\widetilde{H}_{\Phi,x_0}$ consists of the germs H_{Φ,x_0} modulo exponentially small terms, as follows.

Definition 2.17 (Negligible germs and $\widetilde{H}_{\Phi,x_0}$)**.** *An element $u_\hbar \in H_{\Phi,x_0}$ will be called **negligible** if it belongs to the space*

$$\mathcal{N} := \{u_\hbar \in H_{\Phi,x_0} \quad ; \quad \exists c > 0, \exists \Omega \in \mathcal{V}(x_0), \quad u_\hbar \in H_{\Phi-c}^{\mathrm{loc}}(\Omega)\}.$$

The x_0-localized space is the quotient:

$$\widetilde{H}_{\Phi,x_0} := H_{\Phi,x_0}/\mathcal{N}.$$

Thus, two germs $u_\hbar$ and $v_\hbar$ at x_0 are equivalent if $e^{-\Phi/\hbar}(u_\hbar - v_\hbar)$ is exponentially small near x_0 as $\hbar \to 0$. We shall use the notation $u_\hbar \sim v_\hbar$ to indicate that $u_\hbar = v_\hbar \mod \mathcal{N}$. Since $S^0(\Omega) \subset H_0^{\mathrm{loc}}(\Omega)$, the space $\widetilde{H}_{0,x_0}$ contains the subspace $(S^0_{x_0} \mod \mathcal{N})$ of symbols of order zero localized at x_0.

If $\Omega \in \mathcal{V}(x_0)$ and $\hat{a}_\hbar \in \hat{S}^0(\Omega)$, then there exists a unique element $a_\hbar \in \widetilde{H}_{0,x_0}$ that admits, in some $B \in \mathcal{V}(x_0)$, the asymptotic expansion given by $\hat{a}_\hbar$, as follows. Let B be an open ball centred at x_0 and such that $B \Subset \Omega$. Let C be the constant of Definition 2.15 with $K = \overline{B}$, and let, for $z \in B$,

$$a_\hbar^B(z) = \sum_{j=0}^{[1/(eC\hbar)]} a_j(z)\hbar^j.$$

Then, one can check that $a_\hbar^B \in S^0(B)$ and, for any other choice of B, say $\tilde{B}$, there exists a constant $\tilde{C} > 0$ such that

$$a_\hbar^B - a_\hbar^{\tilde{B}} = \mathcal{O}(e^{-1/(\tilde{C}\hbar)}) \quad \text{on } B \cap \tilde{B}.$$

Therefore, $a_\hbar := (a_\hbar^B \mod \mathcal{N}) = (a_\hbar^{\tilde{B}} \mod \mathcal{N})$ is well-defined in $\widetilde{H}_{0,x_0}$. With a slight abuse of notation, $a_\hbar$ will be called a classical analytic symbol at x_0.

Definition 2.18. *A linear operator $R : H_\Phi^{\mathrm{loc}}(\Omega) \to H_\Phi^{\mathrm{loc}}(\Omega)$ will be called **H-negligible at** x_0 (the letter "H" stands here for Holomorphic) if for any $\Omega_1 \in \mathcal{V}(x_0)$ with $\Omega_1 \Subset \Omega$, there exists $\Omega_2 \in \mathcal{V}(x_0)$ with $\Omega_2 \Subset \Omega$, and a continuous function $\Phi_2 < \Phi$ on Ω_2, such that*

$$R : H_\Phi(\Omega_1) \to H_{\Phi_2}(\Omega_2)$$

is uniformly bounded as $\hbar \to 0$, where $H_\Phi(\Omega_1)$ and $H_{\Phi_2}(\Omega_2)$ are equipped with the corresponding L^2_Φ-norm (Definition 4.1). When R_1 and R_2 are two operators such that $R_1 - R_2$ is H-negligible, we will write $R_1 \underset{\mathrm{H}}{\equiv} R_2$.

In particular, if R is H-negligible at x_0 and $u_\hbar \in H_{\Phi,x_0}$, then $Ru_\hbar \sim 0$.

Notation. In the rest of this text, when dealing with germs, we will sometimes use then notation $\mathrm{Neigh}(x_0, E)$, where $x_0 \in E$, to denote "a sufficiently small neighbourhood of x_0 in E".

2.3 Analytic pseudo-differential operators

Classical analytic symbols give rise to a well-behaved pseudo-differential calculus, as shown in the book [28]. We recall here the necessary definitions and properties.

Let $x_0 \in \mathbb{C}^n$, let Φ be a $\mathscr{C}^2$ real-valued function defined in a small neighbourhood Ω of x_0. For $x \in \Omega$, $r > 0$ sufficiently small and $R > 0$, we define the contour in $\mathbb{C}^{2n}$:

$$\Gamma(x) := \left\{ (y, \theta) \in \Omega \times \mathbb{C}^n; \quad \theta = \frac{2}{i} \frac{\partial \Phi}{\partial x}(x) + iR\overline{(x-y)}; \quad |x-y| \leq r \right\}. \tag{2.8}$$

By Taylor's formula $\Phi(y) = \Phi(x) + 2\Re((y-x) \cdot \partial_x \Phi(x) + \mathcal{O}(|x-y|^2))$, we obtain the following estimate when $|x-y|$ is small enough:

$$e^{-\Phi(x)/\hbar} \left| e^{i(x-y)\cdot\theta/\hbar} \right| e^{\Phi(y)/\hbar} \leq e^{-(1/\hbar)(R-C)|x-y|^2}, \tag{2.9}$$

where C is controlled by the $\mathscr{C}^2$-norm of Φ near x_0. Let $R > C$, and let $a_\hbar(x, y, \theta)$ be an analytic symbol defined in a neighbourhood of $(x_0, x_0, \theta_0) \in \mathbb{C}^{3n}$, with $\theta_0 := \frac{2}{i} \frac{\partial \Phi}{\partial x}(x_0)$ (in other terms, $a_\hbar \in H_{0,(x_0,x_0,\theta_0)}$); let us consider, for $u_\hbar \in H_{\Phi,x_0}$, the contour integral:

$$A_\Gamma u(x) = \frac{1}{(2\pi\hbar)^n} \iint_{\Gamma(x)} e^{(i/\hbar)(x-y)\cdot\theta} a_\hbar(x, y, \theta) u(y) \mathrm{d}y \mathrm{d}\theta. \tag{2.10}$$

Using a deformation variant of Stokes' formula (see for instance [28, Lemma 12.2]), one can show that $\overline{\partial}(A_\Gamma u)$ is $\mathcal{O}(e^{-c/\hbar})$, for some $c > 0$, uniformly near x_0. Hence, by solving a $\overline{\partial}$ problem, one can find a holomorphic function v near x_0 such that $A_\Gamma u = v + \mathcal{O}(e^{-c/\hbar})$. Such a v is unique modulo $\mathcal{N}$. Hence we will slightly abuse notation and write $v = A_\Gamma u$.

Note that the size $r > 0$ depends on the domain of definition of $u_\hbar$. However, if $u_\hbar \in H_{\Phi,x_0}$, the choice of r and R only modifies $A_\Gamma u(x)$ by a negligible term in $\mathcal{N}$. Moreover, if $u_\hbar = v_\hbar$ in $\widetilde{H}_{\Phi,x_0}$, then $A_\Gamma u_\hbar = A_\Gamma v_\hbar$ in $\widetilde{H}_{\Phi,x_0}$. Thus, $A = A_\Gamma$ defines an operator on the local space $\widetilde{H}_{\Phi,x_0}$; it is called a complex pseudo-differential operator. If $a_\hbar = 1$, then $A u_\hbar = u_\hbar$ in $\widetilde{H}_{\Phi,x_0}$, which can be viewed as a version of the Fourier inversion formula.

The symbol of A is the function σ_A defined for $(x, \theta) \in \operatorname{Neigh}((x_0, \theta_0); \mathbb{C}^{2n})$ by the formula:

$$\sigma_A(x, \theta) = e^{-ix\cdot\theta/\hbar} A\left(e^{i(\cdot)\cdot\theta/\hbar}\right).$$

Then $\sigma_A \in H_{0,(x_0,\theta_0)}$. If $a_\hbar$ does not depend on the variable y, then $\sigma_A(x,\theta) \sim a_\hbar$ in $H_{0,(x_0,\theta_0)}$. If $\Phi \in C^\infty$ and $a_\hbar$ is a classical analytic symbol of order zero, then by the stationary phase lemma, σ_A is also a classical analytic symbol of order zero. Moreover, if the formal series associated with $a_\hbar$ by (2.7) is zero then the formal series associated with σ_A is also zero. We will see in Section 3.2 that the converse statement holds as well.

An important particular class of analytic pseudo-differential operators concern the case where the symbol $a_\hbar$ has the form $a_\hbar(x,y,\theta) = b^{\mathrm{w}}_\hbar(\frac{x+y}{2},\theta)$, for a classical analytic symbol $b^{\mathrm{w}}_\hbar \in S^0(\mathrm{Neigh}\left(x_0, \theta_0 := \frac{2}{i}\frac{\partial \Phi}{\partial x}(x_0)\right))$. As in Proposition 2.7, we obtain the so-called complex Weyl quantization, namely:

$$\mathrm{Op}^{\mathrm{w}}_\hbar(b^{\mathrm{w}}_\hbar)u(x) = \frac{1}{(2\pi\hbar)^n}\iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} b^{\mathrm{w}}_\hbar\left(\frac{x+y}{2},\theta\right)u(y)\mathrm{d}y\mathrm{d}\theta. \tag{2.11}$$

2.4 Brg-quantization

We introduce here a new quantization scheme which, in view of Formula (2.4), is a natural generalization of Berezin-Toeplitz operators on $\mathbb{C}^n$.

Let $x_0 \in \mathbb{C}^n$, and let Φ be a real-analytic function defined in a neighbourhood of x_0. We view $\mathbb{C}^n$ as a totally real subspace of $\mathbb{C}^{2n}$ via the embedding in the anti-diagonal $\Lambda = \{(x,\overline{x}); x \in \mathbb{C}^n\}$. The map $(x,\bar{x}) \mapsto \Phi(x)$ admits a holomorphic extension to a neighbourhood of $(x_0,\bar{x}_0)$ in $\mathbb{C}^{2n}$. We denote this extension by $\psi(x,w)$; thus $\psi(x,\bar{x}) = \Phi(x)$, and we have

$$\psi(x,w) = \overline{\psi(\bar{w},\bar{x})}\,. \tag{2.12}$$

In fact, the identity holds on the 'real' subspace Λ, on which ψ takes real values. Finally, let us assume that Φ is strictly plurisubharmonic: there exists $m > 0$ such that

$$m\mathrm{Id} \leq (\partial^2_{x_i,\bar{x}_j}\Phi(x_0))^n_{i,j=1}. \tag{2.13}$$

Lemma 2.19. *For x, y near x_0 we have*

$$\Phi(x) + \Phi(y) - 2\Re\left(\psi(x,\bar{y})\right) \asymp |x-y|^2, \tag{2.14}$$

Proof. For $t \in [0,1]$ let $x_t := x + t(y-x)$, and let

$$f(t) := \psi(x,\bar{x}_{1-t}) + \psi(y,\bar{x}_t).$$

We have $f(1)-f(0)=\psi(x,\bar{x})+\psi(y,\bar{y})-\psi(x,\bar{y})-\psi(y,\bar{x})=\Phi(x)+\Phi(y)-2\Re\psi(x,\bar{y})$, see (2.12). On the other hand, the holomorphy of $\tilde{w}\mapsto\psi(x,\tilde{w})$, where $\tilde{w}=\bar{w}$, gives

$$\begin{aligned}
f(1)-f(0) &= \int_0^1 f'(t)\mathrm{d}t \\
&= \int_0^1 \left(\partial_{\tilde{w}}\psi(x,\bar{x}_t)-\partial_{\tilde{w}}\psi(y,\bar{x}_t)\right)\cdot(\overline{x-y})\mathrm{d}t \\
&= \int_0^1\int_0^1 \partial_x\partial_{\tilde{w}}\psi(x_s,\bar{x}_t)\cdot(x-y)\cdot(\overline{x-y})\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

If $\partial_x\partial_{\tilde{w}}\psi$ is constant (for instance if Φ is quadratic), we get the exact formula

$$\Phi(x)+\Phi(y)-2\Re\psi(x,\bar{y})=(\partial^2_{x,\bar{x}}\Phi)(x-y)(\overline{x-y})\geq m\left|x-y\right|^2,$$

where the last inequality is (2.13). In the general case we can write

$$\partial_x\partial_{\tilde{w}}\psi(x_s,\bar{x}_t)=\partial_x\partial_{\tilde{w}}\psi(x_0,\bar{x}_0)+\mathcal{O}(|x-x_0|+|y-x_0|),$$

which gives (2.14). □

Definition 2.20. *Let $\tilde{r}>r>0$. Let $a_\hbar\in L^\infty(B((x_0,\bar{x}_0),\tilde{r}))$. We define the operator $\mathrm{Op}_r^{\mathrm{Brg}}(a_\hbar)$ locally near x_0 by the following integral representation. For $x\in\mathbb{C}^n$ with $|x-x_0|<\tilde{r}-r$, and $u\in L^1(B(x_0,\tilde{r}))$,*

$$[\mathrm{Op}_r^{\mathrm{Brg}}(a_\hbar)u](x)=\int_{B(x,r)}k_\hbar(x,y)u(y)L(\mathrm{d}y), \tag{2.15}$$

where the kernel $k_\hbar$ is defined as follows, for (x,y) such that $|x-y|<r$:

$$k_\hbar(x,y)=\frac{2^n}{(\pi\hbar)^n}e^{\frac{2}{\hbar}(\psi(x,\overline{y})-\Phi(y))}a_\hbar(x,\overline{y})\det\left(\partial_{\tilde{w}}\partial_x\psi\right)(x,\overline{y})$$

Note that $(x,y)\mapsto\det\left(\partial_{\tilde{w}}\partial_x\psi\right)(x,y)$ is the holomorphic extension to a neighbourhood of $(x_0,\bar{x}_0)$ of the real-analytic map $(x,\bar{x})\mapsto\det(\partial^2_{x_i,\bar{x}_j}\Phi(x))_{i,j=1}^n$. Under the assumptions of Lemma 2.19, and choosing a smaller r if necessary, there exists $\epsilon>0$ such that

$$-\Phi(x)+2\Re\left(\psi(x,\overline{y})\right)-\Phi(y)\leq-\left(m-\epsilon\right)|x-y|^2,$$

and hence

$$e^{-\Phi(x)/\hbar} \left|k_\hbar(x,y)\right| e^{\Phi(y)/\hbar} \leq \frac{2^n}{(\pi\hbar)^n} \left|a_\hbar(x,\bar{y})\right| e^{-(1/\hbar)(m-\epsilon)|x-y|^2}, \tag{2.16}$$

which is similar to (2.9). By the same arguments as the ones used there, we see that $\mathrm{Op}^{\mathrm{Brg}}(a_\hbar) = \mathrm{Op}_r^{\mathrm{Brg}}(a_\hbar)$ defines an operator on $\widetilde{H}_{\Phi,x_0}$, which does not depend on r small enough.

This 'Brg-quantization' is a natural generalization of Formula (2.4) when the weight is quadratic: in this special case, we get formally $\Pi_\Phi = \mathrm{Op}_\infty^{\mathrm{Brg}}(1)$, and Berezin-Toeplitz operators can be obtained when $a_\hbar$ only depends on y.

2.5 Analytic Fourier integral operators

In this section, we recall the definition of semiclassical Fourier integral operators in the complex domain, and prove that, under a transversality condition, they act on spaces of germs holomorphic functions $H_{\Psi,y_0} \to H_{\widetilde{\Psi},x_0}$, modulo exponentially small remainders.

We want to give a meaning to the formal expression

$$Au(x) = \iint e^{\frac{i}{\hbar}\varphi(x,y,\theta)} a(x,y,\theta;\hbar) u(y) \mathrm{d}y \mathrm{d}\theta, \tag{2.17}$$

where a is an analytic symbol defined near (x_0, y_0, θ_0), and φ is a non-degenerate holomorphic phase function, as follows. Let $\varphi(x,y,\theta)$ be holomorphic in a neighbourhood of $(x_0,y_0,\theta_0) \in \mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^N$. Assume that $\varphi'_\theta(x_0,y_0,\theta_0) = 0$. Recall that φ is a *non-degenerate phase function* (in the sense of Hörmander) if the map φ'_θ is a local submersion, *i.e.*

$$\mathrm{d}\partial_{\theta_1}\varphi(x_0,y_0,\theta_0), \ldots, \mathrm{d}\partial_{\theta_N}\varphi(x_0,y_0,\theta_0) \text{ are linearly independent.} \tag{2.18}$$

Then

$$C_\varphi := \{(x,y,\theta) \in \mathrm{Neigh}((x_0,y_0,\theta_0), \mathbb{C}^{n+m+N}); \quad \varphi'_\theta(x,y,\theta) = 0)\}$$

is a complex manifold of codimension N. Moreover, the map

$$C_\varphi \ni (x,y,\theta) \mapsto (x, \partial_x\varphi(x,y,\theta); y, -\partial_y\varphi(x,y,\theta)) \in T^*\mathbb{C}^m \times T^*\mathbb{C}^n \tag{2.19}$$

has injective differential and hence the image Λ'_φ is a complex manifold (defined near $(x_0,\xi_0;y_0,\eta_0)$, with $\xi_0 := \partial_x\varphi(x_0,y_0,\theta_0)$ and $\eta_0 := -\partial_y\varphi(x_0,y_0,\theta_0)$)

of dimension $m+n$; thus, in view of (2.19), Λ'_φ is a holomorphic canonical relation. We don't require this relation to be a diffeomorphism; however, in order to have a well defined operator A, we now strengthen the assumption (2.18) to

$$(y,\theta) \mapsto \varphi(x_0,y,\theta) \text{ is a non-degenerate phase function near } (y_0,\theta_0). \tag{2.20}$$

Equivalently, the map

$$\Lambda'_\varphi \ni (x,\xi;y,\eta) \mapsto x$$

is a local submersion (which implies that $\Lambda'_\varphi \cap \{x=x_0\}$ is a complex manifold of dimension n) and the map

$$\Lambda'_\varphi \cap \{x=x_0\} \ni (x_0,\xi;y,\eta) \mapsto (y,\eta) \tag{2.21}$$

is a local immersion. The image of (2.21), namely $(\Lambda'_\varphi)^{-1}(T^*_{x_0}\mathbb{C}^n)$, is a complex Lagrangian manifold in $T^*\mathbb{C}^n$.

Proposition 2.21. *Let Ψ be a pluriharmonic function defined near $y_0 \in \mathbb{C}^n$. Let $\Lambda_\Psi := \{(y, \frac{2}{i}\partial_y\Psi(y)); y \in \mathrm{Neigh}(y_0,\mathbb{C}^n)\}$, and $\eta_0 := \frac{2}{i}\partial_y\Psi(y_0)$. Assume that φ satisfies (2.20), so that $(\Lambda'_\varphi)^{-1}(T^*_{x_0}\mathbb{C}^m)$ and Λ_Ψ both are complex Lagrangian manifolds passing through (y_0,η_0). Assume*

$$(\Lambda'_\varphi)^{-1}(T^*_{x_0}\mathbb{C}^m) \text{ and } \Lambda_\Psi \text{ intersect transversally at } (y_0,\eta_0). \tag{2.22}$$

Then A is a well-defined operator $\widetilde{H}_{\Psi,y_0} \to \widetilde{H}_{\widetilde{\Psi},x_0}$, where $\widetilde{\Psi}$ is a pluriharmonic function defined near x_0 with the property

$$\Lambda_{\widetilde{\Psi}} = \Lambda'_\varphi(\Lambda_\Psi)\,. \tag{2.23}$$

Proof. For x close to x_0, $(\Lambda'_\varphi)^{-1}(T^*_x\mathbb{C}^m)$ and Λ_Ψ intersect transversally at a unique point $(y(x),\eta(x))$, and because of (2.21), there is a corresponding unique point $(x,\xi(x);y(x),\eta(x)) \in \Lambda'_\varphi$. Here $\xi(x)$, $y(x)$, $\eta(x)$ are holomorphic functions of x. Thus $\Lambda'_\varphi(\Lambda_\Psi)$ is a complex manifold of dimension m, given by

$$\Lambda'_\varphi(\Lambda_\Psi) = \{(x,\xi(x));\quad x \in \mathrm{Neigh}(x_0;\mathbb{C}^m)\}.$$

The assumptions (2.20) and (2.21) imply that

$$(y,\theta) \mapsto -\Im\varphi(x,y,\theta) + \Psi(y) \tag{2.24}$$

has a unique non-degenerate critical point $(y(x), \theta(x))$ near (y_0, θ_0), depending holomorphically on x near x_0. Here $y(x)$ is the same as before and if we denote by $\widetilde{\Psi}(x)$ the corresponding critical value:

$$\widetilde{\Psi}(x) := \mathrm{vc}_{(y,\theta)}(-\Im\varphi(x, y, \theta) + \Psi(y)),$$

then we see that $\frac{2}{i}\partial_x \widetilde{\Psi}(x) = \xi(x)$ is the point defined above. Thus (2.23) holds.

Next consider formally Au in (2.17) for $u \in H_{\Psi, y_0}$. Then

$$\left| e^{\frac{i}{\hbar}\varphi(x,y,\theta)} a(x, y, \theta; \hbar) u(x) \right| \leq C_\epsilon e^{\frac{1}{\hbar}(\epsilon - \Im\varphi(x,y,\theta) + \Psi(y))}, \quad \forall \epsilon > 0$$

and since (2.24) has a non-degenerate critical point $(y(x), \theta(x))$, we know that we can find a good contour $\Gamma(x)$, *i.e.* a real submanifold of dimension $n + N$, passing through $(y(x), \theta(x))$ along which

$$-\Im\varphi(x, y, \theta) + \Psi(y) - \widetilde{\Psi}(x) \asymp -|y - y_0|^2 - |\theta - \theta_0|^2 \, .$$

It then suffices to define

$$Au(x) = \iint_{\Gamma(x)} e^{\frac{i}{\hbar}\varphi(x,y,\theta)} a(x, y, \theta; \hbar) u(y) \mathrm{d}y \mathrm{d}\theta \, ,$$

and argue as we did for analytic pseudo-differential operators (Section 2.3) to obtain that $A : \widetilde{H}_{\Psi, y_0} \to \widetilde{H}_{\widetilde{\Psi}, x_0}$ is well-defined. □

Remark 2.22. *The more general case where Ψ is plurisubharmonic can certainly be treated with some additional arguments.*

As an application of this proposition, we can compose Fourier integral operators. If $A : \widetilde{H}_{\Psi, y_0} \to \widetilde{H}_{\widetilde{\Psi}, x_0}$ is as in Proposition 2.21, let K_A be the corresponding canonical relation, so far denoted Λ'_φ. Let $z_0 \in \mathbb{C}^\ell$ and let $B : \widetilde{H}_{\widetilde{\Psi}, x_0} \to \widetilde{H}_{\hat{\Psi}, z_0}$ be a Fourier integral operator which satisfies the same assumption as A with $\widetilde{\Psi}, \Psi$ replaced by $\hat{\Psi}, \widetilde{\Psi}$. Let K_B be the canonical relation of B, and let $(z_0, \zeta_0; x_0, \xi_0) \in K_B$. By Proposition 2.21, the composition $B \circ A : \widetilde{H}_{\Psi, y_0} \to \widetilde{H}_{\hat{\Psi}, z_0}$ is well-defined. The condition (2.22) for B says that

$$K_B^{-1}(T^*_{z_0}\mathbb{C}^\ell) \text{ and } \Lambda_{\widetilde{\Psi}} \text{ intersect transversally at } (x_0, \xi_0), \tag{2.25}$$

and in view of (2.23) we have $\Lambda_{\widetilde{\Psi}} = K_A(\Lambda_\Psi)$. Hence (2.25) is equivalent to

$$\begin{gathered}\big(K_B \cap (T^*_{z_0}\mathbb{C}^\ell \times T^*\mathbb{C}^m)\big) \times (K_A \cap (T^*\mathbb{C}^m \times \Lambda_\Psi)) \text{ and} \\ T^*_{z_0}\mathbb{C}^\ell \times \operatorname{diag}(T^*\mathbb{C}^m \times T^*\mathbb{C}^m) \times \Lambda_\Psi \\ \text{intersect transversally in} \quad T^*\mathbb{C}^\ell \times T^*\mathbb{C}^m \times T^*\mathbb{C}^m \times \Lambda_\Psi\,.\end{gathered}$$

This implies the classical transversality condition for the composition $B \circ A$: $T^*\mathbb{C}^\ell \times \operatorname{diag}(T^*\mathbb{C}^m \times T^*\mathbb{C}^m) \times T^*\mathbb{C}^n$ and $K_B \times K_A$ intersect transversally in $T^*\mathbb{C}^\ell \times T^*\mathbb{C}^m \times T^*\mathbb{C}^m \times T^*\mathbb{C}^n$. Therefore, if in addition to (2.17) we write

$$Bv(z) = \iint e^{\frac{i}{\hbar}\psi(z,x,\omega)} b(z,x,\omega;\hbar) v(x) \mathrm{d}x \mathrm{d}\omega\,,$$

where ψ is a non-degenerate phase function defined near (z_0, x_0, ω_0), then we know that $B \circ A$ is an analytic Fourier integral operator for which $\psi(z,x,\omega) + \varphi(x,y,\theta)$ is a non-degenerate phase function with z, y as base variables and x, ω, θ as fibre variables and that the canonical relation $K_{B\circ A}$ is equal to $K_B \circ K_A$.

3 Equivalence of quantizations

One of the main results of this work is to show that, in the semiclassical limit, operators of the form $\mathrm{Op}^{\mathrm{Brg}}(a_\hbar)$ with an analytic weight Φ can in fact be written, up to exponentially small terms, as analytic pseudo-differential operators.

Theorem 3.1. *Let $\Phi : \mathrm{Neigh}(x_0; \mathbb{C}^n) \to \mathbb{R}$ be a real-analytic and strictly plurisubharmonic function.*

1. *Let $a_\hbar(x,w)$ be a classical analytic symbol of order zero defined in a neighbourhood of $(x_0, \bar{x}_0)$. Then there exists a classical analytic symbol $b^{\mathrm{w}}_\hbar(x,\theta)$ of order zero defined in a neighbourhood of $\big(x_0, \theta_0 := \frac{2}{i}\frac{\partial\Phi}{\partial x}(x_0)\big)$ such that*

$$\mathrm{Op}^{\mathrm{Brg}}(a_\hbar)u(x) \underset{\mathrm{H}}{\equiv} \mathrm{Op}^{\mathrm{w}}_\hbar(b^{\mathrm{w}}_\hbar) \quad : H_{\Phi,x_0} \to H_{\Phi,x_0}\,.$$

2. *Let $b^{\mathrm{w}}_\hbar(x,\theta)$ be a classical analytic symbol of order zero defined in a neighbourhood of $\big(x_0, \theta := \frac{2}{i}\frac{\partial\Phi}{\partial x}(x_0)\big)$. Then there exists a classical analytic symbol $a_\hbar(x,w)$ of order zero defined in a neighbourhood of $(x_0, \bar{x}_0)$ such that*

$$\mathrm{Op}^{\mathrm{w}}_\hbar(b^{\mathrm{w}}_\hbar) \underset{\mathrm{H}}{\equiv} \mathrm{Op}^{\mathrm{Brg}}(a_\hbar) \quad : H_{\Phi,x_0} \to H_{\Phi,x_0}\,.$$

3. *In case (1) (resp. case (2)), the formal symbol associated with $b_\hbar^{\mathrm{w}}(x,\theta)$ (resp. $a_\hbar(x,w)$) is uniquely determined by the formal symbol associated with $a_\hbar(x,w)$ (resp. $b_\hbar^{\mathrm{w}}(x,\theta)$).*

The proof of the first assertion of Theorem 3.1 is divided into two parts (Sections 3.1 and 3.2 below): first, we relate a Brg-operator to a complex pseudo-differential operator in the sense of Equation (2.10) and then we relate this last operator to a complex Weyl pseudo-differential operator.

The second and third assertions of Theorem 3.1 are obtained by showing that the operator $a_\hbar \mapsto b_\hbar$ in the first assertion is in fact an elliptic Fourier Integral Operator and hence can be microlocally inverted in the analytic category; see Sections 3.3 and 3.4.

3.1 From Brg-operators to complex $\hbar$-pseudo-differential operators

Let $\tilde{a}_\hbar(x,w) = a_\hbar(x,\bar{w},w)$, where $a_\hbar(x,y,w)$ is a classical analytic symbol of order zero defined on a neighbourhood of $(x_0,x_0,\bar{x}_0)$. Recall from (2.15) and (2.3) that we have the formula, for $u \in H_{\Phi,x_0}$:

$$\mathrm{Op}^{\mathrm{Brg}}(\tilde{a}_\hbar)u(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathrm{Neigh}(x_0)} e^{\frac{2}{\hbar}\psi(x,\bar{y})}a_\hbar(x,y,\bar{y})u(y)e^{-\frac{2}{\hbar}\Phi(y)}J(x,\bar{y})\,(\mathrm{d}y\wedge \mathrm{d}\bar{y}),$$

where $J(x,\bar{y}) = \det\left(\frac{2}{i}\partial_{\tilde{w}}\partial_x\psi\right)(x,\bar{y})$ (see Section 2.4). We can rewrite this formula as follows, for $u \in H_{\Phi,x_0}$:

$$\mathrm{Op}^{\mathrm{Brg}}(\tilde{a}_\hbar)u(x) = \frac{1}{(2\pi\hbar)^n}\iint_{\tilde{\Gamma}(x_0)} e^{\frac{2}{\hbar}(\psi(x,w)-\psi(y,w))}a_\hbar(x,y,w)u(y)J(x,w)\mathrm{d}y\mathrm{d}w,$$

where $\tilde{\Gamma}(x_0) \subset \mathbb{C}^{2n}$ is the integration contour $\{(y,w)=(y,\bar{y})\}$ for y near x_0, and using the fact that $\Phi(y)=\psi(y,\bar{y})$. We perform Kuranishi's trick and write for $(x,y,w) \in \mathrm{Neigh}(x_0;\mathbb{C}^n)\times\tilde{\Gamma}(x_0)$:

$$2\left(\psi(x,w)-\psi(y,w)\right) = i(x-y)\cdot\theta(x,y,w),$$

where θ is holomorphic on $\mathrm{Neigh}(x_0)\times\tilde{\Gamma}(x_0)$ and satisfies the following equality, for $(x,y,w)\in \mathrm{Neigh}(x_0)\times\tilde{\Gamma}(x_0)$:

$$\theta(x,y,w) = \frac{2}{i}\partial_x\psi(x,w) + \mathcal{O}(|x-y|). \tag{3.1}$$

Although we don't use this here, it is often important to see that, writing θ as

$$\theta(x,y,w) = \int_0^1 \frac{2}{i}\partial_x\psi((1-t)y+tx,w)\mathrm{d}t,$$

then (3.1) improves into:

$$\theta(x,y,w) = \frac{2}{i}\partial_x\psi\left(\frac{x+y}{2},w\right) + \mathcal{O}(|x-y|^2).$$

Therefore, we can rewrite the operator $\mathrm{Op}^{\mathrm{Brg}}(\tilde{a}_\hbar)$ as follows, for $u \in H_{\Phi,x_0}$:

$$\mathrm{Op}^{\mathrm{Brg}}(\tilde{a}_\hbar)u(x) = \frac{1}{(2\pi\hbar)^n}\iint_{\tilde{\Gamma}(x_0)} e^{\frac{i}{\hbar}(x-y)\cdot\theta(x,y,w)}a_\hbar(x,y,w)u(y)J(x,w)\mathrm{d}y\mathrm{d}w.$$

We deduce from (3.1) that for $(x,y,w) \in \mathrm{Neigh}(x_0)\times\tilde{\Gamma}(x_0)$:

$$\partial_w\theta(x,y,w) = \frac{2}{i}\partial_w\partial_x\psi(x,w) + \mathcal{O}(|x-y|), \tag{3.2}$$

whose determinant is non-vanishing because Φ is strictly plurisubharmonic. Thus, according to the holomorphic implicit function theorem, the function $w \mapsto \theta(x,y,w)$ admits a holomorphic inverse in $\mathrm{Neigh}(\bar{x}_0)$. We denote this inverse for $(x,y,\theta) \in \mathrm{Neigh}(x_0,x_0,\theta_0)$ by:

$$w = w(x,y,\theta), \tag{3.3}$$

where

$$\theta_0 := \tfrac{2}{i}\partial_x\psi(x_0,\bar{x}_0) = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x_0)\,.$$

We want to rewrite the operator $\mathrm{Op}^{\mathrm{Brg}}(\tilde{a}_\hbar)$ in terms of the θ-variable. We have, as holomorphic $2n$-forms,

$$\begin{aligned}\mathrm{d}y\wedge\mathrm{d}\theta &= \det\left(\partial_w\theta(x,y,w)\right)\mathrm{d}y\wedge\mathrm{d}w,\\ &= \det\left(\frac{2}{i}\partial_w\partial_x\psi(x,w)+\mathcal{O}(|x-y|)\right)\mathrm{d}y\wedge\mathrm{d}w.\end{aligned}$$

Here, as always in this paper, we use the notation

$$\mathrm{d}y\wedge\mathrm{d}\theta := \bigwedge_{j=1}^n(\mathrm{d}y_j\wedge\mathrm{d}\theta_j)\,,\quad \mathrm{d}y\wedge\mathrm{d}w := \bigwedge_{j=1}^n(\mathrm{d}y_j\wedge\mathrm{d}w_j)\,.$$

Let $\tilde{J}(x,y,\theta)$ be the following quantity, for $(x,y,\theta) \in \mathrm{Neigh}(x_0,x_0,\theta_0)$:

$$\tilde{J}(x,y,\theta) := \frac{J(x,w(x,y,\theta))}{\det\left(\partial_w \theta(x,y,w)\right)} = \left(1+\mathcal{O}(x-y)\right), \tag{3.4}$$

so that:

$$J(x,w)\mathrm{d}y \wedge \mathrm{d}w = \tilde{J}(x,y,\theta)\mathrm{d}y \wedge \mathrm{d}\theta\,.$$

Using (3.1) and (3.2), the image of $\tilde{\Gamma}(x_0)$ under $w \mapsto \theta = \theta(x,y,w)$ can be deformed into $\Gamma(x)$ (see (2.8)) in such a way

$$-\Phi(x) - \Phi(y) + \Re\left(i(x-y)\cdot\theta\right) \asymp -|x-y|^2,$$

uniformly on all the deformed contours. Hence we can rewrite the operator $\mathrm{Op}^{\mathrm{Brg}}(\tilde{a}_\hbar)$ as follows, for $u \in H_{\Phi,x_0}$:

$$\mathrm{Op}^{\mathrm{Brg}}(a_\hbar)u(x) \sim \frac{1}{(2\pi\hbar)^n}\iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} a_\hbar(x,y,w(x,y,\theta))u(y)\tilde{J}(x,y,\theta)\mathrm{d}y\mathrm{d}\theta,$$

which is a complex pseudo-differential operator (in the sense of Equation (2.10)) with symbol, for $(x,y,\theta) \in \mathrm{Neigh}(x_0,x_0,\theta_0 = \frac{2}{i}\partial_x\psi(x_0,\bar{x}_0))$:

$$b_\hbar(x,y,\theta) = a_\hbar(x,y,w(x,y,\theta))\tilde{J}(x,y,\theta)\,. \tag{3.5}$$

Let:

$$\begin{aligned} W : \mathrm{Neigh}(x_0,x_0,\theta_0) &\longrightarrow \mathrm{Neigh}\left(x_0,x_0,\bar{x}_0\right) \\ (x,y,\theta) &\longmapsto (x,y,w(x,y,\theta)). \end{aligned}$$

Then, for $(x,y,\theta) \in \mathrm{Neigh}\left(x_0,x_0,\theta_0\right)$:

$$b_\hbar(x,y,\theta) = \tilde{J}(x,y,\theta)a_\hbar(x,y,w(x,y,\theta)) = \tilde{J}(x,y,\theta)(W^* a_\hbar)(x,y,\theta).$$

Here W^* denotes the pull-back by W, *i.e.* $W^* a_\hbar = a_\hbar \circ W$. To conclude, we have

$$\mathrm{Op}^{\mathrm{Brg}}(\tilde{a}_\hbar)u(x) \underset{\mathrm{H}}{\equiv} B_\Gamma u(x) \quad \text{ in } H_{\Phi,x_0}\,,$$

where B_Γ means the quantization (in the sense of Equation (2.10)) of the classical analytic symbol $b_\hbar$ defined for $(x,y,\theta) \in \mathrm{Neigh}\left(x_0,x_0,\theta_0 = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x_0)\right)$ by:

$$b_\hbar(x,y,\theta) = \left(\tilde{J}W^* a_\hbar\right)(x,y,\theta).$$

3.2 From $\hbar$-pseudo-differential operators to complex Weyl pseudo-differential operators

Our goal is now to replace the symbol $b_\hbar(x,y,\theta)$ defined in a neighbourhood of $\big(x_0,x_0,\theta_0 := \frac{2}{i}\frac{\partial\Phi}{\partial x}(x_0)\big)$ by a symbol of the form $b_\hbar^{\mathrm{w}}\left(\frac{x+y}{2},\theta\right)$ defined in a neighbourhood of (x_0,θ_0) in order to obtain the complex Weyl quantization.

We first recall how to relate the various quantizations of a symbol depending on (y,θ). Let $a_{\hbar,t}(y,\theta)$ be a classical analytic symbol defined in a neighbourhood of (x_0,θ_0). For $t\in[0,1]$, the quantization Op_t is defined, for $u\in H_{\Phi,x_0}$, by

$$\mathrm{Op}_t(a_{\hbar,t})u(x) = \frac{1}{(2\pi\hbar)^n}\iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} a_{\hbar,t}(tx+(1-t)y,\theta)u(y)\mathrm{d}y\mathrm{d}\theta,$$

where $\Gamma(x)$ is defined in Equation (2.8). When $t=\frac{1}{2}$, we recover the complex Weyl quantization (Proposition 2.7). We now look for a symbol $a_{\hbar,t}(y,\theta)$ defined in a neighbourhood of $\big(x_0,\theta_0 := \frac{2}{i}\frac{\partial\Phi}{\partial x}(x_0)\big)$ such that the operator $\mathrm{Op}_t(a_{\hbar,t})$ does not depend on t. Let $u\in H_{\Phi,x_0}$ and denote $y_t(x,y) = tx + (1-t)y$; we have:

$$(2\pi\hbar)^n\hbar D_t\mathrm{Op}_t(a_{\hbar,t})u(x) = \hbar D_t\left(\iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} a_{\hbar,t}(y_t(x,y),\theta)u(y)\mathrm{d}y\mathrm{d}\theta\right),$$

$$\begin{aligned}
&= \iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta}\hbar D_t\left(a_{\hbar,t}(y_t(x,y),\theta)\right)u(y)\mathrm{d}y\mathrm{d}\theta,\\
&= \iint_{\Gamma(x)}\Big(e^{\frac{i}{\hbar}(x-y)\cdot\theta}\hbar D_t a_{\hbar,t} + e^{\frac{i}{\hbar}(x-y)\cdot\theta}(x-y)\cdot\hbar D_y a_{\hbar,t}\Big)(y_t(x,y),\theta)u(y)\mathrm{d}y\mathrm{d}\theta,\\
&= \iint_{\Gamma(x)}\Big(e^{\frac{i}{\hbar}(x-y)\cdot\theta}\hbar D_t a_{\hbar,t} + \hbar D_\theta(e^{\frac{i}{\hbar}(x-y)\cdot\theta})\cdot\hbar D_y a_{\hbar,t}\Big)(y_t(x,y),\theta)u(y)\mathrm{d}y\mathrm{d}\theta,\\
&\sim \iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta}\left(\hbar D_t a_{\hbar,t} - \hbar D_\theta\cdot\hbar D_y a_{\hbar,t}\right)(y_t(x,y),\theta)u(y)\mathrm{d}y\mathrm{d}\theta,
\end{aligned}$$

where the last equality holds modulo a negligible term, see Definition 2.17, and follows from Stokes' formula and (2.9). Consequently, the operator $\mathrm{Op}_t(a_{\hbar,t})$ will be independent of the parameter t if the symbol $a_{\hbar,t}$ satisfies the following condition for $(y,\theta)\in\mathrm{Neigh}\,(x_0,\theta_0)$:

$$\left(\hbar D_t - \hbar D_\theta\cdot\hbar D_y\right)a_{\hbar,t}(y,\theta) = 0.$$

This will hold if the symbol $a_{\hbar,t}$ satisfies the following equality for $(y,\theta) \in \operatorname{Neigh}(x_0,\theta_0)$:

$$a_{\hbar,t}(y,\theta) = e^{\frac{i}{\hbar}(t-s)\hbar D_\theta \cdot \hbar D_y} a_{\hbar,s}(y,\theta)\,.$$

Similarly to the more general case treated below, the propagator $e^{-\frac{it}{\hbar}(-\hbar D_\theta \cdot \hbar D_y)}$ is an analytic Fourier integral operator, with canonical relation:

$$\kappa_t : (y,\theta;\, y^*,\theta^*) \mapsto (y - t\theta^*, \theta - t y^*;\, y^*,\theta^*)\,.$$

Because κ_t sends the zero section $\theta^* = 0$, $y^* = 0$ on itself, we may apply Proposition 2.21 with $\Psi = 0$, which gives that this propagator sends analytic symbols to analytic symbols.

We now wish to generalize this procedure to a symbol of the form $b_{\hbar,t}(x,y,\theta)$, defined for $0 \le t \le 1$ in a neighbourhood of $\big(x_0, x_0, \theta_0 = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x_0)\big)$, and such that

$$b_{\hbar,0}(x,y,\theta) := b_\hbar(x,y,\theta) \quad \text{defined by Equation (3.5).}$$

Let $\operatorname{Op}_t(b_{\hbar,t})$ be the following operator for $u \in H_{\Phi,x_0}$:

$$\begin{aligned}
&\operatorname{Op}_t(b_{\hbar,t})u(x)\\
&= \frac{1}{(2\pi\hbar)^n} \iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} b_{\hbar,t}((1-t)x + ty, tx + (1-t)y, \theta) u(y)\,\mathrm{d}y\,\mathrm{d}\theta
\end{aligned}$$

Remark that, when $t = \frac{1}{2}$, we obtain the complex Weyl quantization (see (2.11)) of the symbol $b_\hbar^{\mathrm{w}}$ defined in $\operatorname{Neigh}(x_0,\theta_0)$ by

$$b_{\hbar,1/2}\left(\frac{x+y}{2}, \frac{x+y}{2}, \theta\right) =: b_\hbar^{\mathrm{w}}\left(\frac{x+y}{2}, \theta\right).$$

In order to lighten notation, let $X_t := ((1-t)x + ty, tx + (1-t)y, \theta)$. Then, for $u \in H_{\Phi,x_0}$, we have

$$(2\pi\hbar)^n \hbar D_t \operatorname{Op}_t(b_{\hbar,t})u(x) = \iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} \hbar D_t\left(b_{\hbar,t}(X_t)\right) u(y)\,\mathrm{d}y\,\mathrm{d}\theta,$$

$$\begin{aligned}
&= \iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} \Big(\hbar D_t b_{\hbar,t}(X_t) - (x-y)\hbar D_x b_{\hbar,t}(X_t)\\
&\qquad\qquad + (x-y)\hbar D_y b_{\hbar,t}(X_t)\Big) u(y)\,\mathrm{d}y\,\mathrm{d}\theta,\\
&= \iint_{\Gamma(x)} \left(e^{\frac{i}{\hbar}(x-y)\cdot\theta} \hbar D_t b_{\hbar,t} - \hbar D_\theta(e^{\frac{i}{\hbar}(x-y)\cdot\theta})\left(\hbar D_x b_{\hbar,t} - \hbar D_y b_{\hbar,t}\right)\right)(X_t) u(y)\,\mathrm{d}y\,\mathrm{d}\theta,\\
&\sim \iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} \left(\hbar D_t b_{\hbar,t} + \hbar D_\theta \cdot \left(\hbar D_x - \hbar D_y\right) b_{\hbar,t}\right)(X_t) u(y)\,\mathrm{d}y\,\mathrm{d}\theta,
\end{aligned}$$

using Stokes' formula. Thus, the operator $\mathrm{Op}_t(b_{\hbar,t})$, acting on $\widetilde{H}_{\Phi,x_0}$, is independent of the parameter t if the symbol $b_{\hbar,t}(x,y,\theta)$ satisfies the following equality for $(x,y,\theta)\in \mathrm{Neigh}\,(x_0,x_0,\theta_0)$:

$$(\hbar D_t + \hbar D_\theta\cdot(\hbar D_x - \hbar D_y))\, b_{\hbar,t}(x,y,\theta) = 0.$$

This leads to

$$b_{\hbar,t}(x,y,\theta) = U_{t-s}\; b_{\hbar,s}(x,y,\theta),$$

where

$$U_t := \exp\left(-\frac{i}{\hbar}t\,(\hbar D_\theta\cdot(\hbar D_x - \hbar D_y))\right).$$

By taking $t=\frac{1}{2}$ and $s=0$, we obtain:

$$b_{\hbar,1/2}\,(x,y,\theta) = U_{1/2} b_\hbar(x,y,\theta),$$

and we recall that the Weyl symbol is defined as $b_\hbar^{\mathrm{w}}(x,\theta) = b_{\hbar,1/2}(x,x,\theta)$. Writing U_t as a Fourier multiplier, *i.e.*

$$U_t = \mathcal{F}_\hbar^{-1}\circ\exp\left(\frac{it}{\hbar}\theta^*\cdot(y^*-x^*)\right)\circ\mathcal{F}_\hbar, \tag{3.6}$$

where $\mathcal{F}_\hbar$ denotes the usual semiclassical Fourier transform, we see that it is formally a semiclassical analytic Fourier integral operator (see Section 2.5); it is the exponential of the differential operator $P=\hbar D_\theta\cdot(\hbar D_x-\hbar D_y)$, acting on formal analytic symbols. Its canonical relation is actually the graph of a symplectic diffeomorphism defined by

$$\begin{aligned}\kappa_t : T^*\mathrm{Neigh}(x_0,x_0,\theta_0) &\longrightarrow T^*\mathrm{Neigh}(x_0,x_0,\theta_0)\\ (x,y,\theta;\,x^*,y^*,\theta^*) &\longmapsto (x,y,\theta;\,x^*,y^*,\theta^*) + t\mathcal{X}_p\end{aligned}$$

where $\mathcal{X}_p$ is the Hamiltonian vector field associated with the symbol p of the differential operator P, namely $p(x,y,\theta;\,x^*,y^*,\theta^*) = \theta^*\cdot(x^*-y^*)$ and $\mathcal{X}_p = \theta^*\cdot\partial_x - \theta^*\cdot\partial_y + (x^*-y^*)\cdot\partial_\theta$. Thus:

$$\kappa_t : (x,y,\theta;\,x^*,y^*,\theta^*)\mapsto(x+t\theta^*, y-t\theta^*, \theta+t(x^*-y^*);\,x^*,y^*,\theta^*).$$

Since κ_t is a diffeomorphism, its phase function is strongly non-degenerate in the sense of (2.20). Because κ_t sends the zero section $x^*=0$, $y^*=0$, $\theta^*=0$ on itself, we may apply Proposition 2.21 with $\Psi=0$, which gives that the

Fourier integral operator U_t sends analytic symbols to analytic symbols of the same order. Besides, using analytic stationary phase lemma, we obtain that U_t sends classical analytic symbols to classical analytic symbols (see also [29]). Let:

$$\begin{aligned} \gamma : \mathrm{Neigh}\,(x_0, \theta_0) &\longrightarrow \mathrm{Neigh}\,(x_0, x_0, \theta_0) \\ (x, \theta) &\longmapsto (x, x, \theta). \end{aligned} \tag{3.7}$$

Then, with γ^* denoting pullback, we have:

$$\gamma^* \left(U_{1/2} b_\hbar(x, y, \theta)\right) = \gamma^* \left(b_{\hbar,1/2}\,(x, y, \theta)\right) = b_\hbar^{\mathrm{w}}\,(x, \theta)\,.$$

$\gamma^* U_{1/2} b_\hbar$ is a classical analytic symbol of order zero that we denote by $b_\hbar^{\mathrm{w}}$. To conclude, taking $b_{\hbar,t}$ such that $\mathrm{Op}_t(b_{\hbar,t})$ is independent of t, gives us:

$$\mathrm{Op}_{1/2}(b_{\hbar,1/2}) = \mathrm{Op}_\hbar^{\mathrm{w}}(b_\hbar^{\mathrm{w}}) = \mathrm{Op}_\hbar^{\mathrm{w}}(\gamma^* U_{1/2} b_\hbar) \underset{\mathrm{H}}{\equiv} \mathrm{Op}_0(b_{\hbar,0}) = B_\Gamma.$$

To summarize, we have the following proposition.

Proposition 3.2. *Let $a_\hbar(x, y, w)$ be a classical analytic symbol of order zero defined on a neighbourhood of $(x_0, x_0, \bar{x}_0)$. Then, on H_{Φ,x_0}, we have:*

$$\mathrm{Op}^{\mathrm{Brg}}(a_\hbar) \underset{\mathrm{H}}{\equiv} \mathrm{Op}_\hbar^{\mathrm{w}}(b_\hbar^{\mathrm{w}}), \quad \textit{where} \quad b_\hbar^{\mathrm{w}} = \gamma^* U_{1/2} \tilde{J} W^* a_\hbar,$$

where:

$$\begin{cases} W : (x, y, \theta) \mapsto (x, y, w(x, y, \theta)), \\ \qquad \textit{with } w \textit{ defined in Equation } (3.3), \\ \gamma : (x, \theta) \mapsto (x, x, \theta), \\ U_{1/2} = \exp\left(\frac{i}{2\hbar} \hbar D_\theta \cdot (\hbar D_y - \hbar D_x)\right), \\ \tilde{J} \textit{ is defined by Equation } (3.4). \end{cases}$$

Besides, $b_\hbar^{\mathrm{w}}$ is a classical analytic symbol of order zero defined on a neighbourhood of $\left(x_0, \theta_0 := \frac{2}{i} \frac{\partial \Phi}{\partial x}(x_0)\right)$. Finally, if $a_\hbar \sim 0$, then $b_\hbar^{\mathrm{w}} \sim 0$.

3.3 Composition of Fourier integral operators

Let $b_\hbar^{\mathrm{w}}(x, \theta)$ be a classical analytic symbol of order zero defined on a neighbourhood of $\left(x_0, \theta_0 := \frac{2}{i} \frac{\partial \Phi}{\partial x}(x_0)\right)$. We want to prove that there exists a classical analytic symbol of order zero $a_\hbar(x, w)$ defined on a neighbourhood of $(x_0, \bar{x}_0)$ (and which does not depend on the y-variable) such that the

Brg-quantization of $a_\hbar$ coincides with the complex Weyl quantization of $b_\hbar^{\mathrm{w}}$ (see (2.11)). Instead of doing this directly, let us consider the map

$$\begin{aligned}\mathbf{S} : \hat{S}^0(\mathrm{Neigh}(x_0, x_0, \bar{x}_0)) &\longrightarrow \hat{S}^0(\mathrm{Neigh}\,(x_0, \theta_0))\\ a_\hbar &\longmapsto b_\hbar^{\mathrm{w}} = \gamma^* U_{1/2} \tilde{J} W^* a_\hbar,\end{aligned}$$

restricted to the subset of classical analytic symbols of order zero which do not depend on the y-variable. We already proved in the previous subsection that this map is well-defined in the sense that it sends a formal classical analytic symbol of order zero to a formal classical analytic symbol of order zero. Consequently, it suffices to prove the following proposition in order to conclude the proof of Theorem 3.1.

Proposition 3.3. *The map* $\mathbf{S}$ *restricted to the set of classical analytic symbols which do not depend on the* y*-variable is an analytic Fourier integral operator associated with a canonical transformation which sends the zero section on itself. Moreover, this Fourier integral operator is elliptic.*

This proposition implies that the map $\mathbf{S}$ is a bijection from the space of classical analytic symbols of order zero defined in a neighbourhood of $(x_0, \bar{x}_0)$ to the space of classical analytic symbols of order zero defined in a neighbourhood of (x_0, θ_0).

Let

$$\begin{aligned}\pi : \mathrm{Neigh}\,(x_0, x_0, \bar{x}_0) &\longrightarrow \mathrm{Neigh}(x_0, \bar{x}_0)\\ (x, y, w) &\longmapsto (x, w).\end{aligned} \tag{3.8}$$

Let $a_\hbar(x, w)$ be a classical analytic symbol of order zero, which we view as a function of (x, y, w) by identifying it with $\pi^* a_\hbar(x, y, w) = a_\hbar \circ \pi(x, y, w)$. We use the maps $W, \gamma, U_{\frac{1}{2}}$ and $\tilde{J}$ from Proposition 3.2. According to this proposition, we introduce

$$b_\hbar^{\mathrm{w}} = \gamma^* U_{1/2} \tilde{J} W^* \pi^* a_\hbar =: A a_\hbar. \tag{3.9}$$

The operator A acting on symbols $a_\hbar \in S^0(\mathrm{Neigh}(x_0, \bar{x}_0))$ is the composition of the five operators (γ^*, $U_{1/2}$, $\tilde{J}$, W^* and π^*). We shall give two independent proofs that this composition is an analytic Fourier integral operator: first by proving that all these operators are good analytic Fourier integral operators and applying Proposition 2.21; in the second proof (Appendix B) we give an explicit computation with stationary phase arguments in order to obtain a simple formula for A (Equation (B.2)).

3.4 Proof of Proposition 3.3

The operator π^*. Recall from (3.8) that $(\pi^* u)(x,y,w) = u(x,w)$. We have

$$(\pi^* u)(x,y,w) = \frac{1}{(2\pi\hbar)^{2n}} \iiiint e^{\frac{i}{\hbar}[(x-\tilde{x})\cdot\theta + (w-\tilde{w})\cdot\omega]} u(\tilde{x},\tilde{w}) \mathrm{d}x \mathrm{d}\theta \mathrm{d}\tilde{w} \mathrm{d}\omega \,.$$

Here $\varphi(x,y,w;\tilde{x},\tilde{w};\theta,\omega) = (x-\tilde{x})\cdot\theta + (w-\tilde{w})\cdot\omega$ is a strongly non-degenerate phase function in the sense of (2.20) (*i.e.*, when (x,y,w) is fixed), with critical variety

$$C_\varphi = \{(x,y,w;\tilde{x},\tilde{w};\theta,\omega); \quad x = \tilde{x}, w = \tilde{w}\} \,.$$

From this we get the canonical relation K_{π^*}:

$$K_{\pi^*} = \{(x,y,w;\theta,0,\omega),(x,w;\theta,\omega)\} = \{((a;{}^t\mathrm{d}\pi_a b^*),\ (\pi(a);b^*))\} \,,$$

where $a = (x,y,w) \in \mathbb{C}^{3n}$ and $b^* = (\theta,\omega) \in (\mathbb{C}^{2n})^*$. It maps the zero section $\{b^* = 0\} \subset T^*\mathbb{C}^{2n}$ to the zero section $\{a^* = 0\} \subset T^*\mathbb{C}^{3n}$, and the inverse image of $T^*_a\mathbb{C}^{3n}$ is

$$\{(\pi(a);b^*); \quad b^* \in (\mathbb{C}^{2n})^*\} = T^*_{\pi(a)}\mathbb{C}^{2n} \,,$$

which intersects the zero section $\{b^* = 0\}$ transversally. Therefore, we may apply Proposition 2.21, and $\pi^* : \widetilde{H}_{0,b} \to \widetilde{H}_{0,a}$ is an analytic Fourier integral operator.

The operator W^*. Recall that $W : \mathbb{C}^{3n} \to \mathbb{C}^{3n}$ is a locally defined diffeomorphism and $W^*u(a) = u(W(a))$, $a \in \mathbb{C}^{3n}$. Here

$$W^*u(a) = \frac{1}{(2\pi\hbar)^{3n}} \iint e^{\frac{i}{\hbar}(W(a)-c)\cdot c^*} u(c) \mathrm{d}c \mathrm{d}c^* \,.$$

The phase $(W(a)-c)\cdot c^*$ is non-degenerate as a function of (c,c^*) with critical manifold $\{(a,c,c^*); W(a) = c\}$. The canonical relation is the graph of the lifted symplectic transformation, *i.e.*

$$K_{W^*} = \{((a;{}^tW'(a)c^*),\ (W(a);c^*))\} \,.$$

It maps the zero section to the zero section, and $K_{W^*}^{-1}(T^*_a\mathbb{C}^{3n}) = T^*_{W(a)}\mathbb{C}^{3n}$, which is transversal to the zero section. Thus we may apply Proposition 2.21.

The operator $\tilde{J}$ is a multiplication operator, $K_{\tilde{J}} = \mathrm{Id}$.

The operator $U_{\frac{1}{2}}$**.** We have seen in (3.6) (and below that) that $U_{\frac{1}{2}}$ is an analytic Fourier integral operator with associated canonical transformation $\kappa_{\frac{1}{2}}$ given by

$$\kappa_{\frac{1}{2}}(a, a^*) = (a + h(a^*),\ a^*),$$

where $a \in \mathbb{C}^{3n}$ and h is the block-matrix $h = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$. It follows that, when $c \in \mathbb{C}^{3n}$ is fixed, $\kappa_{\frac{1}{2}}^{-1}(T_c^*\mathbb{C}^{3n}) = \{(c - h(a^*), a^*),\ a^* \in \mathbb{C}^{3n})\}$ is parametrized by a^* and transversal to the zero section, which permits the application of Proposition 2.21.

The operator γ^***.** Recall from (3.7) that $\gamma^* u(x, \theta) = u(x, x, \theta)$, so

$$\gamma^* u(b) = \frac{1}{(2\pi\hbar)^{3n}} \iint e^{\frac{i}{\hbar}[(\gamma(b) - c)\cdot c^*)]} u(c) \mathrm{d}c \mathrm{d}c^*\,,$$

with $c = (x, y, \theta) \in \mathbb{C}^{3n}$, $c^* \in (\mathbb{C}^{3n})^*$. The phase $\varphi(b, c, c^*)$ is non-degenerate, and since γ^* is a pull-back, we obtain, as for π^* and W^*,

$$K_{\gamma^*} = \{((b, {}^t\mathrm{d}\gamma_b c^*),\ (\gamma(b), c^*))\,;\quad b \in \mathbb{C}^{2n}, c^* \in (\mathbb{C}^{3n})^*\}\,.$$

Again, it maps the zero section $\{c^* = 0\} \subset T^*\mathbb{C}^{3n}$ to the zero section in $T^*\mathbb{C}^{2n}$, and $K_{\gamma^*}^{-1}(T_b^*\mathbb{C}^{2n}) = T_{\gamma(b)}\mathbb{C}^{3n}$, which is transversal to the zero section $\{c^* = 0\}$. Hence Proposition 2.21 can be applied.

To conclude, we have shown that all the compositions involved in the operator A are transverse, making it an analytic Fourier integral operator $\widetilde{H}_{0,(x_0,\bar{x}_0)} \to \widetilde{H}_{0,(x_0,\theta_0)}$, which is elliptic and whose associated canonical transformation $T^*\mathbb{C}^{2n} \to T^*\mathbb{C}^{2n}$ send the zero section to itself, which proves Proposition 3.3. Choosing a Fourier integral operator B associated with the inverse canonical transformation, and applying analytic ellipticity to the pseudo-differential operators AB and BA, we construct in the usual way a local inverse to A, sending $\widetilde{H}_{0,(x_0,\theta_0)}$ to $\widetilde{H}_{0,(x_0,\bar{x}_0)}$, thus proving Theorem 3.1.

4 The approximate Bergman projection

4.1 Functional analysis of L^2_Φ spaces

Definition 4.1. *Let Ω be an open subset of $\mathbb{C}^n$. Let $\Phi \in \mathscr{C}^0(\Omega;\mathbb{R})$. For any $\hbar > 0$, we define the following spaces.*

1. *$L^2_\Phi(\Omega)$ is the L^2-space with weight $e^{-2\Phi/\hbar}$ on Ω; it is a Hilbert space with the norm*
$$\|u\|_{L^2_\Phi(\Omega)} := \|ue^{-\Phi/\hbar}\|_{L^2(\Omega)} = \int |u(x)|^2\, e^{-2\Phi(x)/\hbar} L(\mathrm{d}x).$$

2. *$L^2_{\Phi,\mathrm{loc}}(\Omega)$ is the Fréchet space $L^2_{\mathrm{loc}}(\Omega)$ equipped with the set of seminorms $\|u\|_{L^2_\Phi(\tilde\Omega)}$, where $\tilde\Omega \Subset \Omega$ is an arbitrary open set with compact closure in Ω.*

3. *$L^2_{\Phi,\mathrm{comp}}(\Omega)$ is the space of compactly supported functions in $L^2_\Phi(\Omega)$.*

Since Φ is continuous, for any fixed $\hbar$ we have the set equality $L^2_{\Phi,\mathrm{comp}}(\Omega) = L^2_{\mathrm{comp}}(\Omega)$. Similarly to the space $\mathscr{C}^\infty_0(\Omega)$ in distribution theory, $L^2_{\Phi,\mathrm{comp}}$ is a projective limit of Fréchet spaces, and following the tradition we will only use the convergence of sequences: for a fixed $\hbar$, the sequence $(u_j)_{j\in\mathbb{N}}$ converges to u in $L^2_{\Phi,\mathrm{comp}}$ if the support of all u_j is contained in a fixed subset $\tilde\Omega \Subset \Omega$ and $\|u_j - u\|_{L^2_\Phi(\tilde\Omega)} \to 0$. Thus, the injection $L^2_{\Phi,\mathrm{comp}} \subset L^2_{\Phi,\mathrm{loc}}$ is sequentially continuous; and moreover we have a well-defined pairing on $L^2_{\Phi,\mathrm{loc}} \times L^2_{\Phi,\mathrm{comp}}$ given by
$$(u|v)_{L^2_\Phi} = \int u(x)\overline{v(x)} e^{-2\Phi(x)/\hbar} L(\mathrm{d}x),$$
which is continuous in the first factor and sequentially continuous in the second one.

From the Fréchet topology of $L^2_{\Phi,\mathrm{loc}}(\Omega)$ we obtain that, for a fixed $\hbar > 0$, a linear operator $A : L^2_{\Phi_1,\mathrm{loc}}(\Omega) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega)$ is continuous if and only if for every $\Omega_2 \Subset \Omega$, there exist $\Omega_1 \Subset \Omega$ and a constant $C > 0$ such that
$$\|Au\|_{L^2_{\Phi_2}(\Omega_2)} \le C\, \|u\|_{L^2_{\Phi_1}(\Omega_1)}\,. \tag{4.1}$$

Notice that (4.1) implies that if u vanishes on Ω_1, then Au vanishes on Ω_2; in other words, if we regard the support of the distribution kernel K_A of A as a relation from Ω to itself, we have
$$(\operatorname{supp} K_A)^{-1}(\Omega_2) := \{y \in \Omega; \quad \exists (x,y) \in (\operatorname{supp} K_A) \cap \Omega_2 \times \Omega\} \subset \Omega_1. \tag{4.2}$$

On the other hand, an operator $A : L^2_{\Phi_1,\mathrm{comp}}(\Omega) \to L^2_{\Phi_2,\mathrm{comp}}(\Omega)$ is continuous if and only if for every $\Omega_1 \Subset \Omega$, there exists $\Omega_2 \Subset \Omega$ such that

$$\operatorname{supp} u \subset \Omega_1 \implies \operatorname{supp} Au \subset \Omega_2. \tag{4.3}$$

and A is continuous as an operator from $L^2_\Phi(\Omega_1)$ to $L^2_\Phi(\Omega)$. Then A sends convergent sequences in $L^2_{\Phi_1,\mathrm{comp}}(\Omega)$ to convergent sequences in $L^2_{\Phi_2,\mathrm{comp}}(\Omega)$.

An operator that satisfies both (4.2) and (4.3) is called *properly supported*. Notice that the injections $\mathscr{C}_0^\infty(\Omega) \subset L^2_{\Phi,\mathrm{comp}}(\Omega)$ and $L^2_{\Phi,\mathrm{loc}}(\Omega) \subset \mathscr{D}'(\Omega)$ are (sequentially) continuous. Hence any continuous operator $A : L^2_{\Phi_1,\mathrm{comp}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2)$ admits a Schwartz kernel $K_A \in \mathscr{D}'(\Omega_2 \times \Omega_1)$.

If A has kernel K_A, its formal adjoint, denoted by A^*, is the operator defined by the kernel $(x, y) \mapsto \overline{K_A(y, x)}$. We see that taking formal adjoint swaps properness conditions (4.2) and (4.3). Hence if $A : L^2_{\Phi_1,\mathrm{loc}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2)$ is continuous, then $A^* : L^2_{\Phi_1,\mathrm{comp}}(\Omega_2) \to L^2_{\Phi_2,\mathrm{comp}}(\Omega_1)$ is continuous, and conversely.

We now introduce uniform versions of these remarks, as $\hbar \to 0$.

Definition 4.2. *A linear operator $A = A_\hbar : L^2_{\Phi_1,\mathrm{loc}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2)$ is* ***uniformly continuous****, and we write:*

$$A = \mathcal{O}(1) : L^2_{\Phi_1,\mathrm{loc}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2)$$

if there exists $\hbar_0 > 0$ such that, for every $\tilde\Omega_2 \Subset \Omega_2$, there exist $\tilde\Omega_1 \Subset \Omega_1$, and a constant $C > 0$, both independent of $\hbar$, such that, for all $u \in L^2_{\Phi_1,\mathrm{loc}}(\Omega_1)$,

$$\forall \hbar \in]0, \hbar_0], \qquad \|Au\|_{L^2_{\Phi_2}(\tilde\Omega_2)} \le C \|u\|_{L^2_{\Phi_1}(\tilde\Omega_1)}.$$

Definition 4.3. *A linear operator $A = A_\hbar : L^2_{\Phi_1,\mathrm{loc}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2)$ is* ***uniformly properly supported*** *if the projections from* $\operatorname{supp} K_A$ *to the factors Ω_1 and Ω_2 are uniformly proper, in the following sense: there exists $\hbar_0 > 0$ such that the following properties hold:*

1. *For all $\tilde\Omega_2 \Subset \Omega_2$, there exists $\tilde\Omega_1 \Subset \Omega_1$, independent of $\hbar$, such that*

$$\forall \hbar \in]0, \hbar_0], \qquad (\operatorname{supp} K_A)^{-1}(\tilde\Omega_2) \subset \tilde\Omega_1; \tag{4.4}$$

2. *For all $\tilde\Omega_1 \Subset \Omega_1$, there exists $\tilde\Omega_2 \Subset \Omega_2$, independent of $\hbar$, such that*

$$\forall \hbar \in]0, \hbar_0], \qquad (\operatorname{supp} K_A)(\tilde\Omega_1) \subset \tilde\Omega_2.$$

Proposition 4.4. *If $A : L^2_{\Phi_1,\mathrm{loc}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2)$ satisfies (4.4), then A is uniformly continuous if and only if for all $\chi_j \in \mathscr{C}_0^\infty(\Omega_j)$, $j = 1, 2$, the operator $\chi_2 A \chi_1 : L^2_{\Phi_1}(\Omega_1) \to L^2_{\Phi_2}(\Omega_2)$ is uniformly bounded, as $\hbar \to 0$.*

Proof. If A is uniformly continuous, and $\chi_j \in \mathscr{C}_0^\infty(\Omega_j)$, $j = 1, 2$, are given, let $\tilde{\Omega}_2 \Subset \Omega_2$ contain the support of χ_2. Then there exists $\tilde{\Omega}_1 \Subset \Omega_1$ such that,

$$\begin{aligned}
\|\chi_2 A \chi_1 u\|_{L^2_{\Phi_2}(\Omega_2)} &\le \|\chi_2\|_{L^\infty} \|A \chi_1 u\|_{L^2_{\Phi_2}(\tilde{\Omega}_2)} \le C \|\chi_2\|_{L^\infty} \|\chi_1 u\|_{L^2_{\Phi_1}(\tilde{\Omega}_1)} \\
&\le C \|\chi_2\|_{L^\infty} \|\chi_1\|_{L^\infty} \|u\|_{L^2_{\Phi_1}(\tilde{\Omega}_1)} \\
&\le C \|\chi_2\|_{L^\infty} \|\chi_1\|_{L^\infty} \|u\|_{L^2_{\Phi_1}(\Omega_1)} .
\end{aligned}$$

Conversely, if $\tilde{\Omega}_2$ is given, let $\chi_2 \in \mathscr{C}^\infty(\Omega_2)$ be such that $\chi_2 \equiv 1$ on a neighbourhood of $\overline{\tilde{\Omega}_2}$: we have

$$\|Au\|_{L^2_{\Phi_2}(\tilde{\Omega}_2)} = \|\chi_2 A u\|_{L^2_{\Phi_2}(\tilde{\Omega}_2)} \le \|\chi_2 A u\|_{L^2_{\Phi_2}(\Omega_2)} .$$

Now let $\chi_1 \in \mathscr{C}^\infty(\Omega_1)$ be such that $\chi_1 \equiv 1$ on a neighbourhood of the compact set $(\operatorname{supp} K_A)^{-1}(\operatorname{supp} \chi_2)$ that is independent of $\hbar$ (this is possible thanks to (4.4)). Then

$$\|\chi_2 A u\|_{L^2_{\Phi_2}(\Omega_2)} = \left\|\chi_2 A \chi_1^2 u\right\|_{L^2_{\Phi_2}(\Omega_2)} \le C \|\chi_1 u\|_{L^2_{\Phi_1}(\Omega_1)}$$

Finally we may choose $\tilde{\Omega}_1 \Subset \Omega_1$ containing the support of χ_1, and get $\|\chi_1 u\|_{L^2_{\Phi_1}(\Omega_1)} \le C_1 \|u\|_{L^2_{\Phi_1}(\tilde{\Omega}_1)}$, proving that A is uniformly continuous. □

Corollary 4.4.1. *If $A : L^2_{\Phi_1,\mathrm{loc}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2)$ is uniformly properly supported, then A is uniformly continuous if and only if A^* is uniformly continuous.*

Proof. If $\chi_2 A \chi_1 : L^2_{\Phi_1}(\Omega_1) \to L^2_{\Phi_2}(\Omega_2)$ is uniformly bounded, then its adjoint $\overline{\chi_1} A^* \overline{\chi_2}$ is uniformly bounded. □

In order to discuss exponential decay, it is useful to introduce variations of the weight function Φ.

Proposition 4.5. *Let $A : L^2_{\Phi,\mathrm{loc}}(\Omega) \to L^2_{\Phi,\mathrm{loc}}(\Omega)$ be uniformly continuous. Let $\Phi_1 \in \mathscr{C}(\Omega; \mathbb{R})$ be such that $\Phi_1 < \Phi$. Then there exists $\Phi_2 \in \mathscr{C}(\Omega; \mathbb{R})$, $\Phi_2 < \Phi$, such that*

$$A = \mathcal{O}(1) : L^2_{\Phi_1,\mathrm{loc}}(\Omega) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega)$$

Proof. Let $U_m \Subset \Omega$, $m \in \mathbb{N}$, be open subsets that form a locally finite covering of Ω. For each U_j, by uniform continuity, there exists $V_j \Subset \Omega$ such that

$$\|Au\|_{L^2_\Phi(U_j)} \leq C_j \|u\|_{L^2_\Phi(V_j)},$$

for some uniform constant C_j. Let $\epsilon_j := \inf_{V_j}(\Phi - \Phi_1)$ and

$$\tilde{\epsilon}_m := \min_{j;\, U_j \cap U_m \neq \varnothing} \epsilon_j.$$

We define $\Phi_2 = \Phi - \sum_m \tilde{\epsilon}_m \chi_m$, where $\chi_m \in \mathscr{C}_0^\infty(U_m; [0,1])$ form a partition of unity associated with the covering (U_m). On each U_j we have $\Phi_2 \geq \Phi - \epsilon_j$, and on each V_j we have $\Phi - \epsilon_j \geq \Phi - (\Phi - \Phi_1) = \Phi_1$. Hence

$$\|Au\|_{L^2_{\Phi_2}(U_j)} \leq \|Au\|_{L^2_{\Phi-\epsilon_j}(U_j)} \leq C_j \|u\|_{L^2_{\Phi-\epsilon_j}(V_j)} \leq C_j \|u\|_{L^2_{\Phi_1}(V_j)}.$$

□

Notice that if $\Phi' < \Phi$, then we have a uniformly continuous injection $L^2_{\Phi'} \subset L^2_\Phi$, which is exponentially small, as $\hbar \to 0$, on every compact set. Thus, we introduce the next definition.

Definition 4.6. *Let $A : L^2_{\Phi,\mathrm{loc}}(\Omega) \to L^2_{\Phi,\mathrm{loc}}(\Omega)$ be uniformly properly supported. We say that A is **negligible** and write $A \equiv 0$ if there exists a continuous function Φ_2 on Ω such that $\Phi_2 < \Phi$ and*

$$A = \mathcal{O}(1) : L^2_{\Phi,\mathrm{loc}}(\Omega) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega)$$

If A is negligible, then it is H-negligible in the sense of Definition 2.18.

Proposition 4.7. *If $A : L^2_{\Phi,\mathrm{loc}}(\Omega) \to L^2_{\Phi,\mathrm{loc}}(\Omega)$ is uniformly properly supported, then the following statements are equivalent:*

1. *$A \equiv 0$.*
2. *for all $\chi_j \in \mathscr{C}_0^\infty(\Omega)$, $j = 1, 2$, there exist $\hbar_0 > 0$, $C > 0$ such that, for all $\hbar \in (0, \hbar_0]$,*
$$\|\chi_2 A \chi_1\|_{\mathcal{L}(L^2_\Phi(\Omega), L^2_\Phi(\Omega))} \leq C e^{-\frac{1}{C\hbar}}.$$
3. *there exist $\Phi_j \in \mathscr{C}(\Omega; \mathbb{R})$, $j = 1, 2$ such that $\Phi_2 < \Phi < \Phi_1$ and*
$$A = \mathcal{O}(1) : L^2_{\Phi_1,\mathrm{loc}}(\Omega) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega)$$

Before the proof, we widen the scope and recall a notion of formal exponential estimates of distribution kernels, introduced in Section 2.2 in [18]: Let $\Omega_j \subset \mathbb{C}^{n_j}$ be open for $j = 1, 2$, let $A = A_h : C_0^\infty(\Omega_1) \to \mathcal{D}'(\Omega_2)$, $0 < h \leq h_0$, be a linear operator with distribution kernel $K_A \in \mathcal{D}'(\Omega_2 \times \Omega_1)$, and let $F \in C(\Omega_2 \times \Omega_1; \mathbb{R})$. Then we write

$$K_A(x, y) = \widetilde{\mathcal{O}}(1)e^{F(x,y)/h} = \widetilde{\mathcal{O}}(e^{F(x,y)/h}), \tag{4.5}$$

if for all $(x_0, y_0) \in \Omega_2 \times \Omega_1$ and $\epsilon > 0$, there exist $C > 0$ and open neighborhoods $V_{x_0} \subset \Omega_2$, $V_{y_0} \subset \Omega_1$ of x_0, y_0 respectively, such that $1_{V_{x_0}} \circ A$ is bounded: $L^2_{\mathrm{comp}}(V_{y_0}) \to L^2(V_{x_0})$ with operator norm $\leq Ce^{\epsilon/h}e^{F(x_0,y_0)/h}$. Sometimes, we simply write

$$A = \widetilde{\mathcal{O}}(1)e^{F(x,y)/h} : \; L^2_{\mathrm{comp}}(\Omega_1) \to L^2_{\mathrm{loc}}(\Omega_2). \tag{4.6}$$

We have (4.5) if and only if for all $(x_0, y_0) \in \Omega_2 \times \Omega_1$ and $\epsilon > 0$, there exist $C > 0$ and $\chi_{x_0} \subset C_0^\infty(\Omega_2; [0, +\infty[)$, $\chi_{y_0} \subset C_0^\infty(\Omega_1; [0, +\infty[)$, equal to 1 near x_0 and y_0 respectively, such that $\chi_{x_0} \circ A \circ \chi_{y_0}$ is bounded: $L^2(\mathbb{C}^{n_1}) \to L^2(\mathbb{C}^{n_2})$ with operator norm $\leq Ce^{\epsilon/h}e^{F(x_0,y_0)/h}$.

When (4.5) holds it is then natural to write instead of (4.6):

$$A = \widetilde{\mathcal{O}}(1)e^{F(x,y)/h} : \; L^2_{\mathrm{loc}}(\Omega_1) \to L^2_{\mathrm{loc}}(\Omega_2).$$

Let $\Gamma = \overline{\bigcup_{]0,h_0]} \mathrm{supp}\,(K_{A_h})}$, so that the natural projections $\pi_j : \Gamma \to \Omega_j$ are proper

With F as above, let $\Phi_j \in C(\Omega_j; \mathbb{R})$, $j = 1, 2$. If

$$F(x, y) + \Phi_1(y) \leq \Phi_2(x) \text{ on } \Gamma \tag{4.7}$$

and (4.5) holds, then

$$A = \widetilde{\mathcal{O}}(1) : \; L^2_{\Phi_1,\mathrm{loc}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2) \tag{4.8}$$

in the following sense:

$$\begin{gathered} \text{For every open set } V_2 \Subset \Omega_2, \; \exists \text{ an open set } V_1 \Subset \Omega_1, \\ \text{such that for every } \epsilon > 0, \; \exists C > 0 \text{ such that} \\ \|Au\|_{L^2_{\Phi_2}(V_2)} \leq Ce^{\epsilon/h}\|u\|_{L^2_{\Phi_1}(V_1)}, \; \forall \, u \in L^2_{\Phi_1,\mathrm{loc}}(\Omega_1). \end{gathered} \tag{4.9}$$

Conversely, if (4.8) holds, then

$$K_A(x,y) = \widetilde{\mathcal{O}}(1)e^{(\Phi_2(x)-\Phi_1(y))/h}.$$

If we sharpen (4.7) by assuming strict inequality there, then (4.8) can be replaced by the sharper statement that

$$A = \mathcal{O}(1) : \; L^2_{\Phi_1,\mathrm{loc}}(\Omega_1) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega_2),$$

as defined earlier.

Proof of Proposition 4.7. We show 1 $\Rightarrow$ 2 $\Rightarrow$ 3 $\Rightarrow$ 1.

If 1 holds, then there exists a continuous function Φ_2 on Ω_2 such that $\Phi_2 < \Phi$ and

$$K_A(x,y) = \widetilde{\mathcal{O}}(1)e^{(\Phi_2(x)-\Phi(y))/h}.$$

It follows that if $\chi_j \in C_0^\infty(\Omega)$, $j = 1, 2$ and $\widetilde{C} > 0$ and $1/\widetilde{C}$ is strictly smaller than

$$\inf_{\operatorname{supp}\chi_2 \times \operatorname{supp}\chi_1} (\Phi(x) - \Phi(y)) - (\Phi_2(x) - \Phi(y)) = \inf_{\operatorname{supp}\chi_2 \times \operatorname{supp}\chi_1} (\Phi(x) - \Phi_2(x)),$$

then $\exists\, \widehat{C} > 0$ such that

$$\|\chi_2 A \chi_1\|_{\mathcal{L}(L^2_\Phi(\Omega), L^2_\Phi(\Omega))} \le \widehat{C} e^{-1/(\widetilde{C}h)},$$

and we get 2 with $C = \max(\widetilde{C}, \widehat{C})$.

Assume 2. Let $U_j \Subset \Omega$, $j = 1, 2, \ldots$ be open, forming a locally finite covering of Ω. Let $1_{U_j} \le \chi_j \in C_0^\infty(\Omega; [0, \infty[)$ so that

$$\|\chi_j A \chi_k\|_{\mathcal{L}(L^2_\Phi, L^2_\Phi)} \le C_{j,k} e^{-1/(C_{j,k}h)},$$

for some $C_{j,k} > 0$. Using a partition of unity $\psi_j \in C_0^\infty(U_j; [0,1])$ on Ω subordinated to the covering, and the fact that the norm of $\psi_j A \psi_k$ is bounded from above by that of $\chi_j A \chi_k$, we conclude that

$$A = \mathcal{O}(1) e^{(-F(x,y)+\Phi(x)-\Phi(y))/h} : L^2_{\mathrm{loc}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega), \tag{4.10}$$

if $0 < F(x,y) \in C(\Omega \times \Omega)$ satisfies

$$F(x,y) \le \sup_{j,k} 1_{U_j}(x) 1_{U_k}(y) / C_{j,k}.$$

An example of such a function is given by

$$F(x,y)=\sum_{j,k}\psi_j(x)\psi_k(y)/C_{j,k}. \tag{4.11}$$

Sometimes, we write (4.10) as

$$A=\mathcal{O}(1)e^{-F(x,y)/h}:\ L^2_{\Phi,\mathrm{loc}}(\Omega)\to L^2_{\Phi,\mathrm{loc}}(\Omega),$$

which also has a direct meaning similar to (4.8), (4.9).

It follows that

$$A=\mathcal{O}(1):\ L^2_{\Phi_1,\mathrm{loc}}(\Omega)\to L^2_{\Phi_2,\mathrm{loc}}(\Omega)$$

and hence that 3 holds, provided that we can find continuous functions Φ_1, Φ_2 on Ω such that $\Phi_2<\Phi<\Phi_1$ and

$$\Phi_2(x)>-F(x,y)+\Phi(x)-\Phi(y)+\Phi_1(y),\ (x,y)\in\Gamma,$$

where the last estimate is equivalent to

$$(\Phi(x)-\Phi_2(x))+(\Phi_1(y)-\Phi(y))<F(x,y)\text{ on }\Gamma.$$

It suffices to find continuous functions Φ_j such that

$$0<(\Phi-\Phi_2)(x)<\frac{1}{2}\inf_{y\in\Gamma^{-1}(x)}F(x,y), \tag{4.12}$$

and

$$0<(\Phi_1-\Phi)(y)<\frac{1}{2}\inf_{x\in\Gamma(y)}F(x,y), \tag{4.13}$$

where Γ is viewed as a relation $\Omega_1\to\Omega_2$. By the properness of the projections $\Gamma\ni(x,y)\mapsto x\in\Omega$, $\Gamma\ni(x,y)\mapsto y\in\Omega$ and the continuity of $F>0$ the right hand sides of (4.12), (4.13) are locally bounded from below by constants >0. We can then construct $\Phi-\Phi_2>0$, $\Phi_1-\Phi>0$ as in (4.11) with the difference that we use a partition of unity in y or in x only. We have shown the implication $2\Rightarrow 3$.

Finally, if 3 holds, then

$$A=\mathcal{O}(1):\ L^2_{\Phi,\mathrm{loc}}(\Omega)\to L^2_{\Phi_2,\mathrm{loc}}(\Omega),$$

which implies 1. □

In particular, item 2 above gives the

Corollary 4.7.1. *If A is uniformly continuous and uniformly properly supported, then $A\equiv 0$ if and only if $A^*\equiv 0$.*

4.2 The approximate Bergman projection

Let $\Omega \subset \mathbb{C}^n$ be an open subset, and assume that Φ is real analytic and pointwise strictly plurisubharmonic on Ω:

$$\forall x_0 \in \Omega, \exists m_0 > 0, \qquad m_0 \mathrm{Id} \leq (\partial^2_{x_i,\bar{x}_j}\Phi(x_0))^n_{i,j=1}.$$

In this section we use the Brg quantization (Definition 2.20) to construct an *approximate Bergman projection* on Ω, in the sense of Proposition 4.9 below.

We first work on germs of functions near a point $x_0 \in \mathbb{C}^n$; in this case, the result directly follows from Theorem 3.1. Indeed, we apply the second assertion of Theorem 3.1 to the operator $\mathrm{Op}^{\mathrm{w}}_\hbar(b^{\mathrm{w}}_\hbar)$ whose classical analytic symbol $b^{\mathrm{w}}_\hbar$ is chosen equal to 1. We obtain a classical analytic symbol $a_\hbar$ near $(x_0, \bar{x}_0)$ such that the following holds.

Proposition 4.8. *There exists $r > 0$ and a neighbourhood Ω_0 of x_0 such that the operator $\Pi_{x_0} := \mathrm{Op}^{\mathrm{Brg}}_r(a_\hbar)$ has the following properties.*

1. $\Pi_{x_0} \equiv \Pi^*_{x_0}$ *on* $L^2_\Phi(\Omega_0)$.
2. *For all* $u \in L^2_\Phi(\Omega_0)$, $\Pi_{x_0}u \in H_\Phi(\Omega_0) := \mathrm{Hol}(\Omega_0) \cap L^2_\Phi(\Omega_0)$.
3. *There exists $\Phi_2 \in \mathscr{C}^\infty(\Omega_0;\mathbb{R})$ with $\Phi_2 < \Phi$ and such that for all $\Omega_2 \Subset \Omega_0$, there exists $C > 0$, independent of $\hbar$, with*

$$\forall u \in H_\Phi(\Omega_0), \quad \|\Pi_{x_0}u - u\|_{L^2_{\Phi_2}(\Omega_2)} \leq C \|u\|_{L^2_\Phi(\Omega_0)}.$$

Proof. Using the notation of Definition 2.20, let $\tilde{\Omega}_0 := B((x_0,\bar{x}_0),\tilde{r})$ and $\Omega_0 := B(x_0, r)$. We can write (2.15) as in integral on $\mathbb{C}^n$ by replacing the distribution kernel $k_\hbar(x,y)$ of $\mathrm{Op}^{\mathrm{Brg}}(a_\hbar)$ by $1_{|x-y|<r}1_{(x,\bar{y})\in\tilde{\Omega}_0}k_\hbar(x,y)$. Since $a_\hbar(x,\bar{y})$ is holomorphic in x, this distribution kernel is locally holomorphic in x for almost all y, which gives Item 2.

The symbol $a_\hbar$ given by Theorem 3.1 is such that $\mathrm{Op}^{\mathrm{Brg}}(a_\hbar) \underset{\mathrm{H}}{\equiv} \mathrm{Op}^{\mathrm{w}}_\hbar(1) \underset{\mathrm{H}}{\equiv} \mathrm{Id}$, acting on H_{Φ,x_0}. Thus, $\Pi_{x_0} - \mathrm{Id} \underset{\mathrm{H}}{\equiv} 0$, which gives Item 3 (up to choosing a smaller neighbourhood Ω_0, if necessary). Let $\Pi^*_{x_0}$ be the adjoint of Π_{x_0}, viewed as an operator on $L^2_\Phi(\mathbb{C}^n)$. For any $u \in L^2_\Phi(\Omega_0)$, we have $\Pi^*_{x_0}u = \mathrm{Op}^{\mathrm{Brg}}(\tilde{a}_\hbar)u$, where $\tilde{a}_\hbar(x,y) = \overline{a(y,x)}$. Hence Item 2 holds for $\Pi^*_{x_0}$ as well. Therefore, Item 3 implies

$$\left\|\Pi_{x_0}\Pi^*_{x_0}u - \Pi^*_{x_0}u\right\|_{L^2_{\Phi_2}(\Omega_2)} \leq C \left\|\Pi^*_{x_0}u\right\|_{L^2_\Phi(\Omega_0)} \leq \tilde{C}\|u\|_{L^2_\Phi(\Omega_0)},$$

where the last inequality follows from the fact that all Brg operators with bounded symbols are uniformly bounded in $L^2_\Phi(\Omega_0)$; this is a consequence of (2.16) and the Schur test. In other words, if $\chi_0 \in \mathscr{C}_0^\infty(\mathbb{C}^n)$ is equal to 1 on Ω_0 and $\chi_2 \in \mathscr{C}_0^\infty(\Omega_2)$, we have

$$\chi_2(\Pi_{x_0} - 1)\Pi^*_{x_0}\chi_0 \equiv 0. \tag{4.14}$$

The operator $\chi_2(\Pi_{x_0}-1)\Pi^*_{x_0}\chi_0$ is uniformly properly supported and uniformly continuous on $L^2_\Phi(\mathbb{C}^n)$. Hence, by Corollary 4.7.1, we may take the adjoint:

$$\chi_0\Pi_{x_0}(\Pi^*_{x_0} - 1)\chi_2 \equiv 0\,. \tag{4.15}$$

Assume that $\chi_2 = 1$ on an open neighbourhood Ω_3 of x_0 and let χ_3 be a bounded function with compact support in Ω_3. Multiplying on both sides (4.14) and (4.15) by χ_3, we get

$$\chi_3\Pi^*_{x_0}\chi_3 \equiv \chi_3\Pi_{x_0}\Pi^*_{x_0}\chi_3$$

and

$$\chi_3\Pi_{x_0}\chi_3 \equiv \chi_3\Pi_{x_0}\Pi^*_{x_0}\chi_3\,,$$

and hence

$$\chi_3\Pi^*_{x_0}\chi_3 \equiv \chi_3\Pi_{x_0}\chi_3\,.$$

Up to replacing Ω_0 by a slightly smaller open set $\Omega_0' \Subset \Omega_3$ (which does not impact Items 2 and 3), and letting and $\chi_3 = 1_{\Omega_0'}$, we get Item 1.

$\square$

Next we globalize the operator Π_{x_0} observing that, because of the uniqueness in Theorem 3.1, the formal analytic symbol $\hat{a}_\hbar(x, y) \sim \sum_j a_j(x, y)\hbar^j$ associated with $a_\hbar$ is in fact well defined in a neighbourhood $\Omega^{(2)} \subset \Omega \times \overline{\Omega}$ of the antidiagonal

$$\operatorname{adiag}(\Omega \times \overline{\Omega}) := \{(x, \bar{x});\quad x \in \Omega\}.$$

For each $x_0 \in \Omega$, there exists a small ball Ω_0 around x_0 and a constant $C_{x_0} > 0$ such that

$$\sup_{\Omega_0\times\overline{\Omega_0}} |a_j| \le C_{x_0}^{j+1} j^j.$$

Thus $\hat{a}_\hbar \in \hat{S}^0(\Omega^{(2)})$ in the sense of Definition 2.15. Moreover, by Item 1 of Proposition 4.8, we have $a_j(x, y) = \overline{a_j(y, x)}$ for all j. Using a covering of Ω by such balls, one can construct a smooth function $\mathcal{C} = \mathcal{C}(x,y) \in \mathscr{C}^\infty(\Omega^{(2)}; \mathbb{R}_+^*)$ such that $\mathcal{C}(x,y) = \mathcal{C}(y,x)$ and

$$\forall (x,y) \in \Omega^{(2)}; \quad |a_j(x,y)| \leq \mathcal{C}(x,y)^{j+1} j^j. \tag{4.16}$$

Now put

$$a_{\mathcal{C}}(x,y;\hbar) := \sum_{j\geq 0} \theta(j\hbar\mathcal{C}(x,y)) a_j(x,y)\hbar^j,$$

where $\theta \in \mathscr{C}_0^\infty([0,1[\,;[0,1])$ is equal to 1 on $[0,\frac{1}{2}]$. Then $a_{\mathcal{C}} \in \mathscr{C}^\infty(\Omega^{(2)})$ and

$$a_{\mathcal{C}} - a_\hbar = \mathcal{O}(e^{-1/\hat{C}\hbar}) \quad \text{ in } \Omega_0 \times \overline{\Omega_0}, \tag{4.17}$$

where $\hat{C} > 0$ depends on x_0, $\mathcal{C}$ and θ. Moreover, $a_{\mathcal{C}}$ is 'exponentially close' to a good classical analytic symbol, in that there exists a smooth function $\mathcal{C}_1(x,y) > 0$ such that

$$a_{\mathcal{C}}(x,y) - \sum_{0 \leq j \leq \frac{1}{2\hbar\mathcal{C}(x,y)}} a_j(x,y)\hbar^j = \mathcal{O}(e^{-1/\mathcal{C}_1(x,y)\hbar})$$

and

$$\overline{\partial}_{x,y} a_{\mathcal{C}}(x,y) = \mathcal{O}(e^{-1/\mathcal{C}_1(x,y)\hbar}). \tag{4.18}$$

Let $\chi \in \mathscr{C}^\infty(\Omega \times \overline{\Omega}; \mathbb{R})$ satisfy $\chi(x,\bar{y}) = \chi(y,\bar{x})$, be supported in $\Omega^{(2)}$, and equal to 1 near $\operatorname{adiag}(\Omega \times \overline{\Omega})$. We extend the Brg quantization by putting

$$(\Pi_{\mathcal{C},\chi} u)(x) = \frac{2^n}{(\pi\hbar)^n} \int_\Omega e^{\frac{2}{\hbar}(\psi(x,\bar{y}) - \Phi(y))} a_{\mathcal{C}}(x,\bar{y}) u(y) \chi(x,\bar{y}) \det(\partial^2_{x,w}\psi)(x,\bar{y}) L(\mathrm{d}y). \tag{4.19}$$

Proposition 4.9. *The operator $\Pi_{\mathcal{C},\chi}$ has the following properties.*

1. Continuity: $\Pi_{\mathcal{C},\chi} : L^2_{\Phi,\mathrm{loc}}(\Omega) \to L^2_{\Phi,\mathrm{loc}}(\Omega)$ *is uniformly properly supported and uniformly continuous.*

2. Self-adjointness: $\Pi_{\mathcal{C},\chi} = \Pi^*_{\mathcal{C},\chi}$.

3. Exponential localization: *If $K \subset \Omega$ is closed, there exists $\Phi_2 \in \mathscr{C}^0(\Omega;\mathbb{R})$, $\Phi_2 \leq \Phi$ with*
$$\Phi_2 < \Phi \quad \text{ on } \Omega \setminus K$$
such that
$$\Pi_{\mathcal{C},\chi} = \mathcal{O}(1) : L^2_{\Phi,\mathrm{loc}}(K) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega),$$
where $L^2_{\Phi,\mathrm{loc}}(K) := \{u \in L^2_{\Phi,\mathrm{loc}}(\Omega); \quad \operatorname{supp} u \subset K\}$.

Proof. The fact that $\Pi_{\mathcal{C},\chi}$ is properly supported holds if $\Omega^{(2)}$ is chosen close enough to the antidiagonal. In this case the projections on x or y of any closed subset of $\Omega^{(2)}$ will be proper.

The uniform continuity of $\Pi_{\mathcal{C},\chi}$, as in the proof of Proposition 4.8 above, follows from (2.16). This gives Item 1.

Item 2 (selfadjointness) is deduced from the fact that $\overline{a_{\mathcal{C}}(x,y)} = a_{\mathcal{C}}(y,x)$.

In order to prove Item 3, we remark that if $K_2 \subset \Omega \setminus K$ is compact, then the distance δ between K and K_2 is positive. Hence, the distribution kernel of the restriction of $\Pi_{\mathcal{C},\chi} : L^2_{\Phi,\mathrm{loc}}(K) \to L^2_{\Phi,\mathrm{loc}}(K_2)$ is of the form $1_{|x-y|>\delta} 1_{K_2}(x) k_\hbar(x,y) 1_K(y)$, where $k_\hbar$ is the original kernel of (4.19). In view of (2.16), the norm of this restriction is $\mathcal{O}(e^{-c(K_2)/\hbar})$ for some $c(K_2) > 0$. Using a partition of unity of $\Omega \setminus K$, we construct a function Φ_2 as in the proof of Proposition 4.5, and we obtain Item 3. □

Note that $\Pi_{\mathcal{C},\chi} u$ is no longer holomorphic since the presence of $\mathcal{C}$ and χ destroys holomorphy, but we see that $\hbar\overline{\partial}\Pi_{\mathcal{C},\chi} u$ is 'exponentially small'. To formulate this, we define appropriate spaces.

Definition 4.10. *Let $\Phi_1 \in \mathscr{C}^0(\Omega;\mathbb{R})$. We define*

$$H^{\mathrm{loc}}_{\Phi,\Phi_1}(\Omega) := \{u \in L^2_{\Phi,\mathrm{loc}}(\Omega); \quad \hbar\partial u \in L^2_{\Phi_1,\mathrm{loc}}(\Omega)\}.$$

$H^{\mathrm{loc}}_{\Phi,\Phi_1}$ is a Fréchet space when equipped with the natural semi-norms, which injects uniformly continuously into $L^2_{\Phi,\mathrm{loc}}(\Omega)$.

Proposition 4.11. *There exists $\Phi_1 \in \mathscr{C}^0(\Omega;\mathbb{R})$ with $\Phi_1 < \Phi$ such that*

$$\Pi_{\mathcal{C},\chi} = \mathcal{O}(1) : L^2_{\Phi,\mathrm{loc}}(\Omega) \to H^{\mathrm{loc}}_{\Phi,\Phi_1}(\Omega).$$

Proof. Here the notation $\mathcal{O}(1)$ is used similarly to Definition 4.2. Let $\Omega_2 \Subset \Omega$. Since $\Pi_{\mathcal{C},\chi}$ is properly supported, there exists $\Omega_1 \Subset \Omega$ such that the support of the distribution kernel of $\Pi_{\mathcal{C},\chi} 1_{\Omega_2}$ is contained in $\Omega_1 \times \Omega_2$. Applying the $\overline{\partial}$ operator on (4.19), we get the sum of two terms: one involving $\overline{\partial}_x a_{\mathcal{C}}(x,\bar{y})$, which we estimate uniformly on $\Omega_1 \times \Omega_2$ by (4.18), and another term involving $\overline{\partial}_x \chi(x,\bar{y})$. Since $\overline{\partial}_x \chi(x,\bar{y})$ is supported away from the anti-diagonal, this last term can be uniformly estimated as well by the good contour property (2.16). This finally gives

$$\left\| \hbar\overline{\partial}\Pi_{\mathcal{C},\chi} u \right\|_{L^2_\Phi(\Omega_2)} \leq C e^{-1/C\hbar} \left\| u \right\|_{L^2_\Phi(\Omega_1)}.$$

In other words, $\hbar\overline{\partial}\Pi_{\mathcal{C},\chi} \equiv 0$ (Proposition 4.7). Hence there exists $\Phi_1 < \Phi$ such that

$$\hbar\overline{\partial}\Pi_{\mathcal{C},\chi} = \mathcal{O}(1) : L^2_{\Phi,\mathrm{loc}}(\Omega) \to L^2_{\Phi_1}(\Omega)\,.$$

By Proposition 4.9, the operator $\Pi_{\mathcal{C},\chi}$ is uniformly continuous: $L^2_\Phi(\Omega) \to L^2_\Phi(\Omega)$, which finishes the proof. □

We next turn to the reproducing property: if u is holomorphic, or exponentially close to holomorphic, then $\Pi_{\mathcal{C},\chi}u$ must be exponentially close to u. We first deal with the case of a holomorphic u.

Lemma 4.12. *There exists $\Phi_2 \in \mathscr{C}^\infty(\Omega;\mathbb{R})$ with $\Phi_2 < \Phi$ such that for all $\Omega_2 \Subset \Omega$, there exists $\Omega_1 \Subset \Omega$ and $C > 0$, independent of $\hbar$, with*

$$\forall u \in H^{\mathrm{loc}}_\Phi(\Omega), \quad \|\Pi_{\mathcal{C},\chi}u - u\|_{L^2_{\Phi_2}(\Omega_2)} \leq C\,\|u\|_{L^2_\Phi(\Omega_1)}\,.$$

Proof. Around any x_0 there is a ball Ω_0 such that $1_{\Omega_0}\Pi_{\mathcal{C},\chi}1_{\Omega_0} \equiv 1_{\Omega_0}\Pi_{x_0}1_{\Omega_0}$ (see (4.17)) where Π_{x_0} is as in Proposition 4.8. By Item 3 of that proposition, we have, for any $\Omega_2 \Subset \Omega_0$,

$$\forall u \in H_\Phi(\Omega_0), \quad \|\Pi_{\mathcal{C},\chi}u - u\|_{L^2_{\Phi_2}(\Omega_2)} \leq C\,\|u\|_{L^2_\Phi(\Omega_0)}\,.$$

We may conclude by a partition of unity argument, as in the proof of Proposition 4.5. □

Proposition 4.13. *If $\Phi_1 \in \mathscr{C}^0(\Omega;\mathbb{R})$ satisfies $\Phi_1 < \Phi$, then there exists $\Phi_2 \in \mathscr{C}^0(\Omega;\mathbb{R})$ with $\Phi_2 < \Phi$ such that*

$$\Pi_{\mathcal{C},\chi} - 1 = \mathcal{O}(1) : H^{\mathrm{loc}}_{\Phi,\Phi_1}(\Omega) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega).$$

Proof. Let $z_0 \in \Omega$.

Lemma 4.14. *$\exists$ an open neighborhood $V \Subset \Omega$ of z_0 and $\Phi_0 \in C^\infty(\mathbb{C}^n;\mathbb{R})$ such that:*

$$\Phi_0 = \Phi \textit{ in } V,$$

$$\nabla^\alpha\Phi_0 = \mathcal{O}(1) \textit{ on } \mathbb{C}^n \textit{ when } |\alpha| \geq 2, \tag{4.20}$$

$$\exists\, C > 0 \textit{ such that } \partial_{\bar{z}}\partial_z\Phi_0 \geq 1/C. \tag{4.21}$$

Proof. Let $\Phi^{(2)}$ be the Taylor polynomial of order 2 of Φ at z_0, so that

$$\nabla^\alpha(\Phi - \Phi^{(2)}) = \mathcal{O}(|z - z_0|^{3-|\alpha|}), \ |z - z_0| \text{ small },$$

for $0 \leq |\alpha| \leq 2$. Let $\chi \in C_0^\infty(B_{\mathbb{C}^n}(0,1);[0,1])$ be equal to 1 on $B_{\mathbb{C}^n}(0,1/2)$ and consider for $0 < \epsilon \ll 1$:

$$\Phi_0(z) = \Phi_{0,\epsilon}(z) = \Phi^{(2)}(z) + \chi(|z - z_0|/\epsilon)(\Phi - \Phi^{(2)})(z).$$

Then,

$$\Phi_0(z) = \begin{cases} \Phi(z) \text{ in } B(z_0, \epsilon/2), \\ \Phi^{(2)}(z) \text{ in } \mathbb{C}^n \setminus B(z_0, \epsilon), \end{cases}$$

$$\nabla^\alpha \Phi_0 = \nabla^\alpha \Phi^{(2)} + \mathcal{O}(|z - z_0|^{3-|\alpha|}), \ |\alpha| \leq 2,$$

and in particular

$$\partial_z \partial_{\bar{z}} \Phi_0 = \partial_z \partial_{\bar{z}} \Phi^{(2)} + \mathcal{O}(\epsilon) \geq 1/\mathcal{O}(1),$$

when $\epsilon > 0$ is small enough. The lemma follows with $V = B(x_0, \epsilon/2)$ for some $0 < \epsilon \ll 1$. $\square$

Let $W \Subset V$ be an open neighborhood of z_0 with smooth boundary and let $\chi_W \in C_0^\infty(V;[0,1])$ satisfy:

$$\chi_W > 0 \text{ in } W, \ \operatorname{supp} \chi_W \subset \overline{W}.$$

Then for $\delta > 0$ small enough, the function $\Phi_\delta = \Phi_0 - \delta \chi_W$ satisfies (4.20), (4.21) and

$$\Phi_\delta = \Phi \text{ in } V \setminus W,$$

$$\Phi - \delta \leq \Phi_\delta < \Phi \text{ in } W.$$

We choose $\delta > 0$ small enough so that

$$\Phi_\delta > \Phi_1 \text{ in } \overline{V}.$$

Let $\widetilde{\chi} \in C_0^\infty(V;[0,1])$ be equal to 1 on W and write

$$u = \widetilde{\chi} u + (1 - \widetilde{\chi})u, \ u \in H^{\mathrm{loc}}_{\Phi,\Phi_1}(\Omega).$$

Then

$$\hbar\overline{\partial}(\widetilde{\chi} u) = u \hbar\overline{\partial}\widetilde{\chi} + \widetilde{\chi}\hbar\overline{\partial} u \in L^2_{\Phi_\delta},$$

and

$$\|\hbar\overline{\partial}(\widetilde{\chi}u)\|_{L^2_{\Phi_\delta}(\mathbb{C}^n)} \leq \mathcal{O}(1)\left(\hbar\|u\|_{L^2_\Phi(V)} + \|\hbar\overline{\partial}u\|_{L^2_{\Phi_1}(V)}\right).$$

Moreover $\hbar\overline{\partial}(\widetilde{\chi}u)$ is $\overline{\partial}$-closed, so we can apply Appendix A (cf. (A.11), (A.12)), to find $w \in L^2_{\Phi_\delta}$ such that

$$\hbar\overline{\partial}w = \hbar\overline{\partial}(\widetilde{\chi}u),$$

$$\|w\|_{L^2_{\Phi_\delta}(\mathbb{C}^n)} \leq \mathcal{O}(\hbar^{-1/2})\left(\hbar\|u\|_{L^2_\Phi(V)} + \|\hbar\overline{\partial}u\|_{L^2_{\Phi_1}(V)}\right) \tag{4.22}$$

Write

$$u = (\widetilde{\chi}u - w) + (1-\widetilde{\chi})u + w.$$

Here $\widetilde{\chi}u - w \in H_\Phi(V)$,

$$\|\widetilde{\chi}u - w\|_{H_\Phi(V)} \leq \mathcal{O}(\hbar^{-1/2})\left(\hbar^{1/2}\|u\|_{L^2_\Phi(V)} + \|\hbar\overline{\partial}u\|_{L^2_{\Phi_1}(V)}\right).$$

We may assume without loss of generality (see also a comment below) that $\operatorname{supp}\chi$ is contained in a sufficiently small neighborhood of the diagonal, so that the restriction of $\Pi_{C,\chi}u$ to W only depends on $u_{|V}$. By Lemma 4.12, we get with $\delta > 0$ small enough,

$$\|(\Pi_{C,\chi} - 1)(\widetilde{\chi}u - w)\|_{L^2_{\Phi_\delta}(W)} \leq \mathcal{O}(\hbar^{-1/2})\left(\hbar^{1/2}\|u\|_{L^2_\Phi(V)} + \|\hbar\overline{\partial}u\|_{L^2_{\Phi_1}(V)}\right). \tag{4.23}$$

Let $\widetilde{W} \Subset W$ be another neighborhood of z_0 with smooth boundary and let $\widetilde{\Phi}_\delta \geq \Phi_\delta$ be a new function with the same properties as Φ_δ after replacing W with $\widetilde{W}$. Then (4.23) still holds after replacing $L^2_{\Phi_\delta}(W)$ with $L^2_{\widetilde{\Phi}_\delta}(\widetilde{W})$. From (4.22) we get

$$\|(\Pi_{C,\chi} - 1)w\|_{L^2_{\widetilde{\Phi}_\delta}(\widetilde{W})} \leq \mathcal{O}(\hbar^{-1/2})\left(\hbar\|u\|_{L^2_\Phi(V)} + \|\hbar\overline{\partial}u\|_{L^2_{\Phi_1}(V)}\right)$$

when $\widetilde{\Phi}_\delta$ is close enough to Φ but still $< \Phi$ in $\widetilde{W}$.

Since $\Pi_{C,\chi}$ enjoys the pseudolocal property (item 3 of Proposition 4.9) we get the same estimate for $(\Pi_{C,\chi} - 1)(1-\widetilde{\chi})u$.

Thus we have found a continuous function $\widetilde{\Phi}_\delta \leq \Phi$ in V with $\widetilde{\Phi}_\delta < \Phi$ in $\widetilde{W}$, such that

$$\|(\Pi_{C,\chi} - 1)u\|_{L^2_{\widetilde{\Phi}_\delta}(\widetilde{W})} \leq \mathcal{O}(\hbar^{-1/2})\left(\hbar^{1/2}\|u\|_{L^2_\Phi(V)} + \|\hbar\overline{\partial}u\|_{L^2_{\Phi_1}(V)}\right).$$

After a slight shrinking of $\widetilde{W}$ and increase of $\widetilde{\Phi}_\delta$ we can eliminate the factor $\mathcal{O}(\hbar^{-1/2})$.

Without the shrinking of the support of χ, we get the same estimate after replacing V with some larger domain $\Subset \Omega$. Varying z_0, we get the proposition by means of a partition of unity. $\square$

4.3 Uniqueness of the approximate Bergman projection

In the previous paragraphs, we have constructed an operator $\Pi_0 = \Pi_{\mathcal{C},\chi}$ with the following properties:

1. Π_0 is uniformly continuous: $L^2_{\Phi,\mathrm{loc}}(\Omega) \to H^{\mathrm{loc}}_{\Phi,\Phi_1}(\Omega)$ for some $\Phi_1 \in \mathscr{C}^0(\Omega;\mathbb{R})$ with $\Phi_1 < \Phi$.

2. Π_0 is uniformly properly supported.

3. $\Pi_0 \equiv \Pi_0^*$ (see Definition 4.6).

4. If $\Phi_1 \in \mathscr{C}^0(\Omega;\mathbb{R})$ satisfies $\Phi_1 < \Phi$, then there exists $\Phi_2 \in \mathscr{C}^0(\Omega;\mathbb{R})$ with $\Phi_2 < \Phi$ such that

$$\Pi_0 - 1 = \mathcal{O}(1) : H^{\mathrm{loc}}_{\Phi,\Phi_1}(\Omega) \to L^2_{\Phi_2,\mathrm{loc}}(\Omega).$$

Proposition 4.15. *Assume that* Π_0 *and* $\tilde{\Pi}$ *satisfy 1–4. Then* $\tilde{\Pi} \equiv \Pi_0$.

Proof. Using 3 and 4 for $\tilde{\Pi}$, we see that $\tilde{\Pi}^*$ satisfies 4. Since Π_0 satisfies 1, we get $(\tilde{\Pi}^* - 1)\Pi_0 \equiv 0$, *i.e.*

$$\tilde{\Pi}^*\Pi_0 \equiv \Pi_0. \tag{4.24}$$

By Corollary 4.7.1, we get $\Pi_0 \equiv \Pi_0^* \equiv \Pi_0\tilde{\Pi}$. By (4.24) with Π_0 and $\tilde{\Pi}$ exchanged, we get $\Pi_0^*\tilde{\Pi} \equiv \tilde{\Pi}$ and hence $\Pi_0 \equiv \tilde{\Pi}$ as claimed. $\square$

5 The Bergman projection on $\mathbb{C}^n$

Let $\Phi : \mathbb{C}^n \to \mathbb{R}$ satisfy

$$\begin{gathered}\Phi \text{ has a holomorphic extension to a tubular} \\ \text{neighborhood } T \text{ in the complexification } \mathbb{C} \otimes \mathbb{C}^n\,.\end{gathered}$$

We use 'Φ' also to denote the extension. Also assume that

$$\nabla^2\Phi \text{ is bounded in } T,$$

$$\partial_{\bar{z}}\partial_z\Phi \geq 1/C \text{ on } \mathbb{C}^n, \text{ for some constant } C > 0.$$

Examining the proofs, we see that the formal analytic symbol $\widehat{a}_\hbar(x, y) \sim \sum_j a_j(x, y)\hbar^j$ in the proof of Proposition 4.8 is well defined in a tubular neighborhood Ω_1 of the antidiagonal, $\operatorname{adiag}(\mathbb{C}^n)$ and satisfies the estimates on (4.16) with $C(x, y) = C$ independent of (x, y). Correspondingly, we define a_C simply by

$$a_C(x, y; \hbar) = \sum_{0\leq j\leq 1/(2C\hbar)} a_j(x, y)\hbar^j \tag{5.1}$$

in Ω_1 and a_C is holomorphic. We can define $\Pi_{c,\chi}u$ as in (4.19) with χ of the form $\chi(x - y)$, where $\chi \in C_0^\infty(\mathbb{C}^n)$ is equal to 1 near 0 and with support in a small neighborhood of 0. Choosing χ real and even; $\chi(-y) = \chi(y)$, we get

Proposition 5.1. *$\exists\, C_1 > 0$ such that with $\Phi_1 = \Phi - 1/C_1$,*

i) $\Pi_{C,\chi} = \mathcal{O}(1) : L^2_\Phi(\mathbb{C}^n) \to L^2_\Phi(\mathbb{C}^n)$ *is selfadjoint,*

ii) $\Pi_{C,\chi} = \mathcal{O}(1) : L^2_\Phi(\mathbb{C}^n) \to H_{\Phi,\Phi_1}(\mathbb{C}^n)$,

iii) $\Pi_{C,\chi} - 1 = \mathcal{O}(1) : H_\Phi(\mathbb{C}^n) \to L^2_{\Phi_1}(\mathbb{C}^n)$.

Notice that ii) amounts to i) and

$$\hbar\overline{\partial}\Pi_{c,\chi} = \mathcal{O}(e^{-1/(C_1\hbar)}) : L^2_\Phi \to L^2_\Phi,$$

where we omit to write out '$\mathbb{C}^n$' when there is no risk of confusion.

Since $\hbar\overline{\partial}\Pi_{C,\chi}u$ is $\overline{\partial}$-closed for every $u \in L^2_\Phi$, we can decompose:

$$\Pi_{C,\chi} = (\Pi_{C,\chi} - R) + R =: \widetilde{\Pi} + R,$$

where

$$R = (\hbar\overline{\partial})^{\Phi,*}(\Box^{(1)}_\Phi)^{-1}\hbar\overline{\partial}\Pi_{C,\chi} = \mathcal{O}(\hbar^{-1/2})e^{-1/(C_1\hbar)} : L^2_\Phi \to L^2_\Phi,$$
$$\widetilde{\Pi} = \mathcal{O}(1) : L^2_\Phi \to H_\Phi.$$

Here the box operator is defined in Section A and as there we let the exponent $(\Phi, *)$ indicate that we take adjoints in the L^2_Φ-spaces of scalar or form-valued functions.

Let Π be the orthogonal projection in $L^2_\Phi(\mathbb{C}^n)$ onto $H_\Phi(\mathbb{C}^n)$.

Theorem 5.2. *We have*

$$\Pi - \Pi_{C,\chi} = \mathcal{O}(\hbar^{-1/2})e^{-1/(C_1\hbar)}) : L^2_\Phi \to L^2_\Phi. \tag{5.2}$$

Proof. We have

$$\begin{aligned}\Pi\,\Pi_{C,\chi} &= \Pi\,\widetilde{\Pi} + \Pi R = \widetilde{\Pi} + \Pi\, R \\ &= \Pi_{C,\chi} - (1-\Pi)R = \Pi_{C,\chi} + \mathcal{O}(\hbar^{-1/2})e^{-1/(C_1\hbar)} : L^2_\Phi \to L^2_\Phi.\end{aligned}$$

Taking the adjoints of this relation and using that $\Pi^*_{C,\chi} = \Pi_{C,\chi}$, $\Pi^* = \Pi$, we get

$$\Pi_{C,\chi} = \Pi_{C,\chi}\Pi + \mathcal{O}(\hbar^{-1/2})e^{-1/C_1\hbar}. \tag{5.3}$$

By iii) in Proposition 5.1, we have

$$\Pi_{C,\chi}\Pi = \Pi + \mathcal{O}(1)e^{-1/(C_1\hbar)}. \tag{5.4}$$

(5.2) follows from (5.3) and (5.4). □

We next prove a corresponding result on the level of distribution kernels. Let $\widetilde{k}(x,y)e^{-2\Phi(y)/\hbar}$ denote the distribution kernel of Π. For any $1 \le \nu \le n$, since $\hbar\partial_{\bar{z}_\nu}\Pi = 0$ we know that $\partial_{\bar{z}_\nu}k = 0$. Taking the adjoint of this relation, we get $\Pi(\hbar\partial_{\bar{z}_\nu})^* = 0$ as an operator $C_0^\infty(\mathbb{C}^n) \to \mathcal{D}'(\mathbb{C}^n)$. Here $(\hbar\partial_{\bar{z}_\nu})^* = -\hbar\partial_{z_\nu} + 2\partial_{z_\nu}\Phi$ is the adjoint of $\hbar\partial_{\bar{z}_\nu}$ in for the inner product of L^2_Φ, so we get for every $u \in C_0^\infty(\mathbb{C}^n)$:

$$\begin{aligned}0 &= \int \widetilde{k}(x,y)e^{-2\Phi(y)/\hbar}(-\hbar\partial_{y_\nu} + 2\partial_{y_\nu}\Phi(y))u(y)L(\mathrm{d}y) \\ &= \int \widetilde{k}(x,y)(-\hbar\partial_{y_\nu})(e^{-2\Phi(y)/\hbar}u(y))L(\mathrm{d}y) \\ &= \int \hbar\partial_{y_\nu}(\widetilde{k}(x,y))e^{-2\Phi(y)/\hbar}u(y)L(\mathrm{d}y).\end{aligned}$$

It follows that $\partial_{y_\nu}\widetilde{k}(x,y) = 0$, so we have the elliptic 1st order system for $\widetilde{k}$:

$$\overline{\partial}_{x_\nu}\widetilde{k}(x,y) = 0,\ \partial_{y_\nu}\widetilde{k}(x,y) = 0.$$

From the ellipticity, we conclude that $\widetilde{k}(x,y)$ is a smooth function, holomorphic in x and anti-holomorphic in y. Hence $\widetilde{k}(x,y) = k(x,\bar{y})$ where $k(x,y)$ if holomorphic on $\mathbb{C}^{2n}$. For more details, see [14].

Recall that

$$\begin{aligned}\Phi(y) &= \Phi(y_0) + 2\Re(\partial_y \Phi(y_0) \cdot (y - y_0)) + \mathcal{O}(|y - y_0|^2)\\ &= \Phi(y_0) + 2\Re(\partial_{\bar{y}} \Phi(y_0) \cdot (\overline{y - y_0})) + \mathcal{O}(|y - y_0|^2).\end{aligned}$$

Let $f \in C_0^\infty(\mathbb{C}^n)$ be a radial function with $\int f(y)L(\mathrm{d}y) = 1$ and put

$$e_{x_0}(x) = \hbar^{-n} f\left(\frac{x - x_0}{\hbar^{1/2}}\right) e^{\frac{1}{\hbar}(2\Phi(x) - \Phi(x_0) - 2\partial_{\bar{x}}\Phi(x_0)\cdot\overline{(x - x_0)})}. \tag{5.5}$$

Then

$$|e_{x_0}(x)| = \hbar^{-n} \left| f\left(\frac{x - x_0}{\hbar^{1/2}}\right)\right| e^{\frac{1}{\hbar}(\Phi(x) + \mathcal{O}(|x - x_0|^2))},$$

so

$$\|e_{x_0}\|^2_{L^2_\Phi} \asymp \hbar^{-n}. \tag{5.6}$$

Lemma 5.3. *For x_0, $y_0 \in \mathbb{C}^n$, we have*

$$(\Pi e_{y_0} | e_{x_0})_{L^2_\Phi} = k(x_0, \bar{y}_0) e^{-\frac{1}{\hbar}(\Phi(x_0) + \Phi(y_0))}. \tag{5.7}$$

Proof.

$$\begin{aligned}&e^{\frac{1}{\hbar}(\Phi(x_0)+\Phi(y_0))} (\Pi e_{y_0} | e_{x_0})_{L^2_\Phi} =\\ &\qquad \iint \hbar^{-n} \overline{f\left(\frac{x - x_0}{\hbar^{1/2}}\right)} e^{-\frac{2}{\hbar}\partial_x \Phi(x_0)\cdot(x - x_0)} k(x, \bar{y}) \times\\ &\qquad\qquad \hbar^{-n} f\left(\frac{y - y_0}{\hbar^{1/2}}\right) e^{-\frac{2}{\hbar}\partial_{\bar{y}}\Phi(y_0)\cdot\overline{(y - y_0)}} L(\mathrm{d}x) L(\mathrm{d}y).\end{aligned}$$

Applying the spherical mean-value property for holomorphic and anti-holomorphic functions to the x-integral and y-integral respectively, this boils down to $\widetilde{k}(x_0, y_0)$. □

Remark 5.4. *In [14, Section 3] a somewhat similar argument is given to estimate a distribution kernel in the metaplectic framework and with the spherical mean-value property replaced by the use of the reproducing kernel (known exactly in that case).*

Theorem 5.5. *Let Ω_1 and a_C be as in and around (5.1). Let $k(x, \bar{y}; \hbar) e^{-2\Phi(y)/\hbar}$ be the distribution kernel of Π. There exists a constant $C_2 > 0$ such that*

$$\left| e^{-(\Phi(x)+\Phi(y))/\hbar} \left(k(x, \bar{y}; \hbar) - (1_{\Omega_1} a_C)(x, \bar{y}; \hbar) e^{\psi(x,\bar{y})/\hbar}\right)\right| \leq \mathcal{O}(1) e^{-\frac{1}{C_2 \hbar}},$$

uniformly on $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. For (x_0, y_0) in a small tubular neighborhood of the diagonal, we have $\chi(x-y) = 1$ in a small ball of fixed radius around (x_0, y_0) and by the proof of Lemma 5.3, we get

$$(\Pi_{C,\chi} e_{y_0} | e_{x_0})_{L^2_\Phi} = a_C(x_0, \bar{y}_0; \hbar) e^{(2\Re\psi(x_0,\bar{y}_0) - (\Phi(x_0)+\Phi(y_0)))/\hbar}. \tag{5.8}$$

Recall here that

$$2\Re\psi(x, \bar{y}) - \Phi(x) - \Phi(y) \asymp -|x-y|^2$$

so the right hand side of (5.8) is exponentially decreasing outside any tubular neighborhood of $\operatorname{diag}(\mathbb{C}^n \times \mathbb{C}^n)$. Using this fact, we get by direct estimates that

$$(\Pi_{C,\chi} e_{y_0} | e_{x_0}) = \mathcal{O}(1) e^{-\frac{1}{C_1 \hbar}},$$

for (x_0, y_0) outside any fixed tubular neighborhood of $\operatorname{diag}(\mathbb{C}^n \times \mathbb{C}^n)$. Thus,

$$(\Pi_{C,\chi} e_{y_0} | e_{x_0})_{L^2_\Phi} = 1_{\Omega_1}(x_0, \bar{y}_0) a_C(x_0, \bar{y}_0; \hbar) e^{(2\Re\psi(x_0,\bar{y}_0) - (\Phi(x_0)+\Phi(y_0)))/\hbar} + \mathcal{O}(1) e^{-\frac{1}{C_1\hbar}}, \tag{5.9}$$

where Ω_1 is any small tubular neighborhood of the diagonal and $C_1 = C_1(\Omega) > 0$.

The theorem, now follows from (5.9), (5.7) and the fact that (5.2) provides us with the estimate,

$$((\Pi - \Pi_{C,\chi}) e_{y_0} | e_{x_0})_{L^2_\Phi} = \mathcal{O}(\hbar^{-1/2}) e^{-\frac{1}{C_1\hbar}} \|e_{x_0}\|_{L^2_\Phi} \|e_{y_0}\|_{L^2_\Phi} = \mathcal{O}(\hbar^{-2n-1/2}) e^{-\frac{1}{C_1\hbar}},$$

with some new constant, that can be further increased to absorb the power of $\hbar$. $\square$

6 The Bergman projection for line bundles

In this section we consider a compact complex manifold X, of complex dimension n, and two holomorphic line bundles L and E over X. Both L and E are equipped with Hermitian metrics, denoted respectively by g_L and g_E, giving rise to a metric $g_L^k \otimes g_E$ on the tensor product $F_k := L^k \otimes E$, $k \in \mathbb{N}^*$. We assume that g_L has strictly positive curvature. Then $i/2$ times the curvature of L, which is a closed 2-form whose cohomology class is the Chern class $2\pi c_1(g_L)$, is a Kähler form, and therefore induces a volume form ω_n on X, and hence a scalar product $(\cdot|\cdot)_k$ on the space of sections of F_k.

Notice that F_k is positive if k is large enough. The orthogonal projection Π_k from $L^2(X, F_k)$ onto $\mathcal{H}^0(X, F_k)$, the subspace of holomorphic sections, is called the associated *Bergman projection.* Its distribution kernel is a smooth section $K(\cdot, \cdot; k)$ of the external tensor product $F_k \boxtimes F_k^*$ over $X \times X$ defined by

$$\Pi_k u(x) = \int_X K(x, y; k) u(y) \omega_n(\mathrm{d}y).$$

(Recall that $F_k \boxtimes F_k^* = \pi_1^*(F_k) \otimes \pi_2^*(F_k^*)$, where π_j, $j = 1, 2$ are the coordinate projection maps $X \times X \to X$. Thus $F_k \boxtimes F_k^*$ is the line bundle over $X \times X$ whose fiber over (x, y) is the space of linear maps from $F_k(y)$ to $F_k(x)$.) If $f_1, \ldots, f_{N_k}$ is an orthonormal basis of $\mathcal{H}^0(X, F_k)$ then the formula $\Pi_k u = \sum_{j=1}^{N_k} (u|f_j) f_j$ gives

$$K(x, y; k) = \sum_{j=1}^{N_k} f_j(x; k) \left(\cdot | f_j(y; k)\right)_{F_k(y)}. \tag{6.1}$$

We now fix a point $x_0 \in X$ and use a trivializing holomorphic section s_L of L above a neighborhood Ω_0 of x_0 (which me may identify with an open ball around $0 \in \mathbb{C}^n$) to define the local real-valued analytic function Φ_{x_0} such that the Hermitian norm of s_L is given, for $x \in \Omega_0$, by

$$|s_L(x)|_L = e^{-\Phi_{x_0}(x)}. \tag{6.2}$$

The corresponding Kähler form is $\omega_L = i\partial\overline{\partial}\Phi_{x_0}$. We define similarly the section s_E, and

$$s_k := s_L^k \otimes s_E \in \mathcal{H}^0(X, F_k).$$

Notice that $|s_k(x)|_{F_k} = e^{-k\Phi_{x_0}(x)} G_{x_0}(x)$ for some non-vanishing analytic function $G_{x_0} = |s_E|_E$, and hence, if a local section of F_k has the form $\tilde{u} = us_k$, then

$$\|\tilde{u}\|_k^2 = \int_{\Omega_0} |u|^2 e^{-2k\Phi_{x_0}} G_{x_0}^2 \omega_n.$$

Similarly, if $y_0 \in X$, we construct a trivializing section t_k of F_k on a neighborhood V_0 of y_0, with $|t_k(y)|_{F_k} = e^{-k\Phi_{y_0}(y)} G_{y_0}(y)$, and we can write

$$K(x, y; k) = b(x, \bar{y}; k) s_k(x) \otimes t_k(y)^*, \tag{6.3}$$

for $(x, y) \in \Omega_0 \times V_0$, where $t_k(y)^*$ denotes the adjoint map $(\cdot | t_k(y))$. Then from (6.1) we see that $b(x, y; k)$ is holomorphic both in x and y, and

$$|K(x, y; k)|_{F_{k,x} \otimes F_{k,y}^*} = e^{-k(\Phi_{x_0}(x) + \Phi_{y_0}(y))} G_{x_0}(x) G_{y_0}(y) \left| b(x, \bar{y}; k) \right|.$$

On V_0, we define the 'local Bergman projection' $\tilde{\Pi}_k = \tilde{\Pi}_{k,x_0,y_0}$ by

$$\forall u \in L^2_\Phi(V_0, G^2_{y_0}\omega_n), \qquad \Pi_k(ut_k) = (\tilde{\Pi}_k u)s_k,$$

which means that

$$\tilde{\Pi}_k u(x) = \int_{\Omega_0} b(x,\bar{y})u(y)e^{-2k\Phi(y)}G^2(y)\omega_n(\mathrm{d}y) = \big(uG^2_{y_0}|B_x\big)_{L^2_\Phi(\Omega_0,\omega_n)}$$

with $B_x(y;k) := \overline{b(x,\bar{y};k)}$.

Theorem 6.1. *Assume that g_L and g_E are real-analytic (and g_L has strictly positive curvature). Then the following estimates hold:*

1. *If $x_0 \neq y_0$ then there exists $C > 0$ such that, uniformly in a neighborhood $\Omega_0 \times V_0$ of (x_0, y_0),*

$$|K(x,y;k)|_{F_{k,x}\otimes F^*_{k,y}} \leq Ce^{-\frac{k}{C}}.$$

Equivalently,

$$e^{-k(\Phi_{x_0}(x)+\Phi_{y_0}(y))}\,|b(x,\bar{y};k)| = \mathcal{O}(e^{-\frac{k}{C}}).$$

2. *For any $x_0 \in X$, there exists a neighborhood Ω_0 of x_0, and a classical analytic symbol a on $\Omega_0 \times \overline{\Omega_0}$, such that, for all $(x,y) \in \Omega_0 \times \Omega_0$,*

$$e^{-k(\Phi(x)+\Phi(y))}\left|b_k(x,\bar{y}) - \frac{(2k)^n}{\pi^n}a(x,\bar{y};k^{-1})e^{2k\psi(x,\bar{y})}\right| \leq Ce^{-\frac{k}{C}},$$

for some constant $C > 0$, where $\Phi = \Phi_{x_0}$ is defined in (6.2), and ψ is its polarized form (2.12).

Proof. We first treat the case where the bundle E is trivial. The strategy is the same as in Section 5. Consider a trivialization of F_k in a neighborhood Ω_0 of x_0, as above. As in Proposition 4.9, we construct a classical analytic symbol $a_\hbar$ and the approximate Bergman projection $\Pi_0 = \Pi_{\mathcal{C},\chi}$ obtained by smooth cut-off of $\mathrm{Op}_r^{\mathrm{Brg}}(a_\hbar)$ (see (4.19)), acting on $L^2_\Phi(\Omega_0)$. We now let $\hbar = 1/k$ and define $\hat{\Pi}_0 : L^2_{\mathrm{comp}}(\Omega_0; F_k) \to L^2(\Omega_0; F_k)$ by

$$\hat{\Pi}_0(us_k) = (\Pi_0 u)s_k.$$

One can find a finite cover of X by open sets Ω_j, with $x_j \in \Omega_j$, on which the corresponding operator $\hat{\Pi}_j$ is defined as above. Let $\chi_j \in \mathscr{C}_0^\infty(\Omega_j; \mathbb{R}^+)$ be such that $\sum_j \chi_j = 1$ on X, let $\tilde{\chi}_j \in \mathscr{C}_0^\infty(\Omega_j)$ be equal to 1 on a neighborhood of the support of χ_j, and define

$$\hat{\Pi} := \sum_j \tilde{\chi}_j \hat{\Pi}_j \chi_j \quad = \mathcal{O}(1) : L^2(X; F_k) \to L^2(X; F_k).$$

Let $\Lambda^{(p,q)} \to X$ be the vector bundle of (p,q)-forms on the tangent space of X, equipped with the metric induced from ω_n on X. Let $\overline{\partial}_k$ be the usual Dolbeault operator, mapping sections of $\Lambda^{(0,q)} \otimes F_k$ to sections of $\Lambda^{(0,q+1)} \otimes F_k$. The following analogue of Proposition 5.1 holds.

Proposition 6.2. $\exists\, C_1 > 0$ *such that*

i) $\hat{\Pi}^* - \hat{\Pi} = \mathcal{O}(e^{-kC_1}) : \; L^2(X; F_k) \to L^2(X; F_k)$,

ii) $\overline{\partial}_k \hat{\Pi} = \mathcal{O}(e^{-kC_1}) : \; L^2(X; F_k) \to L^2(X; \Lambda^{(0,1)} \otimes F_k)$,

iii) $\hat{\Pi} - 1 = \mathcal{O}(e^{-kC_1}) : \; \mathcal{H}^0(X; F_k) \to L^2(X; F_k)$.

Proof. We use the exponential locality property of Proposition 4.9 which implies that

$$\forall \chi_1, \chi_2 \in \mathscr{C}_0^\infty(\Omega_j) \text{ with disjoint supports, } \chi_1 \hat{\Pi}_j \chi_2 \equiv 0. \tag{6.4}$$

This gives, for all m,

$$\chi_m \tilde{\chi}_j \hat{\Pi}_j \chi_j \equiv \chi_m \tilde{\chi}_j \hat{\Pi}_j \chi_j \tilde{\chi}_m. \tag{6.5}$$

Let $\Omega_{j,m} \Subset \Omega_j \cap \Omega_m$; the restricted operator on $L^2_\Phi(\Omega_{i,j})$, $1_{\Omega_{j,m}} \hat{\Pi}_j 1_{\Omega_{j,m}}$ is an approximate Bergman projection in the sense of Section 4.3, and so is $1_{\Omega_{j,m}} \hat{\Pi}_m 1_{\Omega_{j,m}}$. Hence, by uniqueness (Proposition 4.15), we have

$$1_{\Omega_{j,m}} \hat{\Pi}_j 1_{\Omega_{j,m}} \equiv 1_{\Omega_{j,m}} \hat{\Pi}_m 1_{\Omega_{j,m}}.$$

Hence we have $\tilde{\chi}_j \chi_m \hat{\Pi}_j \chi_j \tilde{\chi}_m \equiv \tilde{\chi}_j \chi_m \hat{\Pi}_m \chi_j \tilde{\chi}_m$ which, in view of (6.5), gives

$$\chi_m \tilde{\chi}_j \hat{\Pi}_j \chi_j \equiv \chi_m \hat{\Pi}_m \chi_j \tilde{\chi}_m.$$

Thus from item 2 of Proposition 4.9 we have $\hat{\Pi}_m^* \equiv \hat{\Pi}_m$, and hence

$$\hat{\Pi} = \sum_{j,m} \chi_m \tilde{\chi}_j \hat{\Pi}_j \chi_j \equiv \sum_{j,m} \chi_m \hat{\Pi}_m \tilde{\chi}_m \chi_j = \sum_m \chi_m \hat{\Pi}_m \tilde{\chi}_m \equiv \hat{\Pi}^*,$$

which shows item i).

Applying the Dolbeault operator $\overline{\partial}_k$, we obtain

$$\hbar\overline{\partial}_k\hat{\Pi} \equiv \sum_j (\hbar\overline{\partial}\tilde{\chi}_j)\hat{\Pi}_j\chi_j + \tilde{\chi}_j\hbar\overline{\partial}_k\hat{\Pi}_j\chi_j.$$

From Proposition 4.11 and its proof, we have $\hbar\overline{\partial}_k\hat{\Pi}_j \equiv 0$, and from (6.4) we get $(\hbar\overline{\partial}\tilde{\chi}_j)\hat{\Pi}_j\chi_j \equiv 0$. This proves item ii).

Finally, let $u \in \mathcal{H}^0(X; F_k)$. Restricting to Ω_j, Lemma 4.12 gives

$$\chi_j(1-\hat{\Pi}_j)u \sim 0.$$

Hence $\chi_j(1-\hat{\Pi}_j)\tilde{\chi}_j u \sim \chi_j(1-\hat{\Pi}_j)(\tilde{\chi}_j - 1)u$. By exponential localization, $\chi_j(1-\hat{\Pi}_j)(1-\tilde{\chi}_j)u \sim 0$. Hence $\chi_j(1-\hat{\Pi}_j)\tilde{\chi}_j u \sim 0$. By summing and using the selfadjointness of $\hat{\Pi}$, we get

$$\hat{\Pi}u \sim \sum_j \chi_j\tilde{\chi}_j u = \sum_j \chi_j u = u.$$

In addition, we see from Lemma 4.12 that the estimates are actually uniform in u when $\|u\|_k = 1$, since in fact $\chi_j(1-\hat{\Pi}_j) \underset{\mathrm{H}}{\equiv} 0$ on $H_\Phi(\Omega_j)$ (Definition 2.18). Thus, we obtain item iii). $\square$

We now use some basic facts from the Hodge-Kodaira theory in order to prove that $\hat{\Pi}$ is close to Π_k. The Hodge-Kodaira Laplacian, acting on sections of $\Lambda^{(0,1)} \otimes F_k$, is

$$\Box_k := \overline{\partial}_k^*\overline{\partial}_k + \overline{\partial}_k\overline{\partial}_k^*, \tag{6.6}$$

where the adjoint is taken with respect to the scalar product $(\cdot|\cdot)_k$ defined above, extended to differential forms thanks to the metric ω_n on X. In the semiclassical setting $k \to \infty$, it is natural to consider the renormalized operator $\frac{1}{k^2}\Box_k$. The following well-known estimate can be found in [16, Section 7.3].

Lemma 6.3 (Bochner-Kodaira-Nakano inequality)**.** *There exists $c > 0$ such that, for all $\tilde{u} \in \mathscr{C}^\infty(X; \Lambda^{(0,1)} \otimes F_k)$,*

$$\left(\frac{1}{k^2}\Box_k\tilde{u}|\tilde{u}\right)_k \geq \frac{c}{k}\|\tilde{u}\|_k^2.$$

The constant c is related to the curvature of L as follows. In local coordinates where the Hermitian metric of L is $e^{-2\Phi}$, and $\omega_n = i\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$, write $i\partial\overline{\partial}\phi(x) \geq m(x)\mathrm{Id}$. Then $c = 2\min_X m$.

Proposition 6.4.

$$\Pi_k - \hat{\Pi} = \mathcal{O}(e^{-k/C_1}) : \ L^2(X; F_k) \to L^2(X; F_k). \tag{6.7}$$

Proof. The argument is the same as for the proof of Theorem 5.2. For any smooth section $\tilde{u}$, we get from (6.6) that

$$\left(k^{-2}\Box_k \tilde{u}|\tilde{u}\right)_k = \|k^{-1}\overline{\partial}_k \tilde{u}\|_k^2 + \|k^{-1}\overline{\partial}_k^* \tilde{u}\|_k^2.$$

This, together with Lemma 6.3, gives

$$\left(k^{-2}\Box_k \tilde{u}|\tilde{u}\right)_k \geq \tilde{c}\,\|\tilde{u}\|^2_{H^1_k}, \tag{6.8}$$

where

$$\|\tilde{u}\|^2_{H^1_k} := k^{-2}\|\overline{\partial}_k \tilde{u}\|_k^2 + k^{-2}\|\overline{\partial}_k^* \tilde{u}\|_k^2 + k^{-1}\,\|\tilde{u}\|_k^2\,. \tag{6.9}$$

(This is analogous to (A.1).) Since $\Box_k$ is selfadjoint, Lemma 6.3 implies that we can define a bounded operator $(k^{-2}\Box_k)^{-1}$, acting on $(0,1)$-forms, with norm

$$(k^{-2}\Box_k)^{-1} = \mathcal{O}(k) : L^2(X; \Lambda^{(0,1)} \otimes F_k) \to L^2(X; \Lambda^{(0,1)} \otimes F_k).\,,$$

and (6.8) implies that $(k^{-2}\Box_k)^{-1}$ can be extended to

$$(k^{-2}\Box_k)^{-1} = \mathcal{O}(1) : H_k^{-1} \to H_k^1\,, \tag{6.10}$$

where H_k^{-1} is the dual to H_k^1, and the latter is the completion of the space of smooth sections for the norm (6.9). Consider the bounded selfadjoint operator P on $L^2(X; F_k)$ given by the formula:

$$\begin{aligned} P &= 1 - (k^{-1}\overline{\partial}_k^*)(k^{-2}\Box_k)^{-1}(k^{-1}\overline{\partial}_k) \\ &= 1 - \overline{\partial}_k^*\Box_k^{-1}\overline{\partial}_k \quad : L^2(X; F_k) \to L^2(X; F_k)\,. \end{aligned}$$

First, we remark that if $\overline{\partial}_k u = 0$ then $Pu = u$. Next, we have

$$\overline{\partial}_k P = \overline{\partial}_k - \overline{\partial}_k\overline{\partial}_k^*\Box_k^{-1}\overline{\partial}_k = \overline{\partial}_k - (\Box_k - \overline{\partial}_k^*\overline{\partial}_k)\Box_k^{-1}\overline{\partial}_k. \tag{6.11}$$

Since $\overline{\partial}_k^*\overline{\partial}_k$ commutes with $\Box_k$, it also commutes with $\Box_k^{-1}$, and the right-hand side of (6.11) vanishes. Therefore the range of P is contained in $\mathcal{H}^0(X;F_k)$. This entails that P is the orthogonal projection onto $\mathcal{H}^0(X;F_k)$, *i.e.* $P=\Pi_k$.

Let $v \in L^2(X;F_k)$. In order to measure the lack of holomorphy of $\hat{\Pi}v$ we define

$$u = Rv := (1-\Pi_k)\hat{\Pi}v = \overline{\partial}_k^*\Box_k^{-1}\overline{\partial}_k\hat{\Pi}v.$$

From (6.10), we get

$$\|Rv\|_k \leq \frac{C}{\sqrt{k}}\|\overline{\partial}_k\hat{\Pi}v\|_k. \tag{6.12}$$

Since $\overline{\partial}_k(\hat{\Pi}-R) = \overline{\partial}_k\hat{\Pi} - \overline{\partial}_k(1-\Pi_k)\hat{\Pi} = 0$, we have

$$\Pi_k\hat{\Pi} = \Pi_k(\hat{\Pi}-R) + \Pi_k R = \hat{\Pi} - R + \Pi_k R = \hat{\Pi} - (1-\Pi_k)R.$$

Using (6.12) with item (ii) of Proposition 6.2, we get

$$\Pi_k\hat{\Pi} = \hat{\Pi} + \mathcal{O}(k^{-1/2}e^{-k/C_1}).$$

Passing to the adjoints we get, using item *(i)* of Proposition 6.2:

$$\hat{\Pi}\Pi_k = \hat{\Pi} + \mathcal{O}(e^{-k/C_1}). \tag{6.13}$$

On the other hand, using item *(iii)* of Proposition 6.2 we have $\hat{\Pi}\Pi_k = \Pi_k + \mathcal{O}(e^{-k/C_1})$; in view of (6.13), this gives the result. $\Box$

Finally, let $(x_0,y_0) \in X \times X$, and let s_k, t_k be trivializing sections of F_k near x_0 and y_0, respectively, as discussed above, see (6.3). Near x_0 we may define the compactly supported section $\tilde{e}_{x_0} = e_{x_0}s_{x_0}$ where e_{x_0} is given by (5.5), in which $\Phi = \Phi_{x_0}$ is now defined by $|s_{x_0}(x)|_L = e^{-\Phi_{x_0}(x)}$. Similarly, we define $\tilde{e}_{y_0}$, using a function Φ_{y_0} defined near y_0. Lemma 5.3 gives

$$(\Pi_k\tilde{e}_{y_0}|\tilde{e}_{x_0})_k = b(x_0,\bar{y}_0)e^{-k(\Phi_{x_0}(x_0)+\Phi_{y_0}(y_0))}. \tag{6.14}$$

If $x_0 \neq y_0$, one can find smooth cut-off functions χ_{x_0} and χ_{y_0}, with disjoint supports, such that $\chi_{x_0}e_x = e_x$ and $\chi_{y_0}e_y = e_y$, for (x,y) close to (x_0,y_0). By (6.4) we see that $\chi_{x_0}\hat{\Pi}\chi_{y_0} \equiv 0$. Hence

$$\left|\left(\hat{\Pi}\tilde{e}_y|\tilde{e}_x\right)_k\right| \leq Ce^{-k/C}$$

for some $C>0$, and in this case item 1 of the theorem is a consequence of (6.14) and (6.7) (and, of course, (5.6)).

Now let us assume that x_0 and y_0 belong to the same trivializing open set Ω_j, and take $\Phi_{x_0} = \Phi_{y_0}$. Thus, as in the proof of Theorem 5.5, we have

$$\left(\hat{\Pi}_j e_{y_0} | e_{x_0}\right)_{L^2_\Phi} = \frac{(2k)^n}{\pi^n} a(x_0, \bar{y}_0; k^{-1}) e^{2k(\psi(x_0,\bar{y}_0)-(\Phi(x_0)+\Phi(y_0)))}.$$

Using again (6.14), (6.7) and (5.6), we obtain item 2 of the theorem, finishing the proof in the case of a trivial factor E.

If the bundle E is not trivial, we need to replace the local weight $k\Phi(x)$ by $k\Phi(x)+\Phi_G(x)$, where $\Phi_G(x) := -\ln G(x)$. This amounts to replacing the symbol $a_\hbar(x,\bar{y})$ by $a_\hbar^G(x,\bar{y}) := a_\hbar(x,\bar{y})e^{2\psi_G(x,\bar{y})}$, where $\psi_G(x,y)$ is the holomorphic function defined for x close to y by $\psi_G(x,\bar{x}) = \Phi_G(x)$, as in (2.12). Similarly, $b_k(x,\bar{y})$ should be replaced by $b_k(x,\bar{y})e^{2\psi_G(x,\bar{y})}$, see (6.3). Since Φ_G does not depend on k, the ellipticity estimates of Lemma 6.3 hold with ck replaced by $ck - C$ for some $C > 0$. Hence if k is large enough, items 1 and 2 of the theorem still hold true, with a possibly different constant C. $\square$

Remark 6.5. *Using the auxiliary bundle E, one obtains that Theorem 6.1 holds for an arbitrary analytic volume form instead of the natural Kähler one. Indeed, the analytic factor in front of ω_n can be incorporated in the Hermitian metric of E.*

Remark 6.6. *After this article was written, we have received a preliminary work by Deleporte [15], where the author first studies the particular case of Kähler manifolds with constant sectional curvature, for which explicit computations can be done that don't require microlocal analysis, and then obtains a new proof of Theorem 6.1.*

A Quick review of $\overline{\partial}$ on $L^2_\Phi(\mathbb{C}^n)$.

We review Hörmander's approach [23] to the $\overline{\partial}$-problem in the simple case of functions on $\mathbb{C}^n$ and with more explicit reference to the Hodge Laplacian. A more detailed presentation can be found in the appendix of [29] and here we only give a short résumé.

Let $\Phi : \mathbb{C}^n \to \mathbb{R}$ be a function of class C^2 (in [29] we treat the slightly more general case of $C^{1,1}$) such that

$$\nabla^\alpha \Phi \in L^\infty(\mathbb{C}^n), \text{ for } |\alpha| = 2,$$

$$\Phi''_{\bar{x},x} \geq 1/C, \text{ for some constant } C > 0.$$

The problem $\hbar\overline{\partial}u = v$ in the spaces L^2_Φ is equivalent to

$$\overline{\partial}_\Phi(e^{-\Phi/\hbar}u) = e^{-\Phi/\hbar}v$$

in the usual (unweighted) L^2-spaces, where

$$\overline{\partial}_\Phi = e^{-\Phi/\hbar} \circ \hbar\overline{\partial} \circ e^{\Phi/\hbar} = \hbar\overline{\partial} + (\overline{\partial}\Phi)^\wedge,$$

and $\omega^\wedge$ indicates left exterior multiplication with the $(0,1)$-form ω. The corresponding real adjoint operator will be denoted by $\omega^\rfloor$ (contraction with ω). Here we use the standard point-wise real scalar product on real p-forms, extended bilinearly to the complexified space. Recall that

$$\langle \mathrm{d}x_j|\mathrm{d}x_k\rangle = \langle \mathrm{d}\bar{x}_j|\mathrm{d}\bar{x}_k\rangle = 0,\ \langle \mathrm{d}x_j|\mathrm{d}\bar{x}_k\rangle = \delta_{j,k},\quad \mathbb{C}^n = \mathbb{C}^n_{x_1,\dots,x_n}.$$

Write

$$\overline{\partial}_\Phi = \sum Z_j \otimes \mathrm{d}\bar{x}_j^\wedge,\quad \overline{\partial}_\Phi^* = \sum Z_j^* \otimes \mathrm{d}x_j^\rfloor,$$
$$Z_j = \hbar\partial_{\bar{x}_j} + \partial_{\bar{x}_j}\Phi,\quad Z_j^* = -\hbar\partial_{x_j} + \partial_{x_j}\Phi.$$

Recall that $\overline{\partial}$ and $\overline{\partial}_\Phi$ take $(0,q)$-forms to $(0,q+1)$ forms and define complexes: $\overline{\partial}^2 = 0$, $\overline{\partial}_\Phi^2 = 0$. The Hodge Laplacian is

$$\widetilde{\Box}_\Phi = \overline{\partial}_\Phi\overline{\partial}_\Phi^* + \overline{\partial}_\Phi^*\overline{\partial}_\Phi.$$

It preserves $(0,q)$-forms and a standard calculation gives

$$\widetilde{\Box}_\Phi = \left(\sum_1^n Z_j^* Z_j\right) \otimes 1 + \sum_{j,k}[Z_j, Z_k^*]\mathrm{d}\bar{x}_j^\wedge \mathrm{d}x_k^\rfloor,\ [Z_j, Z_k^*] = 2\hbar\partial_{\bar{x}_j}\partial_{x_k}\Phi.$$

Identifying the $(0,1)$-form $\sum u_j \mathrm{d}\bar{x}_j$ with the $\mathbb{C}^n$-valued function $(u_1,\dots,u_n)^{\mathrm{t}}$, we get for the restriction $\widetilde{\Box}_\Phi^{(1)}$ of $\widetilde{\Box}_\Phi$ to $(0,1)$-forms:

$$\widetilde{\Box}_\Phi^{(1)} = \left(\sum_1^n Z_j^* Z_j\right) \otimes 1 + 2\hbar\Phi''_{\bar{x},x}.$$

It follows that

$$\|u\|_{H^1}^2 \leq \mathcal{O}(1)\left(\widetilde{\Box}_\Phi^{(1)}u|u\right),\ u \in C_0^\infty(\mathbb{C}^n; \wedge^{0,1}\mathbb{C}^n), \tag{A.1}$$

where

$$\|u\|_{H^1} := \left(\sum \left(\|Z_j u\|^2 + \|Z_j^* u\|^2\right) + \hbar\|u\|^2\right)^{\frac{1}{2}}.$$

Let $H^1 \subset L^2(\mathbb{C}^n; \wedge^{0,1}\mathbb{C}^n)$ be the Hilbert space obtained as the completion of $C_0^\infty(\mathbb{C}^n; \wedge^{0,1}\mathbb{C}^n)$ for the H^1-norm. Sometimes we drop the notation $\wedge^{0,1}\mathbb{C}^n$, when it is clear that we work with $(0,1)$-forms. The inclusion map $H^1 \to H^0 := L^2$ is of norm $\mathcal{O}(\hbar^{-1/2})$ and the same holds for the dual inclusion $H^0 \to H^{-1}$, where H^{-1} denotes the dual of H^1 for the L^2-inner product.

From (A.1) we get with standard variational arguments that

$$\widetilde{\Box}_\Phi^{(1)} : H^1 \to H^{-1} \tag{A.2}$$

is bijective with with inverse satisfying

$$(\widetilde{\Box}_\Phi^{(1)})^{-1} = \begin{cases} \mathcal{O}(1) : \ H^{-1} \to H^1, \\ \mathcal{O}(\hbar^{-1/2}) : \ H^0 \to H^1, \ H^{-1} \to H^0, \\ \mathcal{O}(\hbar^{-1}) : \ H^0 \to H^0. \end{cases} \tag{A.3}$$

We saw in the appendix of [29] that if $v \in L^2(\mathbb{C}^n; \wedge^{0,1}\mathbb{C}^n)$ satisfies $\overline{\partial}_\Phi v = 0$, then $u = \overline{\partial}_\Phi^*(\widetilde{\Box}_\Phi^{(1)})^{-1} v$ solves

$$\overline{\partial}_\Phi u = v, \tag{A.4}$$

and

$$\|u\|_{H^0} \leq \mathcal{O}(1)\|v\|_{H^{-1}} \leq \mathcal{O}(\hbar^{-1/2})\|v\|_{H^0}. \tag{A.5}$$

If $v \in L^2_\Phi(\mathbb{C}^n; \wedge^{0,1}\mathbb{C}^n)$ and $\overline{\partial} v = 0$, then

$$u = e^{\Phi/\hbar}\overline{\partial}_\Phi^*(\Box_\Phi^{(1)})^{-1} e^{-\Phi/\hbar} v$$

solves,

$$\hbar\overline{\partial} u = v, \tag{A.6}$$

and

$$\|u\|_{L^2_\Phi} \leq \mathcal{O}(\hbar^{-1/2})\|v\|_{L^2_\Phi}. \tag{A.7}$$

The orthogonal projection $\widetilde{\Pi} : L^2(\mathbb{C}^n) \to L^2 \cap \mathcal{N}(\overline{\partial}_\Phi)$ on the level of 0-forms, is given by

$$\widetilde{\Pi} = 1 - \overline{\partial}_\Phi^*(\widetilde{\Box}_\Phi^{(1)})^{-1}\overline{\partial}_\Phi. \tag{A.8}$$

See (A.14) in [29].

We finally translate the results to the setting of L^2_Φ, noting that $L^2 \ni u \mapsto e^{\Phi/\hbar} u \in L^2_\Phi$ is unitary and maps $\mathcal{N}(\overline{\partial}_\Phi)$ to $\mathcal{N}(\hbar\overline{\partial})$. Correspondingly we have the unitary conjugations

$$
\begin{aligned}
\overline{\partial}_\Phi &= e^{-\Phi/\hbar} \hbar \overline{\partial} e^{\Phi/\hbar}, \\
\overline{\partial}_\Phi^* &= e^{-\Phi/\hbar} (\hbar\overline{\partial})^{\Phi,*} e^{\Phi/\hbar},
\end{aligned}
\tag{A.9}
$$

where the exponent $(\Phi, *)$ indicates the adjoint for the L^2_Φ norms. Note that the last relation gives,

$$
(\hbar\overline{\partial})^{\Phi,*} = e^{\Phi/\hbar} \overline{\partial}_\Phi^* e^{-\Phi/\hbar} = e^{2\Phi/\hbar} (\hbar\overline{\partial})^{\Phi,*} e^{-2\Phi/\hbar},
$$

which is easy to show directly.

Also by unitarity,

$$
\Box_\Phi := (\hbar\overline{\partial})^{\Phi,*} \hbar\overline{\partial} + \hbar\overline{\partial} (\hbar\overline{\partial})^{\Phi,*}
$$

fulfills

$$
\Box_\Phi = e^{\Phi/\hbar} \widetilde{\Box}_\Phi e^{-\Phi/h},
\tag{A.10}
$$

hence

$$
(\Box_\Phi^{(1)})^{-1} = e^{\Phi/\hbar} (\widetilde{\Box}_\Phi^{(1)})^{-1} e^{-\Phi/h}.
$$

By unitarity and (A.8) the orthogonal projection

$$
\Pi : \ L^2_\Phi(\mathbb{C}^n) \to L^2_\Phi(\mathbb{C}^n) \cap \mathcal{N}(\hbar\overline{\partial})
$$

is given by

$$
\Pi = e^{\Phi/\hbar} \widetilde{\Pi} e^{-\Phi/\hbar} = 1 - (\hbar\overline{\partial})^{\Phi,*} (\Box_\Phi^{(1)})^{-1} \hbar\overline{\partial}.
$$

In line with the unitary relations (A.9), (A.10) we have

$$
\begin{aligned}
Y_j : &= e^{\Phi/\hbar} Z_j e^{-\Phi/\hbar} = h\partial_{\bar{x}_j}, \\
Y_j^{\Phi,*} &= e^{\Phi/\hbar} Z_j^* e^{-\Phi/\hbar} = -h\partial_{x_j} + 2\partial_x \Phi
\end{aligned}
$$

and the continuity statements (A.1), (A.2), (A.3) remain valid for $\Box_\Phi$ if we redefine the spaces H^k by replacing the unweighted L^2 norms with L^2_Φ norms and replace Z_j, Z_j^* with Y_j, $Y_j^{\Phi,*}$.

If $v \in L^2_\Phi(\mathbb{C}^n; \wedge^{0,1}\mathbb{C}^n)$ and $\overline{\partial} v = 0$, then from (A.4), (A.5) and the unitary conjugations above, we see that

$$
u = (\hbar\overline{\partial})^{\Phi,*} (\Box_\Phi^{(1)})^{-1} v
$$

solves,

$$
\hbar\overline{\partial} u = v,
\tag{A.11}
$$

and

$$
\|u\|_{L^2_\Phi} \le \mathcal{O}(\hbar^{-1/2}) \|v\|_{L^2_\Phi}.
\tag{A.12}
$$

B Direct study of A in (3.9)

Here we perform a more direct study of the operator A in Subsection 3.4. Recall that

$$U_{1/2} = \exp\left(\frac{i}{2\hbar}\hbar D_\theta \cdot (\hbar D_y - \hbar D_x)\right),$$

so that

$$U_{1/2} = \mathcal{F}^{-1} \circ \exp\left(\frac{i}{2\hbar}\theta^* \cdot (y^* - x^*)\right) \circ \mathcal{F},$$

where $\mathcal{F} = \mathcal{F}_\hbar$ denotes the usual semiclassical Fourier transform on $\mathbb{R}^{3n}$. Hence

$$U_{1/2}u = K * u,$$

where

$$K = \mathcal{F}^{-1}\left(\exp\left(\frac{i}{2\hbar}\theta^* \cdot (y^* - x^*)\right)\right).$$

To compute K we first diagonalize the quadratic form $q = \theta^* \cdot (y^* - x^*)$ by means of a real orthogonal change of variables. Writing q as a difference of two squares and adjusting a parameter, we find

$$\theta^* \cdot (y^* - x^*) = \frac{1}{\sqrt{2}}(\xi_1^2 - \xi_2^2),$$

where

$$\begin{aligned}\xi_1 &= \frac{x^*}{2} - \frac{y^*}{2} - \frac{\theta^*}{\sqrt{2}},\\ \xi_2 &= -\frac{x^*}{2} + \frac{y^*}{2} - \frac{\theta^*}{\sqrt{2}}.\end{aligned}$$

Adding the third coordinate

$$\xi_3 = \frac{x^*}{\sqrt{2}} + \frac{y^*}{\sqrt{2}},$$

we get

$$\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = V\begin{pmatrix}x^*\\ y^*\\ \theta^*\end{pmatrix},$$

where

$$V = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

is orthogonal with determinant 1. Thus,

$$q \circ V^{-1}(\xi) = \frac{1}{\sqrt{2}}(\xi_1^2 - \xi_2^2).$$

Let x_1, x_2, x_3 be the coordinates on $\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_\theta$ that are dual to ξ_1, ξ_2, ξ_3. In these coordinates,

$$K = \mathcal{F}^{-1}\left(\exp \frac{i}{2\hbar}\frac{1}{\sqrt{2}}\left(\xi_1^2 - \xi_2^2\right)\right) = \frac{1}{(\pi\hbar)^n} \exp\left(-\frac{i}{\hbar\sqrt{2}}\left(x_1^2 - x_2^2\right)\right)\delta(x_3).$$

In order to get K in the coordinates (x, y, θ) we perform the dual change of variables,

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = V^{\mathrm{t}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = V \begin{pmatrix} x \\ y \\ \theta \end{pmatrix},$$

since V is orthogonal; $(V^{\mathrm{t}})^{-1} = V$. More explicitly,

$$\begin{aligned} x_1 &= \frac{x}{2} - \frac{y}{2} - \frac{\theta}{\sqrt{2}}, \\ x_2 &= -\frac{x}{2} + \frac{y}{2} - \frac{\theta}{\sqrt{2}}, \\ x_3 &= \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}. \end{aligned}$$

Thus,

$$\begin{aligned} K =& \frac{1}{(\pi\hbar)^n} \times \\ & \exp\left(-\frac{i}{\hbar\sqrt{2}}\left(\left(\frac{x}{2} - \frac{y}{2} - \frac{\theta}{\sqrt{2}}\right)^2 - \left(-\frac{x}{2} + \frac{y}{2} - \frac{\theta}{\sqrt{2}}\right)^2\right)\right)\delta\left(\frac{x+y}{\sqrt{2}}\right) \\ =& \frac{1}{(\pi\hbar)^n} \exp\left(\frac{i}{\hbar}(x-y)\cdot\theta\right)\delta\left(\frac{x+y}{\sqrt{2}}\right). \end{aligned}$$

Noticing that $\delta(t/\sqrt{2}) = \sqrt{2}^n \delta(t)$ on $\mathbb{R}^n$, we get for $U_{1/2}u = K * u$:

$$
\begin{aligned}
U_{1/2}u(x,y,\theta) &= \frac{1}{(\pi\hbar)^n} \iiint e^{\frac{i}{\hbar}(x-\widetilde{x}-y+\widetilde{y})\cdot(\theta-\widetilde{\theta})} \delta\left(\frac{x-\widetilde{x}+y-\widetilde{y}}{\sqrt{2}}\right) u(\widetilde{x},\widetilde{y},\widetilde{\theta}) \mathrm{d}\widetilde{x}\mathrm{d}\widetilde{y}\mathrm{d}\widetilde{\theta} \\
&= \left(\frac{\sqrt{2}}{\pi\hbar}\right)^n \iint e^{\frac{i}{\hbar}(x-\widetilde{x}-y+x-\widetilde{x}+y)\cdot(\theta-\widetilde{\theta})} u(\widetilde{x}, x+y-\widetilde{x}, \widetilde{\theta}) \mathrm{d}\widetilde{x}\mathrm{d}\widetilde{\theta},
\end{aligned}
$$

$$
U_{1/2}u(x,y,\theta) = \left(\frac{\sqrt{2}}{\pi\hbar}\right)^n \iint e^{\frac{2i}{\hbar}(x-\widetilde{x})\cdot(\theta-\widetilde{\theta})} u(\widetilde{x}, x+y-\widetilde{x}, \widetilde{\theta}) \mathrm{d}\widetilde{x}\mathrm{d}\widetilde{\theta}. \tag{B.1}
$$

Recall from Proposition 3.2 that

$$
\begin{aligned}
(W^* u)(x,y,\theta) &= u(x,y,w(x,y,\theta)), \\
(\gamma^* u)(x,\theta) &= u(x,x,\theta),
\end{aligned}
$$

and from the beginning of the proof of Proposition 3.3, that

$$
(\pi^* v)(x,y,w) = v(x,w).
$$

This gives first that $W^*\pi^* u(x,y,\theta) = u(x, w(x,y,\theta))$ and then with (B.1) that

$$
\begin{aligned}
&U_{1/2}\widetilde{J}W^*\pi^* u(x,y,\theta) \\
&= \left(\frac{\sqrt{2}}{\pi\hbar}\right)^n \iint e^{\frac{2i}{\hbar}(x-\widetilde{x})\cdot(\theta-\widetilde{\theta})} \widetilde{J}(\widetilde{x}, x+y-\widetilde{x}, \widetilde{\theta}) u(\widetilde{x}, w(\widetilde{x}, x+y-\widetilde{x}, \widetilde{\theta})) \mathrm{d}\widetilde{x}\mathrm{d}\widetilde{\theta},
\end{aligned}
$$

and hence,

$$
\begin{aligned}
Au(x,\theta) &= \gamma^* U_{1/2}\widetilde{J}W^*\pi^* u(x,\theta) \\
&= \left(\frac{\sqrt{2}}{\pi\hbar}\right)^n \iint e^{\frac{2i}{\hbar}(x-\widetilde{x})\cdot(\theta-\widetilde{\theta})} \widetilde{J}(\widetilde{x}, 2x-\widetilde{x}, \widetilde{\theta}) u(\widetilde{x}, w(\widetilde{x}, 2x-\widetilde{x}, \widetilde{\theta})) \mathrm{d}\widetilde{x}\mathrm{d}\widetilde{\theta}
\end{aligned}
$$

In this integral, we replace the integration variable $\widetilde{\theta}$ with $\widetilde{w} := w(\widetilde{x}, 2x-\widetilde{x}, \widetilde{\theta})$, so that

$$
\widetilde{\theta} = \theta(\widetilde{x}, 2x-\widetilde{x}, \widetilde{w}), \quad \mathrm{d}\widetilde{\theta} = \det\left(\frac{\partial\theta}{\partial w}\right)(\widetilde{x}, 2x-\widetilde{x}, \widetilde{w})\mathrm{d}\widetilde{w},
$$

and get

$$Au(x,\theta) = \left(\frac{\sqrt{2}}{\pi\hbar}\right)^n \iint e^{\frac{i}{\hbar}F(x,\theta;\widetilde{x},\widetilde{w})}\widetilde{J}(\widetilde{x}, 2x-\widetilde{x}, \theta(\widetilde{x}, 2x-\widetilde{x}, \widetilde{w}))\times \det\left(\frac{\partial\theta}{\partial w}\right)(\widetilde{x}, 2x-\widetilde{x}, \widetilde{w})u(\widetilde{x},\widetilde{w})\mathrm{d}\widetilde{x}\mathrm{d}\widetilde{w}, \quad \text{(B.2)}$$

where

$$F(x,\theta;\widetilde{x},\widetilde{w}) = 2(\theta - \theta(\widetilde{x}, 2x-\widetilde{x}, \widetilde{w}))\cdot(x-\widetilde{x}).$$

There are no fiber variables present in the representation (B.2) of the Fourier integral operator A, so the phase generates a canonical relation

$$C_A:\ (\widetilde{x},\widetilde{w}; -\partial_{\widetilde{x}}F, -\partial_{\widetilde{w}}F) \mapsto (x,\theta;\partial_x F, \partial_\theta F).$$

Recall from the identity after (3.1) that

$$\theta(x,y,\theta) = \frac{2}{i}\psi'_x((x+y)/2,\widetilde{w}) + \mathcal{O}((x-y)^2),$$

hence

$$\begin{aligned}
\theta(\widetilde{x}, 2x-\widetilde{x}, \widetilde{w}) &= \frac{2}{i}\psi'_x(x,\widetilde{w}) + \mathcal{O}((x-\widetilde{x})^2),\\
F(x,\theta;\widetilde{x},\widetilde{w}) &= 2\left(\theta - \frac{2}{i}\psi'_x(x,\widetilde{w})\right)\cdot(x-\widetilde{x}) + \mathcal{O}((x-\widetilde{x})^3),\\
-\partial_{\widetilde{x}}F &= 2\left(\theta - \frac{2}{i}\psi'_x(x,\widetilde{w})\right) + \mathcal{O}((x-\widetilde{x})^2),\\
-\partial_{\widetilde{w}}F &= \frac{4}{i}\psi''_{\widetilde{w},x}(x,\widetilde{w})(x-\widetilde{x}) + \mathcal{O}((x-\widetilde{x})^3)\\
\partial_x F &= 2\left(\theta - \frac{2}{i}\psi'_x(x,\widetilde{w})\right) - \frac{4}{i}\psi''_{x,x}(x,\widetilde{w})(x-\widetilde{x}) + \mathcal{O}((x-\widetilde{x})^2)\\
\partial_\theta F &= 2(x-\widetilde{x}) + \mathcal{O}((x-\widetilde{x})^3),\\
F''_{x,\theta;\widetilde{x},\widetilde{w}} &= \begin{pmatrix} F''_{x,\widetilde{x}} & F''_{x,\widetilde{w}} \\ F''_{\theta,\widetilde{w}} & F''_{\theta,\widetilde{w}} \end{pmatrix} = \begin{pmatrix} F''_{x,\widetilde{x}} & -\frac{4}{i}\psi''_{x,\widetilde{w}} + \mathcal{O}(x-\widetilde{x}) \\ -2 + \mathcal{O}((x-\widetilde{x})^2) & \mathcal{O}((x-\widetilde{x})^3) \end{pmatrix}.
\end{aligned}$$

Recall that $\psi''_{x,\widetilde{w}} = \Phi''_{x,\bar{x}}$ when $\widetilde{w} = \bar{x}$, so $F''_{x,\theta;\widetilde{x},\widetilde{w}}$ is invertible when $\widetilde{w} - \bar{x}$ and $x - \widetilde{x}$ are small. In this region, C_A is therefore equal to the graph of a canonical transformation locally. Still when $|x-\widetilde{x}|$ is small, we have the equivalences

$$\begin{cases} \partial_{\widetilde{x}} F = 0, \\ \partial_{\widetilde{w}} F = 0 \end{cases} \iff \begin{cases} x - \widetilde{x} = 0, \\ \theta = \frac{2}{i}\psi'_x(x, \widetilde{w}) \end{cases} \iff \begin{cases} \partial_x F = 0, \\ \partial_\theta F = 0 \end{cases},$$

so C_A maps the zero-section $\widetilde{x}^* = \widetilde{w}^* = 0$ to the zero-section $x^* = \theta^* = 0$.

In particular, if we restrict the attention to a neighborhood of a point given by

$$x = \widetilde{x} = x_0,\ \widetilde{w} = \bar{x}_0,\ \theta = \frac{2}{i}\partial_x \Phi(x_0),\ \widetilde{x}^* = \widetilde{w}^* = x^* = \theta^* = 0,$$

we see that

$$C_A :\ (x_0, \bar{x}_0; 0, 0) \mapsto (x_0, (2/i)\partial_x \Phi(x_0); 0, 0)$$

and that in a neighborhood of this point C_A coincides with the graph of a canonical transformation which maps the zero section over a neighborhood of $(x_0, \bar{x}_0)$ to the zero section over a neighborhood of $(x_0, (2/i)\partial_x \Phi(x_0))$.

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