THE AFFINE INVARIANT OF PROPER SEMITORIC INTEGRABLE SYSTEMS

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Abstract. This paper initiates the study of semitoric integrable systems with two degrees of freedom and with proper momentum-energy map, but with possibly nonproper S^1-momentum map. This class of systems includes many standard examples, such as the spherical pendulum. To each such system we associate a subset of $\mathbb{R}^2$, invariant under a natural notion of isomorphism and encoding the integral affine structure of the singular Lagrangian fibration, in the spirit of Delzant polygons for toric systems.

1. Introduction

Let M be a connected symplectic four-manifold and $F := (J, H)\colon M \to \mathbb{R}^2$ an integrable system with two degrees of freedom. This means that J and H are smooth functions on M, functionally independent, whose Poisson bracket vanishes:

$$\{J, H\} = 0.$$

The map F is sometimes called the momentum-energy map, in reference to many physical systems where the dynamics is governed by the Hamiltonian H (the energy), and J is a conserved quantity, such as angular momentum.

If M is compact and F is the momentum map of an effective Hamiltonian 2-torus action, the system is called toric. A classification of such systems, due to Atiyah [At82], Guillemin-Sternberg [GS82], and Delzant [De88], is given by the image of F which is a rational, convex polygon. The classification was then extended to non-compact symplectic manifolds M under the hypothesis that the map $F : M \to \mathbb{R}^2$ is proper [LMTW98], which means that the pre-image by F of a compact set must be compact. In this more general situation, which we shall call *proper toric*, $F(M)$ can be unbounded, but it is still a convex polygonal set, in the sense that it is closed, convex, and its boundary is polygonal with a discrete set of vertices.

Another generalization of toric systems are the so-called *semitoric systems*, see [Vu07, Sy01, LS10]. In this case, F is not required to generate a 2-torus action, but the first component J is assumed to be a proper momentum map for a Hamiltonian S^1-action (and the singularities of F, in the Morse-Bott sense, are somewhat restricted). Although the image $F(M)$ may not be a polygonal set, it was shown in [Vu07] how to canonically construct a rational convex polygon from the system. Semitoric systems were classified in [PV09, PV11], this so-called "semitoric polygon" being a crucial ingredient of the classification.

In these works, the properness of J plays the same role as the properness of F in the toric case: it permits the use of Morse theory in order to rule out disconnected fibers. There are, however, many examples of integrable systems, from theoretical physics and classical mechanics, satisfying all the hypotheses of semitoric systems, except for the properness of the S^1-momentum map J. For instance, the angular momenta in both the spherical pendulum (see Section 6) and in the system with "Champagne bottle" potential are not proper. In this paper, we construct the natural object generalizing both the semitoric [Vu07] and the toric polygons [At82, GS82] which allow us to treat such classical examples.

Integrable systems with proper momentum-energy map lie at the crossroad of geometry (with compact Lie groups techniques) and classical mechanics. The action-angle theorem states that the *generic* dynamics of such systems is universal and consists of quasiperiodic motions on Liouville tori. Nevertheless, the *global* behavior of an integrable system can be very intricate, due to the possible bifurcations of the Liouville tori.

Recently, a systematic study of global properties of finite dimensional integrable systems has been started by several authors; it turns out that the symplectic geometry of *singular leaves* of the system plays a prominent role. (The reader can consult [Au08, BF04, Gu94, Vu06b, PV11a] for an overview and more references on this topic, from the point of view of symplectic geometry and spectral theory, and their interactions via quantization.) The goal of this paper is to contribute to this line of research by investigating the special class of *proper semitoric systems with two degrees of freedom, motivated, in particular, by the results in* [At82, GS82, Vu07, PV09, PV11, PRV15]. For these systems, we construct a tractable object (a subset of $\mathbb{R}^2$ with special properties, generalizing the Delzant polygon of toric manifolds), which is invariant under a natural notion of isomorphism and encodes several topological and dynamical properties of the system. Contrary to the case of toric manifolds, proper semitoric systems may have isolated singularities (see §9.1 and §9.2 for a quick review of the material concerning singularities used in this paper), called *focus-focus singularities*, giving rise to fibers of F which are pinched (or multiply pinched) tori. Focus-focus singularities are often present in simple classical mechanical systems, such as the spin-orbit Hamiltonian [SaZh99], the Jaynes-Cummings system [BCD09, PV12a], the spherical pendulum [AM78, Exercise 4.5F], [CB97, Chapter 4], or the system with "Champagne bottle" potential [Ch98]. Note that focus-focus singularities appear also in algebraic geometry [KS06, GS06] and symplectic topology, e.g. [LS10, Sy01, Vi13], where they are sometimes called *nodes*.

Let us turn now to our precise setting. Let (M, ω) be a symplectic $2n$-manifold. Throughout this paper, we assume that M is connected but not necessarily compact. An *integrable system* on (M, ω) is a map $F\colon M \to \mathbb{R}^n$ whose components $f_1, \ldots, f_n\colon M \to \mathbb{R}$ are Poisson commuting smooth functions which generate Hamiltonian vector fields $\mathcal{X}_{f_1}, \ldots, \mathcal{X}_{f_n}$ (via pairing with ω) that are linearly independent at almost every point. The *singular points* of F are the points in M where the differential (or tangent map) $\mathrm{T}F$ does not have maximal rank or, equivalently, $\mathcal{X}_{f_1}, \dots, \mathcal{X}_{f_n}$ fail to be linearly independent. In this article, we assume that $n = 2$ and use the index free notation $f_1 = J$ and $f_2 = H$. Recall that an S^1-action on (M, ω) is *Hamiltonian* if there exists a smooth map $J\colon M \to \mathbb{R}$, the *momentum map*, such that $\omega(\mathcal{X}_M, \cdot) = -\mathrm{d}J$, where $\mathcal{X}_M$ is the infinitesimal generator of the action. A group action on a manifold is called *effective* or *faithful*, if the intersection of all its stabilizer subgroups is the identity element.

The following definition uses the notions of bifurcation set and non-degenerate singularities for completely integrable systems that we recall in the appendix (Section 9).

Definition 1.1 An integrable system $F := (J, H)\colon M \to \mathbb{R}^2$ on a symplectic 4-dimensional connected manifold (M, ω) is *proper semitoric* if:

(H.i) J is a momentum map of an effective Hamiltonian circle action.
(H.ii) The singularities of F are non-degenerate with no hyperbolic component.
(H.iii) F is a proper map.
(H.iv) J has connected fibers, the bifurcation set of J is discrete, and for any critical value x of J, there exists a neighborhood $V \ni x$ such that the number of connected components of the critical set of J in $J^{-1}(V)$ is finite.

$\oslash$

Convention. *It is worth emphasizing that an integrable system $F\colon M \to \mathbb{R}^2$ is really given by a triple (M, ω, F) and whenever we refer to F, the triple (M, ω, F) is implicitly understood.*

Definition 1.2 Let $(i_1, \dots, i_n)$ be a tuple of indices, where $i_k \in \{0, 1\}$ for all $k \in \{1, \dots, n\}$. An integrable system $F = (f_1, \dots, f_n) : M \to \mathbb{R}^n$ is called $(i_1, \dots, i_n)$-*proper* if, for all $k \in \{1, \dots, n\}$, $f_k\colon M \to \mathbb{R}$ is a proper map when $i_k = 1$, and f_k may or may not be proper when $i_k = 0$. $\oslash$

Of course, any integrable system F is $(0, 0, \dots 0)$-proper; moreover, if F is $(i_1, \dots, i_n)$-proper and at least one $i_k = 1$, then F is proper.

In the terminology of Definition 1.1, a proper semitoric integrable system $F = (J, H)\colon M \to \mathbb{R}^2$ is $(0, 0)$-proper and proper, i.e., F is a proper map but neither $J\colon M \to \mathbb{R}$ nor $H\colon M \to \mathbb{R}$ are necessarily proper. The systems investigated in [Vu06, Vu07, PV09, PV11, PV12], called there semitoric, are proper semitoric and possess a $(1, 0)$-proper map F. To simplify the terminology, for the remainder of this paper, systems that are both proper semitoric and $(1, 0)$-proper will be called $(1, 0)$-*semitoric*. Similarly, proper toric systems which are $(1, 0)$-proper will be called $(1, 0)$-*toric*.

Remark 1.3

- Item (H.iv) implies that the fibers of F are also connected. This follows from [PRV15, Theorem 4.7]; the statement is recalled in the Appendix (Theorem 9.3).

- If J is proper and (H.i) is satisfied, then (H.iv) holds. This follows from [LMTW98].
- In some simple mechanical systems, like the spherical pendulum (Example 6.2), J is not proper but (H.iii) and (H.iv) still hold.

$\oslash$

If follows from [Vu07] that $(1,0)$-semitoric systems are proper semitoric. In particular, if M is compact, then all toric systems are proper semitoric. However, the toric case is very special: all singularities are elliptic and, in particular, toric systems do not possess focus-focus singularities. Note that, for general proper semitoric systems, the absence of focus-focus singularities is a necessary but not sufficient condition for the system to be toric; a toy model for a genuinely proper semitoric system without any focus-focus singularity is provided by Example 1.4 below. It is also interesting to notice that $(1,0)$-semitoric systems have only a finite number of focus-focus singularities, whereas proper semitoric systems allow an infinite number of focus-focus singularities (see Section 2).

Example 1.4 Let $M = S^2 \times S^2$, endowed with the product area symplectic form, and $F = (z_1, z_2)$ be the usual toric momentum map. Let $f\colon [-1,1] \to (-1,1]$ be smooth. Define $M' := F^{-1}(\{(x,y) \,|\, x \in [-1,1],\ y < f(x)\})$. The set M' is an open subset of M and $\mu = z_1$ is a momentum map for a Hamiltonian S^1-action on M'. Furthermore, μ is not proper because $\mu^{-1}(x) = F^{-1}(\{(x,y) \mid y < f(x)\})$ is not closed. Notice that the full map $F\restriction_{M'}$ is also not proper, but we can easily modify it as follows. Let $g(x,y) := (x, 1/(f(x){-}y))$. Then $F' := g \circ F\restriction_{M'}$ is proper and the S^1-momentum map $\mu = z_1$ is not modified. Thus F' is a proper semitoric system with no focus-focus singularities. In addition, F' has unbounded image

$$F'(M') = \left\{ (x,y) \in \mathbb{R}^2 \,|\, x \in [-1,1],\ y \geqslant \frac{1}{f(x)+1} \right\}$$

and is neither proper toric, nor $(1,0)$-semitoric. $\oslash$

We mentioned that in the toric and $(1,0)$-semitoric cases, it is possible to have a good (and, in the toric case, complete) understanding of the system through a combinatorial object: a convex, rational polygon. The motivation of the present work is to extend some ideas from the toric theory to the framework of proper semitoric systems. The third author [Vu07] initiated this program for $(1,0)$-*semitoric systems* $F\colon M \to \mathbb{R}^2$, i.e., (H.i) and (H.ii) in Definition 1.1 hold plus the assumption that $J\colon M \to \mathbb{R}$ is proper. As remarked earlier, this implies that (H.iii) and (H.iv) hold. The main technical tools in this study are Morse theory and ideas related to the Duistermaat-Heckman construction [DH82] for proper momentum maps. Using a cutting procedure along the vertical lines passing through the isolated singularities of the image $F(M)$ of the system, it was possible to construct a convex polygon from it, which only depends on the isomorphism class of F; see Figure 2.1. This polygon turns out to be the first element of the full invariant classifying $(1,0)$-semitoric systems; see [PV09, PV11, PV11a, PV12].

The notion of isomorphism in [PV09], which we continue to use for proper semitoric systems, is the following.

Definition 1.5 Two proper semitoric systems $(M_1, \omega_1, F_1 := (J_1, H_1))$ and $(M_2, \omega_2, F_2 := (J_2, H_2))$ are *isomorphic* if there exists a symplectomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*(J_2, H_2) = (J_1, h(J_1, H_1))$ for a smooth function h satisfying $\frac{\partial h}{\partial H_1} > 0$. ⊘

By the chain rule, it is straightforward to check that this notion of isomorphism defines an equivalence relation on the set of all proper semitoric systems. Notice that this definition extends the natural notion of S^1-equivariant symplectomorphism of Hamiltonian S^1-spaces (M, ω, S^1) used, for instance, in Karshon's classification paper [Ka99].

Our aim in this article is *to construct an invariant for proper semitoric systems up to isomorphism which extends the polygonal invariant in the theory of Atiyah, Guillemin-Sternberg, and the third author*. By 'extending' we mean, of course, that in the particular cases of proper toric and $(1, 0)$-semitoric systems, our invariant coincides with the previous polygonal invariants. An important difficulty for this program is due to the fact that both Morse theory and standard Duistermaat-Heckman techniques fail for non-proper J (Remark 4.6). This has striking consequences, not only for the proofs, but also for the statement of our extension: while the invariant in [Vu07] is a class of convex polygonal sets, ours is a union of planar regions of various types (see Definition 4.4), which looks, in general, like the one in Figure 1.1. This invariant, which we call the *cartographic projection*, encodes the singular affine structure induced by the (singular) Lagrangian fibration $F\colon M \to \mathbb{R}^2$ on the base $F(M)$. Its construction and properties appear in Theorems B, C, and Corollary 4.3. Theorem D shows that there are many simple examples in which the invariant, which is the natural planar representation of the singular affine structure of the system, has a non-polygonal, non-convex, form.

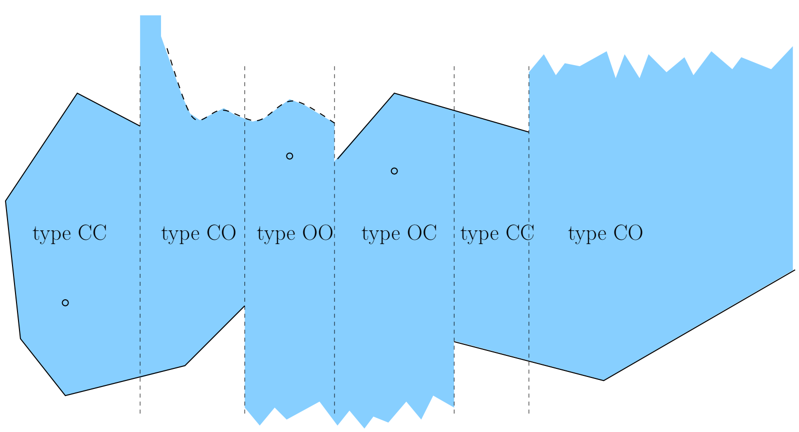

Figure 1.1. A cartographic projection of F. It is a symplectic invariant of F, see Theorem C.

The structure of the article is as follows:

- Section 2 introduces the moduli space of cartographic invariants for proper semitoric systems, which allows for a unified treatment of $(1,0)$-toric, $(1,0)$-semitoric, and proper semitoric systems. We show how the well-known polygon in the standard $(1,0)$-toric and $(1,0)$-semitoric cases can be viewed as such a cartographic invariant.
- In Section 3, we summarize, in an abstract statement, the main results of the paper (Theorem A).
- These abstract results are reformulated in Section 4 in a concrete form, useful for the construction of the cartographic projection (Theorems B, C, D, and Corollary 4.3). The main results of the paper in Section 3 are direct corollaries of the theorems in this section.
- The proofs of the results in Section 4 are given in Sections 5, 6, and 7.
- Section 8 contains three open questions.
- The appendix in Section 9 reviews the essential background material for the paper.

2. The moduli space of semitoric images

In this section, we introduce a general framework to deal in a unified manner with the polygonal invariants of $(1,0)$-toric and $(1,0)$-semitoric systems and the new cartographic invariant, constructed later in this paper, for proper semitoric systems. The main difficulty is to allow for an infinite number of focus-focus singularities; in the $(1,0)$-semitoric case, this number was always finite.

2.1. The semitoric affine group.

Let $\mathcal{P}(\mathbb{R}^2)$ be the power set of $\mathbb{R}^2$. Let

$$T := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \tag{2.1}$$

and consider the subgroup $\mathcal{T}$ of the affine group $\mathrm{Aff}(2,\mathbb{Z})$ (see Section 9.3) whose elements are the matrices T^k, $k \in \mathbb{Z}$, composed with vertical translations. This gives rise to the quotient space $\mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) := \mathcal{P}(\mathbb{R}^2)/\mathcal{T}$.

2.2. Equivalence classes.

The construction of the invariant for proper semitoric systems involves several choices that need to be taken into account in order to define the correct equivalence relations. These choices can be understood as various transformations of the plane $\mathbb{R}^2$, which we introduce in this section.

A vertical line $\mathcal{L} \subset \mathbb{R}^2$ decomposes $\mathbb{R}^2$ into two half-spaces. Let $u \in \mathbb{Z}$. Define a map $t^u_{\mathcal{L}}$ acting on $\mathbb{R}^2$ as follows. On the left half-space defined by $\mathcal{L}$, the map $t^u_{\mathcal{L}}$ acts as the identity. On the right half-space, with an origin placed arbitrarily on $\mathcal{L}$, $t^u_{\mathcal{L}}$ acts as the matrix T^u. The set of all such transformations $t^u_{\mathcal{L}}$ is commutative. Indeed, let $\mathcal{L}_x$ be the vertical line through the point $(x,0)$, where $x \in \mathbb{R}$. Then, if $x_1, x_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathbb{Z}$, we have

$$t^{u_1}_{\mathcal{L}_{x_1}} \circ t^{u_2}_{\mathcal{L}_{x_2}} = t^{u_2}_{\mathcal{L}_{x_2}} \circ t^{u_1}_{\mathcal{L}_{x_1}}, \qquad \text{and} \qquad t^{u_1}_{\mathcal{L}_{x_1}} \circ t^{u_2}_{\mathcal{L}_{x_1}} = t^{u_1+u_2}_{\mathcal{L}_{x_1}}. \tag{2.2}$$

Analogously, define the maps $s^u_{\mathcal{L}}$ which act as the identity on the right half-space limited by $\mathcal{L}$ and as the matrix T^{-u} on the left half-space. It is easy to see that any transformation of the type $s^u_{\mathcal{L}}$ commutes with any transformation of the type $t^{u'}_{\mathcal{L}'}$.

In considering the collective set of vertical lines through focus-focus critical values of an integrable system, it is convenient to use the family of index sets given by:

$$\mathfrak{F} := \Big\{\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-, \{1, \dots, N\}_{N>0}, \varnothing\Big\},$$

where $\mathbb{Z}^+ = \{i \in \mathbb{Z} \mid i \geqslant 0\}$ and $\mathbb{Z}^- = \{i \in \mathbb{Z} \mid i \leqslant 0\}$. Given an index set $Z \in \mathfrak{F}$, a sequence $(x_n)_{n \in Z} \subset \mathbb{R}$ such that for every $c \in \mathbb{R}$ there is a neighborhood of c which contains only a finite number of elements of this sequence is called a *discrete sequence*. Let

$$\mathcal{D}^Z := \{(x_i)_{i \in Z} \in \mathbb{R}^Z \mid (x_i)_{i \in Z} \text{ is a non-decreasing discrete sequence}\}.$$

Fix an index $i_0 \in Z$. Given $\vec{u} = (\vec{u}(i)) \in \mathbb{Z}^Z$ and $\vec{x} = (\vec{x}(i)) \in \mathcal{D}^Z$, define the following (possibly infinite) product:

$$\mathfrak{t}^{\vec{u}}_{i_0, \mathcal{L}_{\vec{x}}} := \prod_{\{i \in Z \mid i \geqslant i_0\}} t^{\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}} \circ \prod_{\{i \in Z \mid i < i_0\}} s^{\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}}. \tag{2.3}$$

Lemma 2.1. *For any point $(a, b) \in \mathbb{R}^2$, only a finite number of terms in the products (2.3) computed at the point (a, b) are not trivial. Thus the map $t^{\vec{u}}_{i_0, \mathcal{L}_{\vec{x}}} : \mathbb{R}^2 \to \mathbb{R}^2$ is well defined.*

Proof. Fix $(a, b) \in \mathbb{R}^2$. First consider the product $\prod_{\{i \in Z \mid i \geqslant i_0\}} t^{\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}}$. Notice that each transformation of the form $t^u_{\mathcal{L}}$ preserves the vertical line $\mathcal{L}_a$. Since the sequence (x_i) is discrete and non-decreasing, the number of indices $i \geqslant i_0$ such that $x_i < a$ is finite. Let us denote by $\mathcal{I}$ this set of indices. For any other index $i' \notin \mathcal{I}$, such that $i \geqslant i_0$, the map $t^{\vec{u}_{i'}}_{\mathcal{L}_{\vec{x}_{i'}}}$ acts as the identity on $\mathcal{L}_a$. Therefore $\prod_{\{i \in Z \mid i \geqslant i_0\}} t^{\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}}(a, b)$ is well defined and equal to the finite product

$$\prod_{i \in \mathcal{I}} t^{\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}}(a, b).$$

Similarly, the number of indices $i < i_0$ such that $x_i > a$ is finite and hence the product $\prod_{\{i \in Z \mid i < i_0\}} s^{\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}}(a, b)$ is finite as well. □

Proposition 2.2. *Fix an index set $Z \in \mathfrak{F}$. Then, given any $i_0 \in Z$, the map*

$$\begin{aligned} \mathbb{Z}^Z \times (\mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \times \mathcal{D}^Z) &\longrightarrow \mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \times \mathcal{D}^Z \\ (\vec{u}, (X, \vec{x})) &\longmapsto \vec{u} \cdot (X, \vec{x}) := \Big(\mathfrak{t}^{\vec{u}}_{i_0, \mathcal{L}_{\vec{x}}}(X),\ \vec{x}\Big) \end{aligned} \tag{2.4}$$

defines an action of the commutative group $\mathbb{Z}^Z$ on $\mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \times \mathcal{D}^Z$.

Moreover, this action does not depend on the choice of i_0.

Proof. The fact that (2.4) defines an action of $\mathbb{Z}^Z$ follows from (2.2) and from the fact that the sequence $\vec{x}$ is preserved by the action defined in (2.4).

Let $i_1 \in Z \setminus \{i_0\}$. We may assume that $i_0 < i_1$. We have

$$(\mathfrak{t}^{\vec{u}}_{i_0, \mathcal{L}_{\vec{x}}})^{-1} \circ \mathfrak{t}^{\vec{u}}_{i_1, \mathcal{L}_{\vec{x}}} = \prod_{\{i \in \mathcal{I} \mid i_0 \leqslant i < i_1\}} t^{-\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}} \circ s^{\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}}.$$

By construction, for any integer u and any vertical line $\mathcal{L}$, one has

$$t^u_{\mathcal{L}} \circ s^{-u}_{\mathcal{L}} = T^u_{\mathcal{L}},$$

where $T^u_{\mathcal{L}}$ is the affine map acting as the matrix T^u with origin on the line $\mathcal{L}$. Hence

$$(\mathfrak{t}^{\vec{u}}_{i_0,\mathcal{L}_{\vec{x}}})^{-1} \circ \mathfrak{t}^{\vec{u}}_{i_1,\mathcal{L}_{\vec{x}}} = \prod_{\{i\in\mathcal{I} \mid i_0 \leqslant i < i_1\}} T^{-\vec{u}(i)}_{\mathcal{L}_{\vec{x}(i)}} = T^n_{\mathcal{L}_0} \circ \tau_y,$$

where $n = \sum_{\{i\in\mathcal{I}|i_0\leqslant i<i_1\}} \vec{u}(i)$, $y = -\sum_{\{i\in\mathcal{I}|i_0\leqslant i<i_1\}} \vec{u}(i)\vec{x}(i)$, and τ_y denotes the translation by the vector $(0, y)$. Thus the transformation $(\mathfrak{t}^{\vec{u}}_{i_0,\mathcal{L}_{\vec{x}}})^{-1} \circ \mathfrak{t}^{\vec{u}}_{i_1,\mathcal{L}_{\vec{x}}}$ belongs to the group $\mathcal{T}$ defined in Section 2.1, which shows that the map (2.4) is independent of the choice of i_0. $\square$

It can be sometimes useful to think of the discrete sequence $\vec{x} \in \mathcal{D}^Z$ as fixed in advance; then, since (2.4) preserves $\vec{x}$, it induces an action of $\mathbb{Z}^Z$ on $\mathcal{P}^{\mathcal{T}}(\mathbb{R}^2)$. However, for the definition of the moduli space of proper semitoric images $\mathfrak{B}_{\mathrm{PST}}$ below (Equation (2.6)), it is more convenient to keep the $\mathcal{D}^Z$ in the definition, because the $\mathbb{Z}^Z$ action does depend on $\vec{x}$.

In order to encode the choice of orientation of the vertical half-lines through focus-focus values, we consider the map:

$$\{-1,1\}^Z \times (\mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \times \mathcal{D}^Z) \times \mathbb{Z}^Z \longrightarrow (\mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \times \mathcal{D}^Z) \times \mathbb{Z}^Z$$

$$(\vec{\epsilon}, ((X, \vec{x}), \vec{k})) \longmapsto \vec{\epsilon} \cdot ((X, \vec{x}), \vec{k}) := \left((\rho(\vec{\epsilon}).\vec{k}) \cdot (X, \vec{x}), \vec{\epsilon}.\vec{k} \right), \tag{2.5}$$

where $\vec{\epsilon}.\vec{k} := (i \mapsto \epsilon(i)k(i))$, and $\rho(\vec{\epsilon}) := \left(i \mapsto \frac{1-\epsilon(i)}{2}\right)$. Notice that the map $\epsilon \mapsto \frac{1-\epsilon}{2}$ is just the standard homomorphism from the multiplicative group $\{-1, 1\}$ to the additive group $\mathbb{Z}_2$. In view of this, one can check that the map (2.5) defines an action of $\{-1, 1\}^Z$ on $\mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \times \mathcal{D}^Z \times \mathbb{Z}^Z$, which acts component-wise. When $Z \neq \varnothing$, denote the $\{-1, 1\}^Z$-orbit space by

$$\mathfrak{B}_{\mathrm{PST}}(Z) := \left(\mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \times \mathcal{D}^Z \times \mathbb{Z}^Z\right) / \{-1, 1\}^Z.$$

When $Z = \varnothing$, we simply let

$$\mathfrak{B}_{\mathrm{PST}}(\varnothing) := \mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \simeq \mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \times \{\varnothing\} \times \{\varnothing\}.$$

Finally, we introduce the disjoint union:

$$\mathfrak{B}_{\mathrm{PST}} := \bigsqcup_{Z\in\mathfrak{F}} \mathfrak{B}_{\mathrm{PST}}(Z). \tag{2.6}$$

2.3. **Affine invariant for $(1, 0)$-semitoric systems.** Let $F = (J, H) \colon M \to \mathbb{R}^2$ be a $(1, 0)$-semitoric system, i.e., in addition to assumptions (H.i) and (H.ii) in Definition 1.1, the S^1-momentum map $J \colon M \to \mathbb{R}$ is proper. As pointed out after Definition 1.1, these hypotheses imply both (H.iii) and (H.iv). There exists a unique $Z \in \mathfrak{F}$ and a unique $\vec{x} \in D^Z$ such that $\vec{x}$ is the tuple of images by J of focus-focus critical points of F ordered in non-decreasing manner. Let $(c_i)_{i\in Z}$ be a sequence in $\mathbb{R}^2$ containing all the focus-focus critical values of F such that $c_i = (x_i, y_i)$, where we use the lighter notation $x_i := \vec{x}(i)$. Let $\vec{k} \in \mathbb{N}^Z$ be such that $k_i := \vec{k}(i)$ is the number of focus-focus critical points in the fiber $F^{-1}(c_i)$.

For each fixed $\vec{\epsilon} \in \{-1,1\}^Z$, the third author constructed [Vu07, Theorem 3.8 and Proposition 4.1] an equivalence class of convex polygonal sets in $\mathbb{R}^2$

$$(\Delta_{\vec{\epsilon}} \mod \mathcal{T}) \in \mathcal{P}^{\mathcal{T}}(\mathbb{R}^2) \tag{2.7}$$

by performing a cutting procedure along the vertical lines $\mathcal{L}_{x_i}$. The choice of cuts is given by $\vec{\epsilon}$, where a positive sign corresponds to an upward cut, and a negative sign corresponds to a downward cut (see Figure 2.1). As explained in [Vu07], an important consequence of this construction is that $|Z|$ (the number of focus-focus points) is finite.

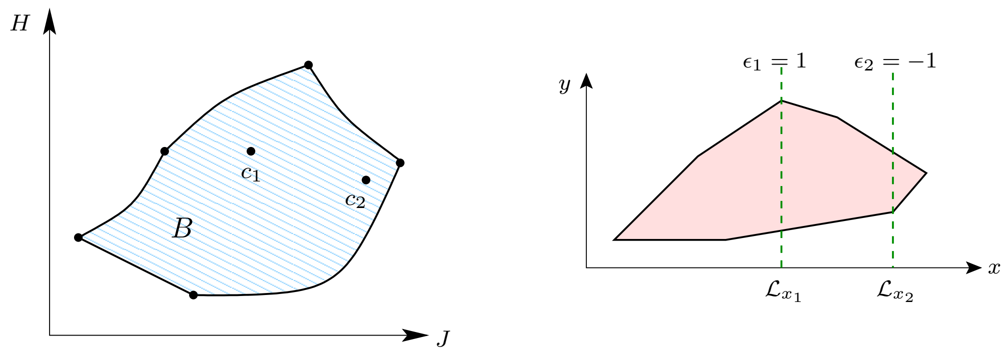

FIGURE 2.1. This figure is taken from [PV09]. The graph on the left depicts the image $B = F(M) \subseteq \mathbb{R}^2$. The interior of $F(M)$ contains two isolated singular values $c_1 = (x_1, y_1)$ and $c_2 = (x_2, y_2)$. The right hand side figure displays the associated polygon $\Delta_{\vec{\epsilon}}$ with the distinguished vertical lines.

For a $(1,0)$-semitoric system, this class of convex sets corresponding to all possible choices of $\vec{\epsilon}$ can be thought of as an element of the general set $\mathfrak{B}_{\mathrm{PST}}$. For this purpose, we use the following notation:

$$\mathcal{C}_{\mathrm{ST}}(M,\omega,F) := (\Delta_{\vec{\epsilon}} \mod \mathcal{T},\, \vec{x},\, \vec{k}) \mod \{-1,1\}^Z \in \mathfrak{B}_{\mathrm{PST}}, \tag{2.8}$$

where $\vec{\epsilon}(i) = 1$ for all $i \in Z$ and the action of $\{-1,1\}^Z$ was defined in (2.5).

Theorem 2.3 ([Vu07]). *If F is $(1,0)$-semitoric, then the class of convex polygonal sets (2.8) is an invariant of the isomorphism type of F.*

The image $F(M)$ itself is, in general, neither convex nor an invariant.

3. Summary result: Theorem A

Let $\mathcal{M}_{\mathrm{T}}$ be the set of $(1,0)$-toric systems, that is, the collection of all triples (M,ω,F) where $F : M \to \mathbb{R}^2$ is a $(1,0)$-proper toric momentum map for an effective Hamiltonian 2-torus action. Let $\mathcal{M}_{\mathrm{ST}}$ and $\mathcal{M}_{\mathrm{PST}}$ be the sets of $(1,0)$-semitoric and proper semitoric systems (M,ω,F).

Note that, contrary to $\mathcal{M}_{\mathrm{ST}}$ and $\mathcal{M}_{\mathrm{PST}}$, the set $\mathcal{M}_{\mathrm{T}}$ is not invariant under isomorphisms, i.e., it is possible to find systems which are not toric, but are isomorphic to a toric system (in the sense of Definition 1.5). In fact, the set of $(1,0)$-toric systems is invariant under a *stronger* notion of isomorphism (a

$\mathbb{T}^2$-equivariant symplectomorphism intertwining the corresponding momentum maps), but this notion is too restrictive for integrable systems. For us, it is more natural to consider the whole equivalence class of toric systems under semitoric isomorphisms, as follows.

Definition 3.1 A proper semitoric system is said to be of $(1,0)$-*toric type* if it is isomorphic to a $(1,0)$-toric system. We denote by $\mathcal{M}_{\mathrm{TT}}$ the set of proper semitoric systems of $(1,0)$-toric type. $\oslash$

Remark 3.2 We have a chain of strict inclusions $\mathcal{M}_{\mathrm{T}} \subsetneq \mathcal{M}_{\mathrm{TT}} \subsetneq \mathcal{M}_{\mathrm{ST}} \subsetneq \mathcal{M}_{\mathrm{PST}}$. $\oslash$

Theorem A below provides a summary of the paper.

To state it, recall that if $(M,\omega,F) \in \mathcal{M}_{\mathrm{T}}$, then F does not possess focus-focus singularities and $F(M)$ is a convex polygonal set [LMTW98]. Let us introduce the map

$$\mathcal{C}_{\mathrm{T}} : \mathcal{M}_{\mathrm{T}} \ni (M,\omega,F) \longmapsto (F(M) \mod \mathcal{T},\, \varnothing, \varnothing) \in \mathfrak{B}_{\mathrm{PST}}. \tag{3.1}$$

In order to define an analogue of this map for systems of toric type, we shall need the following lemma.

Lemma 3.3. *Let (M_1,ω_1,F_1) and (M_2,ω_2,F_2) be two elements of $\mathcal{M}_{\mathrm{T}}$. Assume that they are isomorphic to each other in the sense of Definition 1.5. Then*

$$F_1(M_1) = F_2(M_2) \mod \mathcal{T}.$$

Proof. Since the two systems are isomorphic, we may assume that $M_1 = M_2$. Write $F_i = (J_i, H_i)$, $i = 1,2$. There is a smooth function h such that $H_2 = h(J_1, H_1)$. Therefore, the Hamiltonian vector field of H_2 is $\mathcal{X}_{H_2} = (\partial_1 h)\mathcal{X}_{J_1} + (\partial_2 h)\mathcal{X}_{H_1}$, where the coefficients $\partial_1 h$ and $\partial_2 h$ are constant along the flow. Since the flow of all these Hamiltonians must be 2π-periodic and the F_2-action is free on the principal orbit type, which consists of the Liouville tori, we see that $\partial_1 h$ and $\partial_2 h$ must be integers on all Liouville tori, and hence everywhere by continuity. Hence

$$(\mathcal{X}_{J_1}, \mathcal{X}_{H_1}) = \begin{pmatrix} 1 & k \\ 0 & \ell \end{pmatrix} (\mathcal{X}_{J_2}, \mathcal{X}_{H_2}), \tag{3.2}$$

for some integers k and ℓ. The same argument, switching the roles of F_1 and F_2, shows that the matrix in (3.2) must be invertible, hence $\ell = \pm 1$. The hypothesis that $\partial_2 h > 0$ implies that $\ell = 1$, which means that $F_1 = t \circ F_2$ for some affine transformation $t \in \mathcal{T}$. $\square$

For any $(M,\omega,F) \in \mathcal{M}_{\mathrm{TT}}$, there exists, by Definition 3.1, a $(1,0)$-toric momentum map F' isomorphic to F as a proper semitoric system. Although F' is not unique, Lemma 3.3 shows that the following map is well-defined:

$$\mathcal{C}_{\mathrm{TT}} : \mathcal{M}_{\mathrm{TT}} \ni (M,\omega,F) \longmapsto (F'(M) \mod \mathcal{T},\, \varnothing, \varnothing) \in \mathfrak{B}_{\mathrm{PST}}, \tag{3.3}$$

where for each F one chooses F' to be any $(1,0)$-toric momentum map isomorphic to F.

Definition 3.4 If $\mathcal{F}$ is a family of proper semitoric systems containing $\mathcal{M}_{\mathrm{T}}$, a *cartographic invariant* is any map $\mathcal{C}\colon \mathcal{F} \to \mathfrak{B}_{\mathrm{GST}}$ extending $\mathcal{C}_{\mathrm{T}}$ in (3.1) and invariant under isomorphism, i.e., if (M_1, ω_1, F_1) and (M_2, ω_2, F_2) are isomorphic, then $\mathcal{C}(M_1, \omega_1, F_1) = \mathcal{C}(M_2, \omega_2, F_2)$. $\oslash$

The map $\mathcal{C}_{\mathrm{T}}$ itself is a cartographic invariant, thanks to Lemma 3.3. For the same reason, the map $\mathcal{C}_{\mathrm{TT}}$ is also a cartographic invariant. The fact that $\mathcal{C}_{\mathrm{ST}}$ is a cartographic invariant follows from [Vu07].

Theorem A. *There exists a cartographic invariant $\mathcal{C}_{\mathrm{PST}}\colon \mathcal{M}_{\mathrm{PST}} \to \mathfrak{B}_{\mathrm{PST}}$ such that the diagram*

$$\begin{tikzcd} \mathcal{M}_{\mathrm{T}} \arrow[r, hook] \arrow[drrr, "\mathcal{C}_{\mathrm{T}}"'] & \mathcal{M}_{\mathrm{TT}} \arrow[r, hook] \arrow[drr, "\mathcal{C}_{\mathrm{TT}}"] & \mathcal{M}_{\mathrm{ST}} \arrow[r, hook] \arrow[dr, "\mathcal{C}_{\mathrm{ST}}"] & \mathcal{M}_{\mathrm{PST}} \arrow[d, "\mathcal{C}_{\mathrm{PST}}"] \\ & & & \mathfrak{B}_{\mathrm{PST}} \end{tikzcd} \tag{3.4}$$

is commutative.

Theorem A is a consequence of Theorem B and Corollary 4.3 (formulated and proved in the next sections). These statements, together with Theorem C, are more informative than Theorem A alone, because the cartographic invariant is explicitly constructed.

4. Main results: Theorems B, C, D, and Corollary 4.3

Let (M, ω) be a connected symplectic 4-manifold and $F := (J, H)\colon M \to \mathbb{R}^2$ a proper semitoric system. Define $B := F(M)$ and let $B_r \subseteq B$ be the set of regular values of F. Since F is proper, we know that the set of focus-focus critical values of F is discrete. Denote by $c_i := (x_i, y_i)$, $i \in Z \in \mathfrak{F}$ (see Section 2.2), the focus-focus critical values of F, ordered so that $x_i \leqslant x_{i+1}$, and let k_i be the number of critical points in $F^{-1}(c_i)$. Given $\vec{\epsilon} = (\epsilon_i)_{i \in Z} \in \{-1, +1\}^Z$, define the vertical closed half-line originating at $c_i = (x_i, y_i)$ by

$$\mathcal{L}_i^{\epsilon_i} := \{(x_i, y) \in \mathbb{R}^2 \mid \epsilon_i y \geqslant \epsilon_i y_i\}, \quad \text{for each} \quad i \in Z, \tag{4.1}$$

pointing up from c_i if $\epsilon_i = 1$, and down if $\epsilon_i = -1$. Define $\ell_i^{\epsilon_i} := B \cap \mathcal{L}_i^{\epsilon_i} \subset \mathbb{R}^2$.

For any $c \in \mathbb{R}^2$, define $I_c := \{i \in Z \mid c \in \ell_i^{\epsilon_i}\}$. Let $k : \mathbb{R}^2 \to \mathbb{Z}$ be the map defined by

$$k(c) := \sum_{i \in I_c} \epsilon_i k_i\,, \tag{4.2}$$

with the convention that if $I_c = \varnothing$ then $k(c) = 0$. The sum is finite thanks to (H.iv). Let $\ell^{\vec{\epsilon}}$ be the support of k, *i.e.* $\ell^{\vec{\epsilon}} := \overline{k^{-1}(\mathbb{Z} \setminus \{0\})}$.

Remark 4.1 It may happen that the sum (4.2) vanishes for some values of c for which I_c is not empty. For instance, assume that $B = F(M)$ is a compact set, and that $Z = \{1, 2\}$, $c_1 = (x_1, y_1) = (0, 0)$, $c_2 = (x_2, y_2) = (0, 1)$,

$\epsilon_1 = 1$, $\epsilon_2 = -1$, and $k_1 = k_2 = 1$. Such an example can be realized, for instance, by the quadratic pendulum, see [CVN02]. Then $\mathcal{L}_1^{\epsilon_1} = \{0\} \times [0, \infty)$ and $\mathcal{L}_2^{\epsilon_2} = \{0\} \times (-\infty, 1]$. For any $c \in \mathbb{R}^2$, we have $I_c = \{i \in \{1, 2\} \mid c \in \ell_i^{\epsilon_i}\}$. Therefore, $k\colon \mathbb{R}^2 \to \mathbb{Z}$, defined by $k(c = (x, y)) = \sum_{i \in I_c} \epsilon_i k_i$, is given by

- $k(c) = -1$ if $c \in B$ and $x = 0$ and $y < 0$.
- $k(c) = 0$ if $c \in B$ and $x \neq 0$, or if $c \in B$ and $x = 0$ and $0 \leqslant y \leqslant 1$;
- $k(c) = 1$ if $c \in B$ and $x = 0$ and $y > 1$;

Hence $\ell^{\vec\epsilon} = \overline{k^{-1}(\mathbb{Z} \setminus \{0\})} = B \cap (\{0\} \times ((-\infty, 0] \cup [1, \infty)))$, which is a strict subset of $\ell_1^{\epsilon_1} \cup \ell_2^{\epsilon_2} = B \cap (\{0\} \times \mathbb{R})$.

In general, it is easy to show that when x is fixed, the map $y \mapsto k(x, y)$ is non-decreasing with positive jumps at every y for which there exists a c_i equal to (x, y) (and the number of jumps is finite due to (H.iv)). This implies that $c_i \in \ell^{\vec\epsilon}$ for all $i \in Z$; is also implies that if all k_i's are equal to 1 (which is a generic situation in the sense of [Zu96]), then the map $y \mapsto k(x, y)$ must take the value zero on some positive measure interval (and there is only one such interval), and hence $B \setminus \ell^{\vec\epsilon}$ is connected. However, if in the example above we take $k_1 = 2$, then we have $k(c) = 2$ when $c \in B$, $x = 0$ and $y > 1$, which implies $\ell^{\vec\epsilon} = B \cap (\{0\} \times \mathbb{R})$. In this case, $B \setminus \ell^{\vec\epsilon}$ is not connected. On the other hand, $B \setminus \ell^{\vec\epsilon}$ is always connected if one chooses $\epsilon_i = 1$ for all i. ⊘

For the necessary background on affine manifolds in the discussion below, see the appendix (Section 9.3). Write $\mathbb{A}^2_{\mathbb{Z}}$ for $\mathbb{R}^2$ equipped with its standard integral affine structure with automorphism group $\mathrm{Aff}(2, \mathbb{Z}) := \mathrm{GL}(2, \mathbb{Z}) \ltimes \mathbb{R}^2$. There is a natural integral affine structure on B_r that is defined, for instance, in [Vu07, Section 3] or [HZ94, Appendix A2]; see also Section 9.3: affine charts near regular values are given by action variables $f\colon U \to \mathbb{R}^2$ on open subsets U of B_r. Any two such charts differ by the action of an element of $\mathrm{Aff}(2, \mathbb{Z})$. Note that, in general, this is not the affine structure of $\mathbb{A}^2_{\mathbb{Z}}$.

Let X and Y be smooth manifolds and $A \subset X$. A map $f : A \to Y$ is said to be *smooth*, if every point in A admits an open neighborhood in X to which f can be smoothly extended. The map f is called a *diffeomorphism onto its image* if f is injective, smooth, and its inverse $f^{-1}\colon f(A) \to A$ is smooth as a map $f^{-1} : f(A) \to X$, in the sense above.

The following theorem is a generalization of [Vu07, Theorem 3.8].

Theorem B. *Let $F\colon M \to \mathbb{R}^2$ be a proper semitoric system in $\mathcal{M}_{\mathrm{PST}}$. Then for every $\vec\epsilon \in \{-1, +1\}^Z$ there exists a homeomorphism*

$$f_{\vec\epsilon}\colon B \to f_{\vec\epsilon}(B) \subseteq \mathbb{R}^2$$

of the form $f_{\vec\epsilon}(x, y) = (x, f^{(2)}_{\vec\epsilon}(x, y))$ such that:

(P.i) *the restriction $f_{\vec\epsilon}|_{(B \setminus \ell^{\vec\epsilon})}$ is a diffeomorphism onto its image, with positive Jacobian determinant;*

(P.ii) *the restriction $f_{\vec\epsilon}|_{(B_r \setminus \ell^{\vec\epsilon})}$ sends the integral affine structure of B_r to the standard integral affine structure of $\mathbb{A}^2_{\mathbb{Z}}$;*

(P.iii) *the restriction* $f_{\vec\epsilon}|_{(B_r \setminus \ell^{\vec\epsilon})}$ *extends to a smooth multi-valued map* $B_r \to \mathbb{R}^2$ *and for any* $i \in Z$ *and* $c \in \ell_i^{\epsilon_i} \setminus \{c_j \mid j \in Z\}$, *we have*

$$\lim_{\substack{(x,y)\to c\\ x<x_i}} \mathrm{d}f_{\vec\epsilon}(x,y) = T^{k(c)} \lim_{\substack{(x,y)\to c\\ x>x_i}} \mathrm{d}f_{\vec\epsilon}(x,y), \tag{4.3}$$

where $k(c)$ *is defined in* (4.2) *and* T *in* (2.1).

Such an $f_{\vec\epsilon}$ *is unique modulo a left composition by a transformation in* $\mathcal{T}$.

In this statement, by a smooth multi-valued map $B_r \to \mathbb{R}^2$ we mean a smooth map defined on the universal covering manifold of the open set B_r.

In the proper toric case, the affine structure of B_r coincides with the standard one on $\mathbb{R}^2$, which by uniqueness implies that $f_{\vec\epsilon} \in \mathcal{T}$.

Definition 4.2 The map $f_{\vec\epsilon}$ in Theorem B is a *cartographic map for* F and its image $f_{\vec\epsilon}(B)$ is a *cartographic projection of* F. ⊘

The word 'cartographic' reflects the fact that the map $f_{\vec\epsilon}$ lays out the affine structure of F in two dimensions.

We continue to use the terminology introduced in Sections 2 and 3. In particular, we use the notation $\vec{x} = (x_i)_{i\in Z}$ and $\vec{k} = (k_i)_{i\in Z}$. Given $\vec\epsilon = (\epsilon_i)_{i\in Z}$, with $\epsilon_i = 1$ for all $i \in Z$, define

$$\mathcal{C}_{\mathrm{PST}}(F) := (f_{\vec\epsilon}(B) \bmod \mathcal{T},\, \vec{x},\, \vec{k}) \mod \{-1,1\}^Z \in \mathfrak{B}_{\mathrm{PST}}. \tag{4.4}$$

Corollary 4.3. *The map* $\mathcal{C}_{\mathrm{PST}} : \mathcal{M}_{\mathrm{PST}} \to \mathfrak{B}_{\mathrm{PST}}$ *is a cartographic invariant.*

Proof. Let $F_1\colon M_1 \to \mathbb{R}^2$ and $F_2\colon M_2 \to \mathbb{R}^2$ be proper semitoric systems and $f_{\vec\epsilon,1}$, $f_{\vec\epsilon,2}$ the corresponding cartographic maps defined in Theorem B. If F_1 and F_2 are isomorphic, then we write $\varphi^*(J_2, H_2) = (J_1, h(J_1, H_1))$ with the notation of Definition 1.5. Thus, by the uniqueness of Theorem B, there exists a transformation $t \in \mathcal{T}$ such that $f_{\vec\epsilon,2} \circ g = t \cdot f_{\vec\epsilon,1}$, where $g(x,y) := (x, h(x,y))$. Since $F_2(M_2) = g(F_1(M_1))$, we see from (4.4) that $\mathcal{C}_{\mathrm{PST}}(F_1) = \mathcal{C}_{\mathrm{PST}}(F_2)$.

Now, suppose $F \in \mathcal{M}_{\mathrm{ST}}$, i.e. F is a $(1,0)$-semitoric system. By the uniqueness and [Vu07], $\mathcal{C}_{\mathrm{PST}}(F) = \mathcal{C}_{\mathrm{ST}}(F)$, which means that $\mathcal{C}_{\mathrm{PST}}$ extends $\mathcal{C}_{\mathrm{ST}}$, and hence extends $\mathcal{C}_{\mathrm{T}}$. □

Once we have a cartographic map $f_{\vec\epsilon}$ for F, our next goal is to find a systematic description of the patterns that can occur in the image of $f_{\vec\epsilon}$.

In this paper, we use the usual extended line $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ with the standard topology (a basis of neighborhoods of $-\infty$ is given by the intervals $[-\infty, A]$, $A \in \mathbb{R}$, and similarly at $+\infty$).

Definition 4.4 Let $\mathcal{R}$ be a subset of $\mathbb{R}^2$.

- We say that $\mathcal{R}$ has *type* CC (closed-closed) if there is an interval $I \subseteq \mathbb{R}$ and $f, g\colon I \to \mathbb{R}$ such that f is a piecewise linear continuous convex function, g is a piecewise linear continuous concave function, and

$$\mathcal{R} = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in I \ \text{ and } \ f(x) \leqslant y \leqslant g(x)\Big\}.$$

- We say that $\mathcal{R}$ has *type* CO (closed-open) if there is an interval $I \subseteq \mathbb{R}$ and $f\colon I \to \mathbb{R}$, $g\colon I \to \overline{\mathbb{R}}$ such that f is a piecewise linear continuous convex function, g is lower semicontinuous, and
$$\mathcal{R} = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in I \ \text{ and } \ f(x) \leqslant y < g(x)\Big\}.$$
- We say that $\mathcal{R}$ has *type* OC if there is an interval $I \subseteq \mathbb{R}$ and $f\colon I \to \overline{\mathbb{R}}$, $g\colon I \to \mathbb{R}$ such that f is upper semicontinuous, g is a piecewise linear continuous concave function, and
$$\mathcal{R} = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in I \ \text{ and } \ f(x) < y \leqslant g(x)\Big\}.$$
- We say that $\mathcal{R}$ has *type* OO if there is an interval $I \subseteq \mathbb{R}$ and $f, g\colon I \to \overline{\mathbb{R}}$ such that f is upper semicontinuous, g is lower semicontinuous, and
$$\mathcal{R} = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in I \ \text{ and } \ f(x) < y < g(x)\Big\}.$$

$\oslash$

Notice that $\mathcal{R}$ has type CC if and only if there is a convex polygon $\Delta \subset \mathbb{R}^2$ and an interval $I \subseteq \mathbb{R}$ such that
$$\mathcal{R} = \Delta \cap \Big\{(x,y) \in \mathbb{R}^2 \mid x \in I\Big\}.$$

The following theorem shows that the image of a cartographic map of a proper semitoric system can always be split into regions of type CC, CO, OC, or OO. The theorem also explains how the boundary between these regions is characterized by a bifurcation of the fibers of J. This bifurcation can be caused by singularities of J, or by the lack of compactness of the fiber in either direction $H > 0$ or $H < 0$.

Theorem C. *Let $F = (J,H)\colon M \to \mathbb{R}^2$ be a proper semitoric system and let $f_{\vec{\epsilon}}$ be a cartographic map for F. Let*
$$K^+ := \Big\{x \in J(M) \mid \ J^{-1}(x) \cap H^{-1}([0,+\infty)) \text{ is compact}\Big\}.$$
and
$$K^- := \Big\{x \in J(M) \mid \ J^{-1}(x) \cap H^{-1}((-\infty,0]) \text{ is compact}\Big\}.$$
Then there exist a discrete increasing sequence $\{x_j\}_{j\in\mathbb{Z}} \subset \Sigma_J$ (where $\Sigma_J \subset \mathbb{R}$ is the bifurcation set of J), and sets $\mathcal{C}_j^{\vec{\epsilon}} \subset \mathbb{R}^2$, $j \in \mathbb{Z}$, such that:

(P.1) $f_{\vec{\epsilon}}(B) = \bigcup_{j\in\mathbb{Z}} \mathcal{C}_j^{\vec{\epsilon}}$;

(P.2) *for each $j \in \mathbb{Z}$, the set $\mathcal{C}_j^{\vec{\epsilon}}$ has type* CC, CO, OC, *or* OO *associated to an interval I_j with interior (x_j, x_{j+1}); the set $\mathcal{C}_j^{\vec{\epsilon}}$ has type* CC *(resp.* CO, OC, OO*) if and only if I_j is contained in $K^+ \cap K^-$ (resp. $K^- \setminus K^+$, $K^+ \setminus K^-$, $J(M) \setminus (K^+ \cup K^-)$);*

(P.3) *for every $j \in \mathbb{Z}$ and every regular value $x \in I_j$ of J, the Liouville volume $V(x) \leqslant +\infty$ of $J^{-1}(x)$ is equal to the Euclidean length of the vertical line segment $(\{x\} \times \mathbb{R}) \cap \mathcal{C}_j^{\vec{\epsilon}}$.*

While in this statement we chose, for simplicity, to label the sequence (x_j) by $j \in \mathbb{Z}$, in some cases (for instance if M is compact), only a finite number of the x_j's are relevant. We recall that, given a Hamiltonian J on a symplectic manifold (M, ω) of dimension $2n$, one can define a volume element on any level set of J (near a regular point of J) by taking the quotient of the top-form $\frac{\omega^n}{2\pi n!}$ by the 1-form dJ. If J generates an S^1-action, then the total volume of a regular fiber of J is equal to the volume of the Marsden-Weinstein reduced space; see [DH82] and [Vu07, Section 5.1].

For $(1,0)$-toric and $(1,0)$-semitoric systems, every $\mathcal{C}_j^{\vec{\epsilon}}$ is of type CC. Indeed, suppose that $F : M \to \mathbb{R}^2$ is the momentum map of a Hamiltonian $\mathbb{T}^2$-action on a compact connected symplectic 4-manifold. Then the cartographic projection of F is a compact convex polygonal set in $\mathbb{R}^2$; see [At82] and [GS82]. If $F : M \to \mathbb{R}^2$ is a $(1,0)$-semitoric system (i.e., J is proper), then any cartographic projection of F is a convex polygon in $\mathbb{R}^2$, which may be bounded or unbounded, and which is always a closed subset of $\mathbb{R}^2$; see [Vu07, Theorem 3.8]. If F is a proper toric system, then the identity is a cartographic map, and hence the image $F(M)$ is a cartographic projection.

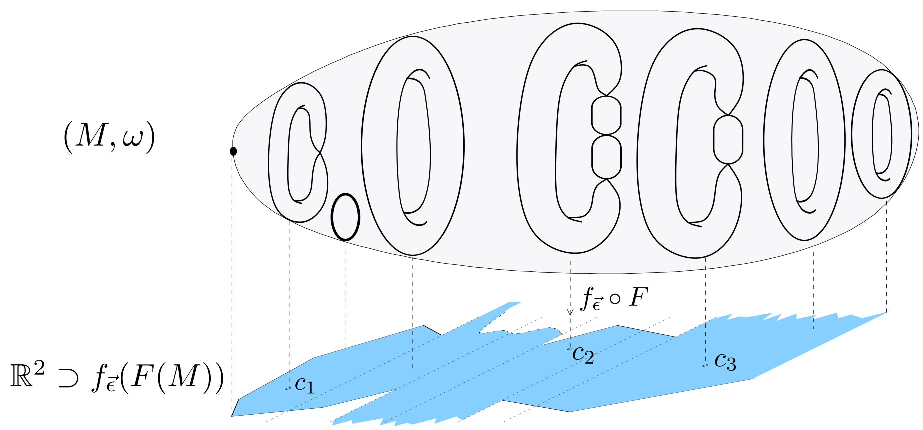

FIGURE 4.1. The singular Lagrangian fibration $F \colon M \to \mathbb{R}^2$ of a proper semitoric system with three isolated singular values c_1, c_2, c_3. The generic fiber is a 2-dimensional torus, the singular fibers are circles, points, or pinched tori.

Example 4.5 Figure 4.1 shows the regular and singular focus-focus fibers of the singular Lagrangian fibration $f_{\vec{\epsilon}} \circ F \colon M \to \bigcup_{j\in\mathbb{Z}} \mathcal{C}_j^{\vec{\epsilon}}$ in Theorem C. There are three focus-focus singular fibers, $F^{-1}(c_i)$, $i = 1, 2, 3$. The value c_1 has multiplicity $k_1 = 2$, c_2 has multiplicity $k_2 = 3$, and c_3 has multiplicity $k_3 = 2$. ⊘

Remark 4.6 Concerning Theorem C(P.3), note that the Duistermaat-Heckman theorem does not hold for non-proper momentum maps. Indeed, consider Example 1.4; let $V(x)$ be the symplectic volume of M'_x where $M'_x = M' \cap \mu^{-1}(x)/S^1 = S^2 \cap \{z_2 < f(x)\}$ (see Figure 4.2). Then $V(x) = \mathrm{vol}(S^2)\ [(1 +$

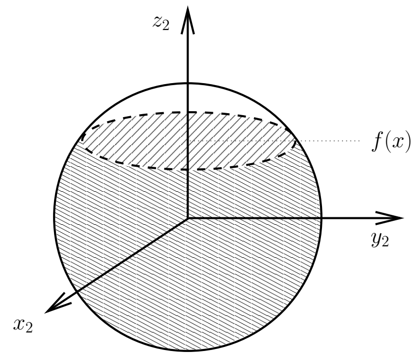

FIGURE 4.2. The reduced manifold M'_x in Example 1.4.

$f(x))/2] = 2\pi(1 + f(x))$. So $V(x)$ is not piecewise linear in general, in contrast with the statement of the Duistermaat-Heckman Theorem [DH82]. $\oslash$

We conclude with a result showing that there are proper semitoric systems with a cartographic projection which may not occur as the cartographic projection of a $(1, 0)$-toric or $(1, 0)$-semitoric system.

Theorem D. *There exists an uncountable family of proper semitoric integrable systems $\Lambda = \{F_\lambda : M \to \mathbb{R}^2\}_{\lambda\in\Lambda}$, with cartographic maps $f_{\lambda,\vec{\epsilon}}$, such that the following properties hold:*

(E.1) *$B_\lambda := F_\lambda(M)$ is unbounded in $\mathbb{R}^2$;*
(E.2) *there are uncountable subfamilies $\Lambda_1 \subset \Lambda$ and $\Lambda_2 \subset \Lambda$ such that $f_{\lambda,\vec{\epsilon}}(B_\lambda)$ is bounded when $\lambda \in \Lambda_1$ and unbounded when $\lambda \in \Lambda_2$;*
(E.3) *$f_{\lambda,\vec{\epsilon}}(B_\lambda)$ is not a convex region;*
(E.4) *$f_{\lambda,\vec{\epsilon}}(B_\lambda)$ is neither open nor closed in $\mathbb{R}^2$;*
(E.5) *F_λ is isomorphic to $F_{\lambda'}$ if and only of $\lambda = \lambda'$;*
(E.6) *for every $i \in \{\mathrm{CC}, \mathrm{CO}, \mathrm{OC}, \mathrm{OO}\}$, there exists λ such that $f_{\lambda,\vec{\epsilon}}(B_\lambda)$, as in Theorem C(P.1), is a union of regions in $\mathbb{R}^2$ of types CC, CO, OC, and OO, in which at least one of them has type i.*

5. Proof of Theorem B

The proof is close to [Vu07], but our construction is more transparent thanks to the use of some recent results in [PRV15], which we recall in the appendix (Section 9.4) for the reader's convenience.

The following lemma is essential in the construction.

Lemma 5.1. *Let $(M, \omega, F = (J, H))$ be a proper semitoric system. Let B_r be the set of regular values of F. Then there exists an oriented affine atlas of B_r such that the first component of all charts from B_r to $\mathbb{R}^2$ is the projection on the first factor.*

Proof. Let $q_0 = (x_0, y_0) \in B_r$. Since the fibers of J are connected by (H.iv), we know from Theorem 9.3 that the fibers of F are also connected. By the action-angle theorem (see [HZ94, Appendix A2]), there exists a diffeomorphism g :

$U \subset B_r \to g(U) \subset \mathbb{R}^2$, with positive Jacobian determinant (so it is orientation preserving), defined on a simply connected open neighborhood U of q_0 such that $a = (a_1, a_2) = g \circ F$ are local action variables. Let us show that one can always choose such action variables satisfying, in addition, $a_1 = J$. The fact that all transition functions belong to the group $\mathcal{T}$ is then a direct consequence.

Let $(a_1, a_2, \theta_1, \theta_2)$ be action-angle variables. Fix $(a_1^0, a_2^0) \in \mathbb{R}^2$ and let $\Lambda_0 = \{(a_1^0, a_2^0, \theta_1, \theta_2) \mid (\theta_1, \theta_2) \in S^1 \times S^1\}$ be the Liouville torus corresponding to these fixed values. Since g is a diffeomorphism, the Hamiltonian vector field $\mathcal{X}_J$ on Λ_0 is of the form $\mathcal{X}_J = \lambda \mathcal{X}_{a_1} + \mu \mathcal{X}_{a_2}$, where λ and μ depend only on (a_1^0, a_2^0), i.e., $\lambda, \mu \in \mathbb{R}$. Therefore the flow of $\mathcal{X}_J$ restricted to Λ_0 is $t \mapsto F^1_{\lambda_t} \circ F^2_{\mu_t}$, where F^i_t is the flow of $\mathcal{X}_{a_i}$. Thus, 2π-periodicity of this flow is equivalent to $F^1_{\lambda 2\pi} \circ F^2_{\mu 2\pi} = \mathrm{Id}$. Since F^i_t is translation by t modulo 2π of the angle θ_i, this relation is equivalent to $\lambda 2\pi = 0 \bmod 2\pi$ and $\mu 2\pi = 0 \bmod 2\pi$, hence $\lambda, \mu \in \mathbb{Z}$.

If λ, μ are co-prime, then we can complete $\mathcal{X}_J$ by another integral vector field (in the sense of the action-angle theorem) to obtain a $\mathbb{Z}$-basis of the integral lattice generated by $\mathcal{X}_{a_1}$ and $\mathcal{X}_{a_2}$: this means that, up to a matrix in $\mathrm{SL}(2, \mathbb{Z})$, we may assume that $A_1 = a_1 = J$, which finishes the proof.

If λ, μ are not co-prime, let $p \geqslant 2$ be the greatest common divisor of λ and μ. The isotropy of the J-action on Λ_0 contains $\mathbb{Z}/p\mathbb{Z}$. The same argument shows that the isotropy of the J-action on all Liouville tori above the open subset U contains $\mathbb{Z}/p\mathbb{Z}$. Since the set of all Liouville tori is open and dense in M, the Principal Orbit Theorem implies that the isotropy of the principal orbit type contains $\mathbb{Z}/p\mathbb{Z}$. This contradicts the effectiveness of the J-action. ☐

Using Lemma 5.1, from now on, we consider a fixed affine atlas of B_r such that the first component of all charts from B_r to $\mathbb{R}^2$ is the projection onto the first factor. Notice that, given such an atlas, all transition functions must belong to the group $\mathcal{T}$ (see Section 2.1).

The proof of Theorem B is divided into five steps: the first four treat the generic case in which the lines in $\ell^{\vec{\epsilon}}$ are pairwise distinct, whereas the last step deals with the non-generic case. We warn the reader that statements (P.i)–(P.iii) are proven in the first three steps, but the claim that $f_{\vec{\epsilon}}$ is a homeomorphism onto its open image is proven in Step 4.

Let Σ_J be the bifurcation set of J. Fix a point $q_0 = (x_0, y_0) \in B_r$, such that $x_0 \notin \Sigma_J$. The homeomorphism $f_{\vec{\epsilon}}$ with the required properties is constructed from the developing map of the universal cover $p_r : \widetilde{B}_r \to B_r$, chosen with q_0 as base point (see Section 9.3). Let $\widetilde{G}_\epsilon : \widetilde{B}_r \to \mathbb{R}^2$ be the unique developing map such that $\widetilde{G}_\epsilon([\gamma]) = g(\gamma(1))$ for paths γ contained in U, such that $\gamma(0) = q_0$. The goal is to use $\widetilde{G}_\epsilon$ in order to extend g to the whole image $B = F(M)$. We assume, for the moment, that the half-lines $\mathcal{L}_i^{\epsilon_i}$ (see (4.1)) that were used to define $\ell^{\vec{\epsilon}}$ do not overlap; this is for instance the case if all x_i's are distinct. We deal with the case of overlapping half-lines in the last step.

Step 1. (*$B_r \setminus \ell^{\vec{\epsilon}}$ is simply connected*).

Let $H^+, H^- \colon J(M) \to \overline{\mathbb{R}}$ be the functions with values in $\overline{\mathbb{R}}$ defined by $H^+(x) := \sup_{J^{-1}(x)} H$ and $H^-(x) := \inf_{J^{-1}(x)} H$. Since J is Morse-Bott with connected fibers (see, e.g., [PR11, Theorem 3]) we may apply Theorem 9.4 which states that H^+, H^- are continuous and $F(M) =$ (hypograph of H^+) $\cap$ (epigraph of H^-).

By Theorem 9.5, B_r is a connected open set given by $B_r = \mathring{B} \setminus \{c_i \mid i \in Z\}$. Therefore,

$$B_r = \left\{(x,y) \in \mathbb{R}^2; \quad x \in I, H^-(x) < y < H^+(x)\right\} \setminus \{c_i \mid i \in Z\}$$

Since $\ell^{\vec{\epsilon}}$ is, by assumption, a union of pairwise disjoint vertical half-lines starting at the c_i's, and $c_i \in \mathring{B}$, it follows that $B_r \setminus \ell^{\vec{\epsilon}}$ is simply connected.

Step 2. (*Proof of* (P.i) *on* $B_r \setminus \ell^{\vec{\epsilon}}$ *and proof of* (P.ii)). Hence, the developing map $\widetilde{G}_{\vec{\epsilon}} : \widetilde{B}_r \to \mathbb{R}^2$ induces a unique affine map $G_{\vec{\epsilon}} : B_r \setminus \ell^{\vec{\epsilon}} \to \mathbb{R}^2$ by the relation

$$G_{\vec{\epsilon}} \circ p_r := \widetilde{G}_{\vec{\epsilon}},$$

i.e., if $c \in B_r \setminus \ell^{\vec{\epsilon}}$ and γ is a smooth path in $B_r \setminus \ell^{\vec{\epsilon}}$ connecting q_0 to c, then $G_{\vec{\epsilon}}(c) := \widetilde{G}_{\vec{\epsilon}}([\gamma])$. Note that $G_{\vec{\epsilon}}|_U = g$.

The definition implies that $G_{\vec{\epsilon}}$ is a local diffeomorphism. We show now that $G_{\vec{\epsilon}}$ is injective. Since $A_1 = J$, $G_{\vec{\epsilon}}|_U$ is of the form $G_{\vec{\epsilon}}(x,y) = (x, h^U_{\vec{\epsilon}}(x,y))$ for some smooth function $h^U_{\vec{\epsilon}} : U \to \mathbb{R}$. Because we have an affine atlas of B_r with transition functions in $\mathcal{T}$, the affine map $G_{\vec{\epsilon}}$ must preserve the first component x, i.e., there exists a smooth function $h_{\vec{\epsilon}} : B_r \setminus \ell^{\vec{\epsilon}} \to \mathbb{R}$, extending $h^U_{\vec{\epsilon}}$, such that

$$G_{\vec{\epsilon}}(x,y) = (x, h_{\vec{\epsilon}}(x,y))$$

for all $(x,y) \in B_r \setminus \ell^{\vec{\epsilon}}$. Since $G_{\vec{\epsilon}}$ is a local diffeomorphism, $\frac{\partial h_{\vec{\epsilon}}}{\partial y}$ never vanishes. By Step 1, if x is fixed, the set $\{y \in \mathbb{R} \mid (x,y) \in B_r \setminus \ell^{\vec{\epsilon}}\}$ is connected. Therefore, for each fixed x, all the maps $y \mapsto h_{\vec{\epsilon}}(x,y)$ are injective. Hence $G_{\vec{\epsilon}}$ is injective and thus a global diffeomorphism $B_r \setminus \ell^{\vec{\epsilon}} \to G_{\vec{\epsilon}}\left(B_r \setminus \ell^{\vec{\epsilon}}\right) \subset \mathbb{R}^2$.

This proves (P.i) on $B_r \setminus \ell^{\vec{\epsilon}}$ by choosing $f_{\vec{\epsilon}} := G_{\vec{\epsilon}}$ and (P.ii) because $G_{\vec{\epsilon}}$ is an affine map.

Step 3. (*Extension of the developing map to* $B \setminus \ell^{\vec{\epsilon}}$ *and proof of* (P.i) *and* (P.iii)). By the description of the image of F in Theorem 9.5, and because the set $\ell^{\vec{\epsilon}}$ contains the focus-focus critical values c_i, we simply need to extend $G_{\vec{\epsilon}}$ at elliptic critical values. But the behavior of the affine structure at an elliptic critical value c is well known (see [MZ04]): there exist a smooth map $a \colon V \to \mathbb{R}^2$, where V is an open neighborhood of $c \in \mathbb{R}^2$, and a symplectomorphism $\varphi \colon F^{-1}(V) \to M_Q$ onto its image such that

$$a \circ F|_{F^{-1}(V)} = Q \circ \varphi : F^{-1}(V) \to \mathbb{R}^2, \tag{5.1}$$

where Q is the "normal form" of the same singularity type as F, given by $Q = (x_1^2 + \xi_1^2, \xi_2)$ (rank 1 case) or $Q = (x_1^2 + \xi_2^2, x_2^2 + \xi_2^2)$ (rank 0 case). Here, $M_Q = \mathbb{R}^2 \times \mathrm{T}^*\mathbb{T}^1 = \mathbb{R}^2 \times \mathbb{T}^1 \times \mathbb{R}$ (rank 1) or $M_Q = \mathbb{R}^4$ (rank 0). It follows from the formula for Q, that Q is generated by a Hamiltonian $\mathbb{T}^2$-action; therefore a is an affine map. On the other hand, since F and Q have the same singularity type, the ranks of $\mathrm{d}F$ and $\mathrm{d}Q$ must be equal, and the dimensions of the spaces spanned by the Hessians must be the same as well. Computing the Taylor expansion of (5.1) shows that $\mathrm{d}a(c)$ has to be invertible. Thus, a is a diffeomorphism onto its image. Therefore $a|_{B_r \cap V}$ is a chart for the affine structure of B_r.

Thus, there exists a unique affine map $A \in \mathrm{Aff}(2,\mathbb{Z})$ such that

$$(G_{\vec{\epsilon}})|_{Br\cap V} = A \circ a|_{Br\cap V}$$

and we may simply extend $G_{\vec{\epsilon}}$ to $B_r \cup V$ by letting

$$(G_{\vec{\epsilon}})|_V = A \circ a.$$

Because a is a diffeomorphism onto its image, we see that $G_{\vec{\epsilon}}$ remains a local diffeomorphism. This proves (P.i) with $f_{\vec{\epsilon}}|_{B\setminus\ell^{\vec{\epsilon}}} := G_{\vec{\epsilon}}$.

The fact that $G_{\vec{\epsilon}}$ extends to a smooth multi-valued map $B_r \to \mathbb{R}^2$ follows from the smoothness of the universal cover as in [Vu07, Section 3]. Formula (4.3) follows from the calculation of the monodromy around focus-focus singularities, which is carried out exactly as in [Vu07, pages 921-922] since it relies only on the properness of F (and not on the properness of J). This proves (P.iii).

Step 4. (*Extension to a homeomorphism $B \to \mathbb{R}^2$*). Here we are still assuming that no half-lines in the definition of $\ell^{\vec{\epsilon}}$ overlap, and we show that $G_{\vec{\epsilon}}$ may be extended to a homeomorphism $f_{\vec{\epsilon}}\colon B \to f_{\vec{\epsilon}}(B) \subset \mathbb{R}^2$. This proves the theorem if no half-lines in the definition of $\ell^{\vec{\epsilon}}$ overlap.

Because of (P.iii), if $c_0 \in \ell^{\vec{\epsilon}}$, but c_0 is not a focus-focus value, it follows that $G_{\vec{\epsilon}}$ has a unique continuation to c_0, from the left, and a unique continuation from the right. As in [Vu07, Proof of Theorem 3.8], these continuations coincide because the affine monodromy around a focus-focus singularity leaves the vertical line through c_0 pointwise invariant. That $G_{\vec{\epsilon}}(c)$ has a limit as c approaches the focus-focus value follows from the $z\log z$ behavior of $G_{\vec{\epsilon}}$, as shown in [Vu03, Section 3].

Let $f_{\vec{\epsilon}}\colon B \setminus \{c_i \mid i \in Z\} \to \mathbb{R}^2$ be this continuous extension of $G_{\vec{\epsilon}}$. Because of (P.iii), the extensions of the vertical derivative $\partial_y f_{\vec{\epsilon}}$ from the left or from the right coincide on $\ell^{\vec{\epsilon}}$. Since any extension of $G_{\vec{\epsilon}}(x,y) = (x, h_{\vec{\epsilon}}(x,y))$ is a local diffeomorphism, $\partial_y h_{\vec{\epsilon}}$ cannot vanish on $\ell^{\vec{\epsilon}}$. Thus, $f_{\vec{\epsilon}}|_{\ell^{\vec{\epsilon}}}$ is injective.

This implies that $f_{\vec{\epsilon}}$ is injective on $B \setminus \{c_i \mid i \in Z\}$.

Extend by continuity the map $f_{\vec{\epsilon}}$ to $\{c_i \mid i \in Z\}$. So far, we have shown that $f_{\vec{\epsilon}} : B \to \mathbb{R}^2$ is a continuous injective map which is an affine diffeomorphism away from $\ell^{\vec{\epsilon}}$. It remains to be shown that $(f_{\vec{\epsilon}})^{-1}$ is continuous on $f_{\vec{\epsilon}}(B)$. Since $f_{\vec{\epsilon}}$ is a diffeomorphism away from $\ell^{\vec{\epsilon}}$, we only have to show that $(f_{\vec{\epsilon}})^{-1}$ is continuous at points of $f_{\vec{\epsilon}}(\ell^{\vec{\epsilon}})$.

Let $c_0 = (x_0, y_0) \in \mathring{\ell}^{\vec{\epsilon}}$ and $\widehat{G}_{\vec{\epsilon}}\colon U \to \widehat{G}_{\vec{\epsilon}}(U)$ be an affine chart which coincides with $f_{\vec{\epsilon}}$ on the left hand-side of c_0 in U, that is, on

$$U_{\text{left}} := \Big\{(x,y) \in U \mid x \leqslant x_0\Big\}.$$

Then,

$$(f_{\vec{\epsilon}})^{-1}|_{f_{\vec{\epsilon}}(U_{\text{left}})} = \widehat{G}_{\vec{\epsilon}}^{-1}|_{f_{\vec{\epsilon}}(U_{\text{left}})}$$

and hence it is continuous on $f_{\vec{\epsilon}}(U_{\text{left}})$. Similarly, it is proved that $(f_{\vec{\epsilon}})^{-1}|_{f_{\vec{\epsilon}}(U_{\text{right}})}$ is continuous on U_{right}, which shows that $(f_{\vec{\epsilon}})^{-1}$ is continuous at $f_{\vec{\epsilon}}(c_0)$ for any $c_0 \in \mathring{\ell}^{\vec{\epsilon}}$.

Finally, we need to prove the continuity of $(f_{\vec{\epsilon}})^{-1}$ at all points $f_{\vec{\epsilon}}(c_i)$, where $c_i = (x_i, y_i)$, $i \in Z$, are the focus-focus values in B. Let ℓ_i be the vertical line containing c_i. Let us use the following local description of the behavior of $f_{\vec{\epsilon}}$ at c_i (see [Vu03], [Vu07, Proof of Theorem 3.8]): for all $(x, y) \in U \setminus \ell_i$,

$$f_{\vec{\epsilon}}(x, y) = (x, \operatorname{Im}(z \log z) + g(x, y)),$$

where $z = (x - x_i) + \mathrm{i}\hat{y}(x, y) \in \mathbb{C}$, g and $\hat{y}$ are smooth functions and $\hat{y}(x_i, y_i) = 0$. It follows that $\frac{\partial f_{\vec{\epsilon}}}{\partial y}$ is continuous near c_0 and hence on a punctured neighborhood of c_i (which is in agreement with (4.3)) and its second component, $\frac{\partial}{\partial y}(\operatorname{Im}(z \log z) + g(x, y))$, is equivalent, as $(x, y) \to c_i$, to the function $(x, y) \mapsto K \log(|z|)$ with $K = \frac{\partial \hat{y}}{\partial y}(c_i) > 0$. Hence we get the lower bound

$$\left| \frac{\partial f_{\vec{\epsilon}}}{\partial y} \right| \geqslant C > 0 \tag{5.2}$$

for some constant C, if (x, y) is in a small punctured neighborhood $V = [x_i - \eta, x_i + \eta] \times [y_i - \eta, y_i + \eta] \setminus c_i$, for some $\eta > 0$. For simplicity of notation, let us assume, for instance, that $\epsilon_i = 1$; the case $\epsilon_i = -1$ is treated similarly. It follows from (5.2) that, for any fixed $x \in [x_i - \eta, x_i + \eta]$, the function $y \mapsto f_{\vec{\epsilon}}(x, y)$ is invertible on $(y_i, y_i + \eta]$ and its inverse has bounded derivative, uniformly for $x \in [x_i - \eta, x_i + \eta]$. Hence, the inverse $(f_{\vec{\epsilon}})^{-1}$ extends by continuity at $f(c_i) = f(x_i, y_i)$. The limit of the inverse at this point must equal y_i since $f_{\vec{\epsilon}}$ is injective. This shows that $(f_{\vec{\epsilon}})^{-1}$ is continuous at the point $f_{\vec{\epsilon}}(c_i)$.

This concludes the proof of Theorem B in case there is no overlap of vertical lines in the definition of $\ell^{\vec{\epsilon}}$.

Step 5. (*Proof in the case of overlapping lines*). In this step we deal with the general case where the half-lines $\mathcal{L}_i^{\epsilon_i}$ may overlap. In this case the set $B_r \setminus \ell^{\vec{\epsilon}}$ may in general not be connected (see Remark 4.1).

For each $c \in B_r \setminus \ell^{\vec{\epsilon}}$, we need to choose a path γ_c joining q_0 to c inside $B_r \setminus \{c_i \mid i \in Z\}$, which we do as follows. In the definition of the half-lines $\mathcal{L}_i^{\epsilon_i}$, we replace the focus-focus critical values c_i which lie in the same vertical line by nearby points $\widetilde{c}_i$, in such a way that their x-coordinates are all pairwise distinct and form an increasing sequence. Let us denote by $\widetilde{\ell}^{\vec{\epsilon}}$ the new union of these half-lines (notice that we don't change the system itself and the true focus-focus values remain the c_i's). This turns the corresponding set $B_r \setminus \widetilde{\ell}^{\vec{\epsilon}}$ into a simply connected set; thus, up to homotopy, there is a unique path γ_c joining q_0 to c inside $B_r \setminus \widetilde{\ell}^{\vec{\epsilon}}$, and we can always assume that this path avoids the true focus-focus values c_i. We define

$$G_{\vec{\epsilon}}(c) := \widetilde{G}_{\vec{\epsilon}}([\gamma_c]).$$

With this $G_{\vec{\epsilon}}$, the previous proof for (P.i) and (P.ii) remains valid. The formula in (P.iii) follows from the fact that the monodromy representation is commutative, due to the global S^1 action (see [CVN02]).

Step 6. (*Uniqueness*). Let $g_{\vec{\epsilon}}$ and $h_{\vec{\epsilon}}$ be two maps satisfying (P.i), (P.ii) and (P.iii). By (P.i) and (P.ii), for each connected component of $B_r \setminus \ell^{\vec{\epsilon}}$, there is a transformation t_C in $\mathcal{T}$ such that $g_{\vec{\epsilon}} = t_C \cdot h_{\vec{\epsilon}}$ on C. Let C and C' be two connected components of $B_r \setminus \ell^{\vec{\epsilon}}$ such that the intersection of their closures is a vertical line through a focus-focus point. By (P.iii), the jumps of the derivatives of $g_{\vec{\epsilon}}$ and $h_{\vec{\epsilon}}$ from the left side to the right side of this vertical must be the same, which implies that the linear parts of the transformation t_C and $t_{C'}$ coincide. Then by continuity of $g_{\vec{\epsilon}}$ and $h_{\vec{\epsilon}}$, the translation parts of these transformations must coincide as well. Therefore there is a unique $t \in \mathcal{T}$ such that $g_{\vec{\epsilon}} = t \cdot h_{\vec{\epsilon}}$ on $B_r \setminus \ell^{\vec{\epsilon}}$. By continuity, this equality holds on B, which finishes the proof of the theorem.

Remark 5.2 If one assumes that there is only one focus-focus point in each focus-focus fiber (*i.e.* $k_i = 1$ for all $i \in Z$), then $B_r \setminus \ell^{\vec{\epsilon}}$ is still connected. Moreover, in this case, the description of $\ell^{\vec{\epsilon}}$ in Step 1 still holds and hence the proof in Steps 1–4 goes through. $\oslash$

6. Proof of Theorem C and the spherical pendulum example

Proof of Theorem C. The proof is divided into four steps.

Step 1. We prove that the topological boundaries ∂K^+ and ∂K^- in $J(M)$ are contained in the bifurcation set Σ_J.

We show the argument for K^+; the result for K^- will follow by replacing H by $-H$. Let $x_0 \notin \Sigma_J$. We have a trivialization:

$$\varphi : I \times \mathcal{F} \to J^{-1}(I),$$

where I is an open interval containing x_0, $\mathcal{F}$ is a smooth manifold, and $J(\varphi(x, f)) = x$. For any $x \in I$, let $H_x : \mathcal{F} \to \mathbb{R}$ be the function defined by $H_x(f) := H(\varphi(x, f))$. For any subset $A \subset \mathbb{R}$,

$$\{x\} \times H_x^{-1}(A) = \varphi^{-1} \circ F^{-1}(\{x\} \times A);$$

therefore the properness of F implies that H_x must be proper as well. Let $I_0 \subset I$ be a compact interval containing x_0 in its interior. When $x \in I_0$, we are interested in the topology of the set

$$J^{-1}(x) \cap H^{-1}([0, +\infty)) = \varphi\left(\{x\} \times H_x^{-1}([0, +\infty))\right),$$

which is diffeomorphic to $N_x := H_x^{-1}([0, +\infty))$; we wish to prove that N_x is either compact for all $x \in I_0$, or non-compact for all $x \in I_0$. Consider the set

$$\Gamma := \{(x, f) \in I_0 \times \mathcal{F} \mid H_x(f) = 0\} = \varphi^{-1}(F^{-1}(I_0 \times \{0\})).$$

Since F is proper and φ is a diffeomorphism, Γ must be compact. Hence we may define, since the map $(x, f) \mapsto H_{x_0}(f)$ is continuous,

$$y_{\max} := \max_{(x,f)\in\Gamma} H_{x_0}(f) < +\infty$$

and

$$y_{\min} := \min_{(x,f)\in\Gamma} H_{x_0}(f) > -\infty.$$

Lemma 6.1. *Let* $Y_1 := \min(0, y_{\min})$ *and* $Y_2 := \max(0, y_{\max})$. *The following inclusions hold, for all* $x \in I_0$*:*

$$H_{x_0}^{-1}((Y_2, +\infty)) \subset N_x \subset H_{x_0}^{-1}([Y_1, +\infty)). \tag{6.1}$$

Proof. First note that, by definition of $y_{\min}$ and $y_{\max}$, if $f \in \mathcal{F}$ is such that $H_{x_0}(f) \notin [y_{\min}, y_{\max}]$, then $H_x(f) \neq 0$ for all $x \in I_0$. In this case, by continuity of the map $x \mapsto H_x(f)$, the sign of $H_x(f)$ is constant for all $x \in I_0$, and hence equal to the sign of $H_{x_0}(f)$.

Thus, if $H_{x_0}(f) > Y_2$, then $H_x(f) > 0$ for all $x \in I_0$, which gives the first inclusion of the lemma. Similarly, if $H_{x_0}(f) < Y_1$, we get $H_x(f) < 0$ for all $x \in I_0$, which gives the second inclusion (by taking the complementary sets). □

Consider the splitting

$$H_{x_0}^{-1}([Y_1, +\infty)) = H_{x_0}^{-1}([Y_1, Y_2]) \cup H_{x_0}^{-1}((Y_2, +\infty))$$

where $H_{x_0}^{-1}([Y_1, Y_2])$ is compact (by properness of H_{x_0}). It implies that $H_{x_0}^{-1}([Y_1, +\infty))$ is bounded if and only if $H_{x_0}^{-1}((Y_2, +\infty))$ is bounded.

Hence, in view of Equation (6.1), we see that N_x is compact if and only if $H_{x_0}^{-1}((Y_2, +\infty))$ is bounded, and this condition does not depend on $x \in I_0$.

Hence, near x_0, $J^{-1}(x) \cap H^{-1}([0, +\infty))$ is either always compact or never compact, which proves that $x_0 \notin \partial K^+$.

Step 2. Let $f_{\vec{\epsilon}}\colon B \to f_{\vec{\epsilon}}(B) \subset \mathbb{R}^2$ be the homeomorphism in Theorem B. We use the notation of Step 1 of the proof of this theorem.

Since H^+, H^- are continuous and F is proper, one can check that the sets K^+, K^- defined in the theorem are open in $J(M)$. Hence we have the following equality of sets, where the four sets on the right hand side are disjoint:

$$J(M) = (K^+ \cap K^-) \cup (K^+ \setminus K^-) \cup (K^- \setminus K^+) \cup (J(M) \setminus (K^+ \cup K^-)).$$

By assumption, ∂K^+ and ∂K^- are discrete, so there exists a countable collection of intervals $\{I_j\}_{j\in\mathbb{Z}}$, whose interiors are pairwise disjoint, such that each I_j is contained in one of the above four sets $(K^+ \cap K^-)$, $(K^+ \setminus K^-)$, $(K^- \setminus K^+)$, or $(J(M) \setminus (K^+ \cup K^-))$, and such that $J(M) = \bigcup_{j\in\mathbb{Z}} I_j$.

Defining $\mathcal{C}_j^{\vec{\epsilon}} := f_{\vec{\epsilon}}((I_j \times \mathbb{R}) \cap F(M)) \subset I_j \times \mathbb{R}$, for every $j \in \mathbb{Z}$, we obtain $f_{\vec{\epsilon}}(F(M)) = \bigcup_{j\in\mathbb{Z}} \mathcal{C}_j^{\vec{\epsilon}}$.

Step 3. (*Proof of* (P.2) *and* (P.1)). We consider the only four possible cases.

(1) If $I_j \subset (K^+ \cap K^-)$, then the fibers of J are compact, so the analysis carried out in [Vu07, Theorem 3.8, (v)] applies. This implies that $\mathcal{C}_j^{\vec{\epsilon}}$ is of type CC.

(2) Consider now $I_j \subset (K^- \setminus K^+)$. Let $x \in I_j$. Since $J^{-1}(x) \cap H^{-1}((-\infty, 0])$ is compact, $H_-(x)$ is finite. On the other hand, $H_+(x)$ must be $+\infty$; otherwise, $F^{-1}(\{x\} \times [0, H_+(x)])$ would be compact, by the properness of F. This would imply that $J^{-1}(x)$ is compact, a contradiction.

Let $y \in H(J^{-1}(x))$. Recall that $f_{\vec{\epsilon}}(x, y) = \big(x, f_{\vec{\epsilon}}^{(2)}(x, y)\big)$ and that $\frac{\partial f_{\vec{\epsilon}}^{(2)}}{\partial y}$ is continuous on $F(M)$ (see (4.3)). Since $\frac{\partial f_{\vec{\epsilon}}^{(2)}}{\partial y} > 0$, the image $f_{\vec{\epsilon}}((I_j \times \mathbb{R}) \cap F(M)) = \mathcal{C}_j^{\vec{\epsilon}}$ has the form

$$\Big\{(x, z) \mid x \in I_j,\ \ h_-^{\vec{\epsilon}}(x) \leqslant z < h_+^{\vec{\epsilon}}(x)\Big\},$$

where

$$h_-^{\vec{\epsilon}}(x) := \min_{y \in J^{-1}(x)} f_{\vec{\epsilon}}^{(2)}(x,y) = f_{\vec{\epsilon}}^{(2)}(x, H_-(x)) \in \mathbb{R}$$
$$h_+^{\vec{\epsilon}}(x) := \sup_{y \in J^{-1}(x)} f_{\vec{\epsilon}}^{(2)}(x,y) = \lim_{y \to +\infty} f_{\vec{\epsilon}}^{(2)}(x,y) \in \overline{\mathbb{R}}.$$

We have used the fact that $f_{\vec{\epsilon}}$ is a homeomorphism, so that the point $(x, h_+^{\vec{\epsilon}}(x))$ cannot belong to $\mathcal{C}_j^{\vec{\epsilon}}$. The function $h_+^{\vec{\epsilon}}$ is a pointwise limit of continuous functions, so it is continuous on a dense set. However, we need to show that it is lower semicontinuous.

The new map

$$\left(J, f_{\vec{\epsilon}}^{(2)}(J,H)\right) = f_{\vec{\epsilon}} \circ F$$

satisfies the hypothesis of the following slight variation of [PRV15, Theorem 5.2] for continuous maps (the proof of which is identical line by line): *Let $\widehat{M}$ be a connected smooth four-manifold. Let $\widehat{F} = \left(\widehat{J}, \widehat{H}\right) : \widehat{M} \to \mathbb{R}^2$ be a continuous map. Suppose that the component $\widehat{J}$ is a smooth non-constant Morse-Bott function with connected fibers. Let $\widehat{H}^+, \widehat{H}^- : \widehat{J}\left(\widehat{M}\right) \to \overline{\mathbb{R}}$ be defined by $\widehat{H}^+(x) := \sup_{\widehat{J}^{-1}(x)} \widehat{H}$, $\widehat{H}^-(x) := \inf_{\widehat{J}^{-1}(x)} \widehat{H}$. Then the functions $\widehat{H}^+$ and $-\widehat{H}^-$ are lower semicontinuous.* This assertion gives the required semicontinuity in the statement of Theorem C.

The analysis of the graph of $h_-^{\vec{\epsilon}}$, which corresponds to the elliptic critical values and possible cuts due to focus-focus singularities, was carried out in [Vu07, Theorem 3.8]: it is continuous, piecewise linear, and convex. Thus, $\mathcal{C}_j^{\vec{\epsilon}}$ is of type CO.

(3) If $I_j \subset (K^+ \setminus K^-)$, then $\mathcal{C}_j^{\vec{\epsilon}}$ is of type OC. This is proved in a similar way to (2).

(4) Finally, let $I_j \subset J(M) \setminus (K^+ \cup K^-)$. Then, for any $x \in I_j$, we have $H_+(x) = +\infty$ and $H_-(x) = -\infty$. Therefore, $f_{\vec{\epsilon}}((I_j \times \mathbb{R}) \cap F(M)) = \mathcal{C}_j^{\vec{\epsilon}}$ has the form

$$\left\{ (x,z) \;\middle|\; x \in I_j, \;\; \lim_{y \to -\infty} f_{\vec{\epsilon}}^{(2)}(x,y) < z < \lim_{y \to +\infty} f_{\vec{\epsilon}}^{(2)}(x,y) \right\},$$

where the limits are understood in $\overline{\mathbb{R}}$. Thus, $\mathcal{C}_j^{\vec{\epsilon}}$ is of type OO.

This proves (P.2).

Step 4. (*Proof of* (P.3)). By the action-angle theorem, $(A_1, A_2) := f_{\vec{\epsilon}} \circ F$ is a set of action variables near $F^{-1}(x,y)$ with

$$A_1 = J, \qquad A_2 = A_2(J,H).$$

We have a symplectomorphism $\mathcal{U} \to \mathbb{T}^2_\theta \times \mathbb{R}^2_A$, where $\mathcal{U}$ is a saturated neighborhood of the fiber $F^{-1}(x,y)$ and the symplectic form on $\mathbb{T}^2_\theta \times \mathbb{R}^2_A$ is given by $\mathrm{d}A_1 \wedge \mathrm{d}\theta_1 + \mathrm{d}A_2 \wedge \mathrm{d}\theta_2$. We have

$$\mathcal{U} \cap J^{-1}(x) = A_1^{-1}(x) = \left\{ (\theta, A) \mid \theta \in \mathbb{T}^2, \; A_1 = x \right\}.$$

Since the normalized Liouville volume form is $(2\pi)^{-2}\mathrm{d}A_1 \wedge \mathrm{d}A_2 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$, the induced volume form on $\mathcal{U} \cap J^{-1}(x)$ is $(2\pi)^{-2}\mathrm{d}A_2 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$. In other words, the push-forward by A_2 of the Liouville measure on $J^{-1}(x)$ is simply the Lebesgue measure $\mathrm{d}A_2$. This gives the result because the set of critical points of H in $J^{-1}(x)$ has zero-measure in $J^{-1}(x)$. This concludes the proof of Theorem C. □

Example 6.2 (Spherical Pendulum) Proper semitoric systems with a genuinely non-proper J include many simple integrable systems from classical mechanics, such as the spherical pendulum, which we now recall. The phase space of the spherical pendulum is $M = \mathrm{T}^*S^2$ with its natural exact symplectic form. Let the circle S^1 act on the sphere $S^2 \subset \mathbb{R}^3$ by rotations about the vertical axis. Identify T^*S^2 with $\mathrm{T}S^2$, using the standard Riemannian metric on S^2, and denote its points by $(q,p) = (q^1,q^2,q^3,p_1,p_2,p_3) \in \mathrm{T}^*S^2 = \mathrm{T}S^2$, $\|q\|^2 = 1$, $q \cdot p = 0$. Working in units in which the mass of the pendulum and the gravitational acceleration are equal to one, the integrable system $F := (J,H)\colon \mathrm{T}S^2 \to \mathbb{R}^2$ is given by the momentum map of the (co)tangent lifted S^1-action on $\mathrm{T}S^2$,

$$J(q^1,q^2,q^3,p_1,p_2,p_3) = q^1p_2 - q^2p_1, \tag{6.2}$$

and the classical Hamiltonian

$$H(q^1,q^2,q^3,p_1,p_2,p_3) = \frac{(p_1)^2+(p_2)^2+(p_3)^2}{2} + q^3, \tag{6.3}$$

the sum of the kinetic and potential energies. The momentum map J is not proper because the sequence $\{(0,0,1,n,n,0)\}_{n\in\mathbb{N}} \subset J^{-1}(0) \subset \mathrm{T}S^2$ does not contain any convergent subsequence. The Hamiltonian H is proper since $H^{-1}([a,b])$ is a closed subset of the compact subset of $\mathrm{T}S^2$ for which $2(a-1) \leqslant \|p\|^2 \leqslant 2(b+1)$. Therefore, F is also proper. In this case, $F(M)$ is depicted in Figure 6.1 and the cartographic invariant of (M,F) is represented in Figure 6.2; we call it $\Delta(F)$.

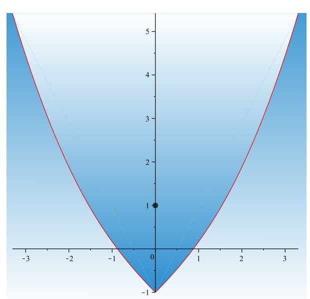

FIGURE 6.1. This figure is taken from [PV12]. It shows the image of of $F := (J,H)$ given by (6.2) and (6.3). The edges are the image of the transversally-elliptic singularities (rank 1), the vertex is the image of the elliptic-elliptic singularity (rank 1), and the dark dot in the interior is the image of the focus-focus singularity (rank 0). All other points are regular (rank 2).

There is precisely one elliptic-elliptic singularity at $((0,0,-1),(0,0,0))$, one focus-focus singularity at $((0,0,1),(0,0,0))$, and uncountably many transversally-elliptic type singularities. The range $F(M)$ and the set of critical values of F, which equals its bifurcation set, are given in Figure 6.1. The image under F of the focus-focus singularity is the point $(0,\,1)$. The image under F of the elliptic-elliptic singularity is the point $(0,\,-1)$. We know that the image by J of critical points of F of rank zero is the singleton $\{0\}$. Hence any representative of $\Delta(F)$ has no vertex in either of the regions $\{(x,y) \mid x<0\}$ or $\{(x,y) \mid x>0\}$. In each of these regions, there is only one connected family of transversally elliptic singular values. This means that the boundary $\Delta(F)$ in these regions consist of a single (semi-infinite) edge. Let us select the representative of $\Delta(F)$ such that, in the region $\{(x,y) \mid x<0\}$, the edge in question is the negative real axis $\{(y,0) \mid y<0\}$. Then we have a vertex at the origin $(x=0,y=0)$.

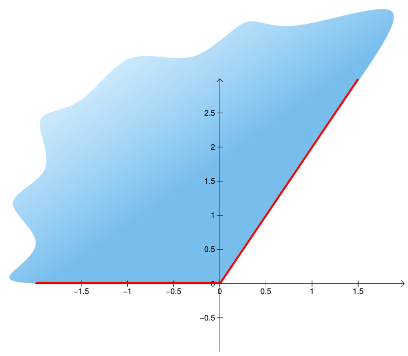

FIGURE 6.2. One of the two cartographic projections of the spherical pendulum. This figure is taken from [PV12].

We still need to compute the slope of the edge corresponding to the region where $\{(x,y) \mid x>0\}$. For this, we apply [Vu07, Theorem 5.3], which states that the change of slope can be deduced from the isotropy weights of the S^1 momentum map J and the monodromy index of the focus-focus point. (We need to include the focus-focus point because its J-value is the same as the J-value of the elliptic-elliptic point.) So we compute these weights now. The vertex of the polygon corresponds to the stable equilibrium at the South Pole of the sphere. We use the variables (q_1,q_2,p_1,p_2) as canonical coordinates on the tangent plane to the South Pole. In these coordinates, the quadratic approximation of J is in fact exact and equal to $J^{(2)} = q^1p_2 - q^2p_1$. Now consider the following change of coordinates:

$$x_1 := \frac{q_2-q_1}{\sqrt{2}}, \quad x_2 := \frac{p_1+p_2}{\sqrt{2}}, \quad \xi_1 := \frac{p_1-p_2}{\sqrt{2}}, \quad \xi_2 := \frac{q_1+q_2}{\sqrt{2}}. \tag{6.4}$$

This is a canonical transformation and the expression of $J^{(2)}$ in these variables is $J^{(2)} = \frac{1}{2}(x_2^2+\xi_2^2) - \frac{1}{2}(x_1^2+\xi_1^2)$. Since the Hamiltonian flows of $\frac{1}{2}(x_2^2+\xi_2^2)$ and $\frac{1}{2}(x_1^2+\xi_1^2)$ are 2π-periodic, this formula implies that the isotropy weights of J at this critical point are -1 and 1. From [Vu07], we know that the difference between the slope of the edge in $J>0$ and the slope of the edge in $J<0$

must be equal to $\frac{-1}{ab} + k$, where a and b are the isotropy weights, and k is the monodromy index. For the spherical pendulum, $k = 1$ because there is only one simple focus-focus point. Thus the new slope is $\frac{-1}{ab} + k = 1 + 1 = 2$. This leads to the polygonal set depicted in Figure 6.2. ⊘

7. Proof of Theorem D

We give here the outline of the construction of a family of integrable systems defined on an open subset of $S^2 \times S^2$, leading to the proof of Theorem D.

Step 1. (*Construction of suitable smooth functions.*) Let

$$\Omega := [-1,\, 1] \times [-1,\, 1] \setminus \{0\} \times [0,\, 1].$$

Let $\chi\colon [-1,\, 1] \to \mathbb{R}$ be any C^∞-smooth function such that $\chi(z_2) \equiv 1$ if $z_2 \leq 0$ and $\chi(z_2) > 0$ if $z_2 > 0$. Define $f\colon \Omega \to \mathbb{R}$ by

$$f(z_1,\, z_2) = \begin{cases} 1 & \text{if } z_1 \leq 0; \\ \chi(z_2) & \text{if } z_1 > 0, \end{cases} \tag{7.1}$$

and note that it is smooth on Ω.

Step 2. (*Definition of a connected smooth* 4*-manifold* M*.*) Let S^2 be the unit sphere in $\mathbb{R}^3$ and

$$\begin{aligned} M &:= S^2 \times S^2 \setminus \{((x_1, y_1, z_1),\, (x_2, y_2, z_2)) \in S^2 \times S^2 \mid z_1 = 0,\, z_2 \geq 0\} \\ &= p^{-1}(\Omega), \end{aligned}$$

where a point in the first sphere has coordinates $(x_1,\, y_1,\, z_1)$, a point in the second sphere has coordinates $(x_2,\, y_2,\, z_2)$, and $p : S^2 \times S^2 \to \mathbb{R}^2$ is defined by $p((x_1, y_1, z_1), (x_2, y_2, z_2)) := (z_1, z_2)$. Since $M \subset S^2 \times S^2$ is an open subset, it is a smooth manifold. Moreover, M is connected.

Step 3. (*Definition of a symplectic* 2*-form* $\omega \in \Omega^2(M)$*.*) Let $\pi_i\colon S^2 \times S^2 \to S^2$ be the projection on the i^{th} copy of S^2, $i = 1,\, 2$. Let $\omega_i := \pi_i^*\omega_{S^2} \in \Omega^2(S^2 \times S^2)$ where ω_{S^2} is the standard area form on S^2. Define the 2-form ω on M by

$$\omega := \iota^*\left(\omega_1 + (p^*f)\omega_2\right) \tag{7.2}$$

where $\iota : M \hookrightarrow S^2 \times S^2$ is the inclusion. Since f is smooth by Step 1, ω is also smooth, i.e., $\omega \in \Omega^2(M)$.

One can check that ω is closed, because $\frac{\partial f}{\partial z_1} = 0$, and that ω is non-degenerate, because $f \neq 0$.

Step 4. ($(M,\, \omega)$ *with* $J := z_1$, $H := z_2$ *satisfies* $\{J,\, H\} = 0$ *and* J *is a momentum map for a Hamiltonian* S^1*-action.*) We let S^1 act on M by rotation about the (vertical) z_1-axis of the first sphere and trivially on the second sphere. The infinitesimal generator of this action equals the vector field $\mathcal{X}((x_1, y_1, z_1), (x_2, y_2, z_2)) = ((-y_1, x_1, 0), (0, 0, 0))$. This immediately shows that $J = z_1$ is a momentum map for this action.

Step 5. (*(M, ω) with $J := z_1$, $H := z_2$ is an integrable system with an effective Hamiltonian S^1-action with momentum map J; $F = (J, H)$ has only non-degenerate elliptic type singularities and F is not proper*). A direct verification shows that the rank zero critical points are precisely (N_1, N_2), (N_1, S_2), (S_1, N_2), and (S_1, S_2), where N_i, S_i are the North and South Poles on the first and second spheres, respectively. One can verify that these critical points are non-degenerate, in the sense that a generic linear combination of the linearization of the vector fields $\mathcal{X}_J$ and $\mathcal{X}_H$ at each of these critical points has four distinct eigenvalues of the form $(\mathrm{i}a, -\mathrm{i}a, \mathrm{i}b, -\mathrm{i}b)$ (here, $\mathrm{i} = \sqrt{-1}$). Therefore, these singularities are of elliptic-elliptic type. The rank one critical points are $(N_1, (x_2, y_2, z_2))$, $(S_1, (x_2, y_2, z_2))$, $((x_1, y_1, z_1), N_2)$ with $z_1 \neq 0$, and $((x_1, y_1, z_1), S_2)$. Another simple computation shows that all of them are non-degenerate and of transversally elliptic type. It follows that $J := z_1$, $H := z_2$ is an integrable system with only non-degenerate singularities, of either elliptic-elliptic or transversally elliptic type.

Since the range

$$F(M) = [-1,\, 1] \times [-1,\, 1] \setminus \{z_1 = 0,\, z_2 \geq 0\}, \tag{7.3}$$

of F is not a closed set (see also Figure 7.1), it follows that F is not a proper map.

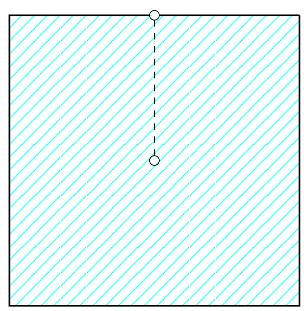

FIGURE 7.1. The image $F(M)$.

Step 6. (*Modify F suitably to turn it into a proper map that now defines a proper semitoric system.*) Let $h : [-1, 1] \to \mathbb{R}$ be any smooth function such that $h(z_2) \geqslant 0$, $h(z_2) = 0$ if and only if $z_2 \geqslant 0$, and $h'(z_2) < 0$ for $z_2 < 0$. Define the smooth function $g : \Omega \to \mathbb{R}^2$ by $g(z_1,\, z_2) = \left(z_1,\, \frac{z_2+2}{z_1^2+h(z_2)}\right)$ and $\widetilde{F} := g \circ F = \left(J,\, \frac{H+2}{J^2+h(H)}\right) : M \to \mathbb{R}^2$. Since the Jacobian of $\widetilde{F}$ is

$$\frac{1}{(z_1^2 + h(z_2))^2}\left(z_1^2 + h(z_2) - h'(z_2)(z_2+2)\right) > 0$$

(recall that $h'(z_2) \leqslant 0$ and $z_1^2 + h(z_2) > 0$ for $(z_1, z_2 \in \Omega)$), it follows that $\widetilde{F}$ is a local diffeomorphism. In order to show that $\widetilde{F}$ is proper, it suffices to prove that $\widetilde{F}^{-1}(K_1 \times K_2)$ is compact if K_1 and K_2 are closed intervals of $\mathbb{R}$; since the second component of g is always positive, we can assume, without loss of generality, that $K_2 = [a, b]$ with $b > 0$. To show that $\widetilde{F}$ is proper, we begin by analyzing $g^{-1}(K_1 \times K_2)$. We have $(z_1, z_2) \in g^{-1}(K_1 \times K_2)$ if and only if $z_1 \in K_1$ and $\frac{z_2+2}{z_1^2+h(z_2)} \leqslant b$, which implies that

$$\frac{1}{b} \leqslant \frac{z_2+2}{b} \leqslant z_1^2 + h(z_2).$$

Hence either $z_1^2 \geqslant 1/2b$ or $h(z_2) \geqslant 1/2b$. Thus $g^{-1}(K_1 \times K_2) \subseteq \Omega_b$ in Figure 7.2. Since $g^{-1}(K_1 \times K_2)$ is closed and obviously bounded, as a subset of the

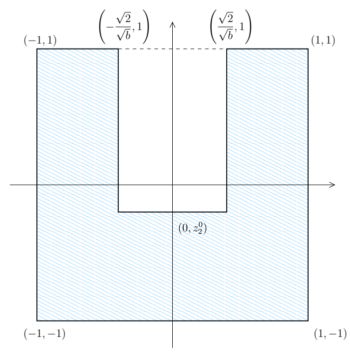

FIGURE 7.2. The set Ω_b, where $z_2^0 < 0$ is uniquely determined by the condition $h(z_2^0) = 1/2b$.

compact set Ω_b, it follows that $g^{-1}(K_1 \times K_2)$ is compact in $\mathbb{R}^2$. Therefore,

$$\widetilde{F}^{-1}(K_1 \times K_2) = F^{-1}\left(g^{-1}(K_1 \times K_2)\right)$$

is compact in $S^2 \times S^2$ and is obviously contained in M, by construction. We conclude that $\widetilde{F}^{-1}(K_1 \times K_2)$ is compact in M, endowed with the subspace topology.

Note that J is not proper because $J^{-1}(0)$ is not compact. However, for $c \neq 0$, $J^{-1}(c)$ is compact. This shows that 0 is a bifurcation point for J. However, $\widetilde{F}$ is a proper semitoric system.

Step 7. (*Determining the image* $\widetilde{F}(M)$.) Let

$$X := \Big([-1,0) \times [-1,1]\Big) \cup \Big((0,1] \times [-1,1]\Big) \cup \Big(\{0\} \times [-1,0)\Big).$$

It follows from (7.3) (see also Figure 7.1) that

$$\widetilde{F}(M) = g(F(M)) = \left\{ \left(z_1, \frac{z_2+2}{z_1^2 + h(z_2)} \right) \,\middle|\, (z_1, z_2) \in X \right\}.$$

Note that the second component of g is an even function of z_1 and hence the range $\widetilde{F}(M)$ is symmetric about the vertical axis in $\mathbb{R}^2$. A straightforward analysis shows that $\widetilde{F}(M)$ is the following region in $\mathbb{R}^2$:

$$\left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, 0 < |x| \leqslant 1, \, \frac{1}{x^2 + h(-1)} \leqslant y \leqslant \frac{3}{x^2} \right\} \bigcup \left(\{0\} \times \left[\frac{1}{h(-1)}, \infty \right) \right);$$

see Figure 7.3.

Note that the closed segment $[-1,1] \times \{-1\} \subset F(M)$ is mapped by g onto the lower curve in Figure 7.3, the two half-open segments $([-1,1] \setminus \{0\}) \times \{1\}$

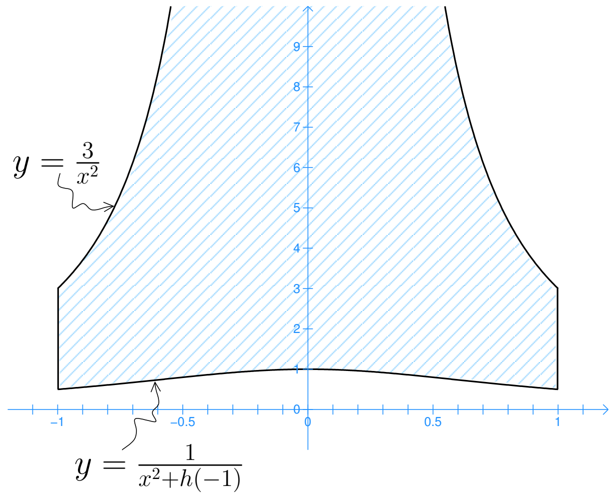

FIGURE 7.3. The set $\widetilde{F}(M)$ with the choice $h(-1) = 1$.

onto the two upper curves, the two closed vertical segments onto the two closed vertical segments, and the half-open interval $\{0\} \times [-1, 0)$ onto the infinite half-open interval $\{0\} \times [1/h(-1), \infty)$.

Step 8. (*Construction of the cartographic representation.*) We construct the cartographic invariant in Theorem C from $\widetilde{F}(M)$ by flattening out the horizontal curves and setting the height between them at the value given by the volume of the corresponding reduced phase space.

For each $|x| \leqslant 1$, let $V(x)$ denote the Liouville volume of the hypersurface $J^{-1}(x)$. (When the reduced space $J^{-1}(x)/S^1$ is smooth, $V(x)$ is equal to the symplectic volume of $J^{-1}(x)/S^1$.) Then, by Theorem C, the cartographic invariant associated to the proper semitoric system $(M, \widetilde{F})$ is given by the formula

$$\Delta = \big\{(x, y) \in \mathbb{R}^2 \mid 0 < |x| \leqslant 1,\, 0 \leqslant y \leqslant V(x)\big\} \cup \{0\} \times [0, 2\pi).$$

Using the definition (7.1) of f, a direct computation shows that if $x < 0$ then $J^{-1}(x)/S^1 = \{x\} \times S^2$, and hence

$$V(x) = \int_{S^2} f(x, z_2) \mathrm{d}\theta \wedge \mathrm{d}z_2 = 2\pi \int_{-1}^{1} f(x, z_2) \mathrm{d}z_2 = 4\pi,$$

because for $x < 0$, we have $f(x, z_2) = 1$ for any $z_2 \in [-1, 1]$. Similarly, if $x > 0$ then, as before, the reduced space is $J^{-1}(x)/S^1 = \{x\} \times S^2$, and hence $V(x) = 2\pi \int_{-1}^{1} \chi(z_2) \mathrm{d}z_2$. If $x = 0$, then the reduced space $J^{-1}(0)/S^1$ is the southern hemisphere of the second factor and hence $V(0) = 2\pi$. Therefore, the cartographic invariant is given in Figure 7.4.

We have so far shown (E.1)-(E.4). Corollary 4.3 implies (E.5). All that is left is to show (E.6).

To conclude the proof, we modify the construction above in order to illustrate the existence of unbounded cartographic invariants with fibers of infinite length.

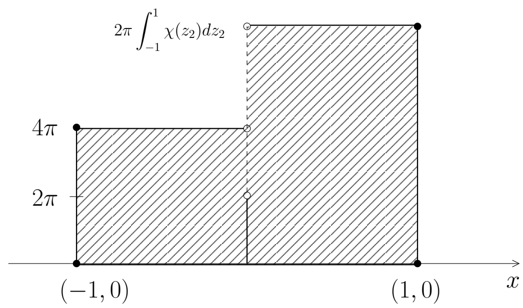

FIGURE 7.4. A representative of $\Delta(M, \widetilde{F})$.

As we shall see, most of the computations of the previous example remain valid. Let

$$\begin{aligned} N &:= p^{-1}\left(\mathbb{R}^2 \setminus \left[\{z_1 = 0,\, z_2 \geqslant 0\} \cup \{z_1 \geqslant 0,\, z_2 = 1\}\right]\right) \\ &= S^2 \times S^2 \setminus \Big[\{((x_1, y_1, z_1),\, (x_2, y_2, z_2)) \in S^2 \times S^2 \mid z_1 = 0,\, z_2 \geqslant 0\} \\ &\qquad\qquad \cup \{((x_1, y_1, z_1),\, (x_2, y_2, z_2)) \in S^2 \times S^2 \mid z_1 \geqslant 0,\, z_2 = 1\}\Big] \end{aligned}$$

As in the previous example, N is open and connected. Consider the form ω given by (7.2), but where the smooth function χ is defined only on the half-open interval $[-1, 1)$. Because N is a subset of M, the restriction of ω is a symplectic form. Similarly, $J = z_1$, $H = z_2$ defines an integrable system on N and J is the momentum map of a Hamiltonian S^1-action. The computations in the previous example show that we have the same singularities, all of them non-degenerate. If $F := (J, H) = p\restriction_N$, its image is

$$F(N) = [-1,\, 1] \times [-1,\, 1] \setminus \big(\{z_1 = 0,\, z_2 \geqslant 0\} \cup \{z_1 \geqslant 0,\, z_2 = 1\}\big) \tag{7.4}$$

(see Figure 7.5) which is not a closed set, and hence F is not a proper map. Define

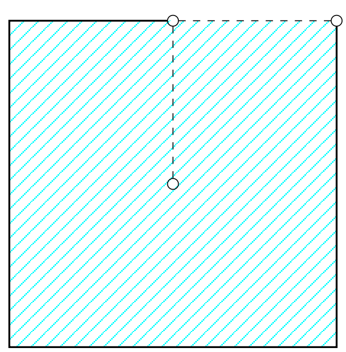

FIGURE 7.5. The image $F(N)$.

$$g(z_1,\, z_2) := \left(z_1,\, \frac{z_2 + 2}{((z_2 - 1)^2 + h(z_1))(z_1^2 + h(z_2))}\right)$$

and $\widetilde{F} := g \circ F$, where h is as in the previous example. To see that g is a local diffeomorphism, it suffices to note that the Jacobian determinant of g has the expression $\left(\Delta - (z_2+2)\frac{\partial \Delta}{\partial z_2}\right)/\Delta^2$, where $\Delta := ((z_2-1)^2 + h(z_1))(z_1^2 + h(z_2))$. Since $\Delta > 0$ and

$$\partial\Delta/\partial z_2 = 2(z_2-1)(z_1^2+h(z_2)) + ((z_2-1)^2+h(z_1))h'(z_2) < 0,$$

it follows that the Jacobian determinant of g is strictly positive. As in the previous example, one can check that $g^{-1}(K_1 \times K_2)$ is a compact subset of $\mathbb{R}^2$, where K_i, $i = 1, 2$, are closed bounded intervals in $\mathbb{R}$. The argument given in the previous example shows then that $\widetilde{F}$ is a proper map. Therefore, $(N, \widetilde{F})$ is a proper semitoric system. The image of $\widetilde{F}$ is given in Figure 7.6.

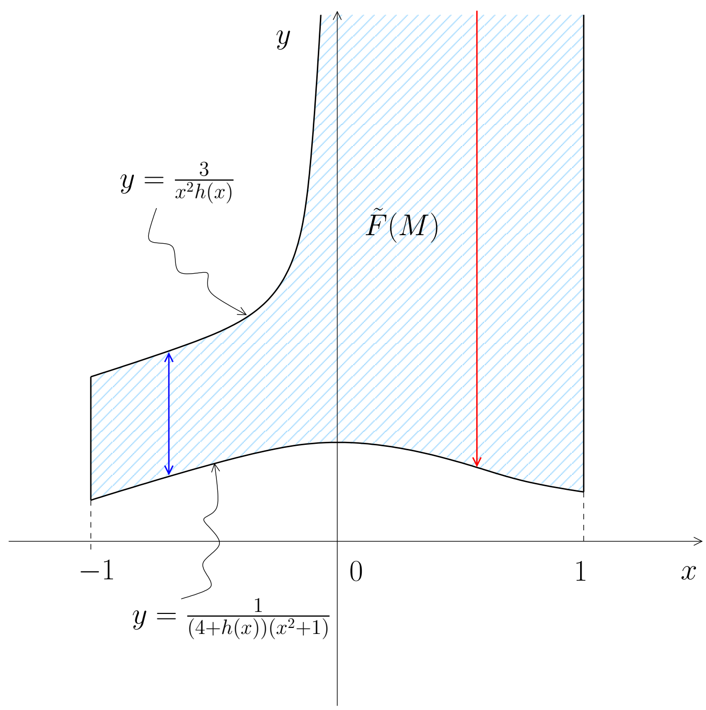

FIGURE 7.6. The image $\widetilde{F}(N)$ with the choice $h(-1) = 1$.

Finally, to determine the possible affine invariants associated to this system, we need to compute $V(x)$, the volume of the reduced manifold $J^{-1}(x)/S^1$. As before, we compute

$$V(x) = \begin{cases} 4\pi, & \text{if} \quad x < 0 \\ 2\pi, & \text{if} \quad x = 0 \\ 2\pi(1+\alpha), & \text{if} \quad x > 0 \end{cases}$$

where $\alpha := \int_0^1 \chi(z_2)\mathrm{d}z_2 > 0$ which, now, can be $+\infty$ for a suitable choice of χ, in which case we have $V(x) = +\infty$. The possible cartographic invariants are given in Figure 7.7. This proves (E.6).

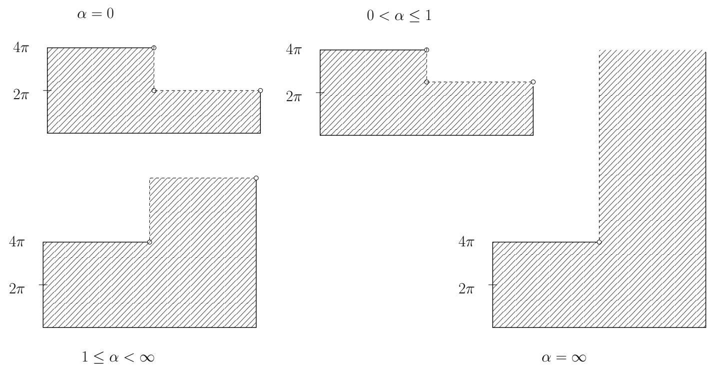

FIGURE 7.7. A representative of the cartographic invariants depending on α.

Remark 7.1 When M is compact (so J, H, F are all proper), the cartographic invariant of F is a polygon which is related to the classification of Hamiltonian S^1-spaces by Karshon [Ka99], as explained in [HSS13]. $\oslash$

8. Questions

It would be interesting to shed some light on the following questions.

Question 8.1 Prove Theorem A (in particular, defining the maps involved) for integrable systems on origami manifolds (see [DGP11]) and on orbifolds (see [LT97]), where, as far as we know, integrable systems have not been studied. $\oslash$

Question 8.2 Concerning the statement of Theorem C, if the points of discontinuity of the functions f and g defining the regions $\mathcal{C}_j^{\vec{\epsilon}}$ are discrete, then one could always refine the intervals I_j in order to ensure that both functions f and g become continuous on their intervals of definition. Find examples where these functions have a non-discrete set of discontinuity points. This question is not directly related to the presence of focus-focus points since, as was shown in [Vu07] in the $(1, 0)$-semitoric case, a focus-focus point affects the derivative of the volume $V(x)$, but not its continuity. $\oslash$

Question 8.3 Motivated by [PV11], the following inverse type question is natural. Let $C := \cup_{j\in\mathbb{N}} C_j$ be a connected set, where $C_j \subset \mathbb{R}^2$ is a region of type CC, CO, OC, or OO. What are the conditions on the C_j's for the existence of a proper semitoric system $F : M \to \mathbb{R}^2$ such that $f_{\vec{\epsilon}}(F(M)) = C$, where $f_{\vec{\epsilon}}$ is a cartographic map for F?

The classifications of Delzant [De88] and [PV11] give partial answers to this question. Note that here we are not claiming uniqueness; in fact, it follows from [PV11] that there are many non-isomorphic proper semitoric systems which realize the same C. $\oslash$

9. Appendix

9.1. **Bifurcation set.** Let M and N be smooth (*i.e.* C^∞) manifolds. A smooth map $f : M \to N$ is said to be *locally trivial* at $n_0 \in f(M)$, if there is an open neighborhood $U \subset N$ of n_0 such that $f^{-1}(n)$ is a smooth submanifold of M for each $n \in U$ and there is a smooth map $h : f^{-1}(U) \to f^{-1}(n_0)$ such that $f \times h : f^{-1}(U) \to U \times f^{-1}(n_0)$ is a diffeomorphism. The *bifurcation set* Σ_f consists of all the points of N where f is not locally trivial. This definition, as introduced by [Sm70], is standard in dynamical systems and geometric mechanics; it does *not* force a change in the topological type of the fibers $f^{-1}(n)$ if n is in a neighborhood of the bifurcation point $n_0 \in f(M) \subseteq N$ and passes through n_0. The classical example is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, where 0 is a bifurcation point but $f^{-1}(n)$ is always a point. In classical bifurcation theory (e.g., [Ha77]), one would request, in the definition, a change of topological type at a bifurcation point. Thus, in this example, from the point of view of classical bifurcation theory, 0 would not be a bifurcation point.

It is known that the set of critical values of f is included in the bifurcation set and that, if f is proper, this inclusion is an equality (see [AM78, Proposition 4.5.1] and the comments following it). In general, the set of critical values of f is strictly contained in the bifurcation set of f. The standard examples when this occurs are the bifurcation sets of the energy-momentum maps for the planar 2-body and 3-body problems (see, [AM78, Corollaries 9.8.2 and 10.4.22], [Sm70]); all points on the coordinate axes of $\mathbb{R}^2$ are bifurcation points, but are not critical values of the energy-momentum map. Another example is given in Step 7 in the proof of Theorem D, since 0 is a bifurcation point for J which is not a critical point.

9.2. **Linearization of singularities.** Let (M, ω) be a connected symplectic 4-manifold, $F = (f_1, f_2)$ an integrable system on (M, ω), and $m \in M$ a critical point of F, i.e., the rank of the derivative (tangent map) $\mathrm{T}_m F : T_m M \to \mathbb{R}^2$ of F is either 0 or 1.

- If $\mathrm{T}_m F = 0$, m is said to be *non-degenerate* if the Hessians $\operatorname{Hess} f_1(m)$ and $\operatorname{Hess} f_2(m)$ span a Cartan subalgebra of the Lie algebra of quadratic forms on the symplectic vector space $(\mathrm{T}_m M, \omega_m)$ equipped with the linearized Poisson bracket.
- If $\operatorname{rank}(\mathrm{T}_m F) = 1$, we may assume that $\mathrm{d}f_1(m) \neq 0$. Let $\iota \colon S \to M$ be an embedded local 2-dimensional symplectic submanifold through m such that $\mathrm{T}_m S \subset \ker(\mathrm{d}f_1(m))$ and $\mathrm{T}_m S$ is transversal to the Hamiltonian vector field $\mathcal{X}_{f_1}$ defined by the function f_1. This is possible by the classical Hamiltonian Flow Box Theorem ([AM78, Theorem 5.2.19]), also known as the Darboux-Carathéodory Theorem ([Vu06, Théorème 3.3.2], [PV11a, Theorem 4.1]). The point m is called *transversally non-degenerate* if $\operatorname{Hess}(\iota^* f_2)(m)$ is a non-degenerate symmetric bilinear form on $\mathrm{T}_m S$. It is easily seen that the definition does not depend on the choice of S.

For the notion of non-degeneracy of a critical point in arbitrary dimensions, see [Ve78] and [Vu06, Section 3]. In this paper, we need the following property of non-degenerate critical points ([El84, El90], [VW10]) in terms of the Williamson normal form ([Wi36]), which we state in any dimension but only use in dimension 4.

Theorem 9.1 (Eliasson)**.** *Let $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ be an integrable system and $m \in M$ a non-degenerate critical point of F. Then there are local symplectic coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ about m, in which m is represented as $(0, \ldots, 0)$, such that $\{f_i, q_j\} = 0$, for all i, j, where the $q_1, \ldots, q_n$ are defined on a neighborhood of 0 in $\mathbb{R}^{2n}$ and have one of the following expressions:*

(a) *Elliptic component: $q_j = (x_j^2 + \xi_j^2)/2$, where $1 \leqslant j \leqslant n$;*
(b) *Hyperbolic component: $q_j = x_j \xi_j$, where $1 \leqslant j \leqslant n$;*
(c) *Focus-focus components: $q_{j-1} = x_{j-1}\,\xi_j - x_j\,\xi_{j-1}$ and $q_j = x_{j-1}\,\xi_{j-1} + x_j\,\xi_j$, where $2 \leqslant j \leqslant n$;*
(d) *Non-singular component: $q_j = \xi_j$, where $1 \leqslant j \leqslant n$.*

Each q_i can belong to any of the cases (a), (b), (c), (d) *and all of them cases can be mixed in the same n-tuple. However, case* (c) *always appears as a whole 2-component block; it cannot be split.*

If m does not have hyperbolic components, then the system of equations $\{f_i, q_j\} = 0$, for all i, j, may be replaced by $(F - F(m)) \circ \varphi = g \circ (q_1, \ldots, q_n)$, where $\varphi = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)^{-1}$ and g is a diffeomorphism from a neighborhood of 0 in $\mathbb{R}^n$ onto another such neighborhood, with $g(0) = 0$.

If M is 4-dimensional and m is a non-degenerate critical point of F with no hyperbolic component, the map (q_1, q_2) given by this theorem can be described as follows. If $\mathrm{T}_m F$ has rank zero, then either

(EE) $q_1 = (x_1^2 + \xi_1^2)/2$ and $q_2 = (x_2^2 + \xi_2^2)/2$, or
(FF) $q_1 = x_1\xi_2 - x_2\xi_1$ and $q_2 = x_1\xi_1 + x_2\xi_2$.

If $\mathrm{T}_m F$ has rank one, then

(XE) $q_1 = (x_1^2 + \xi_1^2)/2$ and $q_2 = \xi_2$.

A non-degenerate critical point of type (EE) is called *elliptic-elliptic*; points of type (FF) are called *focus-focus*, and points of type (XE) are called *transversally-elliptic.*

The following result provides a method to easily check non-degeneracy in dimension 4; see, e.g., [Vu06, Lemme 3.3.6], [DuPe15, Lemma 2.5].

Lemma 9.2. *Let $F := (f_1, f_2) \colon M \to \mathbb{R}^2$ be an integrable system on a symplectic 4-manifold (M, ω). A critical point p of F of rank 0 is non-degenerate if the Hessians $\operatorname{Hess} f_1(p)$ and $\operatorname{Hess} f_2(p)$ are linearly independent and there is a linear combination $\alpha\omega(p) \operatorname{Hess} f_1(p) + \beta\omega(p) \operatorname{Hess} f_2(p)$, $\alpha, \beta \in \mathbb{R}$, which does not have multiple eigenvalues. In particular if $\omega(p) \operatorname{Hess} f_1(p)$ has no multiple eigenvalues, then p is non-degenerate.*

The proof of Lemma 9.2 is based on the fact that a commutative subalgebra of the symplectic algebra in dimension 4 is a Cartan subalgebra if and only if it is two-dimensional and it contains at least one element whose eigenvalues are distinct (see, e.g., [BF04]).

9.3. **Affine manifolds.** An *affine n-dimensional manifold* is a smooth manifold endowed with an atlas whose change of chart maps are in the affine group of $\mathbb{R}^n$, i.e., in

$$\begin{aligned}\mathrm{Aff}(n,\mathbb{R}) &:= \mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n \\ &:= \left\{ \begin{bmatrix} U & u \\ 0 & 1 \end{bmatrix} \,\middle|\, U \in \mathrm{GL}(n,\mathbb{R}),\ \ u \in \mathbb{R}^n \right\} \subset \mathrm{GL}(n+1,\mathbb{R}).\end{aligned}$$

An *integral affine n-dimensional manifold* is an affine manifold endowed with an atlas whose change of chart maps are in $\mathrm{Aff}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \ltimes \mathbb{R}^n$, i.e., $U \in \mathrm{GL}(n,\mathbb{Z})$ in the definition above.

Let M be a connected n-dimensional manifold, $m_0 \in M$, and $p : \widetilde{M} \to M$ its universal covering manifold, i.e., the set of homotopy classes of smooth paths $\lambda : [0,1] \to M$ starting at $\lambda(0) = m_0$ and keeping the endpoints fixed; $p([\lambda]) := \lambda(1)$. Recall that $\widetilde{M}$ is a smooth simply connected n-dimensional manifold and that p is a covering map. The group of deck transformations of p, i.e., all diffeomorphisms $\chi : \widetilde{M} \to \widetilde{M}$ such that $p \circ \chi = p$, is isomorphic to the first fundamental group $\pi_1(M)$ (based at m_0).

If M is, in addition, an affine manifold (see, e.g., [GH84, Section 2.3] for more information), then p induces an affine manifold structure on $\widetilde{M}$ by requiring p to be an affine map, i.e., its local representative is affine in any pair of local charts.

A *developing map* for M is an affine immersion $\zeta : \widetilde{M} \to \mathbb{R}^n$. It is well-known (see, e.g., [GH84, page 641]) that each connected affine manifold M has at least one developing map and that if $\zeta' : \widetilde{M} \to \mathbb{R}^n$ is another developing map, then there is a unique $A \in \mathrm{Aff}(n,\mathbb{R})$ such that $\zeta' = A\zeta$. In addition, for any developing map $\zeta : \widetilde{M} \to \mathbb{R}^n$, there is a unique *monodromy homomorphism* $\mu : \pi_1(M) \to \mathrm{Aff}(n,\mathbb{R})$ for which ζ is equivariant, i.e., $\zeta([\lambda \star \gamma]) = \mu([\lambda])\zeta([\gamma])$, for any $[\lambda] \in \pi_1(M)$ and $[\gamma] \in \widetilde{M}$, where $\star$ denotes composition of paths by concatenation.

9.4. **Almost toric systems.** Our proofs rely on some recent results on almost toric manifolds which we state here for the reader's convenience. If (M,ω) is a connected symplectic four-manifold, an integrable system $F\colon M \to \mathbb{R}^2$ is called *almost-toric* if all the singularities are non-degenerate without hyperbolic components, see Section 9.2. Almost toric systems were first introduced by Symington [Sy01] and then, independently, in [Vu07].

Theorem 9.3 (Theorem 4.7 in [PRV15])**.** *Suppose that (M,ω) is a connected symplectic four-manifold. Let $F = (J, H)\colon M \to \mathbb{R}^2$ be an almost-toric integrable system such that F is a proper map. Suppose that J has connected fibers, or that H has connected fibers. Then the fibers of F are connected.*

Theorem 9.4 (Theorem 5.2 in [PRV15])**.** *Let M be a connected smooth four-manifold. Let $F = (J, H)\colon M \to \mathbb{R}^2$ be a smooth map. Equip $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$*

with the standard topology. Suppose that the component J is a non-constant Morse-Bott function with connected fibers. Let $H^+, H^-\colon J(M) \to \overline{\mathbb{R}}$ be the functions defined by $H^+(x) := \sup_{J^{-1}(x)} H$ and $H^-(x) := \inf_{J^{-1}(x)} H$. The functions H^+, $-H^-$ are lower semicontinuous. Moreover, if $F(M)$ is closed in $\mathbb{R}^2$ then H^+, $-H^-$ are upper semicontinuous (and hence continuous), and $F(M)$ may be described as

$$F(M) = \mathrm{epi}(H^-) \cap \mathrm{hyp}(H^+), \tag{9.1}$$

where $\mathrm{epi}(H^-)$ *(resp.* $\mathrm{hyp}(H^+)$*) is the epigraph of H^- (resp. the hypograph of H^+). In particular, $F(M)$ is contractible.*

Theorem 9.5 (Theorem 3.6 in [PRV15])**.** *Suppose that (M, ω) is a connected symplectic four-manifold. Assume that $F\colon M \to \mathbb{R}^2$ is an almost-toric integrable system with $B := F(M) \subset \mathbb{R}^2$ closed. Then the set of focus-focus critical values is countable, i.e., we may write it as $\{c_i \mid i \in I\}$, where $I \subset \mathbb{N}$. Consider the following statements:*

(i) *the fibers of F are connected;*
(ii) *the set B_r of regular values of F is connected;*
(iii) *for any value c of F, for any sufficiently small disc D centered at c, $B_r \cap D$ is connected;*
(iv) *the set of regular values is $B_r = \mathring{B} \setminus \{c_i \mid i \in I\}$. Moreover, the topological boundary ∂B of B consists precisely of the values $F(m)$, where m is a critical point of elliptic-elliptic or transversally elliptic type.*

Then statement (i) *implies statement* (ii)*, statement* (iii) *implies statement* (iv)*, and statement* (iv) *implies statement* (ii)*. If, in addition, F is proper, then statement* (i) *implies statement* (iv)*.*

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