J. math. fluid mech. 4 (2002) 1–26
1422-6928/02/030001-26 \$ 1.50+0.20/0

Journal of Mathematical Fluid Mechanics

Analysis of an Eddy Viscosity Model for Large Eddy Simulation of Turbulent Flows

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Communicated by G. P. Galdi

Abstract. One simple, computationally attractive yet mathematically intractable eddy viscosity model for turbulent viscous flows reads: $\nabla \cdot w = 0$ and

$$w_t + \nabla \cdot (ww) - \nabla \cdot ((\nu + \beta\delta|w - \bar{w}|)\nabla^s w) + \nabla q = f,$$

where $\bar{w}$ is an average of w, δ a length scale (or grid scale), β is a positive constant and $\nabla^s w$ the deformation tensor, f the locally averagged body forces acting on the fluid. We outline the genesis of the model first in the periodic case, secondly when the fluid flows in a domain with a boundary, making precise the average process. Then we perform the mathematical analysis of the model showing that it leads to a well-posed problem in a weak sense.

Mathematics Subject Classification (2000). 76F65, 76D03.

Keywords. Large eddy simulation, turbulence, weak solution.

1. Introduction

Because computational resources are insufficient for the foreseeable future for the simulation of turbulent flow, it is usual to seek not to approximate the pointwise fluid velocity but rather suitable averages of it. The approach currently thought to be most promising is large eddy simulation (LES) in which the averages are local, spacial averages of the fluid velocity. However, the turbulence models used in LES lead to analytical questions which are critical for evaluation of the models and for designing a proper simulation and whose answers are completely unknown.

This paper considers one such question for the model for the motion of large eddies in a turbulent fluid given as follows: for $\Omega \subset \mathbf{R}^d, d = 2, 3$, a domain with smooth boundary, find

$$w(x,t) : \Omega \times [0,T] \to \mathbf{R}^d,\ q(x,t) : \Omega \times (0,T] \to \mathbf{R}$$

The work of W. J. Layton was partially supported by NSF grants DMS 9972622, INT 9814115 and INT 9805563.

satisfying:

$$\begin{cases} w_t + \nabla \cdot (ww) - \nabla \cdot [(\nu + \nu_T)\nabla^s w] + \nabla q = f(x,t), \text{ in } \Omega, \\ \nabla \cdot w = 0, \quad \text{in } \Omega, \\ w = 0 \text{ on } \partial\Omega \text{ and } w(x,0) = w_0(x), \text{ in } \Omega. \end{cases} \tag{1.1}$$

In (1.1), ν is the (positive) kinematic viscosity, ν_T is the eddy viscosity coefficient, $f(x,t)$ is the (smooth) body force, $w_0(x)$ is the (smooth) initial velocity, and ∇^s is the symmetric part of the gradient tensor (the deformation tensor). The mathematical challenge in (1.1) arises from the presence of a nonlinear and unbounded turbulent viscosity coefficient ν_T, given by:

$$\nu_T = \beta\delta|w - \overline{w}|, \tag{1.2}$$

where $\beta > 0$ is a fixed positive constant, $0 < \delta$ is the (small) eddy length scale of the resolution sought and $\overline{w}$ denotes a local spacial average of w. The model (1.1), (1.2) (from Iliescu and Layton [13]) is particularly simple and compelling. We shall briefly review in section 2 the genesis of (1.1), (1.2) and discuss various options for the averaging operator before proceeding to the analysis of (1.1), (1.2).

Because the diffusion operator

$$\nabla \cdot (\delta|w - \overline{w}|\nabla^s w)$$

in (1.1) is nonlinear, non-monotone and contains an unbounded (for $w \in H^1(\Omega)$) coefficient, the problem (1.1), (1.2) lies outside the usual theory of existence of weak solutions for the Navier–Stokes equations originally due to J. Leray [14]. We prove in this paper an existence result of a solution with finite energy for the system (1.1), (1.2). Here, the obtained solution is only a distributionnal solution in a very weak sense. Thus, it remains to know whether this solution is unique and smooth. The paradox lies in the fact that the model is obtained after filtering the Navier–Stokes equation through a convolution process. Thus, the solution w of (1.1) models a filtered solution of the Navier–Stokes equation and should be smooth and unique. Unfortunately, the problem of uniqueness and regularity of solutions to (1.1), (1.2) is an open problem.

Acknowledgements. The authors thank warmly their colleague Isabelle Gruais for reading this paper carefully and for relevant remarks on it.

2. The model

2.1. Periodic case or whole space

The model (1.1), (1.2) arises very simply as a model of the motion of the eddies of size $\geq O(\delta)$ in a turbulent fluid and is based on modelling the subgrid kinetic energy (see the models summarized in [17], [28]). There are a number of such models, for example Lesieur [16], Horiuti [12], Moeng [24] and Sagaut [29]. The one we study is perhaps the simplest and most direct within this class.

First the desired eddy length scale δ is selected and an averaging operator chosen. There are various possibilities but to fix ideas let us ignore boundaries for the moment and consider averaging via convolution with a gaussian as in the case of periodic boundary conditions or in the whole space. To this end, define

$$g(x) := \left(\frac{6}{\pi}\right)^{3/2} \exp(-6|x|^2), \qquad g_\delta(x) := \delta^{-d} g\left(\frac{x}{\delta}\right),$$

and the averaging operator is:

$$\overline{u}(x) = (g_\delta * u)(x) = \int_{\mathbf{R^d}} g_\delta(x-\xi)u(\xi)d\xi.$$

Convolving the Navier–Stokes equations with $g_\delta(x)$ reveals that $\overline{u}$ satisfies the following (often called the space filtered NSE)

$$\begin{cases} \overline{u}_t + \nabla\cdot(\overline{u}\,\overline{u}) - \nu\Delta\overline{u} + \nabla\overline{p} - \nabla\cdot(\overline{u}\,\overline{u} - \overline{uu}) = \overline{f}, \text{ in } \Omega, \\ \nabla\cdot\overline{u} = 0, \text{ in } \Omega. \end{cases} \tag{2.1}$$

This model is not *closed* due to the presence of the Reynolds stress tensor

$$\tau := \overline{u}\,\overline{u} - \overline{uu},$$

which depends upon u, and not solely $\overline{u}$.

The most common closure model used for $\overline{u}$ is the eddy viscosity hypotheses or Boussinesq assumption. In 1877 Boussinesq [2] postulated that the turbulent fluctuations in a fluid are "dissipative in the mean" based on an analogy between the interaction of small turbulent eddies and elastic collisions of molecules. This is written as:

$$-\nabla\cdot(\tau) \sim -\nabla\cdot(\nu_T \nabla^s \overline{u}) - \frac{2}{3}\,k\,I, \tag{2.2}$$

where the last term is incorporated into an adjusted pressure.

The modelling difficulty is now shifted to determining the turbulent viscosity coefficient ν_T. The most mathematically convenient choice is given by the Smagorinsky–Ladyzhenskaya model:

$$\nu_T = (C_s\delta)^2|\nabla^s\overline{u}|, \qquad C_s \simeq 0.1 \text{ or } 0.17. \tag{2.3}$$

Although widely used, the shortcomings of (2.3) are also widely understood. These include:

(i) (2.2) can introduce large amounts of viscosity for many laminar flow problems (without turbulent fluctuations at all) such as shear flows,

(ii) (2.3) produces far too much damping on the large eddies: ν_T is too large,

(iii) and the general feeling that (2.3) is motivated by mathematical and computational convenience rather than sound physical considerations.

The formula (1.2) for ν_T considered herein arises from physical considerations as follows ([13]). First, accepting Boussinesq's reasoning, ν_T should depend only upon the length scale δ and the local average of the kinetic energy in the turbulent fluctuations, $\nu_T = \nu_T(\delta, \overline{k}')$. The functional dependence is well known and follows from dimensional analysis, giving the Prandtl–Kolmogorov relation:

$$\nu_T = \sqrt{2}\beta\ \delta\sqrt{\overline{k'}}, \quad (\beta > 0 \text{ constant}).$$

As $u = \overline{u} + u'$, the turbulent fluctuation's kinetic energy is easily calculated to be $k' = \frac{1}{2}|u'|^2 = \frac{1}{2}|u - \overline{u}|^2$. Thus, we have

$$\nu_T = \beta\delta\ \sqrt{\overline{|u - \overline{u}|^2}}.$$

This is still not closed. Different closure approximations can be used. The simplest one is consistency on (locally) constant velocities, giving:

$$\overline{|u - \overline{u}|^2}\ \sim\ |\bar{u} - \overline{\overline{u}}|^2. \tag{2.4}$$

This is exactly a scale-similarity type assumption (Sagaut [28]) that can be described physically as follows. Turbulent dissipation is often viewed as occuring through the breakdown of large eddies into a sequence of successively smaller eddies until small enough for molecular viscosity to drive to zero. With this view, the essential step going from resolved scales to unresolved ones is from $O(\delta)$ scales to $O(\delta/2)$ scales:

$$\overline{|u - \overline{u}|^2}\ \sim\ \overline{|\overline{u} - g_{\delta/2} * u|^2}.$$

The scale similarity assumption is that *in the mean*, the breakdown from $O(\delta)$ to $O(\delta/2)$ eddies is comparable to that from $O(2\delta)$ to $O(\delta)$ eddies, or:

$$g_\delta * |g_\delta * u - g_{\delta/2} * u|^2 \sim |g_{2\delta} * u - g_\delta * u|^2,$$

giving (2.4) above.

Remark 2.1. In [28], it is proved that the error made in (2.4) is of order δ^2. The proof holds in a monodimensional case and for smooth functions by using the Taylor formula. It seems that there is an interesting problem in the general case behind this question.

2.2. Parameter selection

The question of parameter selection of β is important for use of (1.1). Like the Smagorinsky model, the parameter β should not be thought of as a data fitting parameter but rather one pre-tuned, as follows, to homogeneous isotropic turbulence.

The usual analytical method of determining the one parameter C_s in the Smagorinsky model, introduced by Lilly [22], is to consider the case of homogeneous isotropic turbulence. Kolmogorov's 1941 theory (see Lesieur [17], Frisch

[9]) predicts an energy spectrum $E(k)$ of the form

$$E(k) \sim \alpha\epsilon^{2/3}\ k^{-5/3},\ \alpha \sim 1.4,$$

$$\epsilon = \text{ energy dissipation rate.}$$

Using this, and under various plausible conditions, the model's energy decay rate ϵ_{model} can be estimated. Equating $\epsilon = \epsilon_{\text{model}}$ yields an equation which is independent of ϵ and gives a universal value of C_s of roughly $C_s = 0.17$.

This same approximate analysis can be applied to the present model to estimate the parameter β. The most direct path to so fitting β to homogeneous, isotropic turbulence is to begin with the fitted value $C_s \simeq 0.17$ for the Smagorinsky constant and equate the net effects of the Smagorinsky turbulent viscosity ν_s and the new one ν_T. Equating $||\nu_s||^2$ and $||\nu_T||^2$ gives

$$(C_s\delta)^4 \int_\Omega |\nabla^s w|^2 dx \doteq (\beta\delta)^2 \int_\Omega |w - \overline{w}|^2\ dx.$$

One assumption behind Lilly's analysis is that the energy spectrum of w is (approximately) that of u, truncated to $0 \leq k \leq k_c$, where k_c the cutoff wave number associated with the length scale $\delta, k_c = \pi/\delta$.

Since this assumption is connected with a sharp Fourier cutoff filter, it gives:

$$(C_s\delta)^4 \int_0^{k_c} k^2 E(k) dk \doteq (\beta\delta)^2 \int_{\frac{1}{2}k_c}^{k_c} E(k) dk.$$

Inserting the above expression for the energy spectrum $E(k)$ and calculating gives:

$$(C_s\delta)^4 \alpha\epsilon^{2/3}\ \frac{3}{4}k_c^{\frac{4}{3}} \doteq (\beta\delta)^2\alpha\epsilon^{2/3}\Big(-\frac{3}{2}\Big)\Big[k_c^{-2/3} - \Big(\frac{1}{2}k_c\Big)^{-2/3}\Big].$$

Algebraic simplification and insertion of the value $k_c = \pi/\delta$ gives:

$$\beta \doteq C_s^2 \pi 2^{-1/2}(2^{2/3} - 1)^{-1/2} \cong 0.17.$$

2.3. Case of a domain with a boundary

As noted above, there are numerous averaging operators used in LES. In the presence of boundaries, and using constant δ, we can extend u by zero off Ω and proceed as above. This leads to nontrivial questions concerning correct boundary conditions for flow averages (which are non-local), see, e.g., Sahin [30], [10] for first steps and noncommutativity of some processes, Dunca, John, Layton and Sahin [7].

We wish to outline here a different averaging procedure which is the natural extension to bounded domains of convolution by a gaussian.

For the Cauchy problem, the solution operator for the heat equation, the time dependent Stokes operator and convolution by gaussian all coincide. Thus, for a bounded domain one natural way to define the averaging operator is as follows. Given $u(x)$, solve the following evolutionary problem for: $u(x,s)$

$$\begin{cases} \dfrac{\partial u(x,s)}{\partial s} - \Delta u(x,s) + \nabla\ r(x,s) = 0, \text{ in } \Omega,\ s>0 \\ \nabla \cdot\ u(x,s) = 0, \text{ in } \Omega,\ s>0, \\ u(x,0) = u(x) \text{ (the given function) in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases} \tag{2.5}$$

Next define

$$\overline{u}(x) = u(x, \delta^2). \tag{2.6}$$

Since δ (hence δ^2) is small, a reasonable approximation is to do one step of backward Euler. This defines $\overline{u}$ to be the solution of the BVP:

$$\begin{cases} -\delta^2 \Delta \overline{u} + \overline{u} + \nabla r = u, \text{ in } \Omega \\ \nabla \cdot \overline{u} = 0, \text{ in } \Omega, \\ \overline{u} = 0, \text{ on } \partial\Omega. \end{cases} \tag{2.7}$$

This averaging commutes with the Stokes operator in the NSE and it preserves both the Dirichlet (no slip) boundary condition and incompressibility. Thus, with this averaging (and unlike convolution, Sahin [30], [7])

$$\nu_T = \beta\delta|u - \overline{u}|(x) \to 0, \text{ as } x \to \partial\Omega,$$

mimicing the same behavior of the Reynolds stresses $\tau = (\overline{u}\ \overline{u} - \overline{uu}) \to 0$ as $x \to \partial\Omega$.

This idea of differential filtering has many connections to the work of Germano [11], Domaradzki and Holm [6], [10], Mullen and Fischer [26] and many others.

In the next section, we shall not specify the exact averaging operator but rather assume some fairly general properties (including preservation of Dirichlet boundary conditions) shared by (2.5), (2.6).

3. Setting the mathematical problem

3.1. Space functions

Let Ω be a bounded domain in $\mathbf{R}^d,\ d = 2,3$, with smooth boundary and $T > 0$ a fixed real number. As usual when one studies the Navier–Stokes equations, we introduce the following spaces functions (see [23], [18], [19] or [33]):

$$\begin{cases} \mathbf{D} = \left\{ u \in (\mathcal{D}(\Omega))^d;\ \nabla \cdot u = 0 \right\}, \\ \mathbf{L}_p = \left\{ u \in (L^p(\Omega))^d;\ u.n|_{\partial\Omega} = 0,\ \nabla \cdot u = 0 \right\}, \\ \mathbf{H} = \left\{ u \in (H_0^1(\Omega))^d;\ \nabla \cdot u = 0 \right\}, \\ \mathbf{W}_{q,p} = \left\{ u \in (W_0^{q,p}(\Omega))^d;\ \nabla \cdot u = 0 \right\}, \end{cases} \tag{3.1}$$

where n denotes the unit outward normal vector to $\partial\Omega$, $p \geq 1$. Notice that for all $1 \leq p < 6$ if $d = 3$, $1 \leq p < \infty$ if $d = 2$, for all q with $1/p + 1/q < 1$,

$$\mathbf{H} \subset \mathbf{L}_p \subset (\mathbf{W}_q)', \tag{3.2}$$

the first embedding being continuous dense and compact, the second being continuous. We shall also introduce

$$\mathbf{E} = L^2([0,T], \mathbf{H}), \quad \mathbf{F}_{q,p} = L^q([0,T], \mathbf{L}_p), \tag{3.3}$$

and

$$\mathbf{G} = \{u \in C^\infty([0,T]), \mathbf{D});\ u(T,.) = 0\}\,. \tag{3.4}$$

These spaces are equipped with their natural topologies. In particular, $\mathbf{E}$ and $\mathbf{F}_{q,p}$ are normed by

$$||w||_{\mathbf{E}} = \left(\int_0^T \int_\Omega |\nabla w|^2 dxdt\right)^{\frac{1}{2}}, \quad ||w||_{\mathbf{F}_{q,p}} = \left(\int_0^T \left(\int_\Omega |w|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}$$

3.2. Variational formulation and main result

The general problem we are concern with is finding a vector field w and a scalar q such that

$$\begin{cases} \partial_t w + \nabla \cdot (ww) - \nabla \cdot [A(w)\nabla w] + \nabla q = f(x,t) \text{ in } \Omega, \\ \nabla \cdot w = 0 \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w(x,0) = w_0(x) \text{ in } \Omega. \end{cases} \tag{3.5}$$

For the sake of the simplicity, we shall work with the operator ∇ unless ∇^s without changing the nature of the mathematical analysis, thanks to Korn's inequality (see [4]). Throughout the paper, we shall assume that

$$f \in [L^2([0,T] \times \Omega)]^d, \quad w_0 \in \mathbf{L}_2. \tag{3.6}$$

We do the following hypotheses on A:

(H1) $\exists\, C \in \mathbb{R}\,;\ \forall\, w \in \mathbf{E} \cap \mathbf{F}_{\infty,2}$,

$$\begin{cases} A(w) \in L^\infty([0,T], L^2(\Omega)), \\ \\ ||A(w)||_{L^\infty([0,T],L^2(\Omega))} \leq C(1 + ||w||_{\mathbf{F}_{\infty,2}}). \end{cases} \tag{3.7}$$

(H2) $\exists\, \rho > 0$ such that $\forall\, w \in \mathbf{E}$,

$$A(w) \geq \rho \quad \text{a.e in } \Omega \tag{3.8}$$

(H3) Given any sequence in $\mathbf{E}$, $(w_n)_{n\in N}$, which converges weakly in $\mathbf{E}$ to some function $w_\infty \in \mathbf{E}$ and strongly in $\mathbf{F}_{2,2}$, then $(A(w_n))_{n\in N}$ converges strongly in $L^2([0,T], L^2(\Omega))$ to $A(w_\infty)$.

Remark 3.1. We shall show in section 6 that the filter introduced by (2.5) satisfies these hypotheses.

Definition 3.1. We shall say that w is a distributional solution to (3.5) if and only if:

$$w \in \mathbf{E} \cap \mathbf{F}_{\infty,2}, \tag{3.9}$$

and if for all $\varphi \in \mathbf{G}$,

$$\begin{cases} \displaystyle\int_\Omega w_0(x)\,\varphi(0,x)\,dx - \int_0^T\int_\Omega w.\partial_t\varphi - \int_0^T\int_\Omega ww : \nabla\varphi \\ \displaystyle + \int_0^T\int_\Omega A(w)\nabla w : \nabla\varphi = \int_0^T\int_\Omega f\varphi. \end{cases} \tag{3.10}$$

Remark 3.2. Arguing as in [18], it is easy to prove that formulation (3.10) is equivalent to: $\forall\, u \in \mathbf{D}$,

$$\frac{d}{dt}\int_\Omega w.u - \int_\Omega ww : \nabla u + \int_\Omega A(w)\nabla w : \nabla u = \int_\Omega f.u, \tag{3.11}$$

where (3.11) holds in $\mathcal{D}'([0,T])$.

Theorem 3.1. *Assume that* (3.6), **(H1), (H2)** *and* **(H3)** *hold. Then there exists a distributional solution w to problem* (3.5).

Remark 3.3. In Definition 3.1 and Theorem 3.1 the pressure q has disappeared from the variational formulation due to the incompressibility constraint $\nabla\cdot\varphi = 0$. Indeed,

$$\forall\,\varphi \in (\mathcal{D}(\Omega))^d \,/\, \nabla\cdot\varphi = 0, \quad \forall\, q \in \mathcal{D}'(\Omega), \quad \langle \nabla q, \varphi\rangle = 0. \tag{3.12}$$

Conversely, the De Rham Theorem says that if a distribution F is such that $\forall\,\varphi \in (\mathcal{D}(\Omega))^d \,/\, \nabla\cdot\varphi = 0$, $\langle F, \varphi\rangle = 0$, then there exists a distribution q verifying $F = \nabla q$ (for the details see [23], [32], [33], [18]). Hence, the pressure is recovered from the variational formulation thanks to the De Rham Theorem. This point will be detailed at the end of the paper.

The proof of Theorem 3.1 will be performed until the end of the paper and follows a classical scheme: construction of approximation and passing to the limit in the equations. This is the aim of the two next sections.

4. Construction of approximations

Approximations are performed in two steps. First the notion of "truncated transport" is introduced, then the approximated problem is introduced.

4.1. Truncated transport

The "truncated transport" was originally introduced in [20] and improved in [18] and is aimed at constructing approximations for turbulent closure models in oceanography.

Let T_n the truncature function at height n:

$$T_n(x) = x \quad \text{if } |x| \le n, \quad T_n(x) = n\,\frac{x}{|x|} \quad \text{if } |x| \ge n. \tag{4.1}$$

Let $w \in \mathbf{H}$, w_i its coordinates. We define the truncated vector field $T_n(w)$ by its truncated coordinates:

$$T_n(w) = (T_n(w_1), T_n(w_2), T_n(w_3)). \tag{4.2}$$

Notice that thanks to a well known result of G. Stampacchia [31], lemma 1.1 page 15, one knows that if $u \in H_0^1(\Omega)$, then $T_n(u) \in H_0^1(\Omega)$.

Remark 4.1. Unfortunately, one does not have generally $\nabla \cdot T_n(w) = 0$ if $\nabla \cdot w = 0$. This is the main obstruction for giving a "renormalized" formulation to the Navier–Stokes equations in the sense of Di Perna–Lions (see [5]), leading to a great theoretical difficulty for improving the mathematical analysis of a lot of turbulence models derived from the Navier–Stokes equations.

Now, one defines the truncated transport term. The transport term in the Navier–Stokes equations is the term $\nabla \cdot (ww)$. Basically, the truncated transport term is defined by $\nabla \cdot (wT_n(w))$, but one prefers to give the definition directly by a variational formulation on the space $\mathbf{E}$. The definition and the main properties are summarized in the following lemma.

Lemma 4.1. *Let $n \in \mathbb{N}^\star$ and for $w \in \mathbf{E}$, let $B_n(w)$ be the form defined by*

$$\forall\, u \in \mathbf{E}, \quad \langle B_n(w), u\rangle = -\int_0^T \int_\Omega w_j T_n(w_i)\partial_j(u_i)\,dxdt. \tag{4.3}$$

Then $B_n(w) \in \mathbf{E}'$ and the following estimate holds

$$||B_n(w)||_{\mathbf{E}'} \le nC_p\,||w||_{\mathbf{E}}, \tag{4.4}$$

C_p being the Poincaré constant. On the other hand,

$$\forall\, w \in \mathbf{E}, \quad \langle B_n(w), w\rangle = 0. \tag{4.5}$$

If a sequence $(w_k)_{k\in\mathbb{N}}$ converges weakly in $\mathbf{E}$ and strongly in $\mathbf{F}_{2,2}$ to some $w \in \mathbf{E}$, then $(B_n(w_k))_{k\in\mathbb{N}}$ converges weakly in $\mathbf{E}'$ to $B_n(w)$ (when $k \to \infty$ and for n fixed). Finally, assume that $w \in \mathbf{E}\cap\mathbf{F}_{\infty,2}$; then $(B_n(w))_{n\in\mathbb{N}}$ converges weakly to $\nabla\cdot(ww)$ when $n \to \infty$ in the space $(L^3([0,T],\mathbf{H}))'$.

Remark 4.2. The last part of the Lemma suggests that $B_n(w)$ is a good approximation of the transport term in some sense. Moreover, equality (4.5) makes sure that if the transport term in the model is replaced by its approximation by truncation, then the new derived model satisfies the right estimates and the same energy balance as the original model.

Proof. Equality (4.4) is obvious. One now proves equality (4.5). Let $w \in \mathbf{E}$. Using the Stokes formula and the fact that $\nabla \cdot w = 0$ yields

$$\langle B_n(w), w\rangle \;= -\int_0^T \int_\Omega w_j T_n(w_i)\partial_j(w_i) = \int_0^T \int_\Omega w_i w_j \partial_j(T_n(w_i)).$$

Still using the result of Stampacchia, one deduces

$$\langle B_n(w), w\rangle \;= \int_0^T \int_\Omega w_j T_n(w_i)\partial_j(T_n(w_i)) = \frac{1}{2}\int_0^T \int_\Omega w_j \partial_j(T_n(w_i)^2).$$

Thus, (4.5) follows by an other integration by parts and using again the constrain $\nabla \cdot w = 0$.

One now proves the sequential weak continuity property. Consider a sequence $(w_k)_{k\in\mathbb{N}}$ converging weakly in $\mathbf{E}$ and strongly in $\mathbf{F}_{2,2}$ to some $w \in \mathbf{E}$ (notice that $\mathbf{E} \subset \mathbf{F}_{2,2}$ with continuous injection). Because obviously $\mathbf{F}_{2,2} \subset (L^2([0,T]\times\Omega))^d$, one applies Theorem IV.9 page 58 in [3] and deduces that there exists $g \in L^2([0,T]\times\Omega)$ such that there exists a subsequence of $(w_k)_{k\in\mathbb{N}}$ (still denoted by $(w_k)_{k\in\mathbb{N}}$) such that $(w_k)_{k\in\mathbb{N}}$ converges almost everywhere in $[0,T]\times\Omega$ to w and

$$\forall\, k \in \mathbb{N}, \quad |w_k| \le g \quad \text{a.e. in } [0,T]\times\Omega. \tag{4.6}$$

Let $u \in \mathbf{E}$ and for (i,j) fixed, consider the sequence

$$(\varphi_k^{i,j})_{k\in\mathbb{N}} = (w_k^j T_n(w_k^i)\partial_j u_i)_{k\in\mathbb{N}},$$

where w_k^j is the j^{th} component of w_k. By (4.6) one has

$$|w_k^j T_n(w_k^i)\partial_j u_i| \le ng|\partial_j u_i| \in L^1([0,T]\times\Omega),$$

and thanks to the continuity of T_n,

$$\varphi_k^{i,j} \to w_j T_n(w_i)\partial_j u_i \quad \text{a.e. in } [0,T]\times\Omega.$$

Thus, one deduces from Lebesgue's Theorem that

$$\langle B_n(w_k), u\rangle \;\to\; \langle B_n(w), u\rangle$$

when $k \to \infty$.

It remains to prove the last statement of the Lemma. Because real turbulent flow is three dimensional, we shall assume $d = 3$ which is the more delicate case but also the more realistic. Notice that from Hölder inequality it is easy to deduce from a couple of lines of calculus that

$$L^2([0,T], L^6(\Omega)) \cap L^\infty([0,T], L^2(\Omega)) \subset L^{\frac{8}{3}}([0,T], L^4(\Omega)). \tag{4.7}$$

Let $w \in \mathbf{E} \cap \mathbf{F}_{\infty,2}$. From Sobolev inequality one deduces $|w| \in L^2([0,T], L^6(\Omega))$. Thus by (4.7) one deduces that each w_i lies in $L^{\frac{8}{3}}([0,T], L^4(\Omega))$. Let $u \in L^3([0,T], \mathbf{H})$. Notice that $L^3([0,T], \mathbf{H}) \subset \mathbf{E}$. Consider the sequence

$$(\psi_n^{i,j})_{n\in\mathbb{N}} = (w_j T_n(w_i) \partial_j u_i)_{n\in\mathbb{N}}.$$

One has

$$|\psi_n^{i,j}| \leq |w_i||w_j||\partial_j u_i| \in L^1([0,T] \times \Omega),$$

thanks to Hölder inequality. Finally, it is obvious that $(\psi_n^{i,j})_{n\in\mathbb{N}}$ converges a.e. in $[0,T] \times \Omega$ to $w_i w_j \partial_j(u_i)$. Thus by Lebesgue Theorem,

$$\langle B_n(w), u\rangle \;\to -\int_0^T \int_\Omega w_i w_j \partial_j(u_i) = \int_0^T \int_\Omega u\,.\nabla \cdot (ww),$$

the last equality having a sense in the distributions space. This completes the proof of the lemma.

4.2. Approximated model and its variational formulation

Formally, one writes the approximated equations as follows:

$$\begin{cases} \partial_t w + B_n(w) - \nabla \cdot [T_n(A(w))\nabla w] + \nabla q = T_n(f(x,t)) \text{ in } \Omega, \\ \nabla \cdot w = 0 \;\text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega \text{ and } w(x,0) = w_0(x) \text{ in } \Omega, \end{cases} \tag{4.8}$$

where $T_n(f)$ is the truncated vector field deduced from f as defined in the previous subsection. The variational formulation is the following:

$$\text{Find } w \in \mathbf{E} \text{ with } \partial_t w \in \mathbf{E}' \text{ and such that } \forall\, u \in \mathbf{E}, \tag{4.9}$$

$$\begin{cases} \langle \partial_t w, u\rangle + \langle B_n(w), u\rangle + \displaystyle\int_0^T \int_\Omega T_n(A(w))\nabla w : \nabla u \\ = \langle T_n(f), u\rangle\,. \end{cases} \tag{4.10}$$

Notice that in this formulation, each term is well defined and that the pressure plays no role (see Remark 3.3).

Remark 4.3. As in remark 3.2, it easy to prove that formulation (4.9), (4.10) is equivalent to

$$\text{Find } w \in \mathbf{E} \text{ with } \partial_t w \in \mathbf{E}' \text{ and such that } \forall\, u \in \mathbf{H}, \tag{4.11}$$

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega w.u - \int_\Omega w_j T_n(w_i)\partial_j u_i + \int_\Omega T_n(A(w))\nabla w : \nabla u \\ = \displaystyle\int_\Omega T_n(f).u, \end{cases} \tag{4.12}$$

where (4.12) holds in $L^2([0,T])$.

Theorem 4.1. *Assume that* (3.6), **(H1), (H2)** *and* **(H3)** *hold, and let* $n \geq \rho$. *Then there exists a solution* w *to problem* (4.9), (4.10) *which satisfies the energy equality at each time* $\tau \in [0, T]$*:*

$$\begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega (|w(\tau, x)|^2 - |w_0(x)|^2) dx + \int_0^\tau \int_\Omega T_n(A(w)) |\nabla w|^2 dx dt \\ = \langle T_n(f), w \rangle . \end{cases} \tag{4.13}$$

Notice that with no additional assumption on A, uniqueness is an open problem. The proof of Theorem 4.1 is performed in the next section.

5. Proof of the existence results

We start by giving a linearized version of problem (4.9), (4.10), and we prove Theorem 4.1 by using the Leray–Schauder fixed point theorem [15]. Then we pass to the limit in the equations and prove Theorem 3.1.

5.1. Linearized version

Basically, one has to consider the following equation, for a given $v \in \mathbf{E}$:

$$\begin{cases} \partial_t w + C_n(v, w) - \nabla \cdot [T_n(A(v)) \nabla w] + \nabla q = T_n(f(x,t)) \text{ in } \Omega, \\ \nabla \cdot w = 0 \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega \text{ and } w(x, 0) = w_0(x) \text{ in } \Omega, \end{cases} \tag{5.1}$$

where $C_n(v, w) \in \mathbf{E}'$ and is defined by

$$\forall\, u \in \mathbf{E}, \quad \langle C_n(v, w), u \rangle = - \int_0^T \int_\Omega w_j T_n(v_i) \partial_j u_i. \tag{5.2}$$

The variational formulation of this problem is

$$\text{Find } w \in \mathbf{E} \text{ with } \partial_t w \in \mathbf{E}' \text{ and such that } \forall\, u \in \mathbf{E}, \tag{5.3}$$

$$\begin{cases} \langle \partial_t w, u \rangle \ + \ \langle C_n(w, v), u \rangle + \displaystyle\int_0^T \int_\Omega T_n(A(v)) \nabla w : \nabla u \\ = \langle T_n(f), u \rangle . \end{cases} \tag{5.4}$$

This problem is a linear problem with a bounded non negative diffusion term. Estimates (5.7), (5.8), (5.9) shown below makes sure that a Galerkin method (see Lions [23]) will converge and thus one clearly have the following.

Theorem 5.1. *Assume that* (3.6), **(H1), (H2)** *and* **(H3)** *hold, and* $n \geq \rho$. *Then there exists a unique solution* w *to problem* (5.3), (5.4).

This allows one to define a map $NS_n : \mathbf{E} \to \mathbf{E}$ which maps each $v \in \mathbf{E}$ to the unique solution of problem (5.3), (5.4). We shall prove in the following that this map has a fixed point in order to solve the problem (4.9), (4.10), after obtaining the right estimates.

Remark 5.1. Only (3.8) and (3.6) are necessary for the result of Theorem 5.1 as soon as $A(v)$ is a measurable function for a given v. Moreover, the pressure which appears in equation (5.1) is recovered from (5.3), (5.4) from De Rham Theorem.

Remark 5.2. In general, $\langle C_n(w,v), w\rangle \neq 0$.

Remark 5.3. As in remarks 3.2 and 4.3, we mention that formulation (5.3), (5.4) is equivalent to

$$\text{Find } w \in \mathbf{E} \text{ with } \partial_t w \in \mathbf{E}' \text{ and such that } \forall\, u \in \mathbf{H}, \tag{5.5}$$

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega w.u - \int_\Omega w_j T_n(v_i)\partial_j u_i + \int_\Omega T_n(A(v))\nabla w : \nabla u \\ = \displaystyle\int_\Omega T_n(f).\, u, \end{cases} \tag{5.6}$$

where (5.6) holds in $L^2([0,T])$.

5.2. Estimates

Throughout this subsection and the next subsection, $n \geq \rho$ is a fixed integer.

Lemma 5.1. *Let $w = NS_n(v)$ be the unique solution to problem (5.3), (5.4). Then $\partial_t w \in \mathbf{E}'$ and the following estimates holds:*

$$\begin{cases} ||w||^2_{\mathbf{F}_{2,\infty}} \\ \leq \left(||w_0||_{(L^2(\Omega))^3} + \dfrac{n^2 \rho\, mes(\Omega)}{\rho + n^2} \right) e^{\frac{\rho+n^2}{2\rho}T} + \dfrac{n^2\rho\, mes(\Omega)}{\rho+n^2} = q_n^2, \end{cases} \tag{5.7}$$

$$||w||^2_{\mathbf{E}} \leq \frac{2}{\rho}\left(\frac{n^2}{2}\, mes(\Omega) + T\left(\frac{\rho+n^2}{2\rho}\right) q_n^2 \right) = r_n^2, \tag{5.8}$$

$$||\partial_t w||_{\mathbf{E}'} \leq n(C_S\sqrt{T} + r_n + \sqrt{T} q_n), \tag{5.9}$$

where C_S is a Sobolev constant. In particular, $NS_n(B(0,r_n)) \subset B(0,r_n)$.

Proof. Notice first that one has $T_n(A(v)) \geq \rho$. As usual, take w as test function in formulation (5.6). This yields:

$$\begin{cases} \dfrac{d}{2dt}\displaystyle\int_\Omega |w|^2 - \int_\Omega w_j T_n(v_i)\partial_j w_i + \int_\Omega T_n(A(v))|\nabla w|^2 \\ = \displaystyle\int_\Omega T_n(f)\, w. \end{cases} \tag{5.10}$$

Combining (5.10) and Young's inequality, one obtains

$$\begin{cases} \dfrac{d}{2dt}\displaystyle\int_\Omega |w|^2 + \int_\Omega T_n(A(v))|\nabla w|^2 \\ \leq \dfrac{n^2}{2}\, mes(\Omega) + \dfrac{1}{2}\displaystyle\int_\Omega |w|^2 + \dfrac{n^2}{2\rho}\int_\Omega |w|^2 + \dfrac{\rho}{2}\int_\Omega |\nabla w|^2, \end{cases} \tag{5.11}$$

from which one deduces

$$\begin{cases} \dfrac{d}{2dt}\displaystyle\int_\Omega |w|^2 + \int_\Omega (T_n(A(v)) - \frac{\rho}{2})\, |\nabla w|^2 \\ \leq \dfrac{n^2}{2}\, mes(\Omega) + \left(\dfrac{1}{2} + \dfrac{n^2}{2\rho}\right)\displaystyle\int_\Omega |w|^2 \,. \end{cases} \tag{5.12}$$

In particular one has

$$\frac{d}{2dt}\int_\Omega |w|^2 \leq \frac{n^2}{2}\, mes(\Omega) + \left(\frac{\rho + n^2}{2\rho}\right)\int_\Omega |w|^2 \,. \tag{5.13}$$

Inequality (5.7) is deduced from (5.13) via the classical Gronwall's Lemma (see [23] for instance). Inequality (5.8) follows form (5.7), when one integrates (5.12) on the interval $[0, T]$.

It remains to prove (5.9). Let

$$D_n(w, v) : u \to \int_0^T \int_\Omega T_n(A(v))\nabla w : \nabla u = \langle D_n(w, v), u\rangle \,.$$

In $\mathbf{D}'$, equation (5.1) reads

$$\partial_t w = T_n(f) - D_n(w, v) - C_n(w, v). \tag{5.14}$$

We have to prove that each term in the right hand side of (5.14) lies in $\mathbf{E}'$. We treat each term separately.

Let $u \in \mathbf{E}$. One has

$$\langle T_n(f), u\rangle = \int_0^T \int_\Omega T_n(f) u.$$

Thus, by the Sobolev and Cauchy–Schwarz inequalities

$$|\langle T_n(f), u\rangle| \leq n C_S \int_0^T ||\nabla u||_{L^2(\Omega)} \leq n C_S \sqrt{T}\, ||u||_{\mathbf{E}}, \tag{5.15}$$

from which one deduces $T_n(f) \in \mathbf{E}'$ and

$$||T_n(f)||_{\mathbf{E}'} \leq n C_S \sqrt{T}. \tag{5.16}$$

In the same way one has obviously,

$$|\langle D_n(w,v), u\rangle| \leq n\, ||w||_{\mathbf{E}} ||u||_{\mathbf{E}},$$

thus $D_n(w,v) \in \mathbf{E}'$ and using (5.8)

$$||D(w,v)||_{\mathbf{E}'} \leq n ||w||_{\mathbf{E}} \leq n\, r_n. \tag{5.17}$$

Finally, using the Cauchy–Schwarz inequality and (5.7),

$$|\langle C_n(w,v), u\rangle| \leq n\sqrt{T} q_n ||u||_{\mathbf{E}},$$

thus $C_n(w,v) \in \mathbf{E}'$ and

$$||C_n(w,v)||_{\mathbf{E}'} \leq n\, \sqrt{T} q_n. \tag{5.18}$$

Then the fact that $\partial_t w \in \mathbf{E}'$ is a consequence of (5.14) and (5.9) is a consequence of (5.16), (5.17) and (5.18).

5.3. Fixed point procedure

We now prove that NS_n has a fixed point, proving by this way Theorem 4.1. We assume of course that (3.6), **(H1)**, **(H2)** and **(H3)** hold. Arguing as in [18], appendix B, for using the Leray–Schauder fixed point Theorem in such kind of situation, we only have to check:

1) there exists a ball $B \subset \mathbf{E}$ such that $NS_n(B) \subset B$,
2) the map NS_n is sequentially weak continuous.

The first point is ensured by Lemma 5.1. Unfortunately, it seems that the point 2 is a source of difficulty due to a lack of compactness in the diffusion term, in view of (**H3**) (see (5.27) below). In order to overcome this difficulty, we shall work in a larger space than $\mathbf{E}$.

Let

$$\mathcal{C} = \{w \in \mathbf{E}\,;\ \partial_t w \in \mathbf{E}'\},$$

where $\partial_t w$ is taken in the sense of distributions, that is

$$\forall\, \varphi \in (\mathcal{D}'([0,T]\times\Omega))^3, \quad \langle \partial_t w, \varphi\rangle = -\int_0^T \int_\Omega w.\partial_t\varphi = -\langle w, \partial_t \varphi\rangle\,.$$

Consider now

$$\mathcal{X} = \{x = (w, \partial_t w)\,;\ w \in \mathcal{C}\}\,. \tag{5.19}$$

The set $\mathcal{X}$ is clearly a subspace of $\mathbf{E}\times\mathbf{E}'$ normed by

$$||x||_{\mathcal{X}} = ||x||_{\mathbf{E}\times\mathbf{E}'} = ||w||_{\mathbf{E}} + ||\partial_t w||_{\mathbf{E}'}. \tag{5.20}$$

Lemma 5.2. *The space $\mathcal{X}$ is a Banach Space, closed in $\mathbf{E}\times\mathbf{E}'$ with respect to its weak topology.*

Proof. We prove that $\mathcal{X}$ is weakly closed in $\mathbf{E}\times\mathbf{E}'$. Let

$$(x_k)_{k\in\mathbb{N}} = (w_k, \partial_t w_k)_{k\in\mathbb{N}}$$

be a sequence in $\mathcal{X}$ which weakly converges to some (w,g) in $\mathbf{E}\times\mathbf{E}'$. We have to prove that $w\in\mathcal{C}$ and $g=\partial_t w$. In fact the first point is a consequence of the second. Let $\varphi\in\mathcal{D}([0,T]\times\Omega)$ with $\nabla\cdot\varphi=0$. Notice that $\varphi\in\mathbf{E}$ and $\partial_t\varphi\in\mathbf{E}$. By definition one has:

$$\langle \partial_t w_k, \varphi\rangle = -\langle w_k, \partial_t\varphi\rangle\,,$$

thus

$$\lim_{k\to\infty}\langle \partial_t w_k, \varphi\rangle = -\langle w, \partial_t\varphi\rangle = \langle \partial_t w, \varphi\rangle = \langle g, \varphi\rangle\,. \tag{5.21}$$

On the other hand, we know that the set made of all $\varphi\in\mathcal{D}([0,T]\times\Omega)$ with $\nabla\cdot\varphi=0$ is dense in $\mathbf{E}$ (see [23]). Thus $\partial_t w\in\mathbf{E}'$ and (5.21) holds for each $\varphi\in\mathbf{E}$, and leads to

$$g=\partial_t w \quad \text{in } \mathbf{E}'. \tag{5.22}$$

It is now clear that $\mathcal{X}$ is a closed subspace of the Banach space $\mathbf{E}\times\mathbf{E}'$ for its strong topology, and thus is itself a Banach space.

Remark 5.4. We have to keep in mind that the rigorous consequence of (5.21) is

$$\exists\, q\in\mathcal{D}'([0,T]\times\Omega)\,;\quad g-\partial_t w=\nabla q.$$

But $\langle\nabla q,\varphi\rangle=0$ when $\nabla\cdot\varphi=0$. Then in fact, (5.22) makes sense in the duality between $\mathbf{E}$ and $\mathbf{E}$', in the same way as we considered (5.14) earlier. There is a quotient space not explicitly written in this reasoning.

We now introduce the map to which the Leray–Schauder fixed point Theorem will be applied. Remark that thanks to Lemma 5.1 and estimate (5.9), for every $v\in\mathbf{E}$, $NS_n(v)\in\mathcal{C}$. Thus it makes sense to introduce

$$\begin{cases} \Psi_n : \mathcal{X}\longrightarrow\mathcal{X} \\ x=(v,\partial_t v)\to (NS_n(v), \partial_t NS_n(v)). \end{cases} \tag{5.23}$$

Any fixed point of the application Ψ_n is of course a solution to problem (4.9), (4.10). As a consequence of Lemma 5.1 and (5.8), (5.9), one can claim:

$$\Psi_n(B(0,\rho_n))\subset B(0,\rho_n),\quad \rho_n=r_n+n(C_S\sqrt{T}+r_n+\sqrt{T}q_n). \tag{5.24}$$

Thus, Ψ_n preserves $B(0,\rho_n)$. It remains to prove that Ψ_n is weakly sequentially continuous.

Lemma 5.3. *Let $(x_k)_{k\in\mathbb{N}} = (v_k,\partial_t v_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{X}$ which converges weakly to some $x=(v,\partial_t v)$. Then the sequence $(\Psi_n(x_k))_{k\in\mathbb{N}} = (w_k,\partial_t w_k)_{k\in\mathbb{N}}$ converges weakly in $\mathcal{X}$ to $(w,\partial_t w)=\Psi_n(x)$.*

Proof. In this proof, hypothesis (**H3**) plays a crucial role. Indeed, one has the following:

$$\mathbf{H} \subset \mathbf{L}_2 \subset \mathbf{H}', \tag{5.25}$$

both of the injections being continuous and dense, and the first being compact. Moreover, $\mathbf{E}' = L^2([0,T],\mathbf{H}')$. Thus, because $(x_k)_{k\in\mathbb{N}}$ converges weakly to x in $\mathcal{X}$, it is bounded in $\mathcal{X}$. In particular, $(v_k)_{k\in\mathbb{N}}$ is bounded in $\mathbf{E}$ and $(\partial_t v_k)_{k\in\mathbb{N}}$ is bounded in **E**'. Applying Aubin–Lions' Lemma (see [23]), one knows that the sequence $(v_k)_{k\in\mathbb{N}}$ is compact in $L^2([0,T],\mathbf{L}_2) = \mathbf{F}_{2,2}$ and in particular

$$(v_k)_{k\in\mathbb{N}} \text{ converges strongly to } v \text{ (up to a subsequence) in } \mathbf{F}_{2,2}. \tag{5.26}$$

Then, applying (**H3**) one claims:

$$(A(v_k))_{k\in\mathbb{N}} \text{ converges strongly to } A(v) \text{ in } L^2([0,T],L^2(\Omega). \tag{5.27}$$

We are now able to pass to the limit in each term of the equation, written under the form, following (5.14):

$$\partial_t w_k = T_n(f) - D_n(w_k,v_k) - C_n(w_k,v_k), \quad w_k = NS_n(v_k). \tag{5.28}$$

Thanks to (5.8), we know that $(w_k)_{k\in\mathbb{N}}$ is bounded in $\mathbf{E}$. Thus one can extract a subsequence (still denoted $(w_k)_{k\in\mathbb{N}}$) which converges weakly in $\mathbf{E}$ to some $w \in \mathbf{E}$. Moreover, thanks to (5.9), $(\partial_t w_k)_{k\in\mathbb{N}}$ is bounded in $\mathbf{E}'$. Applying again Aubin–Lions' Lemma, one deduces that $(w_k)_{k\in\mathbb{N}}$ is compact in $\mathbf{F}_{2,2}$, and in particular converges strongly to w in $\mathbf{F}_{2,2}$ (up to a subsequence). Arguing as in Lemma 5.2, one can still extract another subsequence (without change of notation) such that $(\partial_t w_k)_{k\in\mathbb{N}}$ converges weakly to $\partial_t w$ in $\mathbf{E}'$. It remains to pass to the limit in the transport term and in the diffusion term.

One starts with the transport term. Let $u \in \mathbf{E}$ be any test function. Consider the sequence $(T_n(v_k)\partial_j u_i)_{k\in\mathbb{N}}$. Applying Lebesgue's inverse Theorem (see [3]), one can extract from $(v_k)_{k\in\mathbb{N}}$ a subsequence which converges a.e to v in $[0,T]\times\Omega$. Then

$$\lim_{k\to\infty} T_n(v_k)\partial_j u_i = T_n(v)\partial_j u_i \quad \text{ a.e. in } [0,T]\times\Omega.$$

Moreover,

$$|T_n(v_k)\partial_j u_i| \leq n|\nabla u| \in L^2([0,T]\times\Omega).$$

Applying now Lebesgue's Theorem yields

$$\lim_{k\to\infty} T_n(v_k)\partial_j u_i = T_n(v)\partial_j u_i \quad \text{ strongly in } L^2([0,T]\times\Omega).$$

As one already knows that $(w_k)_{k\in\mathbb{N}}$ converges strongly to w in $L^2([0,T]\times\Omega)$, one obtains

$$\lim_{k\to\infty} \langle C_n(w_k,v_k),u\rangle = \int_0^T\int_\Omega w_j T_n(v_i)\partial_j u_i = \langle C_n(w,v),u\rangle\,. \tag{5.29}$$

It remains to pass to the limit in the diffusion term. Using (5.27) and again Lebesgue's inverse Theorem, from $(A(v_k))_{k\in\mathbb{N}}$ one can extract a subsequence which converges a.e. to $A(v)$ in $[0,T]\times\Omega$. Then

$$\lim_{k\to\infty} T_n(A(v_k))\nabla u = T_n(A(v))\nabla u \quad \text{a.e. in } [0,T]\times\Omega.$$

Moreover,

$$|T_n(A(v_k))\nabla u| \le n|\nabla u| \in L^2([0,T]\times\Omega).$$

Applying now Lebesgue's Theorem yields

$$\lim_{k\to\infty} T_n(A(v_k))\nabla u = T_n(A(v))\nabla u \quad \text{strongly in } L^2([0,T]\times\Omega).$$

One knows that $(\nabla w_k)_{k\in\mathbb{N}}$ converges weakly to ∇w in $(L^2([0,T]\times\Omega))^{d^2}$. Then

$$\begin{cases} \lim\limits_{k\to\infty} \langle D_n(w_k,v_k),u\rangle \\ = \displaystyle\int_0^T \int_\Omega T_n(A(v_k))\nabla u : \nabla w = \langle D_n(w,v),u\rangle\,. \end{cases} \tag{5.30}$$

Combining (5.29), (5.30) with the previous results, one has in $\mathbf{E}'$

$$\partial_t w = T_n(f) - D_n(w,v) - C_n(w,v). \tag{5.31}$$

Because of the uniqueness of the solution to (5.31), (5.31) reads $w = NS_n(v)$, and all the sequence converges, which ends this proof.

In conclusion, thanks to (5.24) and Lemma 5.3 one knows that Ψ_n has a fixed point, and so does NS_n and then (4.9), (4.10) has a solution. The fact that (4.13) holds is proved by taking w as test function and using (4.5). Theorem 4.1 is now completely proved.

Remark 5.5. Uniqueness of the solution to problem (4.9), (4.10) is an open problem.

It remains to pass to the limit in the equation when n goes to infinity.

5.4. Passing to the limit as n goes to infinity

We are now in a position to complete the proof of Theorem 3.1. Of course (3.6), **(H1), (H2)** and **(H3)** hold. Let w_n be a solution to problem (4.9), (4.10). We prove in this subsection that the sequence $(w_n)_{n\in\mathbb{N}}$ converges to a distributional solution to problem 3.5 (see definition 3.1), which will prove the existence result of Theorem 3.1. As usual, one start by a priori estimates and one passes to the limit in each term of the equation.

Taking w_n as test function in (4.12) and using (4.5) combined with (4.7), (3.8), one obtains in the case $d=3$:

$$||w_n||^2_{\mathbf{F}_{\infty,2}} \leq ||w_0||^2_{\mathbf{L}_2} + \frac{C_p^2}{2\rho}||f||^2_{[L^2([0,T]\times\Omega)]^d} = q^2, \tag{5.32}$$

$$||w_n||^2_{\mathbf{E}} \leq \frac{1}{\rho}q^2, \tag{5.33}$$

$$||B_n(w_n)||_{L^{\frac{4}{3}}([0,T],\mathbf{H}')} \leq \frac{C_S}{\rho}q^2, \tag{5.34}$$

where C_p and C_S are Poincaré and Sobolev constants. These estimates are easy to obtain and classical. The details are skipped. Notice that the previous estimates are given for $d = 3$ and do not depend on n. The estimate for the transport term is obtained in the space $L^{\frac{4}{3}}([0,T],\mathbf{H}')$ and nobody knows how to improve this, even in the case of the classical Navier–Stokes equation (see [33], [23], [19]). This is due of course to the nonlinear transport term, and in the case $d = 2$ results are more complete. In particular, it is possible to estimate $B_n(w_n)$ in $\mathbf{E}'$. This later point is left to the reader.

It remains to estimate $\partial_t w_n$. In the following, $p > 0$ is a fixed real number.

Lemma 5.4. *The sequence $(\partial_t w_n)_{n\in\mathbb{N}}$ is bounded in the space $L^{\frac{4}{3}}([0,T],(\mathbf{W}_{2,3+p})')$ and there exists a constant $C = C(f, w_0, \rho, \Omega)$ be such that*

$$||\partial_t w_n||_{L^{\frac{4}{3}}([0,T],(\mathbf{W}_{2,3+p})')} \leq C = C(f, w_0, \rho, \Omega). \tag{5.35}$$

Proof. One starts from the equation which holds in $\mathbf{E}'$:

$$\partial_t w_n = T_n(f) - B_n(w_n) - D_n(w_n), \tag{5.36}$$

where

$$\langle D_n(w_n), u\rangle = \int_0^T \int_\Omega T_n(A(w_n))\nabla w_n : \nabla u.$$

According to (3.7), (5.32) and (5.33), one has

$$A(w_n)\nabla w_n \in L^2([0,T], L^1(\Omega)) \tag{5.37}$$

and

$$||A(w_n)\nabla w_n||_{L^2([0,T],L^1(\Omega))} \leq \frac{C}{\sqrt{\rho}}q(1+q) = r\,. \tag{5.38}$$

Indeed,

$$||A(w_n)\nabla w_n||^2_{L^2([0,T],L^1(\Omega))} = \int_0^T \left(\int_\Omega |A(w_n)\nabla w_n|\right)^2,$$

and by Cauchy–Schwarz inequality

$$\left(\int_\Omega |A(w_n)\nabla w_n|\right)^2 \leq \left(\int_\Omega |A(w_n)|^2\right)\left(\int_\Omega |\nabla w_n|^2\right)$$

and (5.38) follows directly from (3.7), (5.32) and (5.33). Of course, the same estimate holds for the term $T_n(A(w_n))\nabla w_n$ because $|T_n(A(w_n))| \le |A(w_n)|$.

Assume that $u \in L^2([0,T], \mathbf{W}_{2,3+p})$. Then, by the Sobolev embedding Theorem (see [3]), $|\nabla u| \in L^2([0,T], L^\infty(\Omega))$. Thus, using (5.38), there exists $C_S \in \mathbb{R}$ be such that

$$\begin{cases} |\langle D_n(w_n), u\rangle| \\ \le || \, |A(w_n)\nabla w_n| \, ||_{L^2([0,T],L^1(\Omega))} ||\nabla u||_{L^2([0,T],L^\infty(\Omega))} \\ \le C_S \, r \, ||u||_{L^2([0,T],\mathbf{W}_{2,3+p})}. \end{cases} \tag{5.39}$$

Then, $D_n(w_n)$ is bounded in $L^2([0,T], (\mathbf{W}_{2,3+p})')$ and one has

$$||D_n(w_n)||_{L^2([0,T],(\mathbf{W}_{2,3+p})')} \le C_S \, r. \tag{5.40}$$

Finally of course, by the Cauchy–Schwarz inequality and the Poincaré inequality,

$$\left| \int_0^T \int_\Omega T_n(f).u \right| \le C ||f||_{L^2([0,T],L^2(\Omega))} ||u||_{\mathbf{E}}, \tag{5.41}$$

thus $T_n(f)$ is bounded in $\mathbf{E}'$ and one has

$$||T_n(f)||_{\mathbf{E}'} \le C ||f||_{L^2([0,T],L^2(\Omega))}. \tag{5.42}$$

Then, because one has

$$\mathbf{E}' \subset L^{\frac{4}{3}}([0,T], (\mathbf{W}_{2,3+p})'), \quad L^{\frac{4}{3}}([0,T], \mathbf{H}') \subset L^{\frac{4}{3}}([0,T], (\mathbf{W}_{2,3+p})'),$$
$$L^2([0,T], (\mathbf{W}_{2,3+p})') \subset L^{\frac{4}{3}}([0,T], (\mathbf{W}_{2,3+p})'),$$

the claim of Lemma 5.4 and (5.35) is a consequence of (5.34), (5.36), (5.40) and (5.42).

We are now in a position to complete the proof. From (5.33) one deduces that from $(w_n)_{n\in\mathbb{N}}$ one can extract a subsequence (as usual denoted by the same) such that there is some $w \in \mathbf{E}$ satisfying

$$(w_n)_{n\in\mathbb{N}} \text{ converges weakly in } \mathbf{E} \text{ to } w. \tag{5.43}$$

From (5.35), one deduces that from $(\partial_t w_n)_{n\in\mathbb{N}}$ one can extract a subsequence (as usual denoted by the same) such that there is some $g \in L^{\frac{4}{3}}([0,T], (\mathbf{W}_{2,3+p})')$ satisfying

$$(\partial_t w_n)_{n\in\mathbb{N}} \text{ converges weakly in } L^{\frac{4}{3}}([0,T], (\mathbf{W}_{2,3+p})') \text{ to } g. \tag{5.44}$$

Arguing as in Lemma 5.2, it is easy to prove that $g = \partial_t w$. Then

$$(\partial_t w_n)_{n\in\mathbb{N}} \text{ converges weakly in } L^{\frac{4}{3}}([0,T], (\mathbf{W}_{2,3+p})') \text{ to } \partial_t w. \tag{5.45}$$

Moreover, one has

$$\mathbf{H} \subset L^2 \subset (\mathbf{W}_{2,3+p})'.$$

Then applying Aubin–Lions's Lemma, one deduces that $(w_n)_{n\in\mathbb{N}}$ is compact in $L^{\frac{4}{3}}([0,T],L^2(\Omega))$, and in particular in $L^{\frac{4}{3}}([0,T]\times\Omega)$. Thus from the sequence $(w_n)_{n\in\mathbb{N}}$ one can extract a subsequence (still denoted by the same) which converges strongly in $L^{\frac{4}{3}}([0,T]\times\Omega)$ and a.e. in $[0,T]\times\Omega$ to w. Moreover, by using the Sobolev embedding theorem combined with (4.7), one sees that the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in $L^{2+\varepsilon}([0,T]\times\Omega)$ for a suitable $\varepsilon>0$. Thus by a classical argument which combines Egorov Theorem and Hölder inequality (see [27] and [18]), one can conclude that $(w_n)_{n\in\mathbb{N}}$ converges strongly to w in $L^2([0,T]\times\Omega)$. Thus by (**H3**):

$$(A(w_n))_{n\in\mathbb{N}} \text{ converges strongly in } L^2([0,T]\times\Omega) \text{ to } A(w). \tag{5.46}$$

In the following we pass to the limit in each term of the equation, starting with the diffusion term. Let $\varphi\in\mathbf{G}$ be any test function. From the weak convergence in $\mathbf{E}$ of $(w_n)_{n\in\mathbb{N}}$ to w and thanks to (5.46), one has

$$\lim_{n\to\infty}\int_0^T\int_\Omega A(w_n)\nabla w_n:\nabla\varphi=\int_0^T\int_\Omega A(w)\nabla w:\nabla\varphi. \tag{5.47}$$

On the other hand, one has by definition (see [23])

$$\langle\partial_t w_n,\varphi\rangle = \int_\Omega w_0(x)\varphi(0,x)\,dx - \int_0^T\int_\Omega w_n(t,x)\partial_t\varphi(t,x)\,dxdt. \tag{5.48}$$

Then by passing to the limit one has obviously

$$\begin{cases}\lim\limits_{n\to\infty}\langle\partial_t w_n,\varphi\rangle\\ = \displaystyle\int_\Omega w_0(x)\varphi(0,x)\,dx - \int_0^T\int_\Omega w(t,x)\varphi(t,x)\,dxdt.\end{cases} \tag{5.49}$$

It remains now to pass to the limit in the transport term. By definition (see Lemma 4.1 and (4.3))

$$\langle B_n(w_n),\varphi\rangle = -\int_0^T\int_\Omega w_n^j T_n(w_n^i)\partial_j\varphi_i.$$

One already knows that $(w_n)_{n\in\mathbb{N}}$ is compact in $L^2([0,T]\times\Omega)$, and thanks to the the fact that T_n is continuous and $|T_n(g)|\le|g|$ for each function g leads to the fact that $(T_n(w_n))_{n\in\mathbb{N}}$ is also compact in $L^2([0,T]\times\Omega)$. Thus one has obviously

$$\lim_{n\to\infty}\langle B_n(w_n),\varphi\rangle = -\int_0^T\int_\Omega w_i w_j\partial_j\varphi_i = -\int_0^T\int_\Omega ww\nabla\varphi. \tag{5.50}$$

Finally, it is clear that

$$\lim_{n\to\infty}\langle T_n(f),\varphi\rangle = \langle f,\varphi\rangle\,. \tag{5.51}$$

As one knows that w_n satisfies

$$\begin{cases}\langle\partial_t w_n,\varphi\rangle+\langle B_n(w_n),\varphi\rangle+\displaystyle\int_0^T\int_\Omega T_n(A(w_n))\nabla w_n:\nabla\varphi\\ = \langle T_n(f),\varphi\rangle\,,\end{cases} \tag{5.52}$$

one deduces by putting together (5.46), (5.47), (5.50), (5.51) that $w \in \mathbf{E} \cap \mathbf{F}_{\infty,2}$ satisfies for each $\varphi \in \mathbf{G}$:

$$\begin{cases} \displaystyle\int_\Omega w_0(x)\,\varphi(0,x)\,dx \;-\int_0^T\int_\Omega w.\partial_t\varphi \;-\int_0^T\int_\Omega ww\,.\nabla\varphi \\ \displaystyle +\int_0^T\int_\Omega A(w)\nabla w\nabla\varphi = \int_0^T\int_\Omega f\varphi, \end{cases} \tag{5.53}$$

and Theorem 3.1 is completely proved.

6. Consistency with the filter of the LES model and conclusions

6.1. Orientation

We are interested in this part to the case of a boundary domain with Dirichlet boundary conditions. Recall that in the LES model,

$$A(w) = \nu + \beta\delta_0|w - \overline{w}|, \tag{6.1}$$

where δ_0 is the mesh size,

$$\overline{w}(t,x) = u(\delta_0^2, t, x), \tag{6.2}$$

with

$$\begin{cases} \dfrac{\partial u}{\partial \delta} - \Delta u + \nabla q = 0, \\ \nabla \cdot u = 0, \\ u(0,t,x) = w(t,x), \quad u|_{\partial\Omega} = 0. \end{cases} \tag{6.3}$$

The aim of this section is proving that the operator A satisfies the hypotheses (**H1**) and (**H3**), (**H2**) being obviously satisfied with $\rho = \nu$.

Notice at first that for a given $t \in [0,T]$ and for $w \in \mathbf{E}$, $u(\delta,t,x) \in C([0,\delta_0], L^2(\Omega))$ as the solution of a Stokes problem (see [23]), and thus it makes sense to consider $u(\delta_0,t,x) = \overline{w}(t,x)$ as a field of $[L^2(\Omega)]^3$. Consequently, at each time t, $A(w) \in L^2(\Omega)$.

The following subsections are devoted to proving:

Lemma 6.1. *Let A be defined by* (6.1), (6.2), (6.3)*. Then A satisfies* (**H1**)*and* (**H3**).

6.2. Hypothesis (H1)

Let $w \in \mathbf{E} \cap \mathbf{F}_{\infty,2}$ and $u(\delta, t, x)$ be the unique solution of (6.3). Taking u as test function in (6.3) yields

$$\frac{1}{2}\frac{d}{d\delta}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = 0 \tag{6.4}$$

from which is deduced that at each time $t \in [0, T]$

$$\int_\Omega |u(\delta_0, t, x)|^2\, dx \le \int_\Omega |w(t,x)|^2\, dx \le ||w||_{\mathbf{F}_{\infty,2}}. \tag{6.5}$$

Thus, $\overline{w} \in \mathbf{F}_{\infty,2}$, and one can also deduce that $A(w) \in L^\infty([0,T], L^2(\Omega))$ and

$$\begin{cases} ||A(w)||_{L^\infty([0,T],L^2(\Omega))} \le \nu + 2\beta\delta_0 ||w||_{\mathbf{F}_{\infty,2}} \\ \qquad\qquad\qquad\qquad \le \sup(\nu, 2\beta\delta_0)(1 + ||w||_{\mathbf{F}_{\infty,2}}). \end{cases} \tag{6.6}$$

and (3.7) is satisfied with $C = \sup(\nu, 2\beta\delta_0)$.

6.3. Hypothesis (H3)

Let a sequence in $\mathbf{E}$, $(w_n)_{n\in N}$, which converges weakly in $\mathbf{E}$ to some function $w_\infty \in \mathbf{E}$ and strongly in $\mathbf{F}_{2,2}$. One has to prove that $(A(w_n))_{n\in N}$ converges strongly in $L^2([0,T], L^2(\Omega))$ to $A(w_\infty)$.

Of course one has just to prove that the sequence $(\overline{w}_n)_{n\in N}$ converges strongly in $\mathbf{F}_{2,2}$ to $\overline{w}_\infty$, and the claim follows.

Let

$$v_n = w_n - w_\infty.$$

By linearity of the Stokes problem,

$$\overline{v}_n = \overline{w}_n - \overline{w}_\infty,$$

where $\overline{v}_n = u_n(\delta_0, t, x)$, u_n being the solution of the problem

$$\begin{cases} \dfrac{\partial u_n}{\partial \delta} - \Delta u_n + \nabla q_n = 0, \\ \nabla \cdot u_n = 0, \\ u_n(0,t,x) = w_n(t,x) - w_\infty(t,x), \quad u_n|_{\partial\Omega} = 0. \end{cases} \tag{6.7}$$

Taking u_n as test function in (6.7) and integrating in space and with respect to δ for t fixed yields

$$\int_\Omega |u_n(\delta_0, t, x)|^2\, dx \le \int_\Omega |w_n(t,x) - w_\infty(t,x)|^2\, dx \tag{6.8}$$

and integrating again with respect to t:

$$\begin{cases} \displaystyle\int_0^T \int_\Omega |\overline{w}_n(t,x) - \overline{w}_\infty(t,x)|^2 dxdt \\ \displaystyle\leq \int_0^T \int_\Omega |w_n(t,x) - w_\infty(t,x)|^2 dxdt, \end{cases} \tag{6.9}$$

and the claim follows directly from (6.9) because the right hand side of this inequality goes to zero. This ends the proof of Lemma 6.1. Notice that in this proof, the weak convergence of the sequence does not play any role.

6.4. Conclusions

Actually, there are three ways for simulating turbulent flows:
- Direct Numerical Simulation (DNS);
- Closure turbulent models as (k, ε);
- Large Eddy Simulation (LES).

As far as we know, (DNS) is actually valid at Reynolds numbers less than a few hundred in complex applications, and this is far from full developed turbulent flows, even if the power of new computers allows the hope of increasing the Reynolds number in future numerical simulations. The actual limitations lie in the resolution needed to compute fluctuations. Geophysical flows and other complex turbulent flows contain a very large spectrum of scales. With current computers and ones in the foreseeable future, direct numerical simulations of such flows are not possible because of the correspondingly large number of degrees of freedom they would require.

Closure turbulent models (RANS) as (k, ε) and its derived models gave good results in the past (see [25] for a general presentation of the model) and are widely used for industrial flows and large scale turbulent flows (see for example the TKE model for the tropical ocean in [1]). The justification of these models is based on strong hypothesis on the turbulence and the possibility of some terms to vanish because of isotropy of the turbulence, which can be purely wrong in other situations than "grid-turbulence". In these cases, some mathematical justification arises for obtaining a first order closure equation for the kinetic turbulent energy (see [21]). But these strong hypothesis are clearly wrong when symmetry is broken as in the case of a convective boundary layer in the atmosphere (see the results of the Kensas experiments in [8]). Moreover, if existence of solutions to the resulted PDE's system is proved (see [18], [19] and [20]), regularity and uniqueness of the solutions is not known, and therefore, it is not known if these models lead to well posed problems in the sense of Leray–Lions.

LES seems to be more accurate for simulating large scale turbulent flows, and is now intensively used for simulations, because the obtained results match experiments better. Moreover, the concept of "local average" in space is natural. Indeed, LES chooses a cutoff length scale delta (small) and models motions below that scale. Thus, LES has two advantages over RANS. First, a LES model includes more of the time dependent behavior so behavior of real fluids that is inherently nonstationary is more easily matched in LES than in RANS. Second, with the cutoff length scale delta small, the terms that need to be modelled are small. Thus simpler and cruder models can be used in LES than in RANS (a large relative error in a small term can have less impact than a small relative error in a large term).

The work we have done on one of the simplest LES models shows that there exists mathematical solutions to the PDE's deduced from the models. Unfortunately, we are not able for the moment to prove regularity and uniqueness of the solutions and the system we considered is harder to analyse that the classical Navier–Stokes equation; we have to pay the price with hypothesis (2.4) and the way for filtering the equations. Nevertheless, this first mathematical analysis of LES is cheering and ask for new mathematical questions as "uniqueness and regularity of the filtered solution" or "asymptotic behavior when δ goes to zero", as modellization questions as "how to filter the equations in order to use less crude closure models than (2.4)".

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(accepted: ?, 2001)