A Simple and Stable Scale Similarity Model for Large Eddy Simulation: Energy Balance and Existence of Weak Solutions

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Abstract

In averaging the Navier-Stokes equations the problem of closure arises. Scale similarity models address closure by (roughly speaking) extrapolation from the (known) resolved scales to the (unknown) unresolved scales. In a posteriori tests scale similarity models are often the most accurate but can prove to be unstable when used in a numerical simulation. In this report we consider the scale similarity model given by

$$\nabla \cdot w = 0 \text{ and } w_t + \nabla \cdot (\overline{\overline{w}\,\overline{w}}) - \nu \Delta w + \nabla p = f.$$

We prove it is stable (solutions satisfy an energy inequality) and deduce from that existence of weak solutions of the model.

Key Words and Phrases: Large eddy simulation, scale similarity, turbulence, energy inequality, weak solution

1 Introduction

In the numerical solution of turbulent flows one seeks to approximate suitable averages of the pointwise fluid velocity, [MP94]. When the average in question is a local, spacial average the approach is known as large eddy simulation, or LES, [Sag98]. Averaging the nonlinear term in the Navier-Stokes equations leads to the problem of closure modeling. To be specific, let (u, p) denote the pointwise fluid velocity and pressure which are assumed to satisfy the incompressible NSE,

$$(1.1) \qquad \begin{cases} u_t + \nabla \cdot (u\ u) - \nu \Delta u + \nabla p = f & \text{in } \Omega (:= (0, 2\pi)^3) \times (0, T), \\ \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T), \end{cases}$$

subject to 2π-spending periodic boundary conditions (assumed throughout), the initial condition $u(x, 0) = u_0(x)$ and the usual zero mean condition $\int_\Omega u dx = \int_\Omega p dx = \int_\Omega f dx = 0$.
The considered model in this report can be developed for quite general averaging operators. To fix ideas herein we shall specify a differential filter, Germano [Ger86], Mullin and Fischer [MF00], for

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reasons detailed, e.g., in Layton and Lewandowski [LL02], defined as follows. Let the lengthscale $\delta > 0$ be fixed. Given $\phi \in L_0^2(\Omega), \overline{\phi} \in H^2(\Omega) \cap L_0^2(\Omega)$ is the unique solution of

$$-\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi \quad \text{in } (0, 2\pi)^3, \tag{1.2}$$

subject to periodic boundary conditions. Under periodic boundary conditions, this averaging operator commutes with differentiation.
Averaging (1.1) gives the (not closed) space-filtered NSE for $\overline{u}$:

$$\overline{u}_t + \nabla \cdot (\overline{u}\,\overline{u}) - \nu \Delta \overline{u} + \nabla \overline{p} - \nabla \cdot \mathcal{R}(u,\ u) = \overline{f}, \qquad \nabla \cdot \overline{u} = 0, \tag{1.3}$$

subject to $\overline{u}(x, 0) = \overline{u}_0(x)$ and periodic boundary conditions where $\mathcal{R}(u,\ u) := \overline{u\,u} - \overline{u}\,\overline{u}$ is the Reynolds stress tensor. One closure model is scale similarity, introduced by Bardina, Ferziger and Reynolds [BFR80] and well explained in Sagaut [Sag98]:

$$\mathcal{R}(u,\ u) \sim \mathcal{S}(\overline{u},\ \overline{u}) := \overline{\overline{u}\,\overline{u}} - \overline{\overline{u}}\,\overline{\overline{u}}. \tag{1.4}$$

Scale-similarity models typically prove the most accurate in á priori testing (Sagaut [Sag98], Sarghini, Piomelli and Balaras [SPB99], Carati, Winkelman and Jenmart [CWJ00] and Winkelmans, Lund, Carati and Wray [WLCW96]). Also, scale similarity models are reversible; they can provide (so-called) backscatter; they best align the principal axes of the true Reynolds stress tensor $\mathcal{R}$ with its model $\mathcal{S}$. However, stability problems have been reported for them. These have led to more intricate models such as [Lay00], Horiuti's [Hor89] filtered Bardina model, the Liu, Meneveau, Katz [LMK94] model and many 'mixed' models in which additional eddy viscosity terms are added for stability reasons.
In this report we consider a scale similarity model much simpler than (1.4) which has an interesting and a surprisingly strong stability property. To explain the model, first consider an alternate form of the nonlinear terms in (1.3):

$$\nabla \cdot (\overline{u}\,\overline{u}) - \nabla \cdot \mathcal{R}(u,\ u) \equiv \nabla \cdot (\overline{u\,u})$$

and model this via scale similarity by

$$\nabla \cdot (\overline{u\,u}) \sim \nabla \cdot (\overline{\overline{u}\,\overline{u}}). \tag{1.5}$$

This is clearly equivalent to the model of $\mathcal{R}$ given by

$$\mathcal{R}(u,\ u) \sim \tilde{\mathcal{S}}(\overline{u},\ \overline{u}) := \overline{\overline{u}\,\overline{u}} - \overline{u}\,\overline{u}. \tag{1.6}$$

Calling (w,q) the approximations to $(\overline{u}, \overline{p})$ resulting from (1.5), (1.6), (w, q) then satisfies

$$w_t + \nabla \cdot (\overline{w\,w}) - \nu \Delta w + \nabla q = \overline{f}, \qquad \nabla \cdot w = 0, \tag{1.7}$$

subject to $w(x, 0) = \overline{u}_0(x)$ and periodic boundary conditions (with zero means).

Remark 1.1 *Comparing* (1.6) *to the Bardina model* (1.4)*, the Bardina model approximates two terms in* $\mathbf{R}$ *while* (1.6) *only one term in* $\mathbf{R}$*. Furthermore, in the expansion of* $\mathbf{R}$ *into* $O(1), O(\delta^2)$ *and* $O(\delta^4)$ *terms* $\mathbf{R}(u,\ u) = \overline{\overline{u}\,\overline{u}} + (\overline{\overline{u}\ u' + u'\ \overline{u}}) + \overline{u'\ u'}$, (1.6) *is also the simplest* $O(\delta^2)$ *approximation in which the* $O(\delta^2)$ *and higher terms are simply dropped.*

To present our main results on the new model (1.7) we must first introduce some notation (in this we follow the exposition in [Gal00], see also [Gal94], [Ler34], [Lio69] and [CF88]). Let $\Omega = (0, 2\pi)^3$,

$$\mathcal{D}(\Omega) := \{\psi \in C^\infty \text{ periodic fields } (\Omega) : \nabla \cdot \psi = 0 \text{ in } \Omega \text{ and } \int_\Omega \psi \, dx = 0\},$$

and

$$\mathcal{D}_T := \{\phi(t, \cdot) \in \mathcal{D}(\Omega) \text{ for } 0 \le t \le T,\ \phi(T, \cdot) = 0\}$$

The usual $L^2(\Omega)$ inner product is denoted $(\cdot, \cdot)$; $L^2_0(\Omega)$ denotes, as usual, the space of L^2 functions with zero mean over Ω. Moreover, $H(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in the usual $L^2(\Omega)$ norm (denoted $||\cdot||$ herein) while $V = V(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in the usual $H^1(\Omega)$ norm.

Definition 1.1 *Let $\overline{u}_0 \in H(\Omega)$, $\overline{f} \in L^2([0, T] \times \Omega)$. A measurable function $w : [0, T] \times \Omega \to \mathbf{R}^3$ is a weak solution of (1.7) in $[0, T] \times \Omega$ if*

$w \in Y := L^2(0, T; V) \cap L^\infty(0, T; H)$, and w satisfies

$$\left\{ \begin{array}{l} \displaystyle\int_0^T \Big\{ \Big(w, \frac{\partial \phi}{\partial t}\Big) - \nu(\nabla w, \nabla \phi) - (\nabla \cdot (\overline{w\, w}), \phi) \Big\} dt \\ \displaystyle = - \int_0^T (\overline{f}, \phi) dt - (\overline{u}_0, \phi(\cdot, 0)), \text{ for all } \phi \in \mathcal{D}_T. \end{array} \right. \tag{1.8}$$

The main result of this report is that weak solutions of the new LES model (1.7) exist globally in time, for large data and general $\nu > 0$ and that they satisfy an energy equality while the initial data and the source term are smooth enough.

Theorem 1.1 *[Stability and Existence for the Model]. Let $\delta > 0$ be fixed. For any $\overline{u}_0 \in V, \overline{f} \in L^2(0, T; H^1)$ there exists at least one weak solution w to (1.7). That weak solution also belongs to $L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ and $\partial_t w \in L^2((0, T) \times \Omega)$. Moreover, the following energy equality holds for $t \in [0, T]$,*

$$k(t) + \int_0^t \epsilon(t')dt' = k(0) + \int_0^t P(t')dt', \tag{1.9}$$

where

$$\left\{ \begin{array}{l} k(t) = \frac{1}{2}||w(t, \cdot)||^2 + \frac{\delta^2}{2}||\nabla w(t, \cdot)||^2, \quad \epsilon(t) = \nu||\nabla w(t, \cdot)||^2 + \nu\delta^2||\Delta w(t, \cdot)||^2, \\ P(t) = (\overline{f}(t, \cdot), w(t, \cdot)) + \delta^2(\nabla \overline{f}(t, \cdot), \nabla w(t, \cdot)) = (f, w). \end{array} \right. \tag{1.10}$$

Remark 1.2 *Once existence in the large of a weak solution to (1.7) is known. Further theoretical properties of the model, relevant for practical calculations, can then be developed. These are currently under study by the authors and will be presented in a subsequent report.*

Remark 1.3 *The model (1.5), (1.6) is a reversible closure model (unlike eddy viscosity type models) and thus attractive for long time calculations. The reversible/dispersive nature of the model is evidenced in the energy equality.*

Remark 1.4 *In a further paper, the author will prove that the solution w of (1.7) converges in some sense to a solution of (1.1) when δ goes to zero. Finally, pressure is recovered from the weak formulation via the classical De Rham Theorem (see [Lio69]).*

2 Proof of the main result

2.1 Orientation

The proof of Theorem 1.1 follows the classical scheme. One first seeks for *a priori* estimates. Next, one constructs approximated solutions and then one passes to the limit in the equations after having proved compactness properties. In the case of this problem, approximated solutions can be constructed via the classical Galerkin method and then this point will be skipped in this report. We shall focus our attention on the *a priori* estimates and compactness properties of problem (1.7).

2.2 Basic estimates

The main estimate is derived from the multiplication of (1.7) by $w - \delta^2 \Delta w$ in place of the classical multiplication by w. Note that the nonlinear term vanishes since

$$(\nabla \cdot (\overline{w\ w}), (-\delta^2 \Delta + 1)w) = (\nabla \cdot (w\ w), \overline{(-\delta^2 \Delta + 1)w}) = (\nabla \cdot (w,\ w), w) = 0.$$

Integrating the result in time gives immediately

$$\begin{aligned} &\left[\frac{1}{2}||w(t)||^2 + \frac{\delta^2}{2}||\nabla w||^2\right] + \int_0^t \left\{\nu||\nabla w||^2 + \nu\delta^2||\Delta w||^2\right\}dt' = \\ &\left[\frac{1}{2}||\overline{u}_0||^2 + \frac{\delta^2}{2}||\nabla \overline{u}_0||^2\right] + \int_0^t (f, w)dt' + \delta^2 \int_0^t \nabla f : \nabla w\, dt'. \end{aligned} \tag{2.1}$$

By using Cauchy-Schwarz and Young's inequalities, it is deduced from (2.1) that, since $\delta > 0$,

$$w \in L^\infty(0, T; (H^1)^3) \cap L^2(0, T; (H^2)^3). \tag{2.2}$$

In the following, one notes

$$E = \left\{w \in L^\infty(0, T; (H^1)^3) \cap L^2(0, T; (H^2)^3);\ \nabla \cdot w = 0;\ w\ [0, 2\pi]^3 \text{ periodic in space}\right\} \tag{2.3}$$

2.3 Regularity of the nonlinear term and time derivative estimate

Problem (1.7) can be viewed like a stokes problem under the form

$$w_t - \nu \Delta w + \nabla q = f - \nabla \cdot Q_\delta(w) \tag{2.4}$$

where $Q_\delta(w) = \overline{w\ w}$. Of course, one have in view to obtain an estimate for w_t for using Aubin-Lions Lemma (see in [Lio69]). Thus an estimate for Q_δ is necessary.

Lemma 2.1 *One has*

$$\forall\, w \in E, \quad Q_\delta(w) \in L^\infty(0, T; (H^2)^3) \tag{2.5}$$

Proof. Let $w \in E$ and $Q = w\, w$. By using classical Hölder inequality, it is easy checked that $Q \in L^4(0, T; (H^1)^9) \cap L^\infty(0, T; (L^3)^9)$. Now one has

$$Q_\delta - \delta^2 \Delta Q_\delta = Q \tag{2.6}$$

with space periodic conditions. Multiplying (2.6) by Q_δ, integrating in space on Ω and using Cauchy-Schwarz inequality yields

$$\frac{1}{2}\int_\Omega Q_\delta^2 + \delta^2 \int_\Omega |\nabla Q_\delta|^2 \leq \frac{1}{2}\int_\Omega Q^2. \tag{2.7}$$

Then $Q_\delta(w) \in L^\infty(0,T;(H^1)^9)$. Next, one multiplies (2.6) by $\Delta Q_\delta(w)$. One has

$$\int_\Omega Q_\delta.\Delta Q_\delta - \delta^2 \int_\Omega |\Delta Q_\delta|^2 = \int_\Omega Q.\Delta Q_\delta \tag{2.8}$$

Therefore, (2.5) follows from the combination of (2.7), (2.8) with Young's inequality.

Remark 2.1 *Notice that periodic conditions plays a crucial role for obtaining these estimates as well as the next estimates. We do not know wether it remains true with more realistic boundary conditions.*

Now, using equation (1.7) under the form (2.4) combined with (2.5) and taking into account the periodic condition, it is easy proved that

$$w_t \in L^2([0,T]\times\Omega). \tag{2.9}$$

2.4 Passing to the limit in the equations

Consider now a sequence $(w_n)_{n\in\mathbb{N}}$ of solutions to (1.7). We have to prove that this sequence converges to a solution to problem (1.7) (up to a subsequence). Notice that the previous estimates ensures that the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in the space

$$F = \left\{ w \in E;\ w_t \in [L^2([0,T]\times\Omega)]^3 \right\}, \tag{2.10}$$

equipped with its natural norm. Thus from the sequence $(w_n)_{n\in\mathbb{N}}$ one can extract a subsequence (still denoted $(w_n)_{n\in\mathbb{N}}$) converging weakly to some $w \in F$. One have to prove that w is a solution to (1.7).

Because the injection of H^2 onto H^1 is compact (in periodic case) and $(w_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;(H^2)^3)$ whereas $((w_n)_t)_{n\in\mathbb{N}}$ is bounded in $[L^2([0,T]\times\Omega)]^3$, Aubin-Lions's lemma applies and then $(w_n)_{n\in\mathbb{N}}$ is compact in $L^2(0,T;(H^1)^3)$. It is now easely deduced from that fact that the sequence $(w_n\, w_n)_{n\in\mathbb{N}}$ converges strongly to $w\,w$ in the space $L^2([0,T]\times\Omega)]^9$. Thus, $Q_\delta(w_n)$ converges strongly to $Q_\delta(w)$ in $L^2(0,T;(H^1)^9)$. Now the fact that w is a solution to problem (1.7) is a consequence of classical results on Stokes problem when it is viewed as in (2.4). Due to the regularity properties on Stokes problem, L^2 multipliers are "authorized" like $w - \delta^2\Delta w$. This is why the energy budget (1.9) is satisfied. The proof of theorem 1.1 follows now the classical scheme of proof as developped in [Lio69].

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