MAGNETIC WELLS IN DIMENSION THREE

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Abstract. This paper deals with semiclassical asymptotics of the three-dimensional magnetic Laplacian in presence of magnetic confinement. Using generic assumptions on the geometry of the confinement, we exhibit three semiclassical scales and their corresponding effective quantum Hamiltonians, by means of three microlocal normal forms *à la Birkhoff*. As a consequence, when the magnetic field admits a unique and non degenerate minimum, we are able to reduce the spectral analysis of the low-lying eigenvalues to a one-dimensional $\hbar$-pseudo-differential operator whose Weyl's symbol admits an asymptotic expansion in powers of $\hbar^{\frac{1}{2}}$.

1. Introduction

1.1. **Motivation and context.** The analysis of the magnetic Laplacian $(-i\hbar\nabla - \mathbf{A})^2$ in the semiclassical limit $\hbar \to 0$ has been the object of many developments in the last twenty years. The existence of discrete spectrum for this operator, together with the analysis of the eigenvalues, is related to the notion of "magnetic bottle", or quantum confinement by a pure magnetic field, and has important applications in physics. Moreover, motivated by investigations of the third critical field in Ginzburg-Landau theory for superconductivity, there has been a great attention focused on estimates of the lowest eigenvalue. In the last decade, it appears that the spectral analysis of the magnetic Laplacian has acquired a life on its own. For a story and discussions about the subject, the reader is referred to the recent reviews [11, 14, 24].

In contrast to the wealth of studies exploring the semiclassical approximations of the Schrödinger operator $-\hbar^2\Delta + V$, the *classical* picture associated with the Hamiltonian $\|p - \mathbf{A}(q)\|^2$ has almost never been investigated to describe the *semiclassical* bound states (*i.e.* the eigenfunctions of low energy) of the magnetic Laplacian. The paper by Raymond and Vũ Ngọc [25] is to our knowledge the first rigorous work in this direction. In that paper, which deals with the two-dimensional case, the notion of magnetic drift, well known to physicists, is cast in a symplectic framework, and using a semiclassical Birkhoff normal form (see for instance [27, 5, 28]) it becomes possible to describe all the eigenvalues of order $\mathcal{O}(\hbar)$. Independently, the asymptotic expansion of a smaller set of eigenvalues was established in [12, 15] through different methods which act directly on the quantum side: explicit

unitary transforms and a Grushin like reduction are used to reduce the two-dimensional operator to an effective one-dimensional operator.

The three-dimensional case happens to be much harder. The only known results in this case that provide a full asymptotic expansion of a given eigenvalue concern toy models where the confinement is obtained by a boundary carrying a Neumann condition on an half space in [23] or on a wedge in [22]. In the case of smooth confinement without boundary, a construction of quasi-modes by Helffer and Kordyukov in [13] suggests what the expansions of the low lying eigenvalues could be. But, as was expected by Colin de Verdière in his list of open questions in [7], extending the symplectic and microlocal techniques to the three-dimensional case contains an intrinsic difficulty in the fact that the symplectic form cannot be nondegenerate on the characteristic hypersurface. The goal of our paper is to answer this question by fully carrying out this strategy. After averaging the cyclotron motion, the effect of the degeneracy of the symplectic form can be observed on the fact that the reduced operator is only partially elliptic. Hence, the key ingredient will be a separation of scales via the introduction of a new semiclassical parameter for only one part of the variables. These semiclassical scales are reminiscent of the three scales that have been exhibited in the classical picture in the large field limit, see [2, 6]. They are also related to the Born-Oppenheimer type of approximation in quantum mechanics (see for instance [4, 20]). In fact, in a partially semiclassical context and under generic assumptions, a full asymptotic expansion of the first magnetic eigenvalues (and the corresponding WKB expansions) has been recently established in any dimension in the paper by Bonnaillie-Noël–Hérau–Raymond [3].

1.2. **Magnetic geometry.** Let us now describe the geometry of the problem. The configuration space is

$$\mathbb{R}^3 = \{q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3, \quad q_j \in \mathbb{R}, \quad j = 1, 2, 3\},$$

where $(\mathbf{e}_j)_{j=1,2,3}$ is the canonical basis of $\mathbb{R}^3$. The phase space is

$$\mathbb{R}^6 = \{(q, p) \in \mathbb{R}^3 \times \mathbb{R}^3\}$$

and we endow it with the canonical 2-form

$$\omega_0 = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 + \mathrm{d}p_3 \wedge \mathrm{d}q_3. \tag{1.1}$$

We will use the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ and $\| \cdot \|$ the associated norm. In particular, we can rewrite ω_0 as

$$\omega_0((u_1, u_2), (v_1, v_2)) = \langle v_1, u_2 \rangle - \langle v_2, u_1 \rangle, \quad \forall u_1, u_2, v_1, v_2 \in \mathbb{R}^3.$$

The main object of this paper is the magnetic Hamiltonian, defined for all $(q, p) \in \mathbb{R}^6$ by

$$H(q, p) = \|p - \mathbf{A}(q)\|^2, \tag{1.2}$$

where $\mathbf{A} \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

Let us now introduce the magnetic field. The vector field $\mathbf{A} = (A_1, A_2, A_3)$ is associated (via the Euclidean structure) with the following 1-form

$$\alpha = A_1 \mathrm{d}q_1 + A_2 \mathrm{d}q_2 + A_3 \mathrm{d}q_3$$

and its exterior derivative is a 2-form, called magnetic 2-form and expressed as

$$\mathrm{d}\alpha = (\partial_1 A_2 - \partial_2 A_1)\mathrm{d}q_1 \wedge \mathrm{d}q_2 + (\partial_1 A_3 - \partial_3 A_1)\mathrm{d}q_1 \wedge \mathrm{d}q_3 + (\partial_2 A_3 - \partial_3 A_2)\mathrm{d}q_2 \wedge \mathrm{d}q_3 \,.$$

The form $\mathrm{d}\alpha$ may be identified with a vector field. If we let:

$$\mathbf{B} = \nabla \times \mathbf{A} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1) = (B_1, B_2, B_3),$$

then, we can write

$$\mathrm{d}\alpha = B_3 \mathrm{d}q_1 \wedge \mathrm{d}q_2 - B_2 \mathrm{d}q_1 \wedge \mathrm{d}q_3 + B_1 \mathrm{d}q_2 \wedge \mathrm{d}q_3. \tag{1.3}$$

The vector field $\mathbf{B}$ is called the magnetic field. Let us notice that we can express the 2-form $\mathrm{d}\alpha$ thanks to the magnetic matrix

$$M_{\mathbf{B}} = \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}.$$

Indeed we have

$$\mathrm{d}\alpha(U, V) = \langle U, M_{\mathbf{B}} V \rangle = \langle U, V \times \mathbf{B} \rangle = [U, V, \mathbf{B}], \quad \forall (U, V) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{1.4}$$

where $[\cdot, \cdot, \cdot]$ is the canonical mixed product on $\mathbb{R}^3$. We note that $\mathbf{B}$ belongs to the kernels of $M_{\mathbf{B}}$ and $\mathrm{d}\alpha$.

An important role will be played by the characteristic hypersurface

$$\Sigma = H^{-1}(0),$$

which is the submanifold defined by the parametrization:

$$\mathbb{R}^3 \ni q \mapsto j(q) := (q, \mathbf{A}(q)) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

We may notice the relation between Σ, the symplectic structure and the magnetic field in the following relation

$$j^* \omega_0 = \mathrm{d}\alpha \,, \tag{1.5}$$

where $\mathrm{d}\alpha$ is defined in (1.3).

1.3. **Confinement assumptions and discrete spectrum.** This paper is devoted to the semiclassical analysis of the discrete spectrum of the magnetic Laplacian $\mathcal{L}_{\hbar, \mathbf{A}} := (-i\hbar\nabla_q - \mathbf{A}(q))^2$, which is the semiclassical Weyl quantization of H (see (2.1)). This means that we will consider that $\hbar$ belongs to $(0, \hbar_0)$ with $\hbar_0$ small enough.

If $\mathcal{L}$ is a self-adjoint operator, we denote its spectrum by $\mathfrak{s}(\mathcal{L})$. The discrete spectrum of $\mathcal{L}$ consists of the isolated eigenvalues with finite multiplicity. The essential spectrum is by definition the complement in $\mathfrak{s}(\mathcal{L})$ of the discrete spectrum and is denoted by $\mathfrak{s}_{\mathsf{ess}}(\mathcal{L})$. It is empty when $\mathcal{L}$ has compact resolvent.

It is known (see for example [1]) that $\mathcal{L}_{\hbar,\mathbf{A}}$ is essentially self-adjoint and we always consider with the same notation its self-adjoint extension.

Let us recall the assumptions under which discrete spectrum actually exists. In two dimensions, with a non-vanishing magnetic field, a standard estimate (see [1, 8]) gives

$$\hbar \int_{\mathbb{R}^2} |B(q)||u(q)|^2 \mathrm{d}q \leqslant \langle \mathcal{L}_{\hbar,\mathbf{A}} u \,|\, u \rangle \,, \ \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^2) \,. \tag{1.6}$$

This implies that, as $B(q) \to +\infty$, the magnetic Laplacian has compact resolvent. Except in special cases when some components of the magnetic field have constant sign, this is doesn't hold anymore in higher dimension (see [10]). One can give examples where $|B(q)| \to +\infty$ and the operator doesn't have a compact resolvent. We should impose a control of the oscillations of $\mathbf{B}$ at infinity. Under this condition, we get an estimate similar to (1.6) at the price of a small loss. When there exists a constant $C > 0$ such that

$$\|\nabla \mathbf{B}(q)\| \leqslant C \,(1 + b(q)) \,, \ \forall q \in \mathbb{R}^3 \,, \tag{1.7}$$

and $b(q) := \|\mathbf{B}(q)\|$ tends to $+\infty$, one can show again that the magnetic Laplacian has compact resolvent [16].

In the semiclassical context, we would like to consider the case of $\mathbb{R}^3$ and, in addition to (1.7), a confining assumption which allows the presence of essential spectrum above a certain threshold. More precisely, we introduce

Assumption 1.1. We assume that (1.7) holds and

$$b(q) \geqslant b_0 := \inf_{q \in \mathbb{R}^3} b(q) > 0 \,, \tag{1.8}$$

Under Assumption 1.1, it is proven in [16, Theorem 3.1] that there exist $h_0 > 0$ and $C_0 > 0$ such that, for all $\hbar \in (0, h_0)$,

$$\hbar(1 - C_0 \hbar^{\frac{1}{4}}) \int_{\mathbb{R}^3} b(q)|u(q)|^2 \mathrm{d}q \leqslant \langle \mathcal{L}_{\hbar,\mathbf{A}} u \,|\, u \rangle \,, \ \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^3) \,. \tag{1.9}$$

In this case, if we do not assume that $b(q) \to +\infty$, the spectrum is not necessarily discrete, but using this inequality and Persson's theorem (see [21]), we obtain that the bottom of the essential spectrum is asymptotically above $\hbar b_1$, where

$$b_1 := \liminf_{|q| \to +\infty} b(q).$$

More precisely, under Assumption 1.1, there exist $h_0 > 0$ and $C_0 > 0$ such that, for all $\hbar \in (0, h_0)$,

$$\mathfrak{s}_{\mathsf{ess}}(\mathcal{L}_{\hbar,\mathbf{A}}) \subset [\hbar b_1(1 - C_0 \hbar^{\frac{1}{4}}), +\infty) \,. \tag{1.10}$$

Assumption 1.2. We assume that

$$0 < b_0 < b_1 \,. \tag{1.11}$$

Moreover we will assume that there exists a point $q_0 \in \mathbb{R}^3$ and $\varepsilon > 0$, $\tilde{\beta}_0 \in (b_0, b_1)$ such that

$$\{b(q) \leqslant \tilde{\beta}_0\} \subset D(q_0, \varepsilon), \tag{1.12}$$

where $D(q_0, \varepsilon)$ is the Euclidean ball centered at q_0 and of radius ε. For the rest of the article we let $\beta_0 \in (b_0, \tilde{\beta}_0)$. Without loss of generality, we can assume that $q_0 = 0$ and that $\mathbf{A}(0) = 0$ (which can be obtained with a change of gauge).

Note that Assumption 1.2 implies that the minimal value of b is attained inside $D(q_0, \varepsilon)$.

All along this paper, we will strengthen the assumptions on the nature of the point q_0. At some stage of our investigation, q_0 will be the unique minimum of b. Note in particular that (1.12) is satisfied as soon as b admits a unique and non degenerate minimum.

1.4. **Informal description of the results.** Let us now informally walk through the main results of this paper. We will assume (as precisely formulated in (1.11)-(1.12)) that the magnetic field does not vanish and is confining.

Of course, for eigenvalues of order $\mathcal{O}(\hbar)$, the corresponding eigenfunctions are microlocalized in the semi-classical sense near the characteristic manifold Σ (see for instance [26, 31]). Moreover the confinement assumption implies that the eigenfunctions of $\mathcal{L}_{\hbar,\mathbf{A}}$ associated with eigenvalues less that $\beta_0\hbar$ enjoy localization estimates *à la* Agmon. Therefore we will be reduced to investigate the magnetic geometry locally in space near a point $q_0 = 0 \in \mathbb{R}^3$ belonging to the confinement region and which, for notational simplicity, we may assume to be the origin.

Then, in a neighborhood of $(0, \mathbf{A}(0)) \in \Sigma$, there exist symplectic coordinates $(x_1, \xi_1, x_2, \xi_2, x_3, \xi_3)$ such that $\Sigma = \{x_1 = \xi_1 = \xi_3 = 0\}$ and $(0, \mathbf{A}(0))$ has coordinates $0 \in \mathbb{R}^6$. Hence Σ is parametrized by (x_2, ξ_2, x_3).

1.4.1. *First Birkhoff form.* In these coordinates suited for the magnetic geometry, it is possible to perform a semiclassical Birkhoff normal form and microlocally unitarily conjugate $\mathcal{L}_{\hbar,\mathbf{A}}$ to a first normal form $\mathcal{N}_\hbar = \mathrm{Op}^w_\hbar(N_\hbar)$ with an operator valued symbol $N_\hbar$ depending on (x_2, ξ_2, x_3, ξ_3) in the form

$$N_\hbar = \xi_3^2 + b(x_2, \xi_2, x_3)\mathcal{I}_\hbar + f^\star(\hbar, \mathcal{I}_\hbar, x_2, \xi_2, x_3, \xi_3) + \mathcal{O}(|\mathcal{I}_\hbar|^\infty, |\xi_3|^\infty).$$

where $\mathcal{I}_h = \hbar^2 D_{x_1}^2 + x_1^2$ is the first encountered harmonic oscillator and where $(\hbar, I, x_2, \xi_2, x_3, \xi_3) \mapsto f^\star(\hbar, I, x_2, \xi_2, x_3, \xi_3)$ satisfies, for $I \in (0, I_0)$,

$$|f^\star(\hbar, I, x_2, \xi_2, x_3, \xi_3)| \leqslant C\left(|I|^{\frac{3}{2}} + |\xi_3|^3 + \hbar^{\frac{3}{2}}\right).$$

Since we wish to describe the spectrum in a spectral window containing at least the lowest eigenvalues, we are led to replace $\mathcal{I}_\hbar$ by its lowest eigenvalue $\hbar$

and thus, we are reduced to the two-dimensional pseudo-differential operator $\mathcal{N}_\hbar^{[1]} = \mathrm{Op}_\hbar^w\left(N_\hbar^{[1]}\right)$ where

$$N_\hbar^{[1]} = \xi_3^2 + b(x_2,\xi_2,x_3)\hbar + f^\star(\hbar,\hbar,x_2,\xi_2,x_3,\xi_3) + \mathcal{O}(\hbar^\infty,|\xi_3|^\infty).$$

1.4.2. *Second Birkhoff form.* If we want to continue the normalization, we shall assume a new non-degeneracy condition (the first one was the positivity of b).

Now we assume that, for any (x_2,ξ_2) in a neighborhood of $(0,0)$, the function $x_3 \mapsto b(x_2,\xi_2,x_3)$ admits a unique and non-degenerate minimum denoted by $s(x_2,\xi_2)$. Then, by using a new symplectic transformation in order to center the analysis at the partial minimum $s(x_2,\xi_2)$, we get a new operator $\underline{\mathcal{N}}_\hbar^{[1]}$ whose Weyl symbol is in the form

$$\underline{N}_\hbar^{[1]} = \nu^2(x_2,\xi_2)(\xi_3^2+\hbar x_3^2) + \hbar b(x_2,\xi_2,s(x_2,\xi_2)) + \mathsf{remainders},$$

with

$$\nu(x_2,\xi_2) = (\tfrac{1}{2}\partial_3^2 b(x_2,\xi_2,s(x_2,\xi_2)))^{1/4} \tag{1.13}$$

and where the remainders have been properly normalized to be at least formal perturbations of the second harmonic oscillator $\xi_3^2+\hbar x_3^2$. Since the frequency of this oscillator is $\hbar^{-\frac{1}{2}}$ in the classical picture, we are naturally led to introduce the new semiclassical parameter

$$h = \hbar^{\frac{1}{2}}$$

and the new impulsion

$$\xi = \hbar^{\frac{1}{2}}\tilde{\xi}$$

so that

$$\mathrm{Op}_\hbar^w\left(\xi_3^2+\hbar x_3^2\right) = h^2\,\mathrm{Op}_\hbar^w\left(\tilde{\xi}_3^2+x_3^2\right).$$

We therefore get the h-symbol of $\underline{\mathcal{N}}_\hbar^{[1]}$:

$$\underline{\mathsf{N}}_h^{[1]} = h^2\nu^2(x_2,h\tilde{\xi}_2)(\tilde{\xi}_3^2+x_3^2) + h^2 b(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2)) + \mathsf{remainders}.$$

We can again perform a Birkhoff analysis in the space of formal series given by $\mathscr{E} = \mathscr{F}[\![x_3,\tilde{\xi}_3,h]\!]$ where $\mathscr{F}$ is a space of symbols in the form $c(h,x_2,h\tilde{\xi}_2)$. We get the new operator $\mathfrak{M}_h = \mathrm{Op}_h^w(\mathsf{M}_h)$, with

$$\mathsf{M}_h = h^2 b(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2)) + h^2\mathcal{J}_h\,\mathrm{Op}_h^w\,\nu^2(x_2,h\tilde{\xi}_2) + h^2 g^\star(h,\mathcal{J}_h,x_2,h\tilde{\xi}_2) + \mathsf{remainders},$$

where $\mathcal{J}_h = \mathrm{Op}_h^w\left(\tilde{\xi}_3^2+x_3^2\right)$ and $g^\star(h,J,x_2,\xi_2)$ is of order three with respect to $(J^{\frac{1}{2}},h^{\frac{1}{2}})$. Motivated again by the perspective of describing the low lying

eigenvalues, we replace $\mathcal{J}_h$ by h and rewrite the symbol with the old semi-classical parameter $\hbar$ to get the operator $\mathcal{M}_\hbar^{[1]} = \mathrm{Op}_\hbar^w\left(\mathsf{M}_h^{[1]}\right) = \mathrm{Op}_\hbar^w\left(M_\hbar^{[1]}\right)$, with

(1.14)
$$M_\hbar^{[1]} = \hbar b(x_2,\xi_2,s(x_2,\xi_2)) + \hbar^{\frac{3}{2}}\nu^2(x_2,\xi_2) + \hbar g^\star(\hbar^{\frac12},\hbar^{\frac12},x_2,\xi_2) + \mathsf{remainders}.$$

1.4.3. *Third Birkhoff form.* The last generic assumption is the uniqueness and non-degeneracy of the minimum of the new "principal" symbol

$$(x_2,\xi_2) \mapsto b(x_2,\xi_2,s(x_2,\xi_2))$$

that implies that b admits a unique and non-degenerate minimum at $(0,0,0)$. Up to an $\hbar^{\frac12}$-dependent translation in the phase space and a rotation, we are essentially reduced to a standard Birkhoff normal form with respect to the third harmonic oscillator $\mathcal{K}_\hbar = \hbar^2 D_{x_2}^2 + x_2^2$.

Note that all our normal forms may be used to describe the classical dynamics of a charged particle in a confining magnetic field (see Figure 1).

1.4.4. *Microlocalization.* Of course, at each step, we will have to provide accurate microlocal estimates of the eigenfunctions of the different operators to get a good control of the different remainders. In a first approximation, we will get localizations at the following scales $x_1,\xi_1,\xi_3 \sim \hbar^\delta$ ($\delta > 0$ is small enough) and $x_2,\xi_2,x_3 \sim 1$. In a second approximation, we will get $x_3,\tilde{\xi}_3 \sim \hbar^\delta$. In the final step, we will refine the localization by $x_2,\xi_2 \sim \hbar^\delta$.

1.5. **A semiclassical eigenvalue estimate.** Let us already state one of the consequences of our investigation. It will follow from the third normal form that we have a complete description of the spectrum below the threshold $b_0\hbar + 3\nu^2(0,0)\hbar^{\frac32}$. This description is reminiscent of the results *à la* Bohr-Sommerfeld of [17] and [18, Appendix B] (see also [15, Remark 1.4]) obtained in the case of one dimensional semiclassical operators.

Theorem 1.3. *Assume that b admits a unique and non degenerate minimum at q_0. Denote*

$$\sigma = \frac{\mathsf{Hess}_{q_0} b\,(\mathbf{B}(q_0),\mathbf{B}(q_0))}{2b_0^2}, \quad \theta = \sqrt{\frac{\det \mathsf{Hess}_{q_0} b}{\mathsf{Hess}_{q_0} b\,(\mathbf{B}(q_0),\mathbf{B}(q_0))}}. \tag{1.15}$$

There exists a function $k^\star \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with arbitrarily small compact support, and $k^\star(\hbar^{\frac12},Z) = \mathcal{O}((\hbar+|Z|)^{\frac32})$ when $(\hbar,Z)\to(0,0)$, such that the following holds.

For all $c \in (0,3)$, the spectrum of $\mathcal{L}_{\hbar,\mathbf{A}}$ below $b_0\hbar + c\sigma^{\frac12}\hbar^{\frac32}$ coincides modulo $\mathcal{O}(\hbar^\infty)$ with the spectrum of the operator $\mathcal{F}_\hbar$ acting on $L^2(\mathbb{R}_x)$ given by

$$\mathcal{F}_\hbar = b_0\hbar + \sigma^{\frac12}\hbar^{\frac32} - \frac{\zeta}{2\theta}\hbar^2 + \hbar\left(\frac{\theta}{2}\mathcal{K}_\hbar + k^\star(\hbar^{\frac12},\mathcal{K}_\hbar)\right), \quad \mathcal{K}_\hbar = \hbar^2 D_x^2 + x^2,$$

with some constant ζ.

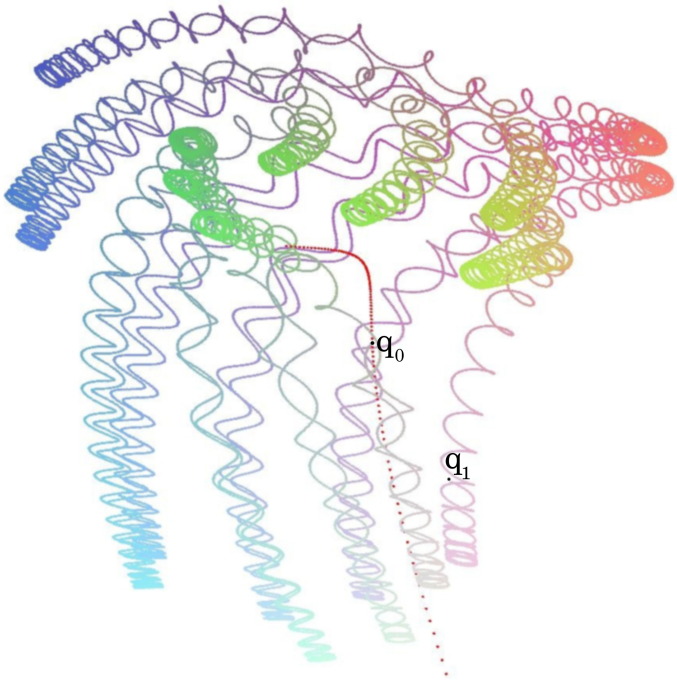


FIGURE 1. The dashed line represents the integral curve of the confining magnetic field $\mathbf{B} = \mathrm{curl}\mathbf{A}$ through $q_0 = (0.5, 0.6, 0.7)$ for $\mathbf{B}(x,y,z) = \left(\frac{y}{2}, \frac{z}{2}, \sqrt{1+x^2}\right)$ and the full line represents the projection in the q-space of the Hamiltonian trajectory with initial condition (q_0, p_0) (with $p_0 = (-0.6, 0.01, 0.2)$) ending at (q_1, p_1). The motion is easier to follow on a video: see
http://blogperso.univ-rennes1.fr/san.vu-ngoc/index.php/post/2015/03/06/Magnetic-flow-in-3D

Remark 1.4. The constant ζ in Theorem 1.3 is given by the formula

$$\zeta = \|\nabla\nu^2(0,0)\|^2,$$

where the function ν is given in (1.13). Observe also that $\sigma = \nu^4(0,0)$.

Corollary 1.5. *Under the hypothesis of Theorem 1.3, let $(\lambda_m(\hbar))_{m\geqslant 1}$ be the non decreasing sequence of the eigenvalues of $\mathcal{L}_{\hbar,\mathbf{A}}$. For any $c \in (0,3)$, let*

$$\mathsf{N}_{\hbar,c} := \{m \in \mathbb{N}^*; \quad \lambda_m(\hbar) \leqslant \hbar b_0 + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}\}.$$

Then the cardinal of $\mathsf{N}_{\hbar,c}$ is of order $\hbar^{-\frac{1}{2}}$, and there exist $\upsilon_1, \upsilon_2 \in \mathbb{R}$ and $\hbar_0 > 0$ such that

$$\lambda_m(\hbar) = \hbar b_0 + \sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}} + \left[\theta(m-\tfrac{1}{2}) - \frac{\zeta}{2\theta}\right]\hbar^2 + \upsilon_1(m-\tfrac{1}{2})\hbar^{\frac{5}{2}} + \upsilon_2(m-\tfrac{1}{2})^2\hbar^3 + \mathcal{O}(\hbar^{\frac{5}{2}}),$$

uniformly for $\hbar \in (0, \hbar_0)$ *and* $m \in \mathsf{N}_{\hbar,c}$.

In particular, the splitting between two consecutive eigenvalues satisfies

$$\lambda_{m+1}(\hbar) - \lambda_m(\hbar) = \theta\hbar^2 + \mathcal{O}(\hbar^{\frac{5}{2}}).$$

Proof. If the support of $k^\star$ is small enough, the hypothesis $k^\star(\hbar^{\frac{1}{2}}, Z) = \mathcal{O}((\hbar + |Z|)^{\frac{3}{2}})$ implies that, when $\hbar$ is small enough,

$$(1+\eta)\mathcal{K}_\hbar \geqslant \mathcal{K}_\hbar + \frac{2}{\theta}k^\star(\hbar^{\frac{1}{2}}, \mathcal{K}_\hbar) \geqslant (1-\eta)\mathcal{K}_\hbar,$$

for some small $\eta > 0$. Therefore, since the eigenvalues of $\mathcal{K}_\hbar$ are $(2m-1)\hbar$, $m \in \mathbb{N}^*$, the variational principle implies that the number of eigenvalues of $\mathcal{K}_\hbar + \frac{2}{\theta}k^\star(\hbar^{\frac{1}{2}}, \mathcal{K}_\hbar)$ below a threshold $C_\hbar$ belongs to $[\frac{1}{2}(\frac{C_\hbar}{\hbar(1+\eta)}+1), \frac{1}{2}(\frac{C_\hbar}{\hbar(1-\eta)}+1)]$. Taking $C_\hbar = \frac{2}{\theta}(c-1)\sigma^{1/2}\hbar^{1/2} + \frac{\zeta}{\theta^2}\hbar$, and applying the theorem, we obtain the estimate for the cardinal of $\mathsf{N}_{\hbar,c}$. The corresponding eigenvalues of $\mathcal{L}_{\hbar,\mathbf{A}}$ are of the form

$$\lambda_m(\hbar) = \hbar b_0 + \sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}} - \frac{\zeta}{2\theta}\hbar^2 + \hbar\left[\theta(m-\tfrac{1}{2})\hbar + k^\star(\hbar^{\frac{1}{2}}, 2m-1)\right] + \mathcal{O}(\hbar^\infty),$$

with $(2m-1)\hbar \leqslant \frac{C_\hbar}{1-\eta}$. Therefore there exists a constant $\tilde{C} > 0$, independent of $\hbar$, such that all $m \in \mathsf{N}_{\hbar,c}$ satisfy the inequality $(2m-1)\hbar \leqslant \tilde{C}\hbar^{1/2}$. Writing

$$k^\star(\hbar^{\frac{1}{2}}, Z) = c_0\hbar^{3/2} + \upsilon_1\hbar^{1/2}(Z/2) + c_1\hbar^2 + \upsilon_2(Z/2)^2 + \upsilon_3\hbar Z + \hbar^{1/2}\mathcal{O}(h+|Z|)^2 + \mathcal{O}(Z^3),$$

we see that, for $m \in \mathsf{N}_{\hbar,c}$,

$$k^\star(\hbar^{\frac{1}{2}}, (2m-1)\hbar) = \upsilon_1\hbar^{3/2}\left(m-\frac{1}{2}\right) + \upsilon_2\hbar^2\left(m-\frac{1}{2}\right)^2 + \mathcal{O}(\hbar^{3/2}),$$

which gives the result. □

Remark 1.6. An upper bound of $\lambda_m(\hbar)$ for fixed $\hbar$-independent m with remainder in $\mathcal{O}(\hbar^{\frac{9}{4}})$ was obtained in [13] through a quasimodes construction involving powers of $\hbar^{\frac{1}{4}}$. To the authors' knowledge, Corollary 1.5 gives the most accurate description of magnetic eigenvalues in three dimensions, in such a large spectral window. Note also that the non-degeneracy assumption on the norm of $\mathbf{B}$ is not purely technical. Indeed, at the quantum level, it appears through microlocal reductions matching with the splitting of the Hamiltonian dynamics into three scales: the cyclotron motion around field lines, the center-guide oscillation along the field lines, and the oscillation within the space of field lines.

1.6. **Organization of the paper.** The paper is organized as follows. In Section 2, we state our main results. Section 3 is devoted to the investigation of the first normal form (see Theorem 2.1 and Corollary 2.4). In Section 4 we analyze the second normal form (see Theorems 2.8 and 2.11 and Corollaries 2.9 and 2.13). Section 5 is devoted to the third normal form (see Theorem 2.15 and Corollary 2.16).

2. Statements of the main results

We recall (see [9, Chapter 7]) that a function $m : \mathbb{R}^d \to [0, \infty)$ is an order function if there exist constants $N_0, C_0 > 0$ such that

$$m(X) \leqslant C_0 \langle X - Y \rangle^{N_0} m(Y)$$

for any $X, Y \in \mathbb{R}^d$. The symbol class $S(m)$ is the space of smooth $\hbar$-dependent functions $a_\hbar : \mathbb{R}^d \to \mathbb{C}$ such that

$$\forall \alpha \in \mathbb{N}^d, \qquad |\partial_x^\alpha a_\hbar(x)| \leqslant C_\alpha m(x), \qquad \forall h \in (0, 1].$$

Throughout this paper, we assume that the components of the vector potential $\mathbf{A}$ belong to a symbol class $S(m)$. Note that this implies that $\mathbf{B} \in S(m)$, and conversely, if $\mathbf{B} \in S(m)$, then there exist a potential $\mathbf{A}$ and another order function m' such that $\mathbf{A} \in S(m')$. Moreover, the magnetic Hamiltonian $H(x, \xi) = \|\xi - \mathbf{A}(x)\|^2$ belongs to $S(m'')$ for an order function m'' on $\mathbb{R}^6$.

We will work with the Weyl quantization; for a classical symbol $a_\hbar = a(x, \xi; \hbar) \in S(m)$, it is defined as:

(2.1)
$$\mathrm{Op}_\hbar^w a\, \psi(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} e^{i\langle x-y,\xi\rangle/\hbar} a\left(\frac{x+y}{2}, \xi\right) \psi(y) \mathrm{d}y \mathrm{d}\xi, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d).$$

The Weyl quantization of H is the magnetic Laplacian $\mathcal{L}_{\hbar,\mathbf{A}} = (-i\hbar\nabla - \mathbf{A})^2$.

2.1. Normal forms and spectral reductions.

Let us introduce our first Birkhoff normal form $\mathcal{N}_\hbar$.

Theorem 2.1. *If $\mathbf{B}(0) \neq 0$, there exists a neighborhood of $(0, \mathbf{A}(0))$ endowed with symplectic coordinates $(x_1, \xi_1, x_2, \xi_2, x_3, \xi_3)$ in which $\Sigma = \{x_1 = \xi_1 = \xi_3 = 0\}$ and $(0, \mathbf{A}(0))$ has coordinates $0 \in \mathbb{R}^6$, and there exist an associated unitary Fourier integral operator $U_\hbar$ and a smooth function, compactly supported with respect to Z and ξ_3, $f^\star(\hbar, Z, x_2, \xi_2, x_3, \xi_3)$ whose Taylor series with respect to $Z, \xi_3, \hbar$ is*

$$\sum_{k \geqslant 3} \sum_{2\ell+2m+\beta=k} \hbar^\ell c^\star_{\ell,m,\beta}(x_2, \xi_2, x_3) Z^m \xi_3^\beta$$

such that

$$U_\hbar^* \mathcal{L}_{\hbar,\mathbf{A}} U_\hbar = \mathcal{N}_\hbar + \mathcal{R}_\hbar, \tag{2.2}$$

with

$$\mathcal{N}_\hbar = \hbar^2 D_{x_3}^2 + \mathcal{I}_\hbar \,\mathrm{Op}_\hbar^w\, b + \mathrm{Op}_\hbar^w\, f^\star(\hbar, \mathcal{I}_\hbar, x_2, \xi_2, x_3, \xi_3),$$

and where

(a) we have $\mathcal{I}_\hbar = \hbar^2 D_{x_1}^2 + x_1^2$,
(b) the operator $\mathrm{Op}_\hbar^w\, f^\star(\hbar, \mathcal{I}_\hbar, x_2, \xi_2, x_3, \xi_3)$ has to be understood as the Weyl quantization of an operator valued symbol,

(c) the remainder $\mathcal{R}_\hbar$ is a pseudo-differential operator such that, in a neighborhood of the origin, the Taylor series of its symbol with respect to $(x_1, \xi_1, \xi_3, \hbar)$ is 0.

Remark 2.2. In Theorem 2.1, the direction of $\mathbf{B}$ considered as a vector field on Σ is $\frac{\partial}{\partial x_3}$ and the function $b \in \mathcal{C}^\infty(\mathbb{R}^6)$ stands for $b \circ j_{|\Sigma}^{-1} \circ \pi$ where $\pi : \mathbb{R}^6 \to \Sigma : \pi(x_1, \xi_1, x_2, \xi_2, x_3, \xi_3) = (0, 0, x_2, \xi_2, x_3, 0)$. In addition, note that the support of $f^\star$ in Z and ξ_3 may be chosen as small as we want.

Remark 2.3. In the context of Weyl's asymptotics, a close version of this theorem appears in [19, Chapter 6].

In order to investigate the spectrum of $\mathcal{L}_{\hbar,\mathbf{A}}$ near the low lying energies, we introduce the following pseudo-differential operator

$$\mathcal{N}_\hbar^{[1]} = \hbar^2 D_{x_3}^2 + \hbar \operatorname{Op}_\hbar^w b + \operatorname{Op}_\hbar^w f^\star(\hbar, \hbar, x_2, \xi_2, x_3, \xi_3),$$

obtained by replacing $\mathcal{I}_\hbar$ by $\hbar$.

Corollary 2.4. *We introduce*

$$\mathcal{N}_\hbar^\sharp = \operatorname{Op}_\hbar^w \left(N_\hbar^\sharp \right), \tag{2.3}$$

with

$$N_\hbar^\sharp = \xi_3^2 + \mathcal{I}_\hbar \underline{b}(x_2, \xi_2, x_3) + f^{\star,\sharp}(\hbar, \mathcal{I}_\hbar, x_2, \xi_2, x_3, \xi_3)$$

and where $\underline{b}$ is a smooth extension of b away from $D(0, \varepsilon)$ such that (1.12) still holds and where $f^{\star,\sharp} = \chi(x_2, \xi_2, x_3) f^\star$, with χ is a smooth cutoff function being 1 in a neighborhood of $D(0, \varepsilon)$. We also define the operator attached to the first eigenvalue of $\mathcal{I}_\hbar$

$$\mathcal{N}_\hbar^{[1],\sharp} = \operatorname{Op}_\hbar^w \left(N_\hbar^{[1],\sharp} \right), \tag{2.4}$$

where $N_\hbar^{[1],\sharp} = \xi_3^2 + \hbar \underline{b}(x_2, \xi_2, x_3) + f^{\star,\sharp}(\hbar, \hbar, x_2, \xi_2, x_3, \xi_3)$.

If ε and the support of $f^\star$ are small enough, then we have

(a) The spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{N}_\hbar^\sharp$ below $\beta_0 \hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

(b) For all $c \in (0, \min(3b_0, \beta_0))$, the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{N}_\hbar^{[1],\sharp}$ below $c\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

Let us now state our results concerning the normal form of $\mathcal{N}_\hbar^{[1]}$ (or $\mathcal{N}_\hbar^{[1],\sharp}$) under the following assumption.

Notation 2.5. If $f = f(\mathbf{z})$ is a differentiable function, we denote by $T_\mathbf{z} f(\cdot)$ its tangent map at the point $\mathbf{z}$. Moreover, if f is twice differentiable, the second derivative of f is denoted by $T_\mathbf{z}^2 f(\cdot, \cdot)$.

Assumption 2.6. We assume that $T_0^2 b(\mathbf{B}(0), \mathbf{B}(0)) > 0$.

Remark 2.7. If the function b admits a unique and positive minimum at 0 and that it is non degenerate, then Assumption 2.6 is satisfied.

Under Assumption 2.6, we have $\partial_3 b(0,0,0) = 0$ and, in the coordinates (x_2, ξ_2, x_3) given in Theorem 2.1,

$$\partial_3^2 b(0,0,0) > 0\,. \tag{2.5}$$

It follows from (2.5) and the implicit function theorem that, for small x_2, there exists a smooth function $(x_2,\xi_2) \mapsto s(x_2,\xi_2)$, $s(0,0)=0$, such that

$$\partial_3 b(x_2,\xi_2,s(x_2,\xi_2)) = 0\,. \tag{2.6}$$

The point $s(x_2,\xi_2)$ is the unique (in a neighborhood of $(0,0,0)$) minimum of $x_3 \mapsto b(x_2,\xi_2,x_3)$. We define

$$\nu(x_2,\xi_2) := (\tfrac{1}{2}\partial_3^2 b(x_2,\xi_2,s(x_2,\xi_2)))^{1/4}.$$

Theorem 2.8. *Under Assumption 2.6, there exists a neighborhood $\mathcal{V}_0$ of 0 and a Fourier integral operator $V_\hbar$ which is microlocally unitary near $\mathcal{V}_0$ and such that*

$$V_\hbar^* \mathcal{N}_\hbar^{[1]} V_\hbar =: \underline{\mathcal{N}}_\hbar^{[1]} = \mathrm{Op}_\hbar^w \left(\underline{N}_\hbar^{[1]}\right),$$

where $\underline{N}_\hbar^{[1]} = \nu^2(x_2,\xi_2)\left(\xi_3^2 + \hbar x_3^2\right) + \hbar b(x_2,\xi_2,s(x_2,\xi_2)) + \underline{r}_\hbar$ and $\underline{r}_\hbar$ is a semiclassical symbol such that $\underline{r}_\hbar = \mathcal{O}(\hbar x_3^3) + \mathcal{O}(\hbar\xi_3^2) + \mathcal{O}(\xi_3^3) + \mathcal{O}(\hbar^2)$.

Corollary 2.9. *Let us introduce*

$$\underline{\mathcal{N}}_\hbar^{[1],\sharp} = \mathrm{Op}_\hbar^w \left(\underline{N}_\hbar^{[1],\sharp}\right),$$

where $\underline{N}_\hbar^{[1],\sharp} = \underline{\nu}^2(x_2,\xi_2)\left(\xi_3^2 + \hbar x_3^2\right) + \hbar \underline{b}(x_2,\xi_2,s(x_2,\xi_2)) + \underline{r}_\hbar^\sharp$, with $\underline{r}_\hbar^\sharp = \chi(x_2,\xi_2,x_3,\xi_3)\underline{r}_\hbar$, and where $\underline{\nu}$ denotes a smooth and constant (with a positive constant) extension of the function ν.

There exists a constant $\tilde{c} > 0$ such that, for any cut-off function χ equal to 1 on $D(0,\varepsilon)$ with support in $D(0,2\varepsilon)$, we have:

(a) The spectra of $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$ and $\mathcal{N}_\hbar^{[1],\sharp}$ below $(b_0 + \tilde{c}\varepsilon^2)\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

(b) For all $c \in (0, \min(3b_0, b_0 + \tilde{c}\varepsilon^2))$, the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$ below $c\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

Notation 2.10 (Change of semiclassical parameter)**.** We let $h = \hbar^{\frac{1}{2}}$ and, if $A_\hbar$ is a semiclassical symbol on $T^*\mathbb{R}^2$, admitting a semiclassical expansion in $\hbar^{\frac{1}{2}}$, we write

$$\mathcal{A}_\hbar := \mathrm{Op}_\hbar^w A_\hbar = \mathrm{Op}_h^w \mathsf{A}_h =: \mathfrak{A}_h,$$

with

$$\mathsf{A}_h(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3) = A_{h^2}(x_2,h\tilde{\xi}_2,x_3,h\tilde{\xi}_3).$$

Thus, $\mathcal{A}_\hbar$ and $\mathfrak{A}_h$ represent the same operator when $h = \hbar^{\frac{1}{2}}$, but the former is viewed as an $\hbar$-quantization of the symbol $A_\hbar$, while the latter is an h-pseudo-differential operator with symbol A_h. Notice that, if $A_\hbar$ belongs to some class $S(m)$, then $\mathsf{A}_h \in S(m)$ as well. This is of course not true the other way around.

Theorem 2.11. *Under Assumption 2.6, there exist a unitary operator W_h and a smooth function $g^\star(h, Z, x_2, \xi_2)$, with compact support as small as we want with respect to Z and with compact support in (x_2, ξ_2), whose Taylor series with respect to Z, h is*

$$\sum_{2m+2\ell\geqslant 3} c_{m,\ell}(x_2,\xi_2) Z^m h^\ell,$$

such that

$$W_h^* \underline{\mathfrak{N}}_h^{[1],\sharp} W_h =: \mathfrak{M}_h = \mathrm{Op}_h^w\left(\mathsf{M}_h\right),$$

with

$$\mathsf{M}_h = h^2 \underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2 \mathcal{J}_h \,\mathrm{Op}_h^w\, \underline{\nu}^2(x_2, h\tilde{\xi}_2) + h^2 g^\star(h, \mathcal{J}_h, x_2, h\tilde{\xi}_2) + h^2 \mathsf{R}_h + h^\infty S(1).$$

where

(a) the operator $\underline{\mathfrak{N}}_h^{[1],\sharp}$ is $\underline{\mathcal{N}}_h^{[1],\sharp}$ (but written in the h-quantization),

(b) we have let $\mathcal{J}_h = \mathrm{Op}_h^w\left(\tilde{\xi}_3^2 + x_3^2\right)$,

(c) the function R_h satisfies $\mathsf{R}_h(x_2, h\tilde{\xi}_2, x_3, \tilde{\xi}_3) = \mathcal{O}((x_3, \tilde{\xi}_3)^\infty)$.

Remark 2.12. Note that the support of $g^\star$ with respect to Z may be chosen as small as we want. Note also that we have used $\underline{\mathfrak{N}}_h^{[1],\sharp}$ instead of $\underline{\mathfrak{N}}_h^{[1]}$: Since W_h is exactly unitary, we get a direct comparison of the spectra.

Corollary 2.13. *We introduce*

$$\mathfrak{M}_h^\sharp = \mathrm{Op}_h^w\left(\mathsf{M}_h^\sharp\right),$$

with

$$\mathsf{M}_h^\sharp = h^2 \underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2 \mathcal{J}_h \underline{\nu}^2(x_2, h\tilde{\xi}_2) + h^2 g^\star(h, \mathcal{J}_h, x_2, h\tilde{\xi}_2).$$

We also define

$$\mathfrak{M}_h^{[1],\sharp} = \mathrm{Op}_h^w\left(\mathsf{M}_h^{[1],\sharp}\right),$$

with

$$\mathsf{M}_h^{[1],\sharp} = h^2 \underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^3 \underline{\nu}^2(x_2, h\tilde{\xi}_2) + h^2 g^\star(h, h, x_2, h\tilde{\xi}_2).$$

If ε and the support of $g^\star$ are small enough, we have

(a) For all $\eta > 0$, the spectra of $\underline{\mathfrak{N}}_h^{[1],\sharp}$ and $\mathfrak{M}_h^\sharp$ below $b_0 h^2 + \mathcal{O}(h^{2+\eta})$ coincide modulo $\mathcal{O}(h^\infty)$.

(b) For $c \in (0,3)$, the spectra of $\mathfrak{M}_h^\sharp$ and $\mathfrak{M}_h^{[1],\sharp}$ below $b_0 h^2 + c\sigma^{\frac{1}{2}} h^3$ coincide modulo $\mathcal{O}(h^\infty)$.

(c) If $c \in (0,3)$, the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{M}_\hbar^{[1],\sharp} = \mathfrak{M}_h^{[1],\sharp}$ below $b_0\hbar + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

Finally, we can perform a last Birkhoff normal form for the operator $\mathcal{M}_\hbar^{[1],\sharp}$ as soon as $(x_2,\xi_2)\mapsto \underline{b}(x_2,\xi_2,s(x_2,\xi_2))$ admits a unique and non degenerate minimum at $(0,0)$. Under this additional assumption, b admits a unique and non degenerate minimum at $(0,0,0)$.

Therefore we will use the following stronger assumption.

Assumption 2.14. The function b admits a unique and positive minimum at 0 and it is non degenerate.

Theorem 2.15. *Under Assumption 2.14, there exist a unitary $\hbar$-Fourier Integral Operator $Q_{\hbar^{\frac12}}$ whose phase admits an expansion in powers of $\hbar^{\frac12}$ such that*

$$Q^*_{\hbar^{\frac12}}\mathcal{M}_\hbar^{[1],\sharp}Q_{\hbar^{\frac12}}=\mathcal{F}_\hbar+\mathcal{G}_\hbar,$$

where

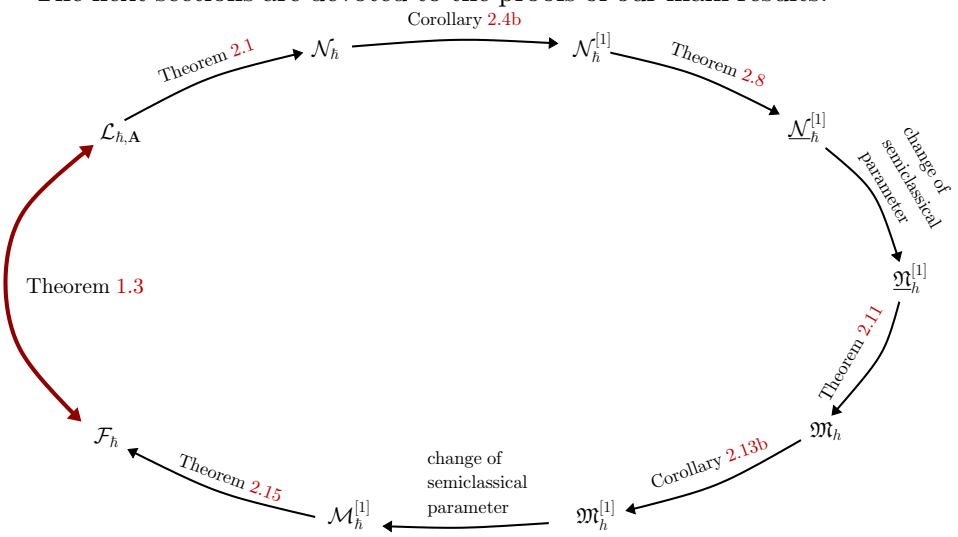
(a) $\mathcal{F}_\hbar$ is defined in Theorem 1.3,
(b) the remainder is in the form $\mathcal{G}_\hbar=\mathrm{Op}^w_\hbar(G_\hbar)$, with $G_\hbar=\hbar\mathcal{O}(|z_2|^\infty)$.

Corollary 2.16. *If ε and the support of $k^\star$ are small enough, we have*

(a) For all $\eta\in\left(0,\frac12\right)$, the spectra of $\mathcal{M}_\hbar^{[1],\sharp}$ and $\mathcal{F}_\hbar$ below $b_0\hbar+\mathcal{O}(\hbar^{1+\eta})$ coincide modulo $\mathcal{O}(\hbar^\infty)$.
(b) For all $c\in(0,3)$, the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{F}_\hbar$ below $b_0\hbar+c\sigma^{\frac12}\hbar^{\frac32}$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

Remark 2.17. Since the spectral analysis of $\mathcal{F}_\hbar$ is straightforward, Item (b) of Corollary 2.16 implies Theorem 1.3.

The next sections are devoted to the proofs of our main results.



3. First Birkhoff normal form

We assume that $\mathbf{B}(0)\neq 0$ so that in some neighborhood Ω of 0 the magnetic field does not vanish. Up to a rotation in $\mathbb{R}^3$ (extended to a symplectic

transformation in $\mathbb{R}^6$) we may assume that $\mathbf{B}(0) = \|\mathbf{B}(0)\|\mathbf{e}_3$. In this neighborhood, we may defined the unit vector:

$$\mathbf{b} = \frac{\mathbf{B}}{\|\mathbf{B}\|} \tag{3.1}$$

and find vectors $\mathbf{c}$ and $\mathbf{d}$ depending smoothly on q such that $(\mathbf{b}, \mathbf{c}, \mathbf{d})$ is a direct orthonormal basis.

3.1. **Symplectic coordinates.**

3.1.1. *Straightening the magnetic vector field.* Let $\hat{\Omega}$ be a small neighborhood of $0 \in \mathbb{R}^2$. We consider the form $\mathrm{d}\alpha$ and we would like to find a diffeomorphism χ, defined on $\hat{\Omega}$, such that $\chi^*(\mathrm{d}\alpha) = \mathrm{d}\hat{q}_1 \wedge \mathrm{d}\hat{q}_2$, where we use the notation $\chi(\hat{q}) = q$. First, it is easy to find a local diffeomorphism φ such that

$$\partial_3\varphi(\tilde{q}) = \mathbf{b}(\varphi(\tilde{q}))$$

and $\varphi(\tilde{q}_1, \tilde{q}_2, 0) = (\tilde{q}_1, \tilde{q}_2, 0)$. This is just the standard straightening lemma for the non-vanishing vector field $\mathbf{b}$.

The vector $\mathbf{e}_3$ is in the kernel of $\varphi^*(\mathrm{d}\alpha)$, which implies that we have $\varphi^*(\mathrm{d}\alpha) = f(\tilde{q})\mathrm{d}\tilde{q}_1 \wedge \mathrm{d}\tilde{q}_2$, for some smooth function f.

But since the form $\varphi^*(\mathrm{d}\alpha)$ is closed, f does not depend on $\tilde{q}_3$. This is then easy to find another diffeomorphism ψ, corresponding to the change of variables

$$\hat{q} = \psi(\tilde{q}) = (\psi_1(\tilde{q}_1, \tilde{q}_2), \psi_2(\tilde{q}_1, \tilde{q}_2), \tilde{q}_3)\,,$$

such that

$$\psi^*(\varphi^*(\mathrm{d}\alpha)) = \mathrm{d}\hat{q}_1 \wedge \mathrm{d}\hat{q}_2\,.$$

We let $\chi = \varphi \circ \psi$ and we notice that

$$\chi^*(\mathrm{d}\alpha) = \mathrm{d}\hat{q}_1 \wedge \mathrm{d}\hat{q}_2 \qquad \partial_3\chi(\hat{q}) = \mathbf{b}(\chi(\hat{q}))\,, \tag{3.2}$$

Remark 3.1. It follows from (3.2) and (1.4) that $\det T\chi = \|\mathbf{B}\|^{-1}$.

3.1.2. *Symplectic coordinates.* Let us consider the new parametrization of Σ given by

$$\begin{aligned} \iota : \hat{\Omega} &\longrightarrow \Sigma \\ \hat{q} &\mapsto (\chi(\hat{q}), A_1(\chi(\hat{q}))\,, A_2(\chi(\hat{q})), A_3(\chi(\hat{q})))\,, \end{aligned}$$

which gives a basis $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ of $T\Sigma$:

$$\mathbf{f}_j = (T\chi(\mathbf{e}_j), T\mathbf{A} \circ T\chi(\mathbf{e}_j)),\ j = 1, 2, 3\,.$$

Using (1.5), and the fact that $\mathbf{f}_3$ is in the kernel of $\mathrm{d}\alpha$, we find $\omega_0(\mathbf{f}_j, \mathbf{f}_3) = 0$, $j = 1, 2$. Finally, $\omega_0(f_1, f_2) = \mathrm{d}\alpha(T\chi\mathbf{e}_1, T\chi\mathbf{e}_2) = \chi^*(\mathrm{d}\alpha)(\mathbf{e}_1, \mathbf{e}_2) = 1$.

The following vectors of $\mathbb{R}^3 \times \mathbb{R}^3$ form a basis of the symplectic orthogonal of $T_{\iota(\hat{q})}\Sigma$:

$$\mathbf{f}_4 = \|\mathbf{B}\|^{-1/2}(\mathbf{c}, ({}^tT_{\chi(\hat{q})}\mathbf{A})\mathbf{c}), \quad \mathbf{f}_5 = \|\mathbf{B}\|^{-1/2}(\mathbf{d}, ({}^tT_{\chi(\hat{q})}\mathbf{A})\mathbf{d}), \tag{3.3}$$

so that

$$\omega_0(\mathbf{f}_4, \mathbf{f}_5) = -1.$$

We let $\mathbf{f}_6 = (0, \mathbf{b}) + \rho_1 \mathbf{f}_1 + \rho_2 \mathbf{f}_2$ where ρ_1 and ρ_2 are determined so that $\omega_0(\mathbf{f}_j, \mathbf{f}_6) = 0$ for $j = 1, 2$. We notice that $\omega_0(\mathbf{f}_j, \mathbf{f}_6) = 0$ for $j = 4, 5$ and $\omega_0(\mathbf{f}_3, \mathbf{f}_6) = -1$.

3.1.3. *Diagonalizing the Hessian.* We recall that

$$H(q,p) = \|p - \mathbf{A}(q)\|^2$$

so that, at a critical point $p = \mathbf{A}(q)$, the Hessian is

$$T^2 H((U_1, V_1), (U_2, V_2)) = 2\langle V_1 - T_q \mathbf{A}(U_1), V_2 - T_q \mathbf{A}(U_2)\rangle.$$

Let us notice that

$$T^2 H(\mathbf{f}_4, \mathbf{f}_5) = 2\|\mathbf{B}\|^{-1} \langle \mathbf{B} \times \mathbf{c}, \mathbf{B} \times \mathbf{d}\rangle = 0,$$
$$T^2 H(\mathbf{f}_4, \mathbf{f}_6) = 2\langle \mathbf{B} \times \mathbf{c}, \mathbf{b}\rangle = 0,$$
$$T^2 H(\mathbf{f}_5, \mathbf{f}_6) = 2\langle \mathbf{B} \times \mathbf{d}, \mathbf{b}\rangle = 0.$$

The Hessian, restricted to the symplectic orthogonal of $T_{\iota(\hat{q})}\Sigma$, is diagonal in the basis $(\mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6)$. Moreover we have

$$T^2 H(\mathbf{f}_4, \mathbf{f}_4) = \mathrm{d}^2 H(\mathbf{f}_5, \mathbf{f}_5) = 2\|\mathbf{B}\|^{-1} \|\mathbf{B} \times \mathbf{c}\|^2 = 2\|\mathbf{B}\|^{-1} \|\mathbf{B} \times \mathbf{d}\|^2 = 2\|\mathbf{B}\|.$$

Finally we have:

$$T^2 H(\mathbf{f}_6, \mathbf{f}_6) = 2.$$

Now we consider the local diffeomorphism:

$$(x, \xi) \mapsto \iota(x_2, \xi_2, x_3) + x_1 \mathbf{f}_4(x_2, \xi_2, x_3) + \xi_1 \mathbf{f}_5(x_2, \xi_2, x_3) + \xi_3 \mathbf{f}_6(x_2, \xi_2, x_3).$$

The Jacobian of this map is a symplectic matrix on Σ. We may apply the Moser-Weinstein argument (see [29]) to make this map locally symplectic near Σ modulo a change of variable which is tangent to the identity.

Near Σ, in these new coordinates, the Hamiltonian H admits the expansion

$$\hat{H} = H^0 + \mathcal{O}(|x_1|^3 + |\xi_1|^3 + |\xi_3|^3), \tag{3.4}$$

where $\hat{H}$ denotes H in the coordinates $(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$, and with

$$H^0 = \xi_3^2 + b(x_2, \xi_2, x_3)(x_1^2 + \xi_1^2), \quad b = \|\mathbf{B}(x_2, \xi_2, x_3)\|. \tag{3.5}$$

3.2. **Semiclassical Birkhoff normal form.**

3.2.1. *Birkhoff procedure in formal series.* Let us consider the space $\mathcal{E}$ of formal power series in $(x_1, \xi_1, \xi_3, \hbar)$ with coefficients smoothly depending on $\tilde{x} = (x_2, \xi_2, x_3)$:

$$\mathcal{E} = C^\infty_{x_2, \xi_2, x_3}[\![x_1, \xi_1, \xi_3, \hbar]\!].$$

We endow $\mathcal{E}$ with the semiclassical Moyal product (with respect to all variables $(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$) denoted by $\star$ and the commutator of two series κ_1 and κ_2 is defined as

$$[\kappa_1, \kappa_2] = \kappa_1 \star \kappa_2 - \kappa_2 \star \kappa_1 .$$

The degree of $x_1^{\alpha_1}\xi_1^{\alpha_2}\xi_3^{\beta}\hbar^{\ell} = z_1^{\alpha}\xi_3^{\beta}\hbar^{\ell}$ is $\alpha_1+\alpha_2+\beta+2\ell = |\alpha|+\beta+2\ell$. $\mathcal{D}_N$ denotes the space of monomials of degree N. $\mathcal{O}_N$ is the space of formal series with valuation at least N. For any $\tau,\gamma\in\mathcal{E}$, we denote $\operatorname{ad}_\tau\gamma = [\tau,\gamma]$.

Proposition 3.2. *Given $\gamma\in\mathcal{O}_3$, there exist formal power series $\tau,\kappa\in\mathcal{O}_3$ such that*

$$e^{i\hbar^{-1}\operatorname{ad}_\tau}(H^0+\gamma) = H^0+\kappa\,,$$

with $[\kappa,|z_1|^2]=0$.

Proof. Let $N\geqslant 1$. Assume that we have, for $\tau_N\in\mathcal{O}_3$,

$$e^{i\hbar^{-1}\operatorname{ad}_{\tau_N}}(H^0+\gamma) = H^0+K_3+\cdots+K_{N+1}+R_{N+2}+\mathcal{O}_{N+3}\,,$$

with $K_i\in\mathcal{D}_i$, $[K_i,|z_1|^2]=0$ and $R_{N+2}\in\mathcal{D}_{N+2}$.

Let $\tau'\in\mathcal{D}_{N+2}$. Then we have

$$e^{i\hbar^{-1}\operatorname{ad}_{\tau_N+\tau'}}(H^0+\gamma) = H^0+K_3+\cdots+K_{N+1}+K_{N+2}+\mathcal{O}_{N+3},$$

with $K_{N+2}\in\mathcal{D}_{N+2}$ such that

$$K_{N+2} = R_{N+2}+i\hbar^{-1}\operatorname{ad}_{\tau'}H^0+\mathcal{O}_{N+3}\,.$$

Let us temporarily admit that (see Lemma 3.3 below):

$$i\hbar^{-1}\operatorname{ad}_{\tau'}H^0 = i\hbar^{-1}b\operatorname{ad}_{\tau'}|z_1|^2+\mathcal{O}_{N+3}\,.$$

We obtain that

$$K_{N+2} = R_{N+2}+b\operatorname{ad}_{\tau'}|z_1|^2\,,$$

that we rewrite as

$$R_{N+2} = K_{N+2}+i\hbar^{-1}b\operatorname{ad}_{|z_1|^2}\tau' = K_{N+2}+b\{|z_1|^2,\tau'\}.$$

Since $b(\tilde{x})\neq 0$, we deduce the existence of τ' and K_{N+2} such that K_{N+2} commutes with $|z_1|^2$. □

Lemma 3.3. *For $\tau'\in\mathcal{D}_{N+2}$, we have*

$$i\hbar^{-1}\operatorname{ad}_{\tau'}H^0 = i\hbar^{-1}b\operatorname{ad}_{\tau'}|z_1|^2+\mathcal{O}_{N+3}\,.$$

Proof. We observe that

$$i\hbar^{-1}\operatorname{ad}_{\tau'}H^0 = i\hbar^{-1}\operatorname{ad}_{\tau'}\xi_3^2+i\hbar^{-1}\operatorname{ad}_{\tau'}(b(\tilde{x})|z_1|^2).$$

Let us write

$$\tau' = \sum_{|\alpha|+\beta+2l=N+2} a_{\alpha,\beta,l}(\tilde{x})z_1^{\alpha}\xi_3^{\beta}\hbar^{l}.$$

Then, for the first term, we have

$$\begin{aligned} i\hbar^{-1}\operatorname{ad}_{\tau'}\xi_3^2 =& \{\tau',\xi_3^2\}\\ =& -2\xi_3\frac{\partial\tau'}{\partial x_3}\\ =& -2\sum_{|\alpha|+\beta+2\ell=N+2}\frac{\partial a_{\alpha,\beta,\ell}}{\partial x_3}(\tilde{x})z_1^{\alpha}\xi_3^{\beta+1}\hbar^{\ell}\in\mathcal{O}_{N+3}\,. \end{aligned}$$

We also have

$$\begin{aligned} i\hbar^{-1}(\mathrm{ad}_{\tau'}\, b(\tilde{x})) =& \{\tau', b\} + \hbar^2 \mathcal{O}_N \\ =& \frac{\partial \tau'}{\partial \xi_3}\frac{\partial b}{\partial x_3} + \frac{\partial \tau'}{\partial \xi_2}\frac{\partial b}{\partial x_2} - \frac{\partial \tau'}{\partial x_2}\frac{\partial b}{\partial \xi_2} + \mathcal{O}_{N+1} \\ =& \sum_{|\alpha|+\beta+2\ell=N+2} \beta a(\tilde{x}) \frac{\partial b}{\partial x_3} z_1^\alpha |z_1|^2 \xi_3^{\beta-1}\hbar^\ell + \mathcal{O}_{N+1} \in \mathcal{O}_{N+1}\,. \end{aligned}$$

Therefore, for the second term, we get

$$\begin{aligned} i\hbar^{-1}\,\mathrm{ad}_{\tau'}(b(\tilde{x})|z_1|^2) =& i\hbar^{-1}(\mathrm{ad}_{\tau'}\, b(\tilde{x}))|z_1|^2 + i\hbar^{-1} b(\tilde{x})\,\mathrm{ad}_{\tau'}\,|z_1|^2 \\ =& i\hbar^{-1} b(\tilde{x})\,\mathrm{ad}_{\tau'}\,|z_1|^2 + \mathcal{O}_{N+3}\,, \end{aligned}$$

that completes the proof of the lemma. □

3.2.2. *Quantizing the formal procedure.* Let us now prove Theorem 2.1. Using (3.4) and applying the Egorov theorem (see [26, 31] or Theorem A.2), we can find a unitary Fourier Integral Operator $U_\hbar$, and such that

$$U_\hbar^* \mathcal{L}_{\hbar,\mathbf{A}} U_\hbar = C_0\hbar + \mathrm{Op}_\hbar^w(H^0) + \mathrm{Op}_\hbar^w(r_\hbar),$$

where the Taylor series (with respect to x_1, ξ_1, ξ_3, $\hbar$) of $r_\hbar$ satisfies $r_\hbar^T = \gamma \in \mathcal{O}_3$ and C_0 is the value at the origin of the sub-principal symbol of $U_\hbar^* \mathcal{L}_{\hbar,\mathbf{A}} U_\hbar$. One can choose $U_\hbar$ such that the subprincipal symbol is preserved by conjugation[1], which implies $C_0 = 0$. Applying Proposition 3.2, we obtain τ and κ in $\mathcal{O}_3$ such that

$$e^{i\hbar^{-1}\,\mathrm{ad}_\tau}(H^0 + \gamma) = H^0 + \kappa,$$

with $[\kappa, |z_1|^2] = 0$.

We can introduce a smooth symbol $a_\hbar$ with compact support such that we have $a_\hbar^T = \tau$ in a neighborhood of the origin. By Proposition 3.2 and Theorem A.4, we obtain that the operator

$$e^{i\hbar^{-1}\,\mathrm{Op}_\hbar^w(a_\hbar)}(\mathrm{Op}_\hbar^w(H^0) + \mathrm{Op}_\hbar^w(r_\hbar))e^{-i\hbar^{-1}\,\mathrm{Op}_\hbar^w(a_\hbar)}$$

is a pseudodifferential operator such that the formal Taylor series of its symbol is $H^0 + \kappa$. In this application of Theorem A.4, we have used the filtration $\mathcal{O}_j$ defined in Section 3.2.1. Since κ commutes with $|z_1|^2$, we can write it as a formal series in $|z_1|^2$:

$$\kappa = \sum_{k\geqslant 3} \sum_{2\ell+2m+\beta=k} \hbar^\ell c_{\ell,m}(x_2,\xi_2,x_3)|z_1|^{2m}\xi_3^\beta.$$

This formal series can be reordered by using monomials $(|z_1|^2)^{\star m}$:

$$\kappa = \sum_{k\geqslant 3} \sum_{2\ell+2m+\beta=k} \hbar^\ell c^\star_{\ell,m}(x_2,\xi_2,x_3)(|z_1|^2)^{\star m}\xi_3^\beta.$$

[1]This is sometimes called the *Improved Egorov Theorem*. It was first discovered by Weinstein in [30], in the homogeneous setting. For the semiclassical case, see for instance [18, Appendix A].

Thanks to the Borel lemma, we may find a smooth function, with a compact support as small as we want with respect to $\hbar$, I and ξ_3, $f^\star(\hbar, I, x_2, \xi_2, x_3, \xi_3)$ such that its Taylor series with respect to $\hbar, I, \xi_3$ is

$$\sum_{k\geqslant 3}\sum_{2\ell+2m+\beta=k} \hbar^\ell c^\star_{\ell,m}(x_2,\xi_2,x_3) I^m \xi_3^\beta.$$

This achieves the proof of Theorem 2.1.

3.3. **Spectral reduction to the first normal form.** This section is devoted to the proof of Corollary 2.4.

3.3.1. *Numbers of eigenvalues.*

Lemma 3.4. *Under Assumption 1.2, there exists $h_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\hbar \in (0, h_0)$, the essential spectrum of $\mathcal{N}_\hbar^\sharp$ admits the following lower bound:*

$$\inf \mathfrak{s}_{\mathsf{ess}}(\mathcal{N}_\hbar^\sharp) \geqslant (\beta_0 + \varepsilon_0)\hbar\,.$$

Proof. By using the assumption we may consider a smooth function χ with compact support and $\varepsilon_0 > 0$ such that

$$\xi_3^2 + b(x_2,\xi_2,x_3) + \chi(x_2,x_3,\xi_2,\xi_3) \geqslant \beta_0 + 2\varepsilon_0.$$

Then, given $\eta \in (0,1)$ and estimating the second term in (2.3) by using that the support of $f^\star$ is chosen small enough and the semiclassical Calderon-Vaillancourt theorem, we notice that, for $\hbar$ small enough,

$$\mathcal{N}_\hbar^\sharp \geqslant (1-\eta)\operatorname{Op}_\hbar^w\left(\xi_3^2 + |z_1|^2 b(x_2,\xi_2,x_3)\right). \tag{3.6}$$

Since the essential spectrum is invariant by (relatively) compact perturbations, we have

$$\mathfrak{s}_{\mathsf{ess}}\left(\mathcal{N}_\hbar^\sharp + (1-\eta)\hbar\operatorname{Op}_\hbar^w \chi(x_2,x_3,\xi_2,\xi_3)\right) = \mathfrak{s}_{\mathsf{ess}}\left(\mathcal{N}_\hbar^\sharp\right).$$

Hence

$$\inf \mathfrak{s}_{\mathsf{ess}}\left(\mathcal{N}_\hbar^\sharp\right) \geqslant \inf \mathfrak{s}\left(\mathcal{N}_\hbar^\sharp + (1-\eta)\hbar\operatorname{Op}_\hbar^w \chi(x_2,x_3,\xi_2,\xi_3)\right).$$

In order to bound the r.h.s. from below, we write

$$\begin{aligned}
&\mathcal{N}_\hbar^\sharp + (1-\eta)\hbar\operatorname{Op}_\hbar^w \chi(x_2,x_3,\xi_2,\xi_3)\\
&\geqslant (1-\eta)\operatorname{Op}_\hbar^w\left(\xi_3^2 + |z_1|^2 b(x_2,\xi_2,x_3)\right) + (1-\eta)\hbar\operatorname{Op}_\hbar^w \chi(x_2,x_3,\xi_2,\xi_3)\\
&\geqslant \hbar(1-\eta)\operatorname{Op}_\hbar^w\left(\xi_3^2 + b(x_2,\xi_2,x_3) + \chi(x_2,x_3,\xi_2,\xi_3)\right)\\
&\geqslant \hbar(1-\eta)(\beta_0 + 2\varepsilon_0 - C\hbar),
\end{aligned}$$

where we have used the semiclassical Gårding inequality. Taking η and then $\hbar$ small enough, this concludes the proof. □

By using the Hilbertian decomposition given by the Hermite functions $(e_{k,\hbar})_{k\geqslant 1}$ associated with $\mathcal{I}_\hbar$, we notice that

$$\mathcal{N}_\hbar^\sharp = \bigoplus_{k\geqslant 1} \mathcal{N}_\hbar^{[k],\sharp},$$

where

$$\mathcal{N}_\hbar^{[k],\sharp} = \hbar^2 D_{x_3}^2 + (2k-1)\hbar\,\mathrm{Op}_\hbar^w\, b + \mathrm{Op}_\hbar^w\, f^{\star,\sharp}(\hbar,(2k-1)\hbar,x_2,\xi_2,x_3,\xi_3), \tag{3.7}$$

acting on $L^2(\mathbb{R}^2)$.

Lemma 3.5. *For all $\eta\in(0,1)$, there exist $C>0$ and $h_0>0$ such that for all $k\geqslant 1$ and $\hbar\in(0,h_0)$, we have $\inf\mathfrak{s}(\mathcal{N}_\hbar^{[k],\sharp})\geqslant(1-2\eta)b_0(2k-1)\hbar$.*

Proof. Applying (3.6) to $\psi(x_1,x_2,x_3)=\varphi(x_2,x_3)e_{k,\hbar}(x_1)$, we infer that

$$\langle \mathcal{N}_\hbar^{[k],\sharp}\varphi,\varphi\rangle \geqslant (2k-1)\hbar(1-\eta)\langle \mathrm{Op}_\hbar^w(b)\varphi,\varphi\rangle.$$

With the Gårding inequality, we get

$$\langle \mathrm{Op}_\hbar^w(b)\varphi,\varphi\rangle \geqslant (b_0-C\hbar)\|\varphi\|^2,$$

and the conclusion follows by the min-max principle. □

We immediately deduce the following proposition.

Proposition 3.6. *We have the following descriptions of the low lying spectrum of $\mathcal{N}_\hbar^\sharp$.*

(a) There exist $\hbar_0>0$ and $K\in\mathbb{N}$ such that, for $\hbar\in(0,\hbar_0)$, the spectrum of $\mathcal{N}_\hbar^\sharp$ lying below $\beta_0\hbar$ is contained in the union $\bigcup_{k=1}^K \mathsf{sp}\left(\mathcal{N}_\hbar^{[k],\sharp}\right)$.

(b) If $c\in(0,\min(3b_0,\beta_0))$, then there exists $\hbar_0>0$ such that for all $\hbar\in(0,\hbar_0)$ the eigenvalues of $\mathcal{N}_\hbar^\sharp$ lying below $c\hbar$ coincide with the eigenvalues of $\mathcal{N}_\hbar^{[1],\sharp}$ below $c\hbar$.

Notation 3.7. If $\mathcal{L}$ is a self-adjoint operator and $E<\inf\mathfrak{s}_{\mathsf{ess}}(\mathcal{L})$, we denote by $\mathsf{N}(\mathcal{L},E)$ the number of eigenvalues of $\mathcal{L}$ lying in $(-\infty,E)$.

We deduce the following proposition.

Corollary 3.8. *Under Assumption 1.11, we have*

$$\mathsf{N}(\mathcal{L}_{\hbar,\mathbf{A}},\beta_0\hbar)=\mathcal{O}(\hbar^{-3/2}),\quad \mathsf{N}\left(\mathcal{N}_\hbar^\sharp,\beta_0\hbar\right)=\mathcal{O}(\hbar^{-2}).$$

Proof. To get the first estimate, we use the Lieb-Thirring inequalities (which provide an upper bound of the number of eigenvalues in dimension three) and the diamagnetic inequality (see [25] and Proposition 1.9). To get the second estimate, we use the first point in Proposition 3.6. Moreover, given $\eta\in(0,1)$, by using $\hbar\in(0,1)$ we infer

$$\langle \mathcal{N}_\hbar^{[k],\sharp}\psi,\psi\rangle \geqslant (1-\eta)\hbar\langle \mathrm{Op}_\hbar^w\left(\xi_3^2+b(x_2,\xi_2,x_3)\right)\psi,\psi\rangle.$$

Note that the last inequality is very rough. By the min-max principle, we deduce that

$$\mathsf{N}\left(\mathcal{N}_\hbar^{[k],\sharp},\beta_0\hbar\right)\leqslant \mathsf{N}\left(\mathrm{Op}_\hbar^w\left(\xi_3^2+b(x_2,\xi_2,x_3)\right),(1-\eta)^{-1}\beta_0\right).$$

Then, we conclude by using the Weyl asymptotics and our confinement assumption:

$$\mathsf{N}\left(\mathrm{Op}_\hbar^w\left(\xi_3^2+b(x_2,\xi_2,x_3)\right),(1-\eta)^{-1}\beta_0\right)=\mathcal{O}(\hbar^{-2}).$$

□

Since $\mathcal{N}_\hbar^\sharp$ commutes with $\mathcal{I}_\hbar$, we also deduce the following corollary.

Corollary 3.9. *For any eigenvalue λ of $\mathcal{N}_\hbar^\sharp$ such that $\lambda\leqslant\beta_0\hbar$ we may consider an orthonormal eigenbasis of the space $\ker\left(\mathcal{N}_\hbar^\sharp-\lambda\right)$ formed with functions in the form $e_{k,\hbar}(x_1)\varphi_\hbar(x_2,x_3)$ with $k\in\{1,\dots K\}$. Moreover we have $\mathbb{1}_{(-\infty,\beta_0\hbar)}(\mathcal{N}_\hbar^\sharp)=\mathcal{O}(\hbar^{-2})$ and each eigenfunction associated with $\lambda\leqslant\beta_0\hbar$ is a linear combination of at most $\mathcal{O}(\hbar^{-2})$ such tensor products.*

3.3.2. *Microlocalization estimates.* The following proposition follows from the same lines as in dimension two (see [16, Theorem 2.1]).

Proposition 3.10. *Under Assumptions 1.1 and 1.2, for any $\epsilon>0$, there exist $C(\epsilon)>0$ and $h_0(\epsilon)>0$ such that for any eigenpair (λ,ψ) of $\mathcal{L}_{\hbar,\mathbf{A}}$ with $\lambda\leqslant\beta_0\,\hbar$ we have for $\hbar\in(0,h_0(\epsilon))$:*

$$\int_{\mathbb{R}^3}e^{2(1-\epsilon)\phi(q)/\hbar^{\frac12}}|\psi|^2\,\mathrm{d}q\leqslant C(\epsilon)\exp(\epsilon\hbar^{-\frac12})\|\psi\|^2,$$

$$\mathcal{Q}_{\hbar,\mathbf{A}}(e^{(1-\epsilon)\phi(q)/\hbar^{\frac12}}\psi)\leqslant C(\epsilon)\exp(\epsilon\hbar^{-\frac12})\|\psi\|^2,$$

where ϕ is the distance to the bounded set $\{\|\mathbf{B}(q)\|\leqslant\beta_0\}$ for the Agmon metric $(\|(\mathbf{B}(q)\|-\beta_0)_+g$, with g the standard metric.

Proposition 3.11. *Under Assumptions 1.1 and 1.2, we consider $0<b_0<\beta_0<b_1$ and there exist $C>0$ and $\hbar_0>0$ such that for any eigenpair (λ,ψ) of $\mathcal{L}_{\hbar,\mathbf{A}}$ with $\lambda\leqslant\beta_0\hbar$ we have for $\hbar\in(0,\hbar_0)$ and $\delta\in(0,\frac12)$:*

$$\psi=\chi_0\left(\hbar^{-2\delta}\mathcal{L}_{\hbar,\mathbf{A}}\right)\chi_1(q)\psi+\mathcal{O}(\hbar^\infty)\|\psi\|,$$

where χ_0 is a cutoff function compactly supported in the ball of center 0 and radius 1 and where χ_1 is a compactly supported smooth cutoff function being 1 in an open neighborhood of $\{\|\mathbf{B}(q)\|\leqslant\beta_0\}$.

Let us now investigate the microlocalization of the eigenfunctions of $\mathcal{N}_\hbar^\sharp$.

Proposition 3.12. *Let χ be a smooth cutoff function being 0 on $\{b\leqslant\beta_0\}$ and 1 on the set $\{b\geqslant\beta_0+\varepsilon\}$. If λ is an eigenvalue of $\mathcal{N}_\hbar^\sharp$ such that $\lambda\leqslant\beta_0\hbar$ and if ψ is an associated eigenfunction, then we have*

$$\mathrm{Op}_\hbar^w\left(\chi(x_2,\xi_2,x_3)\right)\psi=\mathcal{O}(\hbar^\infty)\|\psi\|.$$

Proof. Due to Corollary 3.9, it is sufficient to prove the estimate for a function in the form $\psi(x_1,x_2,x_3) = e_{k,h}(x_1)\varphi(x_2,x_3)$ where k lies in $\{1,\dots,K\}$ and we have

$$\mathcal{N}_\hbar^\sharp \psi = \lambda\psi, \qquad \text{or equivalently} \qquad \mathcal{N}_\hbar^{[k],\sharp}\varphi = \lambda\varphi,$$

where we recall (3.7). Then, we write

$$\mathcal{N}_\hbar^{[k],\sharp} \operatorname{Op}_\hbar^w(\chi)\,\varphi = \lambda \operatorname{Op}_\hbar^w(\chi)\,\varphi + \left[\mathcal{N}_\hbar^{[k],\sharp}, \operatorname{Op}_\hbar^w(\chi)\right]\varphi$$

and it follows that

$$\left\langle \mathcal{N}_\hbar^{[k],\sharp} \operatorname{Op}_\hbar^w(\chi)\,\varphi, \operatorname{Op}_\hbar^w(\chi)\,\varphi\right\rangle = \lambda \|\operatorname{Op}_\hbar^w(\chi)\,\varphi\|^2 + \left\langle \left[\mathcal{N}_\hbar^{[k],\sharp}, \operatorname{Op}_\hbar^w(\chi)\right]\varphi, \operatorname{Op}_\hbar^w(\chi)\,\varphi\right\rangle. \tag{3.8}$$

Rough pseudo-differential estimates imply that there exist $C>0$, $\hbar_0>0$ such that for all $\hbar\in(0,\hbar_0)$,

$$\left|\left\langle \left[\mathcal{N}_\hbar^{[k],\sharp}, \operatorname{Op}_\hbar^w(\chi)\right]\varphi, \operatorname{Op}_\hbar^w(\chi)\,\varphi\right\rangle\right| \leqslant C\hbar^2 \left\|\operatorname{Op}_\hbar^w(\underline{\chi})\,\varphi\right\|^2 + C\hbar \left\|\operatorname{Op}_\hbar^w(\underline{\chi})\,\varphi\right\|^2 + C\hbar\langle \operatorname{Op}_\hbar^w(\partial_3\chi)\,\varphi, \operatorname{Op}_\hbar^w(\xi_3)\operatorname{Op}_\hbar^w(\chi)\,\varphi\rangle. \tag{3.9}$$

Combining (3.9) and (3.8), we get

$$\|\operatorname{Op}_\hbar^w(\xi_3)\operatorname{Op}_\hbar^w(\chi)\,\varphi\| \leqslant C\hbar^{\frac12}\|\operatorname{Op}_\hbar^w(\underline{\chi})\,\varphi\|, \tag{3.10}$$

where $\underline{\chi}$ is a smooth cutoff function living on a slightly larger support than χ. By using (3.10), we can improve the commutator estimate

$$\left|\left\langle \left[\mathcal{N}_\hbar^{[k],\sharp}, \operatorname{Op}_\hbar^w(\chi)\right]\varphi, \operatorname{Op}_\hbar^w(\chi)\,\varphi\right\rangle\right| \leqslant C\hbar^{\frac32}\left\|\operatorname{Op}_\hbar^w(\underline{\chi})\,\varphi\right\|^2.$$

We infer that, there exist $C>0$, $\hbar_0>0$ such that for $\hbar\in(0,\hbar_0)$,

$$\left\langle \mathcal{N}_\hbar^{[k],\sharp} \operatorname{Op}_\hbar^w(\chi)\,\varphi, \operatorname{Op}_\hbar^w(\chi)\,\varphi\right\rangle \leqslant \beta_0\hbar\|\operatorname{Op}_\hbar^w(\chi)\,\varphi\|^2 + C\hbar^{\frac32}\left\|\operatorname{Op}_\hbar^w(\underline{\chi})\,\varphi\right\|^2.$$

By using the semiclassical Gårding inequality and the support of χ, we get

$$\left\langle \mathcal{N}_\hbar^{[k],\sharp} \operatorname{Op}_\hbar^w(\chi)\,\varphi, \operatorname{Op}_\hbar^w(\chi)\,\varphi\right\rangle \geqslant (\beta_0+\varepsilon_0)\hbar\left\|\operatorname{Op}_\hbar^w(\chi)\,\varphi\right\|^2$$

and we deduce

$$\|\operatorname{Op}_\hbar^w(\chi)\,\varphi\|^2 \leqslant C\hbar^{\frac12}\left\|\operatorname{Op}_\hbar^w(\underline{\chi})\,\varphi\right\|^2.$$

The conclusion follows by a standard iteration argument. □

The following proposition is concerned by the microlocalization with respect to ξ_3.

Proposition 3.13. *Let χ_0 be a smooth cutoff function being 0 in a neighborhood of 0 and let $\delta\in\left(0,\frac12\right)$. If λ is an eigenvalue of $\mathcal{N}_\hbar^\sharp$ such that $\lambda\leqslant\beta_0\hbar$ and if ψ is an associated eigenfunction, then we have*

$$\operatorname{Op}_\hbar^w\left(\chi_0\left(\hbar^{-\delta}\xi_3\right)\right)\psi = \mathcal{O}(\hbar^\infty)\|\psi\|.$$

Proof. We write again $\psi(x_1,x_2,x_3)=e_{k,\hbar}(x_1)\varphi(x_2,x_3)$ with $k\in\{1,\dots,K\}$ and we have $\mathcal{N}_\hbar^{[k],\sharp}\varphi=\lambda\varphi$. We use again the formula (3.8) with $\chi_0\left(\hbar^{-\delta}\xi_3\right)$. We get the commutator estimate

$$\left|\left\langle\left[\mathcal{N}_\hbar^{[k],\sharp},\mathrm{Op}_\hbar^w\left(\chi_0\left(\hbar^{-\delta}\xi_3\right)\right)\right]\varphi,\mathrm{Op}_\hbar^w\left(\chi_0\left(\hbar^{-\delta}\xi_3\right)\right)\varphi\right\rangle\right| \leqslant C\hbar^{\frac{3}{2}-\delta}\left\|\mathrm{Op}_\hbar^w\left(\underline{\chi}_0\left(\hbar^{-\delta}\xi_3\right)\right)\varphi\right\|^2.$$

We have

$$\mathrm{Op}_\hbar^w\left((\hbar^{-\delta}\xi_3)^2\chi_0^2\left(\hbar^{-\delta}\xi_3\right)\right)=\mathrm{Op}_{\hbar^{1-\delta}}^w\left(\xi_3^2\chi_0^2\left(\xi_3\right)\right),$$

so that, with the Gårding inequality,

$$\left\langle\mathrm{Op}_\hbar^w\left((\hbar^{-\delta}\xi_3)^2\chi_0^2\left(\hbar^{-\delta}\xi_3\right)\right)\varphi,\varphi\right\rangle\geqslant(1-C\hbar^{1-\delta})\|\varphi\|^2.$$

We infer

$$(\hbar^{2\delta}(1-Ch^{1-\delta})-\beta_0\hbar)\left\|\mathrm{Op}_\hbar^w\left(\chi_0\left(\hbar^{-\delta}\xi_3\right)\right)\varphi\right\|^2 \leqslant C\hbar^{\frac{3}{2}-\delta}\left\|\mathrm{Op}_\hbar^w\left(\underline{\chi}_0\left(\hbar^{-\delta}\xi_3\right)\right)\varphi\right\|^2.$$

□

Using $\mathrm{Op}_\hbar^w f^\star(\hbar,\mathcal{I}_\hbar,x_2,\xi_2,x_3,\xi_3)=\mathrm{Op}_\hbar^w f(\hbar,|z_1|^2,x_2,\xi_2,x_3,\xi_3)$, we deduce the following in the same way.

Proposition 3.14. *Let χ_1 be a smooth cutoff function being 0 in a neighborhood of 0 and let $\delta\in\left(0,\frac{1}{2}\right)$. If λ is an eigenvalue of $\mathcal{N}_\hbar^\sharp$ such that $\lambda\leqslant\beta_0\hbar$ and if ψ is an associated eigenfunction, then we have*

$$\mathrm{Op}_\hbar^w\left(\chi_1\left(\hbar^{-\delta}(x_1,\xi_1)\right)\right)\psi=\mathcal{O}(\hbar^\infty)\|\psi\|.$$

Proposition 3.15. *The spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{N}_\hbar^\sharp$ below $\beta_0\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.*

Proof. We refer to [25, Section 4.3] which contains similar arguments. □

This proposition provides the point (a) in Corollary 2.4. With Proposition 3.6, we deduce the point (b).

4. Second Birkhoff normal form

4.1. **Birkhoff analysis of the first level.** This section is devoted to the proofs of Theorems 2.8 and 2.11.

The goal now is to normalize a $\hbar$-pseudo-differential operator $\mathcal{N}_\hbar^{[1]}$ on $\mathbb{R}^2$ whose Weyl symbol has the form

$$N_\hbar^{[1]}=\xi_3^2+\hbar b(x_2,\xi_2,x_3)+r_\hbar(x_2,\xi_2,x_3,\xi_3),$$

where $r_\hbar$ is a classical symbol with the following asymptotic expansion:

$$r_\hbar=r_0+\hbar r_1+\hbar^2r_2+\cdots$$

(in the symbol class topology), where each r_ℓ has a formal expansion in ξ_3 of the form

$$r_\ell(x_2,\xi_2,x_3,\xi_3) \sim \sum_{2\ell+\beta\geqslant 3} c_{\ell,\beta}(x_2,\xi_2,x_3)\xi_3^\beta. \tag{4.1}$$

The leading terms of $N_\hbar^{[1]}$ are:

$$N_\hbar^{[1]} = \xi_3^2+\hbar b(x_2,\xi_2,x_3)+c_{1,1}(x_2,\xi_2,x_3)\hbar\xi_3+\mathcal{O}(\hbar\xi_3^2)+\mathcal{O}(\xi_3^3)+\mathcal{O}(\hbar^2). \tag{4.2}$$

4.1.1. *First normalization of the symbol.* We consider the following local change of variables $\hat\varphi(x_2,\xi_2,x_3,\xi_3) = (\hat x_2,\hat\xi_2,\hat x_3,\hat\xi_3)$:

$$\begin{cases} \hat x_2 := x_2 + \xi_3\partial_2 s(x_2,\xi_2)\,, \\ \hat\xi_2 := \xi_2 + \xi_3\partial_1 s(x_2,\xi_2)\,, \\ \hat x_3 := x_3 - s(x_2,\xi_2)\,, \\ \hat\xi_3 := \xi_3\,. \end{cases} \tag{4.3}$$

It is easy to check that the differential of $\hat\varphi$ is invertible as soon as ξ_3 is small enough. Moreover, we have

$$\hat\varphi^*\omega_0 - \omega_0 = \mathcal{O}(|\xi_3|).$$

By the Darboux-Weinstein theorem (see for instance [25, Lemma 2.4]), there exists a local diffeomorphism ψ such that

$$\psi = \mathrm{Id} + \mathcal{O}(\xi_3^2) \quad \text{and} \quad \psi^*\hat\varphi^*\omega_0 = \omega_0. \tag{4.4}$$

Using the improved Egorov theorem, one can find a unitary Fourier Integral Operator $V_\hbar$ such that the Weyl symbol of $V_\hbar^*\mathcal{N}_\hbar^{[1]}V_\hbar$ is $\hat N_\hbar := N_\hbar^{[1]}\circ\hat\varphi\circ\psi+\mathcal{O}(\hbar^2)$. From (4.4), and (4.3), we see that $\hat r_\hbar := r_\hbar\circ\hat\varphi\circ\psi$ is still of the form (4.1), with modified coefficients $c_{\ell,\beta}$. Thus, using the new variables and a Taylor expansion in ξ_3, we get

$$\begin{aligned}\hat N_\hbar = \hat\xi_3^2+\hbar b(\hat x_2+\mathcal{O}(\hat\xi_3),\hat\xi_2+\mathcal{O}(\xi_3),\hat x_3+s(\hat x_2+\mathcal{O}(\hat\xi_3),\hat\xi_2+\mathcal{O}(\hat\xi_3))+\mathcal{O}(\hat\xi_3^2)) \\ +\mathcal{O}(\hat\xi_3^3)+\hat r_\hbar+\mathcal{O}(\hbar^2)\end{aligned}$$

and thus

$$\begin{aligned}\hat N_\hbar = \hat\xi_3^2+\hbar b(\hat x_2,\hat\xi_2,\hat x_3+s(\hat x_2,\hat\xi_2))+\hbar\hat\xi_3 g(\hat x_2,\hat\xi_2,\hat x_3) \\ +\mathcal{O}(\hbar\hat\xi_3^2)+\hat r_\hbar+\mathcal{O}(\hat\xi_3^3)+\mathcal{O}(\hbar^2),\end{aligned} \tag{4.5}$$

for some smooth function $g(\hat x_2,\hat\xi_2,\hat x_3)$.
Therefore $\hat N_\hbar$ has the following form:

$$\hat N_\hbar = \hat\xi_3^2+\hbar b(\hat x_2,\hat\xi_2,\hat x_3+s(\hat x_2,\hat\xi_2))+\hat c_{1,1}(x_2,\hat\xi_2,\hat x_3)\hbar\hat\xi_3+\mathcal{O}(\hbar\hat\xi_3^2)+\mathcal{O}(\hat\xi_3^3)+\mathcal{O}(\hbar^2).$$

4.1.2. *Where the second harmonic oscillator appears.* We now drop all the hats off the variables. We use a Taylor expansion with respect to x_3, which, in view of (2.6), yields:

$$b(x_2,\xi_2,x_3+s(x_2,\xi_2)) = b(x_2,\xi_2,s(x_2,\xi_2))+\frac{x_3^2}{2}\partial_3^2 b(x_2,\xi_2,s(x_2,\xi_2))+\mathcal{O}(x_3^3).$$

We let:

$$\nu = (\tfrac{1}{2}\partial_3^2 b(x_2,\xi_2,s(x_2,\xi_2)))^{1/4} \text{ and } \gamma = \ln\nu. \tag{4.6}$$

We introduce the change of coordinates $(\check{x}_2,\check{x}_3,\check{\xi}_2,\check{\xi}_3) = C(x_2,x_3,\xi_2,\xi_3)$ defined by:

$$\begin{cases} \check{x}_3 &= \nu x_3\,,\\ \check{\xi}_3 &= \nu^{-1}\xi_3\,,\\ \check{x}_2 &= x_2+\frac{\partial\gamma}{\partial\xi_2}x_3\xi_3\,,\\ \check{\xi}_2 &= \xi_2-\frac{\partial\gamma}{\partial x_2}x_3\xi_3\,, \end{cases} \tag{4.7}$$

for which one can check that $C^*\omega_0-\omega_0 = \mathcal{O}(x_3\xi_3) = \mathcal{O}(\xi_3)$. As before, we can make this local diffeomorphism symplectic by the Darboux-Weinstein theorem, which modifies (4.7) by $\mathcal{O}(\xi_3^2)$. In the new variables (which we call (x_2,x_3,ξ_2,ξ_3) again), the symbol $\check{N}_\hbar$ has the form:

$$\begin{aligned}\check{N}_\hbar = \nu^2(x_2,\xi_2)\left(\xi_3^2+\hbar x_3^2\right)+\hbar b(x_2,\xi_2,s(x_2,\xi_2))+\check{c}_{1,1}(x_2,\xi_2,x_3)\hbar\xi_3\\ +\mathcal{O}(\hbar x_3^3)+\mathcal{O}(\hbar\xi_3^2)+\mathcal{O}(\xi_3^3)+\mathcal{O}(\hbar^2),\end{aligned}$$

for some smooth function $\check{c}_{1,1}(x_2,\xi_2,x_3)$.

4.1.3. *Normalizing the remainder.* The next step is to get rid of the term $\check{c}_{1,1}(x_2,\xi_2,x_3)\hbar\xi_3$. Let

$$a(x_2,\xi_2,x_3) := -\frac{1}{2}\int_0^{x_3}\check{c}_{1,1}(x_2,\xi_2,t)dt\,.$$

Since $\check{c}_{1,1}$ is compactly supported, a is bounded, and one can form the unitary pseudo-differential operator $\exp(iA)$, $A=\mathrm{Op}_\hbar^w(a)$. We have

$$\exp(-iA)\,\mathrm{Op}_\hbar^w\left(\check{N}_\hbar\right)\exp(iA) = \mathrm{Op}_\hbar^w\left(\check{N}_\hbar\right)+\exp(-iA)[\mathrm{Op}_\hbar^w\left(\check{N}_\hbar\right),\exp(iA)].$$

The symbol of $[\exp(-iA)\,\mathrm{Op}_\hbar^w\left(\check{N}_\hbar\right),\exp(iA)]$ is

$$\frac{\hbar}{i}e^{-ia}\{N,e^{ia}\}+\mathcal{O}(\hbar^2) = \hbar\{\check{N}_\hbar,a\}+\mathcal{O}(\hbar^2) = \hbar\{\check{N}_0,a\}+\mathcal{O}(\hbar^2),$$

where $\check{N}_0$ is the principal symbol of $\check{N}_\hbar$, which satisfies:

$$\check{N}_0 = \xi_3^2+\mathcal{O}(\xi_3^3).$$

Therefore $\{\check{N}_\hbar,a\} = \{\xi_3^2,a\}+\mathcal{O}(\xi_3^2)$. Since

$$\{\xi_3^2,a\} = 2\xi_3\frac{\partial a}{\partial x_3} = -\xi_3\check{c}_{1,1},$$

we get

$$\exp(-iA)\,\mathrm{Op}_\hbar^w\left(\check{N}_\hbar\right)\exp(iA) = \mathrm{Op}_\hbar^w(\check{N}_\hbar-\hbar\xi_3\check{c}_{1,1}+\mathcal{O}(\hbar\xi_3^2)+\mathcal{O}(\hbar^2)),$$

which shows that we can remove the coefficient of $\hbar\xi_3$. The new operator given by the conjugation formula $\underline{\mathcal{N}}_\hbar^{[1]} = \exp(-iA)\,\mathrm{Op}_\hbar^w\left(\check{N}_\hbar\right)\exp(iA)$ has a symbol of the form

$$\underline{N}_\hbar^{[1]} = \nu^2(x_2,\xi_2)\left(\xi_3^2+\hbar x_3^2\right) + \hbar b(x_2,\xi_2,s(x_2,\xi_2)) + \underline{r}_\hbar, \tag{4.8}$$

where $\underline{r}_\hbar = \mathcal{O}(\hbar x_3^3) + \mathcal{O}(\hbar\xi_3^2) + \mathcal{O}(\xi_3^3) + \mathcal{O}(\hbar^2)$.

This proves Theorem 2.8.

4.1.4. *The second Birkhoff normal form.* We now want to perform a Birkhoff normal form for $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$ relative to the "second harmonic oscillator"

$$\underline{\nu}^2(x_2,\xi_2)\left(\xi_3^2+\hbar x_3^2\right).$$

Using Notation 2.10, we introduce the new semiclassical parameter $h=\hbar^{\frac{1}{2}}$, and use the relation

$$\mathrm{Op}_\hbar^w(\underline{N}_\hbar^{[1],\sharp}) = \mathrm{Op}_\hbar^w(\underline{\mathsf{N}}_h^{[1],\sharp}).$$

Thus, let $\tilde{\xi}_j := \hbar^{-1/2}\xi_j$. The new symbol $\underline{\mathsf{N}}_h^{[1],\sharp}$ has the form:

$$\underline{\mathsf{N}}_h^{[1],\sharp}(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3) = h^2\Big(\underline{\nu}^2(x_2,h\tilde{\xi}_2)(\tilde{\xi}_3^2+x_3^2) + \underline{b}(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2)) + h^{-2}\underline{r}_{h^2}^\sharp(x_2,h\tilde{\xi}_2,x_3,h\tilde{\xi}_3)\Big).$$

We introduce momentarily a new parameter μ and define

$$\underline{\mathsf{N}}_h^{[1],\sharp}(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3;\mu) := \underline{\nu}^2(x_2,\mu\tilde{\xi}_2)(\tilde{\xi}_3^2+x_3^2) + \underline{b}(x_2,\mu\tilde{\xi}_2,s(x_2,\mu\tilde{\xi}_2)) + h^{-2}\underline{r}_{h^2}^\sharp(x_2,\mu\tilde{\xi}_2,x_3,h\tilde{\xi}_3).$$

Notice that $\underline{\mathsf{N}}_h^{[1],\sharp}(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3;h) = h^{-2}\underline{\mathsf{N}}_h^{[1],\sharp}(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3)$. We define now a space of functions suitable for the Birkhoff normal form in $(x_3,\tilde{\xi}_3,h)$. Let us now use the notation of the Appendix introduced in (A.4) in the case when the family of smooth linear maps $\mathbb{R}^2\to\mathbb{R}^2$ is given by

$$\varphi_{\mu,\mathbb{R}^2}(x_2,\tilde{\xi}_2) = (x_2,\mu\tilde{\xi}_2).$$

Let

$$\mathscr{F} := \mathcal{C}(1)_{\mathbb{R}^2},$$

where the index $\mathbb{R}^2$ means that we consider symbols on $\mathbb{R}^2$. More explicitly, we have

$$\mathscr{F} = \left\{d \text{ s. t. } \exists c\in S(1;[0,1]\times(0,1])_{\mathbb{R}^2} : d(x_2,\tilde{\xi}_2;\mu,h) = c(\varphi_{\mu,\mathbb{R}^2}(x_2,\tilde{\xi}_2);\mu,h)\right\}.$$

Then we define

$$\mathscr{E} := \mathscr{F}[\![x_3,\tilde{\xi}_3,h]\!],$$

endowed with the full Poisson bracket

$$\mathscr{E}\times\mathscr{E}\ni(f,g)\mapsto\{f,g\} = \sum_{j=2,3}\frac{\partial f}{\partial\tilde{\xi}_j}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial\tilde{\xi}_j}\frac{\partial f}{\partial x_j}\in\mathscr{E},$$

and the corresponding Moyal bracket $[f,g]$. We remark that the formal Taylor series of the symbol $\underline{\mathsf{N}}_h^{[1],\sharp}(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3;\mu)$ with respect to $(x_3,\tilde{\xi}_3,h)$ belongs to $\mathscr{E}$. We may apply the semiclassical Birkhoff normal form relative to the main term $\underline{\nu}^2(x_2,\mu\tilde{\xi}_2)(\tilde{\xi}_3^2+x_3^2)$ exactly as in Section 3.2.1 (and also [25, Proposition 2.7]), where we use the fact that the function

$$(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3;\mu,h)\mapsto(\underline{\nu}^2(x_2,\mu\tilde{\xi}_2))^{-1}$$

belongs to $\mathscr{E}$ because $\underline{\nu}^2>C>0$ uniformly with respect to μ. Let us consider $\gamma\in\mathscr{E}$ the formal Taylor expansion of $h^{-2}\underline{r}_{h^2}^{\sharp}(x_2,\mu\tilde{\xi}_2,x_3,h\tilde{\xi}_3)$ with respect to $(x_3,\tilde{\xi}_3,h)$. The series γ is of valuation 3 and we obtain two formal series $\kappa,\tau\in\mathscr{E}$ of valuation at least 3 such that

$$[\kappa,x_3^2+\tilde{\xi}_3^2]=0$$

and

$$e^{ih^{-1}\mathrm{ad}_\tau}(\underline{\nu}^2(x_2,\mu\tilde{\xi}_2)(\tilde{\xi}_3^2+x_3^2)+\gamma)=\underline{\nu}^2(x_2,\mu\tilde{\xi}_2)(\tilde{\xi}_3^2+x_3^2)+\kappa.$$

The coefficients of τ are in $S(1)$ and one can find a smooth function $\tau_h\in S(1)$ with compact support with respect to $(x_3,\tilde{\xi}_3,h)$ and whose Taylor series in $(x_3,\tilde{\xi}_3,h)$ is τ. By the Borel summation, τ_h will actually lie in $S(m')$ with $m'(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3)=\langle(x_3,\tilde{\xi}_3)\rangle^{-k}$ for any $k>0$, uniformly for small $h>0$ and $\mu\in[0,1]$. Notice that $\underline{\mathsf{N}}_h^{[1],\sharp}\in\mathcal{C}(m)$ with $m=\langle(x_3,\tilde{\xi}_3)\rangle^2\geqslant 1$, and that $mm'=\mathcal{O}(1)$.

Then, we can apply Theorem A.3 with the family of endomorphisms of $\mathbb{R}^4$ defined

$$\varphi_{\mu,\mathbb{R}^4}(x_2,\tilde{\xi}_2,x_3,\tilde{\xi}_3)=(x_2,\mu\tilde{\xi}_2,x_3,\tilde{\xi}_3)\,.$$

Thus, the new operator

$$\mathfrak{M}_h=e^{ih^{-1}\,\mathrm{Op}_h^w\,\tau_h}\underline{\mathfrak{N}}_h^{[1],\sharp}e^{-ih^{-1}\,\mathrm{Op}_h^w\,\tau_h}$$

is a pseudo-differential operator whose Weyl symbol belongs to the class $\mathcal{C}(m)$ modulo $h^\infty S(1)$ (see the notations of Theorem 2.11). Moreover, thanks to Theorem A.4, its symbol M_h admits the following Taylor expansion (with respect to $(x_3,\tilde{\xi}_3,h)$)

$$\tilde{b}(x_2,\mu\tilde{\xi}_2,s(x_2,\mu\tilde{\xi}_2))+\underline{\nu}^2(x_2,\mu\tilde{\xi}_2)(\tilde{\xi}_3^2+x_3^2)+\kappa.$$

We write $\kappa=\sum_{m+2\ell\geqslant 3}c_{m,\ell}(x_2,\mu\tilde{\xi}_2)|\tilde{z}_3|^{\star 2m}h^\ell$ and we may find a smooth function $g^\star(x_2,\mu\tilde{\xi}_2,Z,h)$ such that its Taylor series with respect to Z, h is

$$\sum_{2m+2\ell\geqslant 3}c_{m,\ell}(x_2,\mu\tilde{\xi}_2)Z^mh^\ell.$$

We may now replace μ by h, which achieves the proof of Theorem 2.11.

4.2. **Spectral reduction to the second normal form.** This section is devoted to the proof of Corollary 2.13.

4.2.1. *From $\mathcal{N}_\hbar^{[1],\sharp}$ to $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$.* In this section, we prove Corollary 2.9.

Lemma 4.1. *We have*

$$\mathsf{N}\left(\mathcal{N}_\hbar^{[1],\sharp}, \beta_0\hbar\right) = \mathcal{O}(\hbar^{-2}), \qquad \mathsf{N}\left(\underline{\mathcal{N}}_\hbar^{[1],\sharp}, \beta_0\hbar\right) = \mathcal{O}(\hbar^{-2}).$$

Proof. The first estimate comes from Proposition 3.6 and Corollary 3.8. The second estimate can be obtained by the same method as in the proof of Corollary 3.8. □

Let us now summarize the microlocalization properties of the eigenfunctions of $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$ in the following proposition.

Proposition 4.2. *Let χ_0 be a smooth cutoff function on $\mathbb{R}$ being 0 in a neighborhood of 0 and let $\delta \in \left(0, \frac{1}{2}\right)$. Let χ be a smooth cutoff function being 0 on the bounded set $\{x_3^2 + \underline{b}(x_2, \xi_2, s(x_2, \xi_2)) \leqslant \beta_0\}$ and 1 on the set $\{x_3^2 + \underline{b}(x_2, \xi_2, s(x_2, \xi_2)) \geqslant \beta_0 + \tilde{\varepsilon}\}$, with $\tilde{\varepsilon} > 0$. If λ is an eigenvalue of $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$ such that $\lambda \leqslant \beta_0\hbar$ and if ψ is an associated eigenfunction, then we have*

$$\mathrm{Op}_\hbar^w\left(\chi(x_2, \xi_2, x_3)\right)\psi = \mathcal{O}(\hbar^\infty)\|\psi\|,$$

and

$$\mathrm{Op}_\hbar^w\left(\chi_0\left(\hbar^{-\delta}\xi_3\right)\right)\psi = \mathcal{O}(\hbar^\infty)\|\psi\|.$$

Proof. The proof follows exactly the same lines as for Propositions 3.12 and 3.13. □

Lemma 4.1 and Proposition 4.2 on the one hand and Propositions 3.12 and 3.13 on the other hand are enough to deduce from Theorem 2.8 the point (a) in Corollary 2.9. The point (b) easily follows from Corollary 2.4.

4.2.2. *From $\underline{\mathfrak{N}}_h^{[1],\sharp}$ to $\mathfrak{M}_h^\sharp$.* Let us now prove the point (a) in Corollary 2.13. We get the following rough estimate of the number of eigenvalues.

Lemma 4.3. *We have*

$$\mathsf{N}\left(\underline{\mathfrak{N}}_h^{[1],\sharp}, \beta_0 h^2\right) = \mathsf{N}\left(\mathfrak{M}_h, \beta_0 h^2\right) = \mathcal{O}(h^{-4}), \tag{4.9}$$

$$\mathsf{N}\left(\mathfrak{M}_h^\sharp, \beta_0 h^2\right) = \mathcal{O}(h^{-4}). \tag{4.10}$$

Proof. First, we notice that $\underline{\mathfrak{N}}_h^{[1],\sharp}$ and $\mathfrak{M}_h$ are unitarily equivalent so that (4.9) holds. Then, given $\eta > 0$ and h small enough and up to shrinking the support of $g^\star$ and by using the Calderon-Vaillancourt theorem (as in the proof of Lemma 3.4), $\mathfrak{M}_h^\sharp \geqslant \widetilde{\mathfrak{M}}_h^\sharp$ in the sense of quadratic forms, with

$$\widetilde{\mathfrak{M}}_h^\sharp = \mathrm{Op}_h^w\left(h^2\underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2))\right) + h^2\mathcal{J}_h\,\mathrm{Op}_h^w\left(\left(\underline{\nu}^2(x_2, h\tilde{\xi}_2)\right) - \eta\right).$$

Since $\underline{\nu}^2 \geqslant c > 0$, we get

$$\mathrm{Op}_h^w\left(h^2\underline{b}(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2))\right)+h^2\mathcal{J}_h\,\mathrm{Op}_h^w\left(\left(\underline{\nu}^2(x_2,h\tilde{\xi}_2)\right)-\eta\right)$$
$$\geqslant \mathrm{Op}_h^w\left(h^2\underline{b}(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2))\right)+\frac{c}{2}h^2\mathcal{J}_h\,.$$

We deduce the upper bound (4.10) by separation of variables and the min-max principle. □

The following proposition deals with the microlocal properties of the eigenfunctions of $\underline{\mathfrak{N}}_h^{[1],\sharp}$.

Proposition 4.4. *Let $\eta \in (0,1), \delta \in \left(0,\frac{\eta}{2}\right), C > 0$. Let χ be a smooth cutoff function being 0 on $\{\underline{b}(x_2,\xi_2,s(x_2,\xi_2)) \leqslant \beta_0\}$ and being 1 on the set $\{\underline{b}(x_2,\xi_2,s(x_2,\xi_2)) \geqslant \beta_0+\tilde{\varepsilon}\}$, with $\tilde{\varepsilon} > 0$. Let also χ_1 be a smooth cutoff function on $\mathbb{R}^2$, being 0 in a neighborhood of 0.*

If λ is an eigenvalue of $\underline{\mathfrak{N}}_h^{[1],\sharp}$ such that $\lambda \leqslant \beta_0 h^2$ and if ψ is an associated eigenfunction, we have

$$\mathrm{Op}_h^w\left(\chi(x_2,h\tilde{\xi}_2)\right)\psi = \mathcal{O}(h^\infty)\|\psi\| \tag{4.11}$$

and if λ is an eigenvalue of $\underline{\mathfrak{N}}_h^{[1],\sharp}$ such that $\lambda \leqslant b_0h^2+Ch^{2+\eta}$ and if ψ is an associated eigenfunction, we have

$$\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi = \mathcal{O}(h^\infty)\|\psi\|. \tag{4.12}$$

Proof. The estimate (4.11) is a consequence of Proposition 4.2. Then, let us write the symbol of $\underline{\mathfrak{N}}_h^{[1],\sharp}$:

$$\underline{\mathsf{N}}_h^{[1],\sharp} = h^2\underline{\nu}^2(x_2,h\tilde{\xi}_2)\left(\tilde{\xi}_3^2+x_3^2\right)+h^2\underline{b}(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2))+\underline{R}_{h^2}^\sharp(x_2,h\tilde{\xi}_2,x_3,h\tilde{\xi}_3).$$

We write

$$\left\langle \underline{\mathfrak{N}}_h^{[1],\sharp}\,\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi, \mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\right\rangle$$
$$= \lambda\|\,\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi\|^2$$
$$+\left\langle\left[\underline{\mathfrak{N}}_h^{[1],\sharp},\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\right],\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi\right\rangle.$$

We get

$$\left\langle\left[\underline{\mathfrak{N}}_h^{[1],\sharp},\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\right],\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi\right\rangle$$
$$\leqslant Ch^3\left\|\mathrm{Op}_h^w\left(\underline{\chi}_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi\right\|^2,$$

where we have used (4.11). Then, we use that

$$\underline{b}(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2)) \geqslant b_0, \qquad \underline{\nu}^2(x_2,h\tilde{\xi}_2) \geqslant c_0 > 0, \qquad \lambda \leqslant b_0h^2+Ch^{2+\eta},$$

and the Gårding inequality to deduce

$$h^2\left(Ch^{2\delta}-Ch^{\eta}\right)\left\|\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi\right\|^2 \leqslant Ch^3\left\|\mathrm{Op}_h^w\left(\underline{\chi}_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi\right\|^2.$$

The desired estimate follows by an iteration argument. □

In the same way we can deal with $\mathfrak{M}_h^\sharp$.

Proposition 4.5. *Let $\eta\in(0,1),\delta\in\left(0,\frac{\eta}{2}\right),C>0$. Let χ be a smooth cutoff function being 0 on $\{\underline{b}(x_2,\xi_2,s(x_2,\xi_2))\leqslant\beta_0\}$ and being 1 on the set $\{\underline{b}(x_2,\xi_2,s(x_2,\xi_2))\geqslant\beta_0+\tilde{\varepsilon}\}$, with $\tilde{\varepsilon}>0$. If λ is an eigenvalue of $\mathfrak{M}_h^\sharp$ such that $\lambda\leqslant\beta_0h^2$ and if ψ is an associated eigenfunction, we have*

$$\mathrm{Op}_h^w\left(\chi(x_2,h\tilde{\xi}_2)\right)\psi=\mathcal{O}(h^\infty)\|\psi\| \tag{4.13}$$

and if λ is an eigenvalue of $\mathfrak{M}_h^\sharp$ such that $\lambda\leqslant b_0h^2+Ch^{2+\eta}$ and if ψ is an associated eigenfunction, we have

$$\mathrm{Op}_h^w\left(\chi_1(h^{-\delta}(x_3,\tilde{\xi}_3))\right)\psi=\mathcal{O}(h^\infty)\|\psi\|. \tag{4.14}$$

Proof. In order to get (4.13), it is enough to go back to the representation with semiclassical $\hbar$, that is $\mathfrak{M}_h^\sharp=\mathcal{M}_\hbar^\sharp$. Indeed the microlocal estimate follows by the same arguments as in Propositions 3.12 and 3.13. Then, (4.14) follows as in Proposition 4.4. □

Propositions 4.4 and 4.5 and Theorem 2.11 standardly imply the point (a) in Corollary 2.13.

4.2.3. *From $\mathfrak{M}_h^\sharp$ to $\mathfrak{M}_h^{[1],\sharp}$.* Let us now prove the point (b) in Corollary 2.13. Note that the point (c) is just a reformulation of (b).

Let us consider the Hilbertian decomposition $\mathfrak{M}_h^\sharp=\bigoplus_{k\geqslant1}\mathfrak{M}_h^{[k],\sharp}$, where the symbol $\mathsf{M}_h^{[k],\sharp}$ of $\mathfrak{M}_h^{[k],\sharp}$ is

$$h^2\underline{b}(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2))+(2k-1)h^3\underline{\nu}^2(x_2,h\tilde{\xi}_2)+h^2g^\star(h,(2k-1)h,x_2,h\tilde{\xi}_2).$$

There exists $h_0>0$ such that for all $k\geqslant1$ and $h\in(0,h_0)$,

$$\langle\mathfrak{M}_h^{[k],\sharp}\psi,\psi\rangle \geqslant\langle\mathrm{Op}_h^w\left(h^2\underline{b}(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2))+(2k-1)h^3(\underline{\nu}^2(x_2,h\tilde{\xi}_2)-\varepsilon)\right)\psi,\psi\rangle.$$

Since each eigenfunction of $\mathfrak{M}_h^{[k],\sharp}$ associated with an eigenvalue less than β_0h^2 provides an eigenfunction of $\mathfrak{M}_h^\sharp$, we infer that the eigenfunctions of

$\mathfrak{M}_h^{[k],\sharp}$ are uniformly microlocalized in a (x_2,ξ_2)-neighborhood of $(0,0)$ as small as we want. Therefore, on the range of $\mathbb{1}_{(-\infty,b_0h^2)}(\mathfrak{M}_h^{[k],\sharp})$, we have

$$\begin{aligned}&\langle \mathfrak{M}_h^{[k],\sharp}\psi,\psi\rangle\\ &\quad\geqslant \langle \mathrm{Op}_h^w\left(h^2\underline{b}(x_2,h\tilde{\xi}_2,s(x_2,h\tilde{\xi}_2))+(2k-1)h^3(\nu^2(0,0)-2\varepsilon)\right)\psi,\psi\rangle.\end{aligned}$$

and, with the Gårding inequality in the $\hbar$-quantization, we get

$$\langle \mathfrak{M}_h^{[k],\sharp}\psi,\psi\rangle \geqslant \langle \mathrm{Op}_h^w\left(h^2b_0+(2k-1)h^3(\nu^2(0,0)-\varepsilon)-Ch^4\right)\psi,\psi\rangle.$$

This implies the point (b) in Corollary 2.13.

5. Third Birkhoff normal form

5.1. **Birkhoff analysis of the first level.** In this section we prove Theorem 2.15.

We consider $\mathcal{M}_\hbar^{[1],\sharp}=\mathrm{Op}_\hbar^w\left(M_\hbar^{[1],\sharp}\right)$, with

$$M_\hbar^{[1],\sharp}=\hbar\underline{b}(x_2,\xi_2,s(x_2,\xi_2))+\hbar^{\frac{3}{2}}\underline{\nu}^2(x_2,\xi_2)+\hbar g^\star(\hbar^{\frac12},\hbar^{\frac12},x_2,\xi_2).$$

By using a Taylor expansion, we get,

(5.1)
$$M_\hbar^{[1],\sharp}=\hbar b_0+\frac{\hbar}{2}\mathsf{Hess}_{(0,0)}\underline{b}(x_2,\xi_2,s(x_2,\xi_2))+\hbar^{\frac32}\nu^2(0,0)+cx_2\hbar^{\frac32}+d\xi_2\hbar^{\frac32}+\hbar\mathcal{O}((\hbar^{\frac12},z_2)^3),$$

where $c=\partial_{x_2}\nu^2(0,0)$ and $d=\partial_{\xi_2}\nu^2(0,0)$, and we have identified the Hessian with its quadratic form in (x_2,ξ_2).

Then, there exists a linear symplectic change of variables that diagonalizes the Hessian, so that, if $L_\hbar$ is the associated unitary transform,

$$L_\hbar^*\mathcal{M}_\hbar^{[1],\sharp}L_\hbar=\mathrm{Op}_\hbar^w\left(\hat{M}_\hbar^{[1],\sharp}\right),$$

with

$$\hat{M}_\hbar^{[1],\sharp}=\hbar b_0+\frac{\hbar}{2}\theta(x_2^2+\xi_2^2)+\hbar^{\frac32}\nu^2(0,0)+\hat{c}x_2\hbar^{\frac32}+\hat{d}\xi_2\hbar^{\frac32}+\hbar\mathcal{O}((\hbar^{\frac12},z_2)^3),$$

where

$$\theta=\sqrt{\det\mathsf{Hess}_{(0,0)}b(x_2,\xi_2,s(x_2,\xi_2))}\,.$$

Since $(\partial_{x_3}b)(x_2,\xi_2,s(x_2,\xi_2))=0$ and $(0,0)$ is a critical point of s, we notice that $\partial^2_{x_2x_3}b(0,0,0)=\partial^2_{\xi_2x_3}b(0,0,0)=0$. Thus

$$\det\mathsf{Hess}_{(0,0,0)}b(0,0,0)=\theta^2\partial^2_{x_3}b(0,0,0).$$

Using that b is identified with $b\circ\chi$ (see Remarks 2.2 and 3.1), this provides the expression given in (1.15).

Note that $\hat{c}^2+\hat{d}^2 = \|(\nabla_{x_2,\xi_2}\nu^2)(0,0)\|^2$ since the symplectic transform is in fact a rotation. Moreover we have

$$\theta(x_2^2+\xi_2^2)+\hat{c}x_2\hbar^{\frac12}+\hat{d}\xi_2\hbar^{\frac12} = \theta\left(\left(x_2-\frac{\hat{c}\hbar^{\frac12}}{\theta}\right)^2+\left(\xi_2-\frac{\hat{d}\hbar^{\frac12}}{\theta}\right)^2\right)-\hbar\frac{\hat{c}^2+\hat{d}^2}{\theta}.$$

Thus, there exists a unitary transform $\hat{U}_{\hbar^{\frac12}}$, which is in fact an $\hbar$-Fourier Integral Operator whose phase admits a Taylor expansion in powers of $\hbar^{\frac12}$, such that

$$\hat{U}^*_{\hbar^{\frac12}} L^*_\hbar \mathcal{M}^{[1],\sharp}_\hbar L_\hbar \hat{U}_{\hbar^{\frac12}} =: \underline{\mathcal{F}}_\hbar = \mathrm{Op}^w_\hbar\left(\underline{F}_\hbar\right),$$

where

$$\underline{F}_\hbar = \hbar b_0 + \hbar^{\frac32}\nu^2(0,0) - \frac{\|(\nabla_{x_2,\xi_2}\nu^2)(0,0)\|^2}{2\theta}\hbar^2 + \hbar\left(\frac{\theta}{2}|z_2|^2+\mathcal{O}((\hbar^{\frac12},z_2)^3)\right).$$

Now we perform a semiclassical Birkhoff normal form in the space of formal series $\mathbb{R}[\![x_2,\xi_2,\hbar^{\frac12}]\!]$ equipped with the degree such that $x_2^\ell\xi_2^m\hbar^{\frac{n}{2}}$ is $\ell+m+n$ and endowed with the Moyal product. Let $\underline{F}^T_\hbar$ be the full Taylor series of $\underline{F}_\hbar$. We find a formal series $\tau(x_2,\xi_2,\hbar^{\frac12})$ with a valuation at least 3 such that

$$e^{i\hbar^{-1}\mathrm{ad}_\tau}\underline{F}^T_\hbar = F^T_\hbar,$$

where $F^T_\hbar$ is a formal series of the form

$$F^T_\hbar = \hbar b_0 + \hbar^{\frac32}\nu^2(0,0) - \frac{\|(\nabla_{x_2,\xi_2}\nu^2)(0,0)\|^2}{2\theta}\hbar^2 + \frac{\theta}{2}\hbar|z_2|^2 + \hbar k^T(\hbar^{\frac12},|z_2|^2),$$

and k^T is a formal series in $\mathbb{R}[\![\hbar^{\frac12},|z_2|^2]\!]$ (and that can be also written as a formal series in Moyal power of $|z_2|^2$, say $(k^T)^\star$).

Let $\tilde{\tau}(x_2,\xi_2,\mu)$ be a compactly supported function whose Taylor expansion at $(0,0,0)$ is equal to $\tau(x_2,\xi_2,\mu)$. By the Egorov theorem A.2, uniformly with respect to the parameter μ, we obtain that

$$e^{-i\hbar^{-1}\mathrm{Op}^w_\hbar(\tilde{\tau})}\,\mathrm{Op}^w_\hbar(\underline{F}_{\mu^2})e^{i\hbar^{-1}\mathrm{Op}^w_\hbar(\tilde{\tau})} =: \mathrm{Op}^w_\hbar(\tilde{F}_\mu)$$

is an $\hbar$-pseudo-differential operator depending smoothly on μ. Expanding $\tilde{F}_\mu$ in powers of μ in the $S(1)$ topology, and letting $\mu=\sqrt{\hbar}$, we see that $\tilde{F}_{\sqrt{\hbar}} = F_\hbar + \tilde{G}_\hbar$, where

$$F_\hbar = \hbar b_0 + \hbar^{\frac32}\nu^2(0,0) - \frac{\|(\nabla_{x_2,\xi_2}\nu^2)(0,0)\|^2}{2\theta}\hbar^2 + \frac{\theta}{2}\hbar|z_2|^2 + \hbar k(\hbar^{\frac12},|z_2|^2),$$

with k a smooth function with a support as small as desired w.r.t. its second variable, and $\tilde{G}_h = \hbar\mathcal{O}(|z_2|^\infty)$. It remains to notice that $\mathrm{Op}^w_\hbar\left(k(\hbar^{\frac12},|z_2|^2)\right)$ can be written as $k^\star(\hbar^{\frac12},\mathcal{K}_\hbar)$ modulo $\mathrm{Op}^w_\hbar\left(\mathcal{O}(|z_2|^\infty)\right)$. This achieves the proof of Theorem 2.15.

5.2. **Spectral reduction to the third normal form.** Corollary 2.16 is a consequence of the following lemma and proposition.

Lemma 5.1. *We have*

$$\mathsf{N}\left(\mathcal{M}_\hbar^{[1],\sharp}, \beta_0\hbar\right) = \mathcal{O}(\hbar^{-2}), \qquad \mathsf{N}\left(\mathcal{F}_\hbar, b_0\hbar + C\hbar^{1+\eta}\right) = \mathcal{O}(\hbar^{-1+\eta}).$$

Proof. The first estimate follows from Lemma 4.3 and the second one from a comparison with the harmonic oscillator in x_2. □

The last proposition concerns the microlocalization of the eigenfunctions.

Proposition 5.2. *Let $\eta \in (0,1), \delta \in \left(0, \frac{\eta}{2}\right), C > 0$. Let χ be a smooth cutoff function being 0 in a bounded neighborhood of $(0,0)$ and 1 outside a bounded neighborhood of $(0,0)$. If λ is an eigenvalue of $\mathcal{M}_\hbar^{[1],\sharp}$ or of $\mathcal{F}_\hbar$ such that $\lambda \leqslant b_0\hbar + C\hbar^{1+\eta}$ and if ψ is an associated eigenfunction, we have*

$$\mathrm{Op}_\hbar^w\left(\chi(\hbar^{-\delta}(x_2,\xi_2))\right)\psi = \mathcal{O}(\hbar^\infty).$$

Proof. The proof is similar with the one of Proposition 4.4. □

Appendix A. Egorov theorems

We start with the classical result (see for instance [31, Theorem 11.1] and [26, Théorème IV.10]).

Theorem A.1 ([31, Theorem 11.1, Remark (ii)]). *Let P and Q be h-pseudo-differential operators on $\mathbb{R}^d$, with $P \in \mathrm{Op}_h^w(S(1))$ and $Q \in \mathrm{Op}_h^w(S(1))$. Then the operator $e^{\frac{i}{\hbar}Q}Pe^{-\frac{i}{\hbar}Q}$ is a pseudo-differential operator in $\mathrm{Op}_h^w(S(1))$, and*

$$e^{\frac{i}{\hbar}Q}Pe^{-\frac{i}{\hbar}Q} - \mathrm{Op}_h^w(p\circ\kappa) \in h\,\mathrm{Op}_h^w(S(1)) .$$

Here p is the Weyl symbol of P, and the canonical transformation κ is the time-1 Hamiltonian flow associated with principal symbol of Q.

From this classical version of Egorov's theorem, one can deduce the following refinement that is useful when p does not belong to $S(1)$ (as it is the case in this paper).

Theorem A.2. *Let P and Q be h-pseudo-differential operators on $\mathbb{R}^d$, with $P \in \mathrm{Op}_h^w(S(m))$ and $Q \in \mathrm{Op}_h^w(S(m'))$, where m and m' are order functions such that:*

$$m' = \mathcal{O}(1); \quad mm' = \mathcal{O}(1). \tag{A.1}$$

Then the operator $e^{\frac{i}{\hbar}Q}Pe^{-\frac{i}{\hbar}Q}$ is a pseudo-differential whose symbol is in $S(m)$, and $e^{\frac{i}{\hbar}Q}Pe^{-\frac{i}{\hbar}Q} - \mathrm{Op}_h^w(p\circ\kappa) \in h\,\mathrm{Op}_h^w(S(1))$.

Proof. The proof is based on the following observation. In order to compare $\mathrm{Op}_h^w(p\circ\kappa^t)$ and $e^{\frac{it}{h}Q}Pe^{-\frac{it}{h}Q}$, we consider the derivative:

$$\frac{d}{d\tau}\left(e^{\frac{i\tau}{h}Q}\,\mathrm{Op}_h^w(p\circ\kappa^{t-\tau})e^{-\frac{i\tau}{h}Q}\right) = e^{\frac{i\tau}{h}Q}\left(\frac{i}{h}[Q,\mathrm{Op}_h^w(p\circ\kappa^{t-\tau})]+\frac{d}{d\tau}\,\mathrm{Op}_h^w(p\circ\kappa^{t-\tau})\right)e^{-\frac{i\tau}{h}Q}.$$

From Hypothesis (A.1), the term $[Q,\mathrm{Op}_h^w(p\circ\kappa^{t-\tau})]$ belongs to $\mathrm{Op}_h^w(S(1))$; moreover, if we denote by q_0 the principal symbol of Q, we have

$$\frac{d}{d\tau}\,\mathrm{Op}_h^w(p\circ\kappa^{t-\tau}) = -\,\mathrm{Op}_h^w(\{q_0,p\circ\kappa^{t-\tau}\}),$$

which implies that this term is also in $\mathrm{Op}_h^w(S(1))$. By symbolic calculus, we see that

$$\frac{i}{h}[Q,\mathrm{Op}_h^w(p\circ\kappa^{t-\tau})]+\frac{d}{d\tau}\,\mathrm{Op}_h^w(p\circ\kappa^{t-\tau})\in h\,\mathrm{Op}_h^w(S(1)), \tag{A.2}$$

uniformly for t,τ in compact sets. It follows by integration from 0 to t that

$$e^{\frac{it}{h}Q}Pe^{-\frac{it}{h}Q} = \mathrm{Op}_h^w(p\circ\kappa^t)+h\int_0^t e^{\frac{is}{h}Q}P_1(s)e^{-\frac{is}{h}Q}ds, \tag{A.3}$$

for some $P_1(s)\in\mathrm{Op}_h^w(S(1))$, uniformly for $s\in[0,t]$. Applying Theorem A.1 to the integrand, we see that $e^{\frac{it}{h}Q}Pe^{-\frac{it}{h}Q}-\mathrm{Op}_h^w(p\circ\kappa^t)\in h\,\mathrm{Op}_h^w(S(1))$. □

In order to quantize the formal Birkhoff procedure of Section 4.1.4, one needs to consider symbols in a class $\mathcal{C}$ stable under the Moyal product. For that purpose we first define the families of symbols $S(m;[0,1]\times(0,1])$, that is of smooth functions $a:\mathbb{R}^{2d}\times[0,1]\times(0,1]\to\mathbb{C}$ such that, for any $\alpha\in\mathbb{N}^{2d}$, there exists C_α such that, $\forall(z;\mu,h)\in\mathbb{R}^{2d}\times[0,1]\times(0,1]$,

$$|\partial_z^\alpha a(z;\mu,h)|\leqslant C_\alpha m(z)$$

and where m is an order function on $\mathbb{R}^{2d}$. The pair (μ,h) is considered as a parameter.

Then, let $(\varphi_\mu)_{\mu\in[0,1]}$ be a smooth family of linear maps $\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ and define the following families of symbols on $\mathbb{R}^{2d}$ by

$$\mathcal{C}(m)=\Big\{a\in S(m;[0,1]\times(0,1]);\quad a(z;\mu,h)=\tilde a(\varphi_\mu(z);\mu,h) \text{ with } \tilde a\in S(m;[0,1]\times(0,1])\Big\}. \tag{A.4}$$

Theorem A.3. *Let P and Q be h-pseudo-differential operators on $\mathbb{R}^d$, with $P\in\mathrm{Op}_h^w(\mathcal{C}(m))$ and $Q\in\mathrm{Op}_h^w(\mathcal{C}(m'))$, where m and m' are order functions such that:*

$$m\geqslant 1;\quad m'=\mathcal{O}(1);\quad mm'=\mathcal{O}(1).$$

Then $e^{\frac{i}{h}Q}Pe^{-\frac{i}{h}Q}=\tilde P+R$, where $\tilde P\in\mathrm{Op}_h^w(\mathcal{C}(m))$, $R\in h^\infty\,\mathrm{Op}_h^w(S(1))$, and with $\tilde P-\mathrm{Op}_h^w(p\circ\kappa)\in h\,\mathrm{Op}_h^w(\mathcal{C}(1))$.

Proof. Since φ_μ is linear, one can see (using for instance [31, Theorem 4.17]) that $\mathcal{C}$ is stable under the formal Moyal product, *i.e.* for all order functions m_1 and m_2, we have

$$(\mathcal{C}(m_1)) \star (\mathcal{C}(m_2)) \subset \mathcal{C}(m_1 m_2) + h^\infty S(1).$$

Let κ be the canonical transformation associated with Q. Then, since $m \geqslant 1$, we have $p \circ \kappa \in \mathcal{C}(m)$; indeed, if we write the Hamiltonian flow of Q in terms of the variable $\tilde{z} = \varphi_\mu(z)$, we see from the linearity of φ_μ that the components of the transformed vector field belong to $\mathcal{C}(m')$. Therefore $\varphi_\mu \circ \kappa$ is of the form $\tilde{\kappa}_\mu \circ \varphi_\mu$, for some diffeomorphism $\tilde{\kappa}_\mu$ depending smoothly on μ.

Therefore, both terms in (A.2) belong to $\mathrm{Op}_h^w(\mathcal{C}(1))$. Applying this argument inductively in (A.3), we may write, for any $k > 0$,

$$e^{\frac{i}{h}Q} P e^{-\frac{i}{h}Q} - \mathrm{Op}_h^w(p \circ \kappa) - (h\tilde{P}_1 + h^2 \tilde{P}_2 + \cdots + h^k \tilde{P}_k) \in h^{k+1} \mathrm{Op}_h^w(S(1)),$$

with $\tilde{P}_j \in \mathrm{Op}_h^w(\mathcal{C}(1))$. By a Borel summation in h, parametrized by $\tilde{z} = \varphi_\mu(z)$, we can find a symbol $\hat{P} \in \mathrm{Op}_h^w(\mathcal{C}(1))$ such that we have the asymptotic expansion in $\mathrm{Op}_h^w(S(1))$:

$$\hat{P} \sim h\tilde{P}_1 + h^2 \tilde{P}_2 + \cdots$$

We conclude by letting $\tilde{P} = \mathrm{Op}_h^w(p \circ \kappa) + \hat{P}$. □

We will also need to examine how the Egorov theorem behaves with respect to taking formal power series of symbols. For this, it is convenient to introduce a filtration of $S(m)$.

Theorem A.4. *Let m be an order function on $\mathbb{R}^{2d}$, and let $(\mathcal{O}_j)_{j\in\mathbb{N}}$ be a filtration of $S(m)$,* i.e.*:*

$$\mathcal{O}_0 = S(m), \quad \mathcal{O}_{j+1} \subset \mathcal{O}_j.$$

Let $P = \mathrm{Op}_h^w p$ and $Q = \mathrm{Op}_h^w q$ be h-pseudo-differential operators on $\mathbb{R}^d$, with $p \in S(m)$ and $q \in S(m')$, where m' is an order function such that m' and mm' are bounded.

Assume that:

$$\frac{i}{h} \mathrm{ad}_q(\mathcal{O}_j) \subset \mathcal{O}_{j+1}; \qquad \forall j \geqslant 0. \tag{A.5}$$

Then for any $k \geqslant 0$, the Weyl symbol of the pseudo-differential operator $e^{\frac{i}{h}Q} P e^{-\frac{i}{h}Q} - \sum_{j=0}^k \frac{1}{j!}(\frac{i}{h}\mathrm{ad}_Q)^j P$ belongs to $\mathrm{Op}_h^w(\mathcal{O}_{k+1})$. In other words, the series of $\exp(\frac{i}{h}\mathrm{ad}_Q)P$ converges to $e^{\frac{i}{h}Q} P e^{-\frac{i}{h}Q}$ for the filtration $(\mathcal{O}_j)_{j\in\mathbb{N}}$.

Proof. By the Taylor formula, we can write

$$e^{\frac{i}{h}Q} P e^{-\frac{i}{h}Q} = \sum_{j=0}^{k} \frac{1}{j!}(\mathrm{ad}_{ih^{-1}Q})^j P + \frac{1}{k!}(\mathrm{ad}_{ih^{-1}Q})^{k+1} \int_0^1 (1-t)^k e^{\frac{it}{h}Q} P e^{-\frac{it}{h}Q} dt.$$

By Theorem A.2, we see that the integral belongs to $\mathrm{Op}_h^w(S(m)) = \mathrm{Op}_h^w(\mathcal{O}_0)$. Therefore, by Assumption (A.5), the remainder in the Taylor formula lies in $\mathrm{Op}_h^w(\mathcal{O}_{k+1})$. □

Acknowledgments. The authors would like to thank Yves Colin de Verdière for stimulating discussions. This work was partially supported by the ANR (Agence Nationale de la Recherche), project NOSEVOL n° ANR-11-BS01-0019 and by the Centre Henri Lebesgue (program "Investissements d'avenir" – n° ANR-11-LABX-0020-01). Y. K. was partially supported by the Russian Foundation of Basic Research, project 13-01-91052-NCNI-a. During the completion of this work, B. H. was Simons foundation visiting fellow at the Isaac Newton Institute in Cambridge.

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