A model for two coupled turbulent fluids
Part I: analysis of the system

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Abstract: We consider a system of equations that modelizes the stationary flow of two immiscible turbulent fluids on adjacent subdomains. The equations are coupled by non-linear boundary conditions on the interface which is here a fixed given surface. The aim of this paper is to prove the existence of a solution for this system, together with some further regularity properties in a specific geometry.

Résumé: Nous étudions un système d'équations qui modélise l'écoulement stationnaire de deux fluides turbulents dans des domaines adjacents. Les équations sont couplées par des conditions aux limites non linéaires sur l'interface qui est ici une surface fixe donnée. Le but de cet article est de prouver l'existence d'une solution de ce système, ainsi que quelques propriétés de régularité dans une géométrie particulière.

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1. Introduction.

This paper is devoted to the analysis of the following system which modelizes two stationary turbulent fluids coupled by boundary conditions on the interface:

$$\begin{cases} -\mathrm{div}\,\big(\alpha_i(k_i)\,\nabla \boldsymbol{u}_i\big) + \mathbf{grad}\,p_i = \boldsymbol{f}_i & \text{in } \Omega_i,\ 1 \le i \le 2, \\ \mathrm{div}\,\boldsymbol{u}_i = 0 & \text{in } \Omega_i,\ 1 \le i \le 2, \\ -\mathrm{div}\,\big(\gamma_i(k_i)\,\nabla k_i\big) = \alpha_i(k_i)\,|\nabla \boldsymbol{u}_i|^2 & \text{in } \Omega_i,\ 1 \le i \le 2, \\ \boldsymbol{u}_i = \mathbf{0} & \text{on } \Gamma_i,\ 1 \le i \le 2, \\ k_i = 0 & \text{on } \Gamma_i,\ 1 \le i \le 2, \\ \alpha_i(k_i)\,\partial_{n_i}\boldsymbol{u}_i - p_i\,\boldsymbol{n}_i + (\boldsymbol{u}_i - \boldsymbol{u}_j)\,|\boldsymbol{u}_i - \boldsymbol{u}_j| = \mathbf{0} & \text{on } \Gamma,\ 1 \le i \ne j \le 2, \\ k_i = |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 & \text{on } \Gamma,\ 1 \le i \le 2, \end{cases} \tag{1.1}$$

where each triple $(\boldsymbol{u}_i, k_i, p_i)$ is defined in the domain Ω_i, $1 \le i \le 2$.

In what follows, Ω_1 and Ω_2 stand for disjoint bounded domains in $\mathbb{R}^d$, $d = 2$ or 3, which are either convex or of class $\mathscr{C}^{1,1}$. The generic point in $\mathbb{R}^2$, resp. in $\mathbb{R}^3$, is denoted by $\boldsymbol{x} = (x, z)$, resp. $\boldsymbol{x} = (x, y, z)$. We assume for simplicity that the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ coincides with the intersection of both $\overline{\Omega}_1$ and $\overline{\Omega}_2$ with the hyperplane $z = 0$, while Ω_1 and Ω_2 are contained in the half-spaces $z > 0$ and $z < 0$ respectively, see the following figure where each Γ_i is equal to $\partial\Omega_i \setminus \Gamma$. Note also that, in the physical context, the heights of the domains are much smaller than their horizontal diameters.

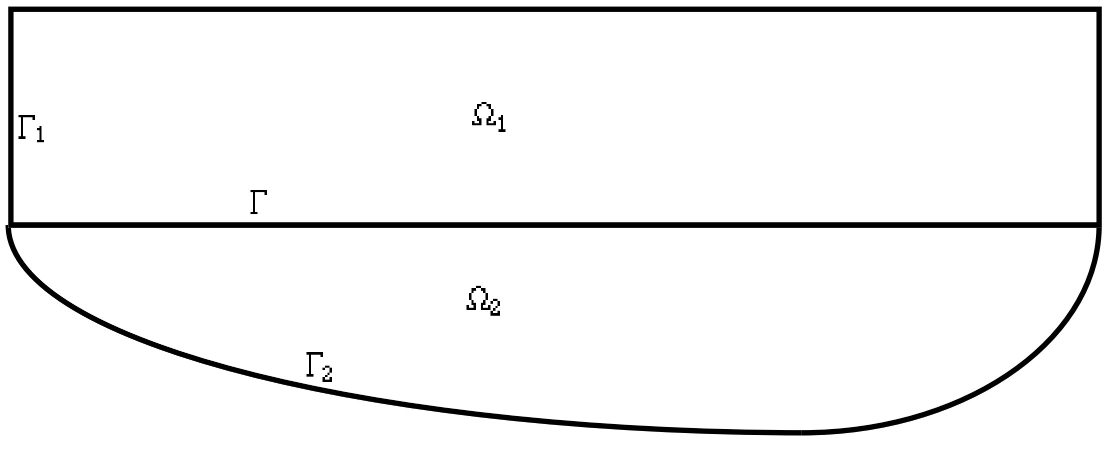

Figure 1

The vector field $\boldsymbol{u}_i$ stands for the velocity of a turbulent fluid in Ω_i, p_i represents its pressure and k_i its turbulent kinetic energy (TKE in what follows). The quantity $\alpha_i(k_i)$ is the eddy viscosity, and we shall assume throughout this paper that the functions α_i and γ_i satisfy

$$\begin{cases} \alpha_i \in \mathscr{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) & \text{and} & \forall k \in \mathbb{R}, & \alpha_i(k) \ge \nu, & 1 \le i \le 2, \\ \gamma_i \in \mathscr{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) & \text{and} & \forall k \in \mathbb{R}, & \gamma_i(k) \ge \nu, & 1 \le i \le 2, \end{cases} \tag{1.2}$$

for some positive constant ν. The functions $\boldsymbol{f}_i$ are given, with

$$\boldsymbol{f}_i \in L^2(\Omega_i)^d, \qquad 1 \leq i \leq 2. \tag{1.3}$$

System (1.1) is motivated by the coupling of two turbulent fluids F_i, $i = 1$ and 2, such as in the framework ocean/atmosphere or in the case of two layers of a stratified fluid (see *e.g.* [13, Chap. 1 & 3] or [17]). These fluids F_i are coupled through the interface condition in the sixth line of (1.1), on their common boundary Γ (which is supposed to be fixed). Indeed, we assume that the so-called "rigid lid hypothesis" holds, an hypothesis which is standard in geophysics and oceanography. Actually, Γ is a mean interface and the values of $\boldsymbol{u}_i$, p_i and k_i on Γ are in fact mean values of the velocity, pressure and TKE. So, the turbulent mixed layer of the two turbulent fluids is modelized by the sixth and seventh lines in (1.1) which summarize the information related to a realistic interface ocean/atmosphere (see e.g. [13, §1.4] for more details about this modelization).

The main goal of this paper is to prove the existence of a solution $(\boldsymbol{u}_i, k_i, p_i)_{1 \leq i \leq 2}$ of system (1.1) (see Corollary 5.3). We start by giving a sense to the equations. The two first lines of system (1.1) are the Stokes equations in Ω_i, equipped with the eddy viscosity $\alpha_i(k_i)$ which is the quantity of interest. The third line is a scalar equation which allows one to compute the k_i.

Following the ideas of [15], one can write a weak mixed formulation of the first two lines (see also [17] for the case of coupling ocean/atmosphere without turbulence). The third line and the corresponding boundary conditions are more complex: the main difficulty comes from the fact that the right-hand side only belongs to $L^1(\Omega_i)$. In the case of homogeneous boundary conditions and with only one turbulent fluid, such type of equations has already been studied (see *e.g.* [4], [10], [13, Chap. 4], [14]). In those references, the equation for the TKE is taken in the renormalized sense of Lions and Murat (see [18], [20], [21]), or in the equivalent entropy sense of Benilan *et al.* (see [3]), and *a priori* estimates of Boccardo–Gallouët type [5] are used. However, because of the boundary conditions at the interface Γ, this renormalization does not seem an easy way for the study of the TKE equations in the present problem, and one cannot hope to use directly the results of [5]. For this reason, we make Kirchoff's change of unknown in order to replace the operator $\operatorname{div}\big(\gamma_i(k_i)\,\nabla\big)$ by a simpler Laplace operator. We consider the corresponding new equation in the sense of transposition, according to the ideas of Stampacchia [23] and of Lions and Magenes [16, Chap. 2, §6]. We are then able to prove the existence of a solution of the global system (1.1).

Under some rather restrictive assumptions on the variations of the functions α_i, we prove a uniqueness result for the solution of (1.1). Some further regularity properties of this solution are also derived with the same assumptions, when the domains Ω_i are rectangles.

Most results of this paper have been announced in [1], however with a slightly different proof relying on Leray–Schauder fixed point theorem. The discretization of system (1.1) by spectral and finite element methods is presently under consideration, from both points of view of numerical analysis and experiments. It must also be observed that the present analysis can be extended to slightly different models,

- by adding convection terms in the momentum equations for the velocities,
- by replacing the transmission conditions for the velocities on Γ (sixth line in (1.1)) by

Manning's law

$$\begin{cases} \alpha_i(k_i)\,\partial_{n_i}\boldsymbol{u}_{iH} + (\boldsymbol{u}_{iH} - \boldsymbol{u}_{jH})\,|\boldsymbol{u}_i - \boldsymbol{u}_j| = 0 & \text{on } \Gamma,\ 1 \le i \ne j \le 2, \\ u_{iV} = 0 & \text{on } \Gamma,\ 1 \le i \le 2, \end{cases}$$

where $\boldsymbol{u}_{iH}$ and u_{iV} respectively stand for the horizontal and vertical components of the velocities,

- by adding the term due to Coriolis acceleration.

An outline of the paper is as follows. In Section 2, we write a system which is equivalent to (1.1) through a change of unknowns. Sections 3 and 4 are devoted to the proof of the existence of a solution for the equations on the velocities and the turbulent energies, respectively. In Section 5, we state and establish our main result, namely that the global system (1.1) admits a solution. In Section 6, we prove the conditional uniqueness result and, in Section 7, we derive the optimal regularity of this solution when the domains Ω_i are rectangles.

2. Transformation of the system.

Let us define the functions G_i, $1 \le i \le 2$, by

$$G_i(k) = \int_0^k \gamma_i(\kappa)\, d\kappa. \tag{2.1}$$

In view of (1.2), the G_i are increasing functions of class $\mathscr{C}^1$, so that they admit an inverse G_i^{-1} from $\mathbb{R}$ into $\mathbb{R}$. Moreover, the functions $\tilde{\alpha}_i$, $i = 1$ and 2, defined by

$$\tilde{\alpha}_i = \alpha_i \circ G_i^{-1}, \tag{2.2}$$

satisfy the same properties as the α_i, namely

$$\tilde{\alpha}_i \in \mathscr{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) \qquad \text{and} \qquad \forall \ell \in \mathbb{R}, \quad \tilde{\alpha}_i(\ell) \ge \nu, \qquad 1 \le i \le 2. \tag{2.3}$$

The idea is to introduce the new unknowns ℓ_i by

$$\ell_i = G_i(k_i), \qquad 1 \le i \le 2. \tag{2.4}$$

From the formula $\nabla \ell_i = \gamma_i(k_i)\, \nabla k_i$, it is readily checked that system (1.1) is equivalent to

$$\begin{cases} -\mathrm{div}\,\big(\tilde{\alpha}_i(\ell_i)\, \nabla \boldsymbol{u}_i\big) + \mathbf{grad}\, p_i = \boldsymbol{f}_i & \text{in } \Omega_i,\ 1 \le i \le 2, \\ \mathrm{div}\, \boldsymbol{u}_i = 0 & \text{in } \Omega_i,\ 1 \le i \le 2, \\ -\Delta \ell_i = \tilde{\alpha}_i(\ell_i)\, |\nabla \boldsymbol{u}_i|^2 & \text{in } \Omega_i,\ 1 \le i \le 2, \\ \boldsymbol{u}_i = \mathbf{0} & \text{on } \Gamma_i,\ 1 \le i \le 2, \\ \ell_i = 0 & \text{on } \Gamma_i,\ 1 \le i \le 2, \\ \tilde{\alpha}_i(\ell_i)\, \partial_{n_i} \boldsymbol{u}_i - p_i\, \boldsymbol{n}_i + (\boldsymbol{u}_i - \boldsymbol{u}_j)\, |\boldsymbol{u}_i - \boldsymbol{u}_j| = \mathbf{0} & \text{on } \Gamma,\ 1 \le i \ne j \le 2, \\ \ell_i = G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) & \text{on } \Gamma,\ 1 \le i \le 2. \end{cases} \tag{2.5}$$

The interest is that now each function ℓ_i is a solution of a nearly standard Laplace equation.

The goal of this paper is to prove the existence of a solution of (2.5), see Theorem 5.2 below. Throughout the paper, we use the spaces $L^p(\Omega_i)$, $1 \le p \le \infty$, and the Sobolev spaces $H^s(\Omega_i)$ and $H_0^s(\Omega_i)$ for any real number s, provided with the standard norm $\|\cdot\|_{H^s(\Omega_i)}$ and semi-norm $|\cdot|_{H^s(\Omega_i)}$, together with their analogues on Γ. We also need the Sobolev spaces $W^{1,p}(\Omega_i)$ and the special space $H_{00}^{\frac{1}{2}}(\Gamma)$, defined e.g. in [16, Chap. 1, Th. 11.7].

3. The equations on the velocities.

Throughout this section, we assume that the functions ℓ_1 and ℓ_2 are given such that

$$\ell_i \in L^1(\Omega_i), \qquad 1 \le i \le 2, \tag{3.1}$$

even if they turn out to be smoother in what follows. For $1 \le i \le 2$, we introduce the spaces

$$X_i = \big\{ \boldsymbol{v}_i \in H^1(\Omega_i)^d;\ \boldsymbol{v}_i = \boldsymbol{0} \text{ on } \Gamma_i \big\}. \tag{3.2}$$

We now write correctly (and not only formally) the equation on each velocity $\boldsymbol{u}_i$ through its variational formulation:

Find $(\boldsymbol{u}_i, p_i)$, $1 \le i \le 2$, *in* $X_i \times L^2(\Omega_i)$ *such that, for* $1 \le i \ne j \le 2$*:*

$$\begin{aligned} &\forall \boldsymbol{v}_i \in X_i, \quad \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, \nabla \boldsymbol{u}_i \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x} + b_i(\boldsymbol{v}_i, p_i) \\ &\qquad\qquad + \int_\Gamma |\boldsymbol{u}_i - \boldsymbol{u}_j|\, (\boldsymbol{u}_i - \boldsymbol{u}_j) \cdot \boldsymbol{v}_i \, d\tau = \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i \, d\boldsymbol{x}, \\ &\forall q_i \in L^2(\Omega_i), \quad b_i(\boldsymbol{u}_i, q_i) = 0, \end{aligned} \tag{3.3}$$

where the form $b_i(\cdot, \cdot)$ is defined by

$$b_i(\boldsymbol{v}_i, q_i) = -\int_{\Omega_i} q_i (\operatorname{div} \boldsymbol{v}_i)\, d\boldsymbol{x}. \tag{3.4}$$

Note that the first bilinear form in (3.3) depends on ℓ_i.

As standard for the Stokes problem, we consider the kernel

$$V_i = \big\{ \boldsymbol{v}_i \in X_i;\ \operatorname{div} \boldsymbol{v}_i = 0 \text{ in } \Omega_i \big\},$$

and we observe that, for each solution $(\boldsymbol{u}_i, p_i)$ of problem (3.3), the velocity $\boldsymbol{u}_i$ is a solution of:

Find $\boldsymbol{u}_i$ *in* V_i, $1 \le i \le 2$, *such that, for* $1 \le i \ne j \le 2$*:*

$$\begin{aligned} &\forall \boldsymbol{v}_i \in V_i, \quad \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, \nabla \boldsymbol{u}_i \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x} \\ &\qquad\qquad + \int_\Gamma |\boldsymbol{u}_i - \boldsymbol{u}_j|\, (\boldsymbol{u}_i - \boldsymbol{u}_j) \cdot \boldsymbol{v}_i \, d\tau = \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i \, d\boldsymbol{x}. \end{aligned} \tag{3.5}$$

The converse property relies on the following inf-sup condition of Babuška and Brezzi type.

Lemma 3.1. *For* $i = 1$ *and* 2*, there exists a constant* $\beta_i > 0$ *such that*

$$\forall q_i \in L^2(\Omega_i), \quad \sup_{\boldsymbol{v}_i \in X_i} \frac{b_i(\boldsymbol{v}_i, q_i)}{\|\boldsymbol{v}_i\|_{H^1(\Omega_i)^d}} \ge \beta_i\, \|q_i\|_{L^2(\Omega_i)}. \tag{3.6}$$

Proof: The argument here is due to Boland and Nicolaides [6]. Let us write any function q_i in $L^2(\Omega_i)$ as

$$q_i = \tilde{q}_i + \overline{q}_i, \qquad \text{with} \quad \overline{q}_i = \frac{1}{\operatorname{meas}(\Omega_i)} \int_{\Omega_i} q_i \, d\boldsymbol{x}.$$

Indeed, from the standard inf-sup condition [11, Chap. I, Cor. 2.4], there exists a function $\tilde{\boldsymbol{v}}_i$ in $H_0^1(\Omega_i)^d$ such that

$$\operatorname{div} \tilde{\boldsymbol{v}}_i = -\tilde{q}_i \quad \text{and} \quad \|\tilde{\boldsymbol{v}}_i\|_{H^1(\Omega_i)^d} \le c_i \, \|\tilde{q}_i\|_{L^2(\Omega_i)},$$

where the constant c_i only depends on the geometry of Ω_i. On the other hand, there exist an open interval or disk $\mathcal{D}$ such that its closure is contained in the interior of Γ and positive real numbers ε_i such that the cylinders $\mathcal{C}_1 = \mathcal{D}\times]0, \varepsilon_1[$ and $\mathcal{C}_2 = \mathcal{D}\times] - \varepsilon_2, 0[$ are contained in Ω_1 and Ω_2, respectively. If ρ denotes a smooth function that is positive on $\mathcal{D}$ and vanishes on $\partial\mathcal{D}$, the function with horizontal components equal to zero and vertical component v_{iz} defined on the cylinder $\mathcal{C}_i$ by (in dimension $d = 3$ for instance)

$$v_{iz}(x, y, z) = \overline{q}_i \, \rho(x, y) \, (\pm\varepsilon_i - z),$$

can be extended by zero into a function $\overline{\boldsymbol{v}}_i$ of X_i which satisfies

$$b_i(\overline{\boldsymbol{v}}_i, \overline{q}_i) = c_i' \, \|\overline{q}_i\|^2_{L^2(\Omega_i)} \quad \text{and} \quad \|\overline{\boldsymbol{v}}_i\|_{H^1(\Omega_i)^d} \le c_i'' \, \|\overline{q}_i\|_{L^2(\Omega_i)}.$$

Finally, taking $\boldsymbol{v}_i = \tilde{\boldsymbol{v}}_i + \mu_i \, \overline{\boldsymbol{v}}_i$ yields

$$\begin{aligned} b_i(\boldsymbol{v}_i, q_i) &= \|\tilde{q}_i\|^2_{L^2(\Omega_i)} + \mu_i c_i' \, \|\overline{q}_i\|^2_{L^2(\Omega_i)} + \mu_i \, b_i(\overline{\boldsymbol{v}}_i, \tilde{q}_i) \\ &\ge \|\tilde{q}_i\|^2_{L^2(\Omega_i)} + \mu_i c_i' \, \|\overline{q}_i\|^2_{L^2(\Omega_i)} - \mu_i c_i'' \, \|\tilde{q}_i\|_{L^2(\Omega_i)} \|\overline{q}_i\|_{L^2(\Omega_i)} \\ &\ge \frac{1}{2} \|\tilde{q}_i\|^2_{L^2(\Omega_i)} + \mu_i \big(c_i' - \frac{\mu_i c_i''^2}{2}\big) \, \|\overline{q}_i\|^2_{L^2(\Omega_i)}, \end{aligned}$$

and also

$$\|\boldsymbol{v}_i\|_{H^1(\Omega_i)^d} \le c_i \, \|\tilde{q}_i\|_{L^2(\Omega_i)} + \mu_i c_i'' \, \|\overline{q}_i\|_{L^2(\Omega_i)} \le \big(c_i^2 + \mu_i^2 c_i''^2\big)^{\frac{1}{2}} \, \|q_i\|_{L^2(\Omega_i)}.$$

So choosing $\mu_i = \frac{c_i'}{c_i''^2}$ leads to the desired result.

The next corollary is now a direct consequence of Lemma 3.1, see [11, Chap. I, Lemma 4.1].

Corollary 3.2. *For $i = 1$ and 2 and for any data $\boldsymbol{f}_i$ in $L^2(\Omega_i)^d$,*
(i) for any solution $(\boldsymbol{u}_i, p_i)$ of problem (3.3), the velocity $\boldsymbol{u}_i$ is a solution of problem (3.5),
(ii) for any solution $\boldsymbol{u}_i$ of problem (3.5), there exists a unique pressure p_i in $L^2(\Omega_i)$ such that the pair $(\boldsymbol{u}_i, p_i)$ is a solution of problem (3.3).

We now prove the existence of a solution of problem (3.5). We begin with an *a priori* estimate.

Lemma 3.3. *For every ℓ_i in $L^1(\Omega_i)$ and $\boldsymbol{f}_i$ in $L^2(\Omega_i)^d$, $1 \le i \le 2$, any solution $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ of problem (3.5) satisfies*

$$\|\boldsymbol{u}_1\|_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2\|_{H^1(\Omega_2)^d} \le \frac{c}{\nu} \big(\|\boldsymbol{f}_1\|_{L^2(\Omega_1)^d} + \|\boldsymbol{f}_2\|_{L^2(\Omega_2)^d}\big). \tag{3.7}$$

Proof: Taking $\boldsymbol{v}_i$ equal to $\boldsymbol{u}_i$ in (3.5), using (2.3) and summing up on i give

$$\nu \, \big(|\boldsymbol{u}_1|^2_{H^1(\Omega_1)^d} + |\boldsymbol{u}_2|^2_{H^1(\Omega_2)^d}\big) + \int_\Gamma |\boldsymbol{u}_i - \boldsymbol{u}_j|^3 \, d\tau \le \int_{\Omega_1} \boldsymbol{f}_1 \, \boldsymbol{u}_1 \, d\boldsymbol{x} + \int_{\Omega_2} \boldsymbol{f}_2 \, \boldsymbol{u}_2 \, d\boldsymbol{x}.$$

Since the term integrated on Γ is nonnegative, the desired estimate follows from the Cauchy–Schwarz and Poincaré–Friedrichs inequalities.

Proposition 3.4. *For every ℓ_i in $L^1(\Omega_i)$ and $\boldsymbol{f}_i$ in $L^2(\Omega_i)^d$, $1 \le i \le 2$, problem* (3.5) *has a solution* $(\boldsymbol{u}_1, \boldsymbol{u}_2)$. *Moreover, this solution satisfies* (3.7).

Proof: Since both spaces V_1 and V_2 are separable (indeed, they are closed subspaces of the spaces $H^1(\Omega_i)$ which are separable), there exist increasing sequences of finite-dimensional Hilbert subspaces V_i^m of V_i, $i = 1$ and 2, such that

$$V_i = \bigcup_{m \ge 0} V_i^m, \qquad 1 \le i \le 2.$$

We define a mapping Φ_m from $V_1^m \times V_2^m$ into itself by

$$\begin{aligned} &\forall (\boldsymbol{u}_1, \boldsymbol{u}_2) \in V_1^m \times V_2^m, \quad \forall (\boldsymbol{v}_1, \boldsymbol{v}_2) \in V_1^m \times V_2^m, \\ &\big(\Phi_m(\boldsymbol{u}_1, \boldsymbol{u}_2), (\boldsymbol{v}_1, \boldsymbol{v}_2)\big) = \int_{\Omega_1} \tilde{\alpha}_1(\ell_1)\, \nabla \boldsymbol{u}_1 \cdot \nabla \boldsymbol{v}_1 \, d\boldsymbol{x} + \int_{\Omega_2} \tilde{\alpha}_2(\ell_2)\, \nabla \boldsymbol{u}_2 \cdot \nabla \boldsymbol{v}_2 \, d\boldsymbol{x} \\ &\qquad\qquad + \int_\Gamma |\boldsymbol{u}_1 - \boldsymbol{u}_2|\, (\boldsymbol{u}_1 - \boldsymbol{u}_2)\, (\boldsymbol{v}_1 - \boldsymbol{v}_2)\, d\tau \\ &\qquad\qquad - \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1 \, d\boldsymbol{x} - \int_{\Omega_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}_2 \, d\boldsymbol{x}, \end{aligned} \tag{3.8}$$

where $(\cdot, \cdot)$ stands for the scalar product on $V_1 \times V_2$. Since the function $\tilde{\alpha}_i$ is bounded and the traces of functions in V_i on Γ belong to $H^{\frac{1}{2}}(\Gamma)^d$, hence at least to $L^4(\Gamma)^d$ from the Sobolev embedding (recall that $d \le 3$), each mapping Φ_m is well-defined and continuous on $V_1^m \times V_2^m$. Moreover, as for estimate (3.7), it is readily checked that

$$\begin{aligned} \big(\Phi_m(\boldsymbol{u}_1, \boldsymbol{u}_2), (\boldsymbol{u}_1, \boldsymbol{u}_2)\big) &\ge \nu \left(|\boldsymbol{u}_1|^2_{H^1(\Omega_1)^d} + |\boldsymbol{u}_2|^2_{H^1(\Omega_2)^d} \right) \\ &\quad - c \left(\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)^d} + \|\boldsymbol{f}_2\|^2_{L^2(\Omega_2)^d} \right)^{\frac{1}{2}} \left(|\boldsymbol{u}_1|^2_{H^1(\Omega_1)^d} + |\boldsymbol{u}_2|^2_{H^1(\Omega_2)^d} \right)^{\frac{1}{2}}. \end{aligned}$$

So the right-hand side is nonnegative on the sphere of radius μ defined by

$$\mu = \frac{c}{\nu} \left(\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)^d} + \|\boldsymbol{f}_2\|^2_{L^2(\Omega_2)^d} \right)^{\frac{1}{2}}.$$

Applying the Brouwer's fixed point theorem (see e.g. [11, Chap. IV, Cor. 1.1]) yields the existence of a pair $(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)$ in $V_1^m \times V_2^m$, with norm less than μ, such that

$$\Phi_m(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m) = 0.$$

Since the sequence $(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)_m$ is bounded by μ in $V_1 \times V_2$, there exists a subsequence which converges weakly to $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in $V_1 \times V_2$. Using the compactness of the imbedding of $H^{\frac{1}{2}}(\Gamma)$ into $L^3(\Gamma)$, we obtain that this pair $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is a solution of problem (3.5).

We conclude the study of problem (3.5) by a uniqueness result.

Proposition 3.5. *For every ℓ_i in $L^1(\Omega_i)$ and $\boldsymbol{f}_i$ in $L^2(\Omega_i)^d$, $1 \le i \le 2$, the solution* $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ *of problem* (3.5) *is unique.*

Proof: Let $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and $(\overline{\boldsymbol{u}}_1, \overline{\boldsymbol{u}}_2)$ be two solutions of this problem. By setting $\boldsymbol{w}_i = \boldsymbol{u}_i - \overline{\boldsymbol{u}}_i$, $1 \le i \le 2$, we observe that each $\boldsymbol{w}_i$ belongs to V_i and satisfies for $1 \le i \le 2$

$$\forall \boldsymbol{v}_i \in V_i, \quad \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, \nabla \boldsymbol{w}_i \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x} + \int_\Gamma \big(|\boldsymbol{u}_i - \boldsymbol{u}_j|\, (\boldsymbol{u}_i - \boldsymbol{u}_j) - |\overline{\boldsymbol{u}}_i - \overline{\boldsymbol{u}}_j|\, (\overline{\boldsymbol{u}}_i - \overline{\boldsymbol{u}}_j) \big)\, \boldsymbol{v}_i \, d\tau = 0.$$

Taking $\boldsymbol{v}_i$ equal to $\boldsymbol{w}_i$ in this equation and summing on i yield, since $\tilde{\alpha}_i$ is $\geq \nu$,

$$\nu \left(|\boldsymbol{w}_1|^2_{H^1(\Omega_1)^d} + |\boldsymbol{w}_2|^2_{H^1(\Omega_2)^d}\right) \\ + \int_\Gamma \left(|\boldsymbol{u}_1 - \boldsymbol{u}_2|\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) - |\overline{\boldsymbol{u}}_1 - \overline{\boldsymbol{u}}_2|\,(\overline{\boldsymbol{u}}_1 - \overline{\boldsymbol{u}}_2)\right)\left((\boldsymbol{u}_1 - \boldsymbol{u}_2) - (\overline{\boldsymbol{u}}_1 - \overline{\boldsymbol{u}}_2)\right) d\tau \leq 0.$$

From the inequality, valid for all real numbers λ and $\overline{\lambda}$,

$$(|\lambda|\,\lambda - |\overline{\lambda}|\,\overline{\lambda})\,(\lambda - \overline{\lambda}) \geq 0, \tag{3.9}$$

we deduce that both $|\boldsymbol{w}_i|_{H^1(\Omega_i)^d}$ are zero, whence the result.

Note as a conclusion of this section that, for any ℓ_i in $L^1(\Omega_i)$ and $\boldsymbol{f}_i$ in $L^2(\Omega_i)^d$, system (3.3) has a unique solution $\big((\boldsymbol{u}_1, p_1), (\boldsymbol{u}_2, p_2)\big)$ in $\big(X_1 \times L^2(\Omega_1)\big) \times \big(X_2 \times L^2(\Omega_2)\big)$, which is bounded as a function of the norms of the data $\boldsymbol{f}_i$.

4. The equations on the turbulent kinetic energies (TKE).

Throughout this section, we assume that the functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are given such that

$$\boldsymbol{u}_i \in H^1(\Omega_i)^d, \qquad 1 \le i \le 2. \tag{4.1}$$

The correct formulation of the equations on the TKE is by transposition, following the ideas of Stampacchia [23] and Lions and Magenes [16].

Let us first perform a formal computation. If φ_i, $1 \le i \le 2$, are functions in $\mathscr{C}^\infty(\overline{\Omega}_i)$ which vanish on $\partial\Omega_i$, we obtain by multiplying the third line of (2.5) by these functions and integrating twice by parts

$$-\int_{\Omega_i} \ell_i \, \Delta\varphi_i \, d\boldsymbol{x} = -\int_\Gamma G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) \, \partial_{n_i}\varphi_i \, d\tau + \int_{\Omega_i} \tilde{\alpha}_i(\ell_i) \, |\nabla \boldsymbol{u}_i|^2 \, \varphi_i \, d\boldsymbol{x}.$$

So, from now on, we look for a solution of the following problem:

$$\begin{aligned} &\textit{Find } \ell_i \textit{ in } L^2(\Omega_i),\ 1 \le i \le 2, \textit{ such that, for } 1 \le i \le 2\textit{:}\\ &\forall \varphi_i \in H^2(\Omega_i) \cap H^1_0(\Omega_i),\\ &\quad -\int_{\Omega_i} \ell_i \, \Delta\varphi_i \, d\boldsymbol{x} = -\int_\Gamma G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) \, \partial_{n_i}\varphi_i \, d\tau + \int_{\Omega_i} \tilde{\alpha}_i(\ell_i) \, |\nabla \boldsymbol{u}_i|^2 \, \varphi_i \, d\boldsymbol{x}. \end{aligned} \tag{4.2}$$

Our existence proof relies on the fact that, since the domain Ω_i is convex or of class $\mathscr{C}^{1,1}$, the Laplace operator $\mathcal{L}_i$ which, with data g_i in $H^{-1}(\Omega)$, associates the solution $\varphi_i = \mathcal{L}_i g_i$ in $H^1_0(\Omega_i)$ of the problem

$$\begin{cases} -\Delta\varphi_i = g_i & \text{in } \Omega_i, \\ \varphi_i = 0 & \text{on } \partial\Omega_i, \end{cases} \tag{4.3}$$

is continuous from $L^2(\Omega_i)$ into $H^2(\Omega_i)$ (see e.g. [12, Thm. 3.2.1.2]). Combining this property with an interpolation argument [16, Chap. 1, Th. 5.1] leads to the following result.

Lemma 4.1. *Let t be a real number, $1 \le t \le 2$. For $1 \le i \le 2$, the operator $\mathcal{L}_i$ is continuous from $H^{t-2}(\Omega_i)$ into $H^t(\Omega_i)$ and satisfies*

$$\forall g_i \in H^{t-2}(\Omega_i), \quad \|\mathcal{L}_i g_i\|_{H^t(\Omega_i)} \le c \, \|g_i\|_{H^{t-2}(\Omega_i)}, \tag{4.4}$$

for a constant c independent of t.

We are now in a position to prove the *a priori* estimate.

Lemma 4.2. *Let s be a real number, $0 < s < \frac{1}{2}$. For $1 \le i \le 2$ and for every pair $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in $H^1(\Omega_1)^d \times H^1(\Omega_2)^d$, there exists a constant c depending on s and on the maximum of α_i and γ_i such that every solution ℓ_i of equation (4.2) satisfies*

$$\|\ell_i\|_{H^s(\Omega_i)} \le c \left(\|\boldsymbol{u}_1\|^2_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2\|^2_{H^1(\Omega_2)^d} \right). \tag{4.5}$$

Proof: Let us introduce the linear form

$$F_i\,\varphi_i = -\int_\Gamma G_i(|\boldsymbol{u}_1-\boldsymbol{u}_2|^2)\,\partial_{n_i}\varphi_i\,d\tau + \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2\,\varphi_i\,d\boldsymbol{x},$$

and check that it is continuous on $H^{2-s}(\Omega_i)$. Firstly, since γ_i is bounded, it follows from the definition (2.1) of G_i that

$$G_i(|\boldsymbol{u}_1-\boldsymbol{u}_2|^2) \le c\,|\boldsymbol{u}_1-\boldsymbol{u}_2|^2.$$

Moreover the trace $\boldsymbol{u}_1-\boldsymbol{u}_2$ belongs to $H^{\frac{1}{2}}(\Gamma)^d$, hence to $L^4(\Gamma)^d$ thanks to the Sobolev imbedding in dimension $d\le 3$. On the other hand, since Γ as a part of an hyperplane is smooth and φ_i belongs to $H^{2-s}(\Omega_i)$, $\partial_{n_i}\varphi_i$ belongs to $H^{\frac{1}{2}-s}(\Gamma)$. So we have

$$|\int_\Gamma G_i(|\boldsymbol{u}_1-\boldsymbol{u}_2|^2)\,\partial_{n_i}\varphi_i\,d\tau| \le c\,\big(\int_\Gamma |\boldsymbol{u}_1-\boldsymbol{u}_2|^4\,d\tau\big)^{\frac{1}{2}}\,\|\partial_{n_i}\varphi_i\|_{L^2(\Gamma)}.$$

Similarly, since $\tilde{\alpha}_i$ is bounded and φ_i belongs to $H^{2-s}(\Omega_i)$, hence to $L^\infty(\Omega_i)$, we obtain

$$|\int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\,|\nabla\boldsymbol{u}_i|^2\,\varphi_i\,d\boldsymbol{x}| \le c\,\big(\|\boldsymbol{u}_1\|^2_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2\|^2_{H^1(\Omega_2)^d}\big)\|\varphi_i\|_{L^\infty(\Omega_i)}.$$

Combining all this leads to

$$F_i\,\varphi_i \le c\,\big(\|\boldsymbol{u}_1\|^2_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2\|^2_{H^1(\Omega_2)^d}\big)\|\varphi_i\|_{H^{2-s}(\Omega_i)}. \tag{4.6}$$

When combined with (4.4) for $t=2-s$, this yields that the linear form $F_i\circ\mathcal{L}_i$ is continuous on $H^{-s}(\Omega_i)$. Note [16, Chap. 1, Thm 11.1] that, for $0\le s<\frac{1}{2}$, the dual space of $H^s(\Omega_i)$ coincides with $H^{-s}(\Omega_i)$ and that these spaces are reflexive. So any function ℓ_i satisfying (4.2), hence

$$\forall g_i\in H^{-s}(\Omega_i),\quad < g_i,\ell_i > = F_i\,\circ\mathcal{L}_i\,g_i,$$

belongs to $H^s(\Omega_i)$ and satisfies (4.5).

From Lemma 4.2, we derive that equation (4.2) with the integral in the left-hand side replaced by the duality pairing is satisfied for any φ_i in $H^{2-s}(\Omega_i)\cap H^1_0(\Omega_i)$. Proving the existence result is now easy, by using the same arguments as for Proposition 3.4, with a further regularization of the data.

Proposition 4.3. *For $1\le i\le 2$, and for any pair $(\boldsymbol{u}_1,\boldsymbol{u}_2)$ in $X_1\times X_2$, problem* (4.2) *has a solution ℓ_i. Moreover, this solution belongs to $H^s(\Omega_i)$, for every $s<\frac{1}{2}$.*

Proof: We firstly consider the problem with more regular data (λ_i,ρ_i) in $L^2(\Omega_i)\times H^{\frac{1}{2}}_{00}(\Gamma)$:

$$\begin{cases} -\Delta\ell_i = \tilde{\alpha}_i(\ell_i)\,\lambda_i & \text{in } \Omega_i,\\ \ell_i = 0 & \text{on } \Gamma_i,\\ \ell_i = \rho_i & \text{on } \Gamma. \end{cases} \tag{4.7}$$

Let us still denote by ρ_i a function in $H^1(\Omega_i)$, the trace of which vanishes on Γ_i and coincides with ρ_i on Γ. The idea is to set $\ell_i^0 = \ell_i - \rho_i$. So, we now consider the problem:

Find ℓ_i^0 in $H_0^1(\Omega_i)$, $1 \leq i \leq 2$, such that:

$$\forall g_i \in H_0^1(\Omega_i), \quad \int_{\Omega_i} \nabla \ell_i^0 \cdot \nabla g_i \, d\boldsymbol{x} = \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^0 + \rho_i)\, \lambda_i \, g_i \, d\boldsymbol{x} - \int_{\Omega_i} \nabla \rho_i \cdot \nabla g_i \, d\boldsymbol{x}. \tag{4.8}$$

Since $H_0^1(\Omega_i)$ is separable, there exists an increasing sequence of finite-dimensional Hilbert subspaces Z_i^m of $H_0^1(\Omega_i)$ such that

$$H_0^1(\Omega_i) = \bigcup_{m \geq 0} Z_i^m.$$

We then define a mapping Ψ_{im} from Z_i^m into itself by

$$\begin{aligned} &\forall \ell_i^0 \in Z_i^m, \forall g_i \in Z_i^m, \\ &\big(\Psi_{im}(\ell_i^0), g_i\big) = \int_{\Omega_i} \nabla \ell_i^0 \cdot \nabla g_i \, d\boldsymbol{x} - \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^0 + \rho_i)\, \lambda_i \, g_i \, d\boldsymbol{x} + \int_{\Omega_i} \nabla \rho_i \cdot \nabla g_i \, d\boldsymbol{x}. \end{aligned} \tag{4.9}$$

It is readily checked that

$$\big(\Psi_{im}(\ell_i^0), \ell_i^0\big) \geq |\ell_i^0|^2_{H^1(\Omega_i)} - \big(c\, \|\lambda_i\|_{L^2(\Omega_i)} + |\rho_i|_{H^1(\Omega_i)}\big)\, |\ell_i^0|_{H^1(\Omega_i)},$$

where c only depends on the maximum of the function $\tilde{\alpha}_i$ and on the Poincaré–Friedrichs constant. So, this quantity is nonnegative when the function ℓ_i^0 belongs to a sphere with appropriate radius

$$\mu_i = c\, \|\lambda_i\|_{L^2(\Omega_i)} + |\rho_i|_{H^1(\Omega_i)}.$$

Using once more the Brouwer's fixed point theorem (see e.g. [11, Chap. IV, Cor. 1.1]) yields the existence of a solution ℓ_i^{0m} of the equation $\Psi_{im}(\ell_i^{0m}) = 0$ with norm less than μ_i.

Since the sequence $(\ell_i^{0m})_{m \geq 0}$ is bounded in $H^1(\Omega_i)$, it admits a subsequence which tends to ℓ_i^0 weakly in $H^1(\Omega_i)$. Due to the compactness of $H^1(\Omega_i)$ into $L^2(\Omega_i)$, this subsequence converges strongly in $L^2(\Omega_i)$. Hence by the inverse Lebesgue theorem (see e.g. [7, Th. IV.9]), there exists a further subsequence, still denoted by $(\ell_i^{0m})_{m \geq 0}$, which tends to ℓ_i^0 a.e. on Ω_i. Since the function $\tilde{\alpha}_i$ is continuous and bounded, the subsequence $\big(\tilde{\alpha}_i(\ell_i^{0m} + \rho_i)\, g_i\big)_{m \geq 0}$ is dominated by $c\, g_i$ which belongs to $L^2(\Omega_i)$, and tends to $\tilde{\alpha}_i(\ell_i^0 + \rho_i)\, g_i$ a.e. in Ω_i, hence in $L^2(\Omega_i)$. So, the function ℓ_i^0 is a solution of (4.8) and finally the function $\ell_i = \ell_i^0 + \rho_i$ is a solution of (4.7).

Since we have proved the existence of a solution of problem (4.7), we now pass to the case of real data. We approximate each $\boldsymbol{u}_i$ in X_i, $1 \leq i \leq 2$, by a sequence $(\boldsymbol{u}_i^n)_{n \geq 0}$ of $\mathscr{C}^\infty(\overline{\Omega}_i)^d \cap X_i$ which converges towards $\boldsymbol{u}_i$ in $H^1(\Omega_i)^d$. Then, it is readily checked that the functions $\lambda_i^n = |\nabla \boldsymbol{u}_i^n|^2$, resp. $\rho_i^n = G_i(|\boldsymbol{u}_1^n - \boldsymbol{u}_2^n|^2)$, belong to $L^2(\Omega_i)$, resp. $H_{00}^{\frac{1}{2}}(\Gamma)$ (see [16, Chap. 1, Th. 11.7] for the characterization of this space). Let ℓ_i^n be a solution of problem (4.7) with data $\lambda_i = \lambda_i^n$ and $\rho_i = \rho_i^n$. Since it belongs to $H^1(\Omega_i)$, by integration by parts, it is readily checked that it satisfies

$$\begin{aligned} &\forall \varphi_i \in H^2(\Omega_i) \cap H_0^1(\Omega_i), \\ &\qquad - \int_{\Omega_i} \ell_i^n \, \Delta \varphi_i \, d\boldsymbol{x} = - \int_\Gamma G_i(|\boldsymbol{u}_1^n - \boldsymbol{u}_2^n|^2)\, \partial_{n_i} \varphi_i \, d\tau + \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^n)\, |\nabla \boldsymbol{u}_i^n|^2 \, \varphi_i \, d\boldsymbol{x}. \end{aligned}$$

Let now s_0 be such that $0 < s_0 < \frac{1}{2}$. From Lemma 4.2, the sequence $(\ell_i^n)_{n\geq 0}$ is bounded in $H^{s_0}(\Omega_i)$, so that it admits a subsequence which converges to ℓ_i weakly in $H^{s_0}(\Omega_i)$ (and in fact in all $H^s(\Omega_i)$, $s < \frac{1}{2}$). From the compactness of $H^{s_0}(\Omega_i)$ into $L^2(\Omega_i)$ and the boundedness of the function $\tilde{\alpha}_i$, by the same arguments as previously, we deduce the existence of subsequences, still denoted by $(\ell_i^n)_{n\geq 0}$ and $(\boldsymbol{u}_i^n)_{n\geq 0}$, such that the sequence $(\tilde{\alpha}_i(\ell_i^n)\,|\nabla \boldsymbol{u}_i^n|^2)_{n\geq 0}$ is dominated by a function of $L^1(\Omega_i)$ and tends to $\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2$ a.e. in Ω_i, hence in $L^1(\Omega_i)$. Similarly, there exists a subsequence $(G_i(|\boldsymbol{u}_1^n - \boldsymbol{u}_2^n|^2))_{n\geq 0}$ which tends to the term $G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)$ in $L^2(\Gamma)$. So, the limit ℓ_i is a solution of (4.2), which completes the proof.

Remark: Consider a solution ℓ_i of problem (4.2) which further belongs to $H^1(\Omega_i)$. Then, if the pair $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ belongs to $W^{1,\frac{12}{5}}(\Omega_1)^d \times W^{1,\frac{12}{5}}(\Omega_2)^d$, it solves the following variational problem:

$$
\begin{aligned}
&\textit{Find } \ell_i \textit{ in } H^1(\Omega_i),\ 1 \leq i \leq 2, \textit{ with}\\
&\qquad \ell_i = 0 \quad \textit{on } \Gamma_i \qquad \textit{and} \qquad \ell_i = G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) \quad \textit{on } \Gamma,\\
&\textit{such that, for } 1 \leq i \leq 2\textit{:}\\
&\forall \varphi_i \in H_0^1(\Omega_i), \quad \int_{\Omega_i} \nabla \ell_i \cdot \nabla \varphi_i \, d\boldsymbol{x} = \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2 \,\varphi_i \, d\boldsymbol{x}.
\end{aligned}
\tag{4.10}
$$

Note that (4.10) is now a classical formulation, which could be discretized in an easy way.

5. The global system.

Proving the existence of a solution for the full system (2.5) follows from the same arguments as previously, however we begin by working with a truncated problem. For each positive integer n, we introduce the function T_n defined from $\mathbb{R}$ onto $\mathbb{R}$ by

$$T_n(x) = \begin{cases} -n & \text{if } \ x \le -n, \\ x & \text{if } \ -n \le x \le n, \\ n & \text{if } \ x \ge n, \end{cases}$$

and we consider the problem

$$\begin{cases} -\text{div}\,\big(\tilde{\alpha}_i(\ell_i)\,\nabla \boldsymbol{u}_i\big) + \mathbf{grad}\, p_i = \boldsymbol{f}_i & \text{in } \Omega_i,\ 1 \le i \le 2, \\ \text{div}\, \boldsymbol{u}_i = 0 & \text{in } \Omega_i,\ 1 \le i \le 2, \\ -\Delta \ell_i = T_n\big(\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2\big) & \text{in } \Omega_i,\ 1 \le i \le 2, \\ \boldsymbol{u}_i = \mathbf{0} & \text{on } \Gamma_i,\ 1 \le i \le 2, \\ \ell_i = 0 & \text{on } \Gamma_i,\ 1 \le i \le 2, \\ \tilde{\alpha}_i(\ell_i)\,\partial_{n_i} \boldsymbol{u}_i - p_i\,\boldsymbol{n}_i + (\boldsymbol{u}_i - \boldsymbol{u}_j)\,|\boldsymbol{u}_i - \boldsymbol{u}_j| = \mathbf{0} & \text{on } \Gamma,\ 1 \le i \ne j \le 2, \\ \ell_i = T_n\big(G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)\big) & \text{on } \Gamma,\ 1 \le i \le 2. \end{cases} \tag{5.1}$$

We first write its reduced variational formulation (where the word "reduced" means that the pressures p_i do not appear in it):

Find $\boldsymbol{u}_i$ in V_i, $1 \le i \le 2$, such that, for $1 \le i \ne j \le 2$:

$$\forall \boldsymbol{v}_i \in V_i, \quad \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\,\nabla \boldsymbol{u}_i \cdot \nabla \boldsymbol{v}_i\, d\boldsymbol{x} + \int_\Gamma |\boldsymbol{u}_i - \boldsymbol{u}_j|\,(\boldsymbol{u}_i - \boldsymbol{u}_j) \cdot \boldsymbol{v}_i\, d\tau = \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i\, d\boldsymbol{x}.$$

Find ℓ_i in $H^1(\Omega_i)$, $1 \le i \le 2$, with

$$\ell_i = 0 \quad \text{on } \Gamma_i \qquad \text{and} \qquad \ell_i = T_n\big(G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)\big) \quad \text{on } \Gamma, \tag{5.2}$$

such that, for $1 \le i \le 2$:

$$\forall \varphi_i \in H_0^1(\Omega_i), \quad \int_{\Omega_i} \nabla \ell_i \cdot \nabla \varphi_i\, d\boldsymbol{x} = \int_{\Omega_i} T_n\big(\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2\big)\,\varphi_i\, d\boldsymbol{x}.$$

The next lemma states that problem (5.1) is well-posed.

Lemma 5.1. *For any positive integer n, for any pair $(\boldsymbol{f}_1, \boldsymbol{f}_2)$ in $L^2(\Omega_1)^d \times L^2(\Omega_2)^d$, problem (5.1) admits the variational formulation (5.2). System (5.2) has a solution (U_1, U_2) with each $U_i = (\boldsymbol{u}_i, \ell_i)$ in $V_i \times H^1(\Omega_i)$. Moreover the functions ℓ_1 and ℓ_2 are nonnegative.*

The proof of this lemma is rather complex, it is performed in five steps.

Proof (I): Liftings of traces.

For $i = 1$ and 2, we introduce the continuous lifting operator L_i from $H^{\frac{1}{2}}_{00}(\Gamma)$ into harmonic functions in X_i (note that the extension by zero of a function in $H^{\frac{1}{2}}_{00}(\Gamma)$ to $\partial\Omega_i$ belongs to $H^{\frac{1}{2}}(\partial\Omega_i)$). Next, with each pair $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in $X_1 \times X_2$, we associate a function $\rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)$ defined on Ω_i by

$$\rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2) = T_n\big(G_i(|L_i(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2)\big).$$

Since the trace of each $\boldsymbol{u}_i$ on Γ belongs to $H^{\frac{1}{2}}_{00}(\Gamma)^d$, the function $\boldsymbol{g}_i = L_i(\boldsymbol{u}_1 - \boldsymbol{u}_2)$ belongs to $H^1(\Omega_i)^d$ and satisfies

$$\|\boldsymbol{g}_i\|_{H^1(\Omega_i)^d} \le c\left(\|\boldsymbol{u}_1\|^2_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2\|^2_{H^1(\Omega_2)^d}\right)^{\frac{1}{2}}.$$

Let us now prove that the function $T_n\big(G_i(|\boldsymbol{g}_i|^2)\big)$ belongs to $H^1(\Omega_i)$. We have [23, Lemme 1.1]

$$\textbf{grad}\, T_n\big(G_i(|\boldsymbol{g}_i|^2)\big) = 2T'_n\big(G_i(|\boldsymbol{g}_i|^2)\big)\,\gamma_i(|\boldsymbol{g}_i|^2)\,\textbf{grad}\,\boldsymbol{g}_i \cdot \boldsymbol{g}_i.$$

Note that γ_i is bounded and that $\textbf{grad}\,\boldsymbol{g}_i$ belongs at most to $L^2(\Omega_i)^{d^2}$. Moreover, $T'_n\big(G_i(|\boldsymbol{g}_i|^2)\big)$ is either 0 or 1, and, when it is not zero,

$$\nu\,|\boldsymbol{g}_i|^2 \le \int_0^{|\boldsymbol{g}_i|^2} \gamma_i(\kappa)\,d\kappa = G_i(|\boldsymbol{g}_i|^2) \le n,$$

so that $T'_n\big(G_i(|\boldsymbol{g}_i|^2)\big)\,\boldsymbol{g}_i$ is bounded and $T'_n\big(G_i(|\boldsymbol{g}_i|^2)\big)\,\gamma_i(|\boldsymbol{g}_i|^2)\,\textbf{grad}\,\boldsymbol{g}_i \cdot \boldsymbol{g}_i$ belongs to $L^2(\Omega_i)^d$. As a consequence, the function $\rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)$ belongs to $H^1(\Omega_i)$ and satisfies

$$\|\rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{H^1(\Omega_i)} \le c\,n^{\frac{1}{2}}\left(\|\boldsymbol{u}_1\|^2_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2\|^2_{H^1(\Omega_2)^d}\right)^{\frac{1}{2}}. \tag{5.3}$$

Next we work with the new unknowns $(\boldsymbol{u}_i, \ell_i^0 = \ell_i - \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2))$ in $V_i \times H^1_0(\Omega_i)$.

Proof (II): Existence of a solution for a finite-dimensional system.

For $i = 1$ and 2, as in the proof of Proposition 3.4, we introduce an increasing sequence of finite-dimensional Hilbert subspaces V_i^m of V_i, $i = 1$ and 2, such that

$$V_i = \bigcup_{m\ge 0} V_i^m, \qquad 1 \le i \le 2,$$

and, as in the proof of Proposition 4.3, we introduce an increasing sequence of finite-dimensional Hilbert subspaces Z_i^m of $H^1_0(\Omega_i)$ such that

$$H^1_0(\Omega_i) = \bigcup_{m\ge 0} Z_i^m.$$

We now define the mappins Φ_m from $V_1^m \times V_2^m$ into itself by

$$\forall(\boldsymbol{u}_1, \boldsymbol{u}_2) \in V_1^m \times V_2^m, \quad \forall(\boldsymbol{v}_1, \boldsymbol{v}_2) \in V_1^m \times V_2^m,$$

$$\begin{aligned}\big(\Phi_m(\boldsymbol{u}_1, \boldsymbol{u}_2), (\boldsymbol{v}_1, \boldsymbol{v}_2)\big) = &\int_{\Omega_1} \tilde{\alpha}_1(\ell_1^0 + \rho_1(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m))\,\nabla\boldsymbol{u}_1 \cdot \nabla\boldsymbol{v}_1\,d\boldsymbol{x}\\ &+ \int_{\Omega_2} \tilde{\alpha}_2(\ell_i^0 + \rho_2(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m))\,\nabla\boldsymbol{u}_2 \cdot \nabla\boldsymbol{v}_2\,d\boldsymbol{x}\\ &+ \int_\Gamma |\boldsymbol{u}_1 - \boldsymbol{u}_2|\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)\,(\boldsymbol{v}_1 - \boldsymbol{v}_2)\,d\tau\\ &- \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1\,d\boldsymbol{x} - \int_{\Omega_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}_2\,d\boldsymbol{x},\end{aligned}$$

and, similarly, for $i = 1$ and 2, a mapping Ψ_{im} from Z_i^m into itself by

$$\forall \ell_i^0 \in Z_i^m, \forall g_i \in Z_i^m,$$
$$\big(\Psi_{im}(\ell_i^0), g_i\big) = \int_{\Omega_i} \nabla \ell_i^0 \cdot \nabla g_i \, d\boldsymbol{x} - \int_{\Omega_i} T_n\big(\tilde{\alpha}_i(\ell_i^0 + \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2))\, |\nabla \boldsymbol{u}_i|^2\big)\, g_i \, d\boldsymbol{x} + \int_{\Omega_i} \nabla \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2) \cdot \nabla g_i \, d\boldsymbol{x}.$$

Finally, we consider the mapping Ξ_m from $V_1^m \times Z_1^m \times V_2^m \times Z_2^m$ defined by

$$\forall (\boldsymbol{u}_1, \ell_1^0, \boldsymbol{u}_2, \ell_2^0) \in V_1^m \times Z_1^m \times V_2^m \times Z_2^m,\ \forall (\boldsymbol{v}_1, g_1, \boldsymbol{v}_2, g_2) \in V_1^m \times Z_1^m \times V_2^m \times Z_2^m,$$
$$\big(\Xi_m(\boldsymbol{u}_1, \ell_1^0, \boldsymbol{u}_2, \ell_2^0), (\boldsymbol{v}_1, g_1, \boldsymbol{v}_2, g_2)\big) = \big(\Phi_m(\boldsymbol{u}_1, \boldsymbol{u}_2), (\boldsymbol{v}_1, \boldsymbol{v}_2)\big) + \big(\Psi_{1m}(\ell_1^0), g_1\big) + \big(\Psi_{2m}(\ell_2^0), g_2\big).$$

Taking $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ on the sphere with radius μ, for the same μ as in the proof of Proposition 3.4, yields that $\Phi_m\big((\boldsymbol{u}_1, \boldsymbol{u}_2), (\boldsymbol{u}_1, \boldsymbol{u}_2)\big)$ is nonnegative. This combined with the arguments of the proof of Proposition 4.3 implies

$$\big(\Xi_m(\boldsymbol{u}_1, \ell_1^0, \boldsymbol{u}_2, \ell_2^0), (\boldsymbol{u}_1, \ell_1^0, \boldsymbol{u}_2, \ell_2^0)\big) \geq \sum_{i=1}^{2} \big(|\ell_i^0|^2_{H^1(\Omega_i)} - \big(c'\, n + |\rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)|_{H^1(\Omega_i)}\big)\, |\ell_i^0|_{H^1(\Omega_i)}\big).$$

So using the bound (5.3) for $\|\rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{H^1(\Omega_i)}$ and taking each ℓ_i^0 on the sphere with radius

$$\mu_i = c'\, n + c\, n^{\frac{1}{2}}\, \mu,$$

we deduce that the previous quantity is nonnegative. Thus, applying the Brouwer's fixed point theorem yields the existence of a solution $(\boldsymbol{u}_1^m, \ell_1^{0m}, \boldsymbol{u}_2^m, \ell_2^{0m})$ of

$$\Xi_m(\boldsymbol{u}_1^m, \ell_1^{0m}, \boldsymbol{u}_2^m, \ell_2^{0m}) = 0, \tag{5.4}$$

which satisfies

$$\|\boldsymbol{u}_1^m\|_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2^m\|_{H^1(\Omega_2)^d} \leq \mu, \quad \|\ell_i^{0m}\|_{H^1(\Omega_i)} \leq \mu_i, \quad i = 1 \text{ and } 2.$$

Since the sequence $(\boldsymbol{u}_1^m, \ell_1^{0m}, \boldsymbol{u}_2^m, \ell_2^{0m})_m$ is bounded, there exists a subsequence, still denoted by $(\boldsymbol{u}_1^m, \ell_1^{0m}, \boldsymbol{u}_2^m, \ell_2^{0m})_m$, which converges to $(\boldsymbol{u}_1, \ell_1^0, \boldsymbol{u}_2, \ell_2^0)$ weakly in $V_1 \times H_0^1(\Omega_1) \times V_2 \times H_0^1(\Omega_2)$. Next, for a fixed $(\boldsymbol{v}_1, g_1, \boldsymbol{v}_2, g_2)$, we pass to the limit in problem (5.4).

Proof (III): The limit on the equations for the velocities.
We start from the equation $\Phi_m(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m) = 0$. For $1 \leq i \leq 2$, there exists a subsequence still denoted by $(\ell_i^{0m})_m$ which converges to ℓ_i^0 strongly in $L^2(\Omega_i)$, hence a.e. in Ω_i. On the other hand, due to the continuity of L_i, the sequence $\big(\boldsymbol{g}_i^m = L_i(\boldsymbol{u}_1^m - \boldsymbol{u}_2^m)\big)_m$ converges to $L_i(\boldsymbol{u}_1 - \boldsymbol{u}_2)$ weakly in $H^1(\Omega_i)$. Due to the formula

$$\textbf{grad}\ \rho_i^m = \textbf{grad}\ T_n\big(G_i(|\boldsymbol{g}_i^m|^2)\big) = 2T_n'\big(G_i(|\boldsymbol{g}_i^m|^2)\big)\, \gamma_i(|\boldsymbol{g}_i^m|^2)\, \textbf{grad}\ \boldsymbol{g}_i^m \cdot \boldsymbol{g}_i^m,$$

there exists another subsequence $(\rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m))_m$ which converges to $\rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)$ weakly in $H^1(\Omega_i)^d$ and strongly in $L^2(\Omega_i)^d$. So, the corresponding subsequence $(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m))_m$

converges to $\ell_1^0 + \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)$ a.e. in Ω_i and, since $\tilde{\alpha}_i$ is continuous and bounded, for all fixed $\boldsymbol{v}_i$ in X_i, the sequence $(\tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)) \nabla \boldsymbol{v}_i)_m$ tends to $\tilde{\alpha}_i(\ell_i^0 + \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)) \nabla \boldsymbol{v}_i$ a.e. in Ω_i and is bounded in $L^2(\Omega_i)^{d^2}$, hence converges strongly in $L^2(\Omega_i)^{d^2}$. This yields the convergence of the first two integrals in the definition of $\Phi_m(\cdot, \cdot)$: for $i = 1$ and 2,

$$\lim_{m\to\infty} \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)) \nabla \boldsymbol{u}_i^m \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x} = \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^0 + \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)) \nabla \boldsymbol{u}_i \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x}.$$

The convergence of the third one follows from the compact imbedding of $H^{\frac{1}{2}}(\Gamma)$ into $L^3(\Gamma)$:

$$\lim_{m\to\infty} \int_\Gamma |\boldsymbol{u}_1^m - \boldsymbol{u}_2^m| \, (\boldsymbol{u}_1^m - \boldsymbol{u}_2^m) \, (\boldsymbol{v}_1 - \boldsymbol{v}_2) \, d\tau = \int_\Gamma |\boldsymbol{u}_1 - \boldsymbol{u}_2| \, (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, (\boldsymbol{v}_1 - \boldsymbol{v}_2) \, d\tau.$$

Combining all this implies that the desired equation is satisfied by $(\boldsymbol{u}_1, \boldsymbol{u}_2)$.

Proof (IV): A stronger convergence result.
For $i = 1$ and 2, we start from the formula

$$\begin{aligned}\int_{\Omega_i} \tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)) \nabla (\boldsymbol{u}_i^m - \boldsymbol{u}_i) \cdot \nabla (\boldsymbol{u}_i^m - \boldsymbol{u}_i) \, d\boldsymbol{x} \\ = \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)) \nabla \boldsymbol{u}_i^m \cdot \nabla \boldsymbol{u}_i^m \, d\boldsymbol{x} \\ - 2 \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)) \nabla \boldsymbol{u}_i^m \cdot \nabla \boldsymbol{u}_i \, d\boldsymbol{x} \\ + \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)) \nabla \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i \, d\boldsymbol{x}.\end{aligned}$$

From equation (5.4), the first term in the right-hand side is equal to

$$- \int_\Gamma |\boldsymbol{u}_i^m - \boldsymbol{u}_j^m| \, (\boldsymbol{u}_i^m - \boldsymbol{u}_j^m) \cdot \boldsymbol{u}_i^m \, d\tau + \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{u}_i^m,$$

and using once more the compactness of $H^{\frac{1}{2}}$ into $L^3(\Gamma)$ implies its convergence. The convergence of the second and third term follows from the weak convergence of $(\nabla \boldsymbol{u}_i^m)_m$ in $L^2(\Omega_i)^{d^2}$ and the strong convergence of $(\tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)) \nabla \boldsymbol{v}_i)_m$ in $L^2(\Omega_i)$. So, we obtain that

$$\lim_{m\to\infty} \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)) \nabla (\boldsymbol{u}_i^m - \boldsymbol{u}_i) \cdot \nabla (\boldsymbol{u}_i^m - \boldsymbol{u}_i) \, d\boldsymbol{x} = 0,$$

which yields the strong convergence of $(\boldsymbol{u}_i^m)_m$ towards $\boldsymbol{u}_i$ in $H^1(\Omega_i)^d$.

Proof (V): The limit on the equations for the TKE.
Next, we consider each equation $\Psi_m(\ell_i^{0m}) = 0$. As previously, there exist a subsequence $(\tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m)))_m$ which tends to $\tilde{\alpha}_i(\ell_i^0 + \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2))$ strongly in $L^2(\Omega_i)$ and, from part IV of the proof, a subsequence $(|\nabla \boldsymbol{u}_i^m|^2)_m$ which tends to $|\nabla \boldsymbol{u}_i|^2$ strongly in $L^1(\Omega_i)$.

Hence, the sequence $(\tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m))\,|\nabla \boldsymbol{u}_i^m|^2)_m$ converges a.e. in Ω_i and, since T_n is continuous and bounded, the sequence

$$(T_n(\tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m))\,|\nabla \boldsymbol{u}_i^m|^2))_m,$$

converges towards $T_n(\tilde{\alpha}_i(\ell_i^0 + \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2))\,|\nabla \boldsymbol{u}_i|^2)$ strongly in $L^2(\Omega_i)$. This yields

$$\lim_{m\to\infty} \int_{\Omega_i} T_n\big(\tilde{\alpha}_i(\ell_i^{0m} + \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m))\,|\nabla \boldsymbol{u}_i^m|^2\big)\, g_i\, d\boldsymbol{x} = \int_{\Omega_i} T_n\big(\tilde{\alpha}_i(\ell_i^0 + \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2))\,|\nabla \boldsymbol{u}_i|^2\big)\, g_i\, d\boldsymbol{x}.$$

Also, from the weak convergence of a subsequence $(\rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m))_m$ to $\rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in $H^1(\Omega_i)^d$, we deduce

$$\lim_{m\to\infty} \int_{\Omega_i} \nabla \rho_i(\boldsymbol{u}_1^m, \boldsymbol{u}_2^m) \cdot \nabla g_i\, d\boldsymbol{x} = \int_{\Omega_i} \nabla \rho_i(\boldsymbol{u}_1, \boldsymbol{u}_2) \cdot \nabla g_i\, d\boldsymbol{x}.$$

So the desired equation is satisfied by ℓ_i. Finally, the nonnegativity of the ℓ_i follows from the standard maximum principle [7, Prop. IX.29].

We are now in a position to state the main result of this section. There also, we write the reduced variational formulation of system (2.5), where the equation on the ℓ_i has now the same "transposed" form as in Section 4:

Find $\boldsymbol{u}_i$ in V_i, $1 \le i \le 2$, such that, for $1 \le i \ne j \le 2$:

$$\forall \boldsymbol{v}_i \in V_i, \quad \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, \nabla \boldsymbol{u}_i \cdot \nabla \boldsymbol{v}_i\, d\boldsymbol{x} + \int_{\Gamma} |\boldsymbol{u}_i - \boldsymbol{u}_j|\,(\boldsymbol{u}_i - \boldsymbol{u}_j) \cdot \boldsymbol{v}_i\, d\tau = \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i\, d\boldsymbol{x}. \tag{5.5}$$

Find ℓ_i in $L^2(\Omega_i)$, $1 \le i \le 2$, such that, for $1 \le i \le 2$:

$$\forall \varphi_i \in H^2(\Omega_i) \cap H_0^1(\Omega_i), \quad - \int_{\Omega_i} \ell_i\, \Delta \varphi_i\, d\boldsymbol{x} = - \int_{\Gamma} G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)\, \partial_{n_i} \varphi_i\, d\tau + \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, |\nabla \boldsymbol{u}_i|^2\, \varphi_i\, d\boldsymbol{x}.$$

Theorem 5.2. *For any $\boldsymbol{f}_i$ in $L^2(\Omega_i)^d$, $i = 1$ or 2, problem* (2.5) *admits the formulation* (5.5). *System* (5.5) *has a solution (U_1, U_2) with each $U_i = (\boldsymbol{u}_i, \ell_i)$ in $X_i \times L^2(\Omega_i)$. Moreover, each function ℓ_i, $i = 1$ and 2, is nonnegative and belongs to $H^s(\Omega_i)$ for all $s < \frac{1}{2}$, and this solution satisfies* (3.7) *and* (4.5).

Proof: For each integer n, let us now denote by (U_1^n, U_2^n), with $U_i^n = (\boldsymbol{u}_i^n, \ell_i^n)$, a solution of problem (5.1) (its existence is proven in Lemma 5.1). As previously, see (3.7) and (4.5), it satisfies, for a fixed number $s < \frac{1}{2}$ and for a constant c independent of n,

$$\|\boldsymbol{u}_1^n\|_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2^n\|_{H^1(\Omega_2)^d} \le \frac{c}{\nu} \big(\|\boldsymbol{f}_1\|_{L^2(\Omega_1)^d} + \|\boldsymbol{f}_2\|_{L^2(\Omega_2)^d}\big),$$
$$\|\ell_i^n\|_{H^s(\Omega_i)} \le c\, \big(\|\boldsymbol{u}_1^n\|^2_{H^1(\Omega_1)^d} + \|\boldsymbol{u}_2^n\|^2_{H^1(\Omega_2)^d}\big), \quad i = 1 \text{ and } 2.$$

So, there exists a subsequence, still denoted by $(\boldsymbol{u}_1^n, \ell_1^n, \boldsymbol{u}_2^n, \ell_2^n)_n$, which converges towards $(\boldsymbol{u}_1, \ell_1, \boldsymbol{u}_2, \ell_2)$ weakly in $V_1 \times H^s(\Omega_1) \times V_2 \times H^s(\Omega_2)$. We must now prove that $(\boldsymbol{u}_1, \ell_1, \boldsymbol{u}_2, \ell_2)$

satisfies (5.5), which is performed in three steps.
1) We start from the variational formulation, for $i = 1$ and 2:

$$\forall \boldsymbol{v}_i \in V_i, \quad \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^n)\, \nabla \boldsymbol{u}_i^n \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x} + \int_\Gamma (\boldsymbol{u}_i^n - \boldsymbol{u}_j^n)\, |\boldsymbol{u}_i^n - \boldsymbol{u}_j^n|\, \boldsymbol{v}_i \, d\tau = < \boldsymbol{f}_i, \boldsymbol{v}_i > . \quad (5.6)$$

Next, from the convergence properties of the sequence $(\ell_i^n)_n$, there exists a subsequence, still denoted by $(\ell_i^n)_n$, which converges to ℓ_i strongly in $L^2(\Omega_i)$ and a.e. in Ω_i. So, thanks to the continuity and boundedness of the function $\tilde{\alpha}_i$, for any fixed $\boldsymbol{v}_i$ in V_i, the sequence $(\tilde{\alpha}_i(\ell_i^n)\, \nabla \boldsymbol{v}_i)_n$ tends to $\tilde{\alpha}_i(\ell_i)\, \nabla \boldsymbol{v}_i$ a.e. in Ω_i and is bounded in $L^2(\Omega_i)^{d^2}$ by $c\, \|\nabla \boldsymbol{v}_i\|_{L^2(\Omega_i)^{d^2}}$, hence it converges strongly in $L^2(\Omega_i)^{d^2}$. Since $(\nabla \boldsymbol{u}_i^n)_n$ converges to $\nabla \boldsymbol{u}_i$ weakly in $L^2(\Omega_i)^{d^2}$, this yields

$$\lim_{n\to\infty} \int_{\Omega_i} \tilde{\alpha}_i(\ell_i^n)\, \nabla \boldsymbol{u}_i^n \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x} = \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, \nabla \boldsymbol{u}_i \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x}.$$

Moreover, due to the compactness of the embedding of $H^{\frac{1}{2}}(\Gamma)$ into $L^3(\Gamma)$, there exists two subsequences, still denoted by $(\boldsymbol{u}_1^n)_n$ and $(\boldsymbol{u}_2^n)_n$, so that $((\boldsymbol{u}_i^n - \boldsymbol{u}_j^n)\, |\boldsymbol{u}_i^n - \boldsymbol{u}_j^n|)_n$ converges to $(\boldsymbol{u}_i - \boldsymbol{u}_j)\, |\boldsymbol{u}_i - \boldsymbol{u}_j|$ strongly in $L^{\frac{3}{2}}(\Gamma)$. Consequently, $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ satisfies the first equation in (5.5) for $i = 1$ and 2.
2) Taking $\boldsymbol{v}_i = \boldsymbol{u}_i^n$ in (5.2) yields

$$\int_{\Omega_1} \tilde{\alpha}_i(\ell_1^n)\, |\nabla \boldsymbol{u}_1^n|^2 \, d\boldsymbol{x} + \int_{\Omega_2} \tilde{\alpha}_i(\ell_2^n)\, |\nabla \boldsymbol{u}_2^n|^2 \, d\boldsymbol{x} + \int_\Gamma |\boldsymbol{u}_1^n - \boldsymbol{u}_2^n|^3 \, d\tau = < \boldsymbol{f}_1, \boldsymbol{u}_1^n > + < \boldsymbol{f}_2, \boldsymbol{u}_2^n >,$$

so that passing to the limit yields

$$\lim_{n\to\infty} \int_{\Omega_1} \tilde{\alpha}_i(\ell_1^n)\, |\nabla \boldsymbol{u}_1^n|^2 \, d\boldsymbol{x} + \int_{\Omega_2} \tilde{\alpha}_i(\ell_2^n)\, |\nabla \boldsymbol{u}_2^n|^2 \, d\boldsymbol{x} \\ = < \boldsymbol{f}_1, \boldsymbol{u}_1 > + < \boldsymbol{f}_2, \boldsymbol{u}_2 > - \int_\Gamma |\boldsymbol{u}_1 - \boldsymbol{u}_2|^3 \, d\tau.$$

We also derive from the first equation in (5.5) that

$$\int_{\Omega_1} \tilde{\alpha}_i(\ell_1)\, |\nabla \boldsymbol{u}_1|^2 \, d\boldsymbol{x} + \int_{\Omega_2} \tilde{\alpha}_i(\ell_2)\, |\nabla \boldsymbol{u}_2|^2 \, d\boldsymbol{x} = < \boldsymbol{f}_1, \boldsymbol{u}_1 > + < \boldsymbol{f}_2, \boldsymbol{u}_2 > - \int_\Gamma |\boldsymbol{u}_1 - \boldsymbol{u}_2|^3 \, d\tau,$$

whence

$$\lim_{n\to\infty} \int_{\Omega_1} \tilde{\alpha}_1(\ell_1^n)\, |\nabla \boldsymbol{u}_1^n|^2 \, d\boldsymbol{x} + \int_{\Omega_2} \tilde{\alpha}_2(\ell_2^n)\, |\nabla \boldsymbol{u}_2^n|^2 \, d\boldsymbol{x} \\ = \int_{\Omega_1} \tilde{\alpha}_1(\ell_1)\, |\nabla \boldsymbol{u}_1|^2 \, d\boldsymbol{x} + \int_{\Omega_2} \tilde{\alpha}_2(\ell_2)\, |\nabla \boldsymbol{u}_2|^2 \, d\boldsymbol{x}. \quad (5.7)$$

On the other hand, let us set: $\boldsymbol{h}_i^n = \sqrt{\tilde{\alpha}_i(\ell_i^n)}\, \nabla \boldsymbol{u}_i^n$. Since the sequence $(\boldsymbol{h}_i^n)_n$ is bounded in $L^2(\Omega_i)^{d^2}$, there exists a subsequence, still denoted by $(\boldsymbol{h}_i^n)_n$, which converges to $\boldsymbol{h}_i$ weakly in $L^2(\Omega_i)^{d^2}$. In order to identify $\boldsymbol{h}_i$, we introduce a function φ in $L^2(\Omega_i)^{d^2}$. Since the previous subsequence $(\sqrt{\tilde{\alpha}_i(\ell_i^n)})_n$ converges to $\sqrt{\tilde{\alpha}_i(\ell_i)}$ a.e. in Ω_i, the subsequence

$(\sqrt{\tilde{\alpha}_i(\ell_i^n)}\,\varphi)_n$ also converges to $\sqrt{\tilde{\alpha}_i(\ell_i)}\,\varphi$ a.e. in Ω_i and, since it is obviously bounded by a function in $L^2(\Omega_i)$, it tends to $\sqrt{\tilde{\alpha}_i(\ell_i)}\,\varphi$ in $L^2(\Omega_i)^{d^2}$. Since $(\nabla \boldsymbol{u}_i^n)_n$ converges to $\nabla \boldsymbol{u}_i$ weakly in $L^2(\Omega_i)^{d^2}$, this yields

$$\lim_{n\to\infty}\int_{\Omega_i}\sqrt{\tilde{\alpha}_i(\ell_i^n)}\,\nabla \boldsymbol{u}_i^n\,\cdot\,\varphi\,d\boldsymbol{x}=\int_{\Omega_i}\sqrt{\tilde{\alpha}_i(\ell_i)}\,\nabla \boldsymbol{u}_i\,\cdot\,\varphi\,d\boldsymbol{x},$$

so that $\boldsymbol{h}_i$ is equal to $\sqrt{\tilde{\alpha}_i(\ell_i)}\,\nabla \boldsymbol{u}_i$. Moreover, from the weak convergence of $(\boldsymbol{h}_i^n)_n$, we deduce that

$$\int_{\Omega_i}\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2\,d\boldsymbol{x}\le\liminf_{n\to\infty}\int_{\Omega_i}\tilde{\alpha}_i(\ell_i^n)\,|\nabla \boldsymbol{u}_i^n|^2\,d\boldsymbol{x},$$

and combining this inequality with (5.7) implies

$$\lim_{n\to\infty}\int_{\Omega_i}\tilde{\alpha}_i(\ell_i^n)\,|\nabla \boldsymbol{u}_i^n|^2\,d\boldsymbol{x}=\int_{\Omega_i}\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2\,d\boldsymbol{x},\quad i=1\text{ and }2. \tag{5.8}$$

Equivalently, the sequence $(\sqrt{\tilde{\alpha}_i(\ell_i^n)}\,\nabla \boldsymbol{u}_i^n)_n$ tends to $\sqrt{\tilde{\alpha}_i(\ell_i)}\,\nabla \boldsymbol{u}_i$ strongly in $L^2(\Omega_i)^{d^2}$, so the sequence $(\tilde{\alpha}_i(\ell_i^n)\,|\nabla \boldsymbol{u}_i^n|^2)_n$ tends to $\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2$ strongly in $L^1(\Omega_i)$.
3) We observe that the solution ℓ_i^n, $i=1$ and 2, of the second equation in (5.2) also satisfies the "transposed" formulation, where φ_i is a smooth enough function on Ω_i:

$$-\int_{\Omega_i}\ell_i^n\,\Delta\varphi_i\,d\boldsymbol{x}=-\int_\Gamma T_n\big(G_i(|\boldsymbol{u}_1^n-\boldsymbol{u}_2^n|^2)\big)\,\partial_{n_i}\varphi_i\,d\tau+\int_{\Omega_i}T_n\big(\tilde{\alpha}_i(\ell_i^n)\,|\nabla \boldsymbol{u}_i^n|^2\big)\,\varphi_i\,d\boldsymbol{x}.$$

The convergence of the last term follows from part 2) of the proof together with the definition of T_n. The convergence $\big(G_i(|\boldsymbol{u}_1^n-\boldsymbol{u}_2^n|^2)\big)_n$ can easily be deduced from the sublinearity of G_i. So, for $i=1$ and 2, each ℓ_i satisfies the second part of (5.5), which ends the proof.

To conclude, we go back to the initial system (1.1) and we write its full variational formulation:

Find $(\boldsymbol{u}_i,p_i)$ in $X_i\times L^2(\Omega_i)$, $1\le i\le 2$, such that, for $1\le i\ne j\le 2$:

$$\begin{aligned}
&\forall \boldsymbol{v}_i\in X_i,\quad \int_{\Omega_i}\alpha_i(k_i)\,\nabla \boldsymbol{u}_i\,\cdot\,\nabla \boldsymbol{v}_i\,d\boldsymbol{x}-\int_{\Omega_i}p_i(\operatorname{div}\boldsymbol{v}_i)\,d\boldsymbol{x}\\
&\qquad\qquad+\int_\Gamma|\boldsymbol{u}_i-\boldsymbol{u}_j|\,(\boldsymbol{u}_i-\boldsymbol{u}_j)\,\cdot\,\boldsymbol{v}_i\,d\tau=\int_{\Omega_i}\boldsymbol{f}_i\,\cdot\,\boldsymbol{v}_i\,d\boldsymbol{x}.\\
&\forall q_i\in L^2(\Omega_i),\quad -\int_{\Omega_i}q_i(\operatorname{div}\boldsymbol{u}_i)\,d\boldsymbol{x}=0,\\
&\qquad \textit{Find } k_i \textit{ in } L^2(\Omega_i),\ 1\le i\le 2, \textit{ such that, for } 1\le i\le 2:\\
&\forall\varphi_i\in H^2(\Omega_i)\cap H_0^1(\Omega_i),\\
&\qquad -\int_{\Omega_i}G_i(k_i)\,\Delta\varphi_i\,d\boldsymbol{x}=-\int_\Gamma G_i(|\boldsymbol{u}_1-\boldsymbol{u}_2|^2)\,\partial_{n_i}\varphi_i\,d\tau\\
&\qquad\qquad+\int_{\Omega_i}\alpha_i(k_i)\,|\nabla \boldsymbol{u}_i|^2\,\varphi_i\,d\boldsymbol{x}.
\end{aligned}\tag{5.9}$$

Here, the argument is due to [24].

Corollary 5.3. *For any $\boldsymbol{f}_i$ in $L^2(\Omega_i)^d$, $i = 1$ or 2, system* (1.1) *admits the formulation* (5.9)*. System* (5.9) *has a solution (W_1, W_2) with each $W_i = (\boldsymbol{u}_i, p_i, k_i)$ in $X_i \times L^2(\Omega_i) \times L^2(\Omega_i)$. Moreover, each function k_i, $i = 1$ and 2, is nonnegative and belongs to $H^s(\Omega_i)$ for all $s < \frac{1}{2}$.*

Proof: Since the existence of p_i in $L^2(\Omega_i)$ is a consequence of Lemma 3.1, it suffices to check that the mapping: $\ell \mapsto k = G_i^{-1}(\ell)$ is continuous from $H^s(\Omega_i)$ into itself. This follows by an interpolation argument: indeed, it is continuous from $L^2(\Omega_i)$ into itself and from $H^1(\Omega_i)$ into itself thanks to the inequalities

$$k \leq \nu^{-1}\,\ell, \quad |\nabla k| \leq \nu^{-1}\,|\nabla \ell|.$$

6. A uniqueness result for smooth solutions.

The aim of this section is to prove that any solution of system (1.1) which is slightly more regular than in the existence theorem, is unique when a further condition holds: the data must be small enough in comparison of the relative variation of the functions α_i. So, we assume that, for some data $\boldsymbol{f}_i$ in $L^2(\Omega_i)^d$, system (1.1) has two solutions $(\boldsymbol{u}_i, p_i, k_i)_{i=1,2}$ and $(\overline{\boldsymbol{u}}_i, \overline{p}_i, \overline{k}_i)_{i=1,2}$, we define the corresponding functions $\ell_i = G_i(k_i)$ and $\overline{\ell}_i = G_i(\overline{k}_i)$ as in Section 2 and we set:

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \overline{\boldsymbol{u}}_i, \qquad m_i = \ell_i - \overline{\ell}_i, \quad i = 1, 2. \tag{6.1}$$

We assume that the functions α_i are continuously differentiable with bounded derivatives and, for simplicity, we also introduce the notation

$$\nu^* = \max_{i=1,2} \sup_{k\in\mathbb{R}} \sup \{\alpha_i(k), \gamma_i(k)\}, \qquad \nu' = \max_{i=1,2} \sup_{k\in\mathbb{R}} |\alpha_i'(k)|.$$

Note that the maximum of $\tilde{\alpha}_i'$ is smaller than $\frac{\nu'}{\nu}$.

In the next two lemmas, we treat separately the equations on the velocities and on the turbulent energies.

Lemma 6.1. *Assume that the pair $(\overline{\boldsymbol{u}}_1, \overline{\boldsymbol{u}}_2)$ belongs to $W^{1,p}(\Omega_1)^d \times W^{1,p}(\Omega_2)^d$ for a real number $p > 2$, and let q be such that: $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. The following estimate holds for the functions $\boldsymbol{w}_i$, $i = 1$ and 2:*

$$\begin{aligned}\big(|\boldsymbol{w}_1|^2_{H^1(\Omega_1)^d} + |\boldsymbol{w}_2|^2_{H^1(\Omega_2)^d}\big)^{\frac{1}{2}} \\ \le \frac{\nu'}{\nu^2} \big(\|\overline{\boldsymbol{u}}_1\|^p_{W^{1,p}(\Omega_1)^d} + \|\overline{\boldsymbol{u}}_2\|^p_{W^{1,p}(\Omega_2)^d}\big)^{\frac{1}{p}} \big(\|m_1\|^q_{L^q(\Omega_1)} + \|m_2\|^q_{L^q(\Omega_2)}\big)^{\frac{1}{q}}.\end{aligned} \tag{6.2}$$

Proof: Subtracting formulation (3.5) for $\boldsymbol{u}_i$ and $\overline{\boldsymbol{u}}_i$, we obtain

$$\begin{aligned}\forall \boldsymbol{v}_i \in V_i, \quad \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, \nabla \boldsymbol{w}_i \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x} \\ + \int_\Gamma (\boldsymbol{u}_i - \boldsymbol{u}_j)\, |\boldsymbol{u}_i - \boldsymbol{u}_j|\, \boldsymbol{v}_i \, d\tau - \int_\Gamma (\overline{\boldsymbol{u}}_i - \overline{\boldsymbol{u}}_j)\, |\overline{\boldsymbol{u}}_i - \overline{\boldsymbol{u}}_j|\, \boldsymbol{v}_i \, d\tau \\ = - \int_{\Omega_i} \big(\tilde{\alpha}_i(\ell_i) - \tilde{\alpha}_i(\overline{\ell}_i)\big)\big)\, \nabla \overline{\boldsymbol{u}}_i \cdot \nabla \boldsymbol{v}_i \, d\boldsymbol{x}.\end{aligned}$$

Since $\boldsymbol{w}_i$ belongs to V_i, we take each $\boldsymbol{v}_i$ equal to $\boldsymbol{w}_i$, we sum up on i and we observe from (3.9) that, as in the proof of Proposition 3.5, the quantity that is integrated on Γ is nonnegative. This yields

$$\sum_{i=1}^2 \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, |\nabla \boldsymbol{w}_i|^2 \, d\boldsymbol{x} \le \sum_{i=1}^2 \Big| \int_{\Omega_i} \big(\tilde{\alpha}_i(\ell_i) - \tilde{\alpha}_i(\overline{\ell}_i)\big)\, \nabla \overline{\boldsymbol{u}}_i \cdot \nabla \boldsymbol{w}_i \, d\boldsymbol{x} \Big|,$$

whence

$$\nu \big(\sum_{i=1}^2 |\boldsymbol{w}_i|^2_{H^1(\Omega_i)^d}\big) \le \sum_{i=1}^2 \|\tilde{\alpha}_i(\ell_i) - \tilde{\alpha}_i(\overline{\ell}_i)\|_{L^q(\Omega_i)}\, |\overline{\boldsymbol{u}}_i|_{W^{1,p}(\Omega_i)^d}\, |\boldsymbol{w}_i|_{H^1(\Omega_i)^d}.$$

We also have

$$\|\tilde{\alpha}_i(\ell_i) - \tilde{\alpha}_i(\overline{\ell}_i)\|_{L^q(\Omega_i)} \le \frac{\nu'}{\nu}\, \|m_i\|_{L^q(\Omega_i)},$$

so that the desired estimate follows by applying Hölder's inequality.

Lemma 6.2. *Assume that the pair $(\overline{\boldsymbol{u}}_1, \overline{\boldsymbol{u}}_2)$ belongs to $W^{1,p}(\Omega_1)^d \times W^{1,p}(\Omega_2)^d$ for a real number $p > 2$, and let q be such that: $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Let s be a real number, $0 < s < \frac{1}{2}$. The following estimate holds for the functions m_i, $i = 1$ and 2:*

$$\begin{aligned} \|m_i\|_{H^s(\Omega_i)} \le c\,\frac{\nu^*}{\nu}\,\big(\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)^d} + \|\boldsymbol{f}_2\|^2_{L^2(\Omega_2)^d}\big)^{\frac{1}{2}}\,\big(\|\boldsymbol{w}_1\|^2_{H^1(\Omega_1)^d} + \|\boldsymbol{w}_2\|^2_{H^1(\Omega_2)^d}\big)^{\frac{1}{2}} \\ + c'\,\frac{\nu'}{\nu}\,\|\overline{\boldsymbol{u}}_i\|^2_{W^{1,p}(\Omega_i)}\,\|m_i\|_{L^{\frac{q}{2}}(\Omega_i)}. \end{aligned} \tag{6.3}$$

Proof: Subtracting formulation (4.2) for ℓ_i and $\overline{\ell}_i$ gives

$$\begin{aligned} &\forall \varphi_i \in H^2(\Omega_i) \cap H^1_0(\Omega_i), \\ &\quad -\int_{\Omega_i} m_i\,\Delta\varphi_i\,d\boldsymbol{x} = -\int_\Gamma \big(G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) - G_i(|\overline{\boldsymbol{u}}_1 - \overline{\boldsymbol{u}}_2|^2)\big)\,\partial_{n_i}\varphi_i\,d\tau \\ &\qquad\qquad + \int_{\Omega_i} \big(\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2 - \tilde{\alpha}_i(\overline{\ell}_i)\,|\nabla \overline{\boldsymbol{u}}_i|^2\big)\,\varphi_i\,d\boldsymbol{x}. \end{aligned}$$

As in the proof of Lemma 4.2, relying on the analogue of (4.6), with any function g_i in $H^{-s}(\Omega_i)$, we associate the function $\varphi_i = \mathcal{L}_i g_i$ defined by (4.3) and we recall that

$$\|\partial_n \varphi_i\|_{L^2(\Gamma)} + \sup_{\boldsymbol{x}\in\Omega_i} |\varphi_i(\boldsymbol{x})| \le c_i\,\|g_i\|_{H^{-s}(\Omega_i)},$$

for a constant c_i only depending on Ω_i. We now estimate the right-hand side of the previous equation for such a φ_i. We observe that

$$|G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) - G_i(|\overline{\boldsymbol{u}}_1 - \overline{\boldsymbol{u}}_2|^2)| \le \nu^*\,|\boldsymbol{u}_1 + \boldsymbol{u}_2 + \overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{u}}_2|\,|\boldsymbol{w}_1 - \boldsymbol{w}_2|,$$

so that denoting by c'_i the norm of the embedding of $H^1_0(\Omega_i)$ into $L^4(\Omega_i)$, we obtain

$$\begin{aligned} &|\int_\Gamma \big(G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) - G_i(|\overline{\boldsymbol{u}}_1 - \overline{\boldsymbol{u}}_2|^2)\big)\,\partial_{n_i}\varphi_i\,d\tau| \\ &\qquad \le c_i {c'_i}^2\,\nu^*\,\big(\sum_{i=1}^2 |\boldsymbol{u}_i|^2_{H^1(\Omega_i)^d} + |\overline{\boldsymbol{u}}_i|^2_{H^1(\Omega_i)^d}\big)^{\frac{1}{2}}\,\big(\sum_{i=1}^2 |\boldsymbol{w}_i|^2_{H^1(\Omega_i)^d}\big)^{\frac{1}{2}}\,\|g_i\|_{H^{-s}(\Omega_i)}. \end{aligned}$$

The norms $|\boldsymbol{u}_i|^2_{H^1(\Omega_i)^d}$ and $|\overline{\boldsymbol{u}}_i|^2_{H^1(\Omega_i)^d}$ are bounded from (3.7). Similarly, we have

$$|\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2 - \tilde{\alpha}_i(\overline{\ell}_i)\,|\nabla \overline{\boldsymbol{u}}_i|^2| \le \nu^*\,|\nabla(\boldsymbol{u}_i + \overline{\boldsymbol{u}}_i)|\,|\nabla \boldsymbol{w}_i| + \frac{\nu'}{\nu}\,|m_i|\,|\nabla \overline{\boldsymbol{u}}_i|^2,$$

whence

$$\begin{aligned} &|\int_{\Omega_i} \big(\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2 - \tilde{\alpha}_i(\overline{\ell}_i)\,|\nabla \overline{\boldsymbol{u}}_i|^2\big)\,\varphi_i\,d\boldsymbol{x}| \\ &\qquad \le c_i\,\Big(\nu^*\,\big(\sum_{i=1}^2 |\boldsymbol{u}_i|^2_{H^1(\Omega_i)^d} + |\overline{\boldsymbol{u}}_i|^2_{H^1(\Omega_i)^d}\big)^{\frac{1}{2}}\,\big(\sum_{i=1}^2 |\boldsymbol{w}_i|^2_{H^1(\Omega_i)^d}\big)^{\frac{1}{2}} \\ &\qquad\qquad + \frac{\nu'}{\nu}\,\|\overline{\boldsymbol{u}}_i\|^2_{W^{1,p}(\Omega_i)}\,\|m_i\|_{L^{\frac{q}{2}}(\Omega_i)}\Big)\|g_i\|_{H^{-s}(\Omega_i)}. \end{aligned}$$

Combining all this yields the desired result.

We are now in a position to prove the uniqueness result.

Theorem 6.3. *Assume that the data $\boldsymbol{f}_i$, $i = 1$ and 2, belong to $L^2(\Omega_i)^d$. If system* (1.1) *admits a solution $(\overline{W}_1, \overline{W}_2)$, with $\overline{W}_i = (\overline{\boldsymbol{u}}_i, \overline{p}_i, \overline{k}_i)$, such that the $\overline{\boldsymbol{u}}_i$, $i = 1$ and 2, belong to $W^{1,p}(\Omega_i)^d$ for a real number $p > 2d$ and that the following condition holds for appropriate constants c and c'*

$$c\,\frac{\nu'}{\nu}\,\overline{\Lambda}\left(\overline{\Lambda} + c'\,\frac{\nu^*}{\nu^2}\left(\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)^d} + \|\boldsymbol{f}_2\|^2_{L^2(\Omega_2)^d}\right)^{\frac{1}{2}}\right) < 1,$$
$$\text{with}\quad \overline{\Lambda} = \left(\|\overline{\boldsymbol{u}}_1\|^p_{W^{1,p}(\Omega_1)^d} + \|\overline{\boldsymbol{u}}_2\|^p_{W^{1,p}(\Omega_2)^d}\right)^{\frac{1}{p}}, \tag{6.4}$$

then it is the unique solution of system (1.1) *in the sense that the pair (W_1, W_2) is the unique solution of problem* (5.9).

Proof: Since p is $> 2d$, there exists an $s < \frac{1}{2}$ such that $H^s(\Omega_i)$ is imbedded in $L^q(\Omega_i)$. So replacing the $\|m_i\|_{L^q(\Omega_i)}$ and $\|m_i\|_{L^{\frac{q}{2}}(\Omega_i)}$ in (6.2) and (6.3), respectively, by an appropriate constant times $\|m_i\|_{H^s(\Omega_i)}$, summing up the estimate (6.3) on i and inserting (6.2) in the sum, we derive from (6.4) that both m_i cancel. The complete uniqueness result follows.

So, the uniqueness mainly depends the parameter $\frac{\nu'}{\nu}$ which represents the variation of the function $\tilde{\alpha}_i$. When the α_i are constant, we recover in the previous theorem the nearly obvious and unconditional uniqueness result which also follows from Proposition 3.5.

7. Some further regularity properties in dimension 2.

We intend to study the regularity of the solution of system (1.1) in the simple case of dimension $d = 2$, when the Ω_i are rectangles. Indeed this geometry is the key one for most discretizations. We need several lemmas.

The first one deals with the Stokes problem with mixed boundary conditions, and results from [22, Cor. 4.2], extended to less smooth data thanks to the arguments in [9, Chap. 8]. From now on, we set:

$$s_0 = 1.5946. \tag{7.1}$$

Lemma 7.1. *For $i = 1$ and 2, if the domain Ω_i is a rectangle, the mapping: $\boldsymbol{f}_i \mapsto (\boldsymbol{v}_i, q_i)$, where $(\boldsymbol{v}_i, q_i)$ is the solution of the Stokes problem*

$$\begin{cases} -\Delta \boldsymbol{v}_i + \mathbf{grad}\, q_i = \boldsymbol{f}_i & \text{in } \Omega_i, \\ \operatorname{div} \boldsymbol{v}_i = 0 & \text{in } \Omega_i, \\ \boldsymbol{v}_i = \mathbf{0} & \text{on } \Gamma_i, \\ \partial_{n_i} \boldsymbol{v}_i - q_i\, \boldsymbol{n}_i = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{7.2}$$

is continuous from $H^{s-2}(\Omega_i)^2$ into $H^s(\Omega_i)^2 \times H^{s-1}(\Omega_i)$ for all s, $\frac{3}{2} < s \le s_0$.

The second lemma extends this result to the case where the Neumann boundary conditions are not zero.

Lemma 7.2. *For $i = 1$ and 2, if the domain Ω_i is a rectangle, the Stokes mapping $\mathcal{S}_i$: $(\boldsymbol{f}_i, \boldsymbol{g}_i) \mapsto (\boldsymbol{v}_i, q_i)$, where $(\boldsymbol{v}_i, q_i)$ is the solution of the Stokes problem*

$$\begin{cases} -\Delta \boldsymbol{v}_i + \mathbf{grad}\, q_i = \boldsymbol{f}_i & \text{in } \Omega_i, \\ \operatorname{div} \boldsymbol{v}_i = 0 & \text{in } \Omega_i, \\ \boldsymbol{v}_i = \mathbf{0} & \text{on } \Gamma_i, \\ \partial_{n_i} \boldsymbol{v}_i - q_i\, \boldsymbol{n}_i = \boldsymbol{g}_i & \text{on } \Gamma, \end{cases} \tag{7.3}$$

is continuous from $H^{s-2}(\Omega_i)^2 \times H^{s-\frac{3}{2}}(\Gamma)^2$ into $H^s(\Omega_i)^2 \times H^{s-1}(\Omega_i)$ for all s, $\frac{3}{2} < s \le s_0$.

Proof: We want to construct a divergence-free function $\boldsymbol{w}_i$ and a function r_i with the required regularity such that:

$$\boldsymbol{w}_i = \mathbf{0} \quad \text{on } \Gamma_i \qquad \text{and} \qquad \partial_{n_i} \boldsymbol{w}_i - r_i\, \boldsymbol{n}_i = \boldsymbol{g}_i \quad \text{on } \Gamma,$$

and to apply the previous Lemma 7.1 to the pair $(\boldsymbol{v}_i - \boldsymbol{w}_i, q_i - r_i)$. So, denoting by g_{ix} and g_{iz} the components of $\boldsymbol{g}_i$, we look for a function $\boldsymbol{w}_i$ of the form $\mathbf{curl}\ \psi$, so that the desired boundary conditions are written

$$\psi = \partial_{n_i} \psi = 0 \quad \text{on } \Gamma_i \qquad \text{and} \qquad \partial_z \psi = \pm \int_a^x (g_{iz} - r_i)(\xi)\, d\xi, \quad \partial_z^2 \psi = \pm(-g_{ix}) \quad \text{on } \Gamma,$$

where $(a, 0)$ stands for the left endpoint of Γ and the sign $\pm$ depends on i. The idea is to take:

$$r_i = \frac{1}{\operatorname{meas}\Gamma} \int_\Gamma g_{iz}(x)\,dx$$

(such a constant is well-defined since g_{iz} belongs to $L^1(\Gamma)$). Indeed, thanks to this choice, the right compatibility conditions are satisfied at the corners of Ω_i. So, the existence of a function ψ in $H^{s+1}(\Omega_i)$ follows from [2, Thm. 2.d.2], together with the desired bound on its norm. This, combined with Lemma 7.1, yields the result.

Let us now check how the previous results extend to smaller values of s, namely $1 < s < \frac{3}{2}$.

Corollary 7.3. *For $i = 1$ and 2, if the domain Ω_i is a rectangle, the Stokes mapping $\mathcal{S}_i$: $(\boldsymbol{f}_i, \boldsymbol{g}_i) \mapsto (\boldsymbol{v}_i, q_i)$, where $(\boldsymbol{v}_i, q_i)$ is the solution of (7.3), is continuous from $(X_i^{2-s})' \times H^{s-\frac{3}{2}}(\Gamma)^2$ into $H^s(\Omega_i)^2 \times H^{s-1}(\Omega_i)$ for all s, $1 < s < \frac{3}{2}$, where $(X_i^{2-s})'$ stands for the dual space of*

$$X_i^{2-s} = \big\{\boldsymbol{v}_i \in H^{2-s}(\Omega_i)^2;\ \boldsymbol{v}_i = \boldsymbol{0} \text{ on } \Gamma_i\big\}. \tag{7.4}$$

Proof: From Lemma 7.2, the mapping is continuous from $H^{s_0-2}(\Omega_i)^2 \times H^{s_0-\frac{3}{2}}(\Gamma)^2$ into $H^{s_0}(\Omega_i)^2 \times H^{s_0-1}(\Omega_i)$. When writing the variational formulation of problem (7.3), we observe that it is also continuous from $X_i' \times \big(H_{00}^{\frac{1}{2}}(\Gamma)\big)'^2$ (the $'$ denotes the dual space) into $H^1(\Omega)^2 \times L^2(\Omega)$. So, if $H^{s_0-2}(\Omega)^2$ is dense in X_i' and if $H^{s_0-\frac{3}{2}}(\Gamma)$ is dense in $\big(H_{00}^{\frac{1}{2}}(\Gamma)\big)'$, applying the main theorem of interpolation [16, Chap. 1, Th. 5.1] yields that the mapping is continuous from $F_s \times G_s$ into $H^s(\Omega_i)^2 \times H^{s-1}(\Omega_i)$, with:

$$F_s = [H^{s_0-2}(\Omega_i)^2, X_i']_\theta, \qquad G_s = [H^{s_0-\frac{3}{2}}(\Gamma)^2, \big(H_{00}^{\frac{1}{2}}(\Gamma)\big)'^2]_\theta, \qquad \text{with} \quad \theta = \frac{s_0 - s}{s_0 - 1}.$$

So, it remains to check the density results and to characterize the spaces F_s and G_s.
1) First, since $2 - s_0$ is $< \frac{1}{2}$, $\mathscr{D}(\Omega)$ is dense in $H^{2-s_0}(\Omega)$. Since $\mathscr{D}(\Omega)^2$ is contained in X_i, X_i is dense in $H^{2-s_0}(\Omega)^2$, so that $H^{s_0-2}(\Omega)^2$ is dense in X_i'. Moreover the following characterization holds [16, Chap. 1, Th. 6.2]:

$$F_s = \Big([X_i, H^{2-s_0}(\Omega_i)^2]_{1-\theta}\Big)'.$$

And the interpolate space in the previous line coincides with X_i^{2-s}. This follows from one-dimensional interpolation results, combined with the tensorization property (where the rectangle Ω_i is the product of the two intervals Λ_i^x and Λ_i^z)

$$X_i^{2-s} = H_0^{2-s}\big(\Lambda_i^x; L^2(\Lambda_i^z)\big) \cap L^2\big(\Lambda_i^x; H_*^{2-s}(\Lambda_i^z)\big),$$

where $H_*^{2-s}(\Lambda_i^z)$ stands for the space of functions in $H^{2-s}(\Lambda_i^z)$ vanishing in the upper endpoint.
2) Next, we observe that $H_{00}^{\frac{1}{2}}(\Gamma)$ is dense in $L^2(\Gamma)$ which is dense in $H^{\frac{3}{2}-s_0}(\Gamma)$, so that $H^{s_0-\frac{3}{2}}(\Gamma)$ is dense in $\big(H_{00}^{\frac{1}{2}}(\Gamma)\big)'$ and the space G_s is well-defined. Moreover, applying [16, Chap. 1, Th. 6.2] gives

$$G_s = \Big([H_{00}^{\frac{1}{2}}(\Gamma)^2, H^{\frac{3}{2}-s_0}(\Gamma)^2]_{1-\theta}\Big)'.$$

Next, relying on [16, Chap. 1, Rem. 12.6], we observe that $[H_{00}^{\frac{1}{2}}(\Gamma), H^{\frac{3}{2}-s_0}(\Gamma)]_{1-\theta}$ coincides with $H_0^{\frac{3}{2}-s}(\Gamma)$, which coincides with $H^{\frac{3}{2}-s}(\Gamma)$ since s is > 1.
This ends the proof.

We also need some regularity properties of problem (4.2) when the functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are smoother than in the existence result of Theorem 5.2.

Lemma 7.4. *Let s be a real number, $1 < s \le \frac{3}{2}$. For $i = 1$ or 2, and for any pair $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in $H^s(\Omega_1)^2 \times H^s(\Omega_2)^2$, the solution ℓ_i of problem (4.2) belongs to $H^s(\Omega_i)$ and satisfies, for a constant κ_i only depending on Ω_i,*

$$\|\ell_i\|_{H^s(\Omega_i)} \le \kappa_i \left(\|\boldsymbol{u}_1\|^2_{H^s(\Omega_1)^2} + \|\boldsymbol{u}_2\|^2_{H^s(\Omega_2)^2} \right). \tag{7.5}$$

The same properties holds for $\frac{3}{2} < s < 2$ if the functions γ_i are continuously differentiable with bounded derivatives

Proof: Thanks to Lemma 4.1, it suffices to check that $\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2$ belongs to $H^{s-2}(\Omega_i)$ and that $G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)$, extended by zero on Γ_i, belongs to $H^{s-\frac{1}{2}}(\partial\Omega_i)$.
1) On one hand, the functions $\boldsymbol{u}_i$ belong to $H^s(\Omega_i)^2$, hence to $W^{1,\frac{2}{2-s}}(\Omega_i)^2$ by the Sobolev imbedding theorem. So, $|\nabla \boldsymbol{u}_i|^2$ belongs to $L^{\frac{1}{2-s}}(\Omega_i)$ and, since $\tilde{\alpha}_i$ is bounded, the same property holds for $\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2$.
For $s \ge \frac{3}{2}$, we obtain that $\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2$ belongs to $L^2(\Omega_i)$, hence to $H^{s-2}(\Omega_i)$. For $s \le \frac{3}{2}$, since the imbedding of $H^{3-2s}(\Omega_i)$ into $L^{\frac{1}{s-1}}(\Omega_i)$ yields the imbedding of $L^{\frac{1}{2-s}}(\Omega_i)$ into $H^{2s-3}(\Omega_i)$, we derive that $\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2$ belongs to $H^{2s-3}(\Omega_i)$, hence to $H^{s-2}(\Omega_i)$. Moreover, in both cases, we have the following estimate

$$\|\tilde{\alpha}_i(\ell_i)\,|\nabla \boldsymbol{u}_i|^2\|_{H^{s-2}(\Omega_i)} \le c\,\|\boldsymbol{u}_i\|^2_{H^s(\Omega_i)^2}. \tag{7.6}$$

2) On the other hand, since $G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)$ vanishes at the two endpoints of Γ, it suffices to check that it belongs to $H^{s-\frac{1}{2}}(\Gamma)$. First, we observe that $\boldsymbol{u}_1 - \boldsymbol{u}_2$ belongs to $H^{s-\frac{1}{2}}(\Gamma)^2$ and, since $H^{s-\frac{1}{2}}(\Gamma)$ is an algebra [12, Thm 1.4.4.2], $|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2$ also belongs to $H^{s-\frac{1}{2}}(\Gamma)$. When s is $\le \frac{3}{2}$, relying on [24] and using similar arguments as for Corollary 5.3, we derive that $G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)$ belongs to $H^{s-\frac{1}{2}}(\Gamma)$ and satisfies

$$\|G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)\|_{H^{s-\frac{1}{2}}(\Gamma)} \le c\left(\|\boldsymbol{u}_1\|^2_{H^s(\Omega_i)^2} + \|\boldsymbol{u}_2\|^2_{H^s(\Omega_i)^2} \right). \tag{7.7}$$

The same result is derived when s is $> \frac{3}{2}$, by applying the same arguments to the derivative of $G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)$.
This ends the proof.

We are in a position to prove the main result of this section.

Theorem 7.5. *Assume that the domains Ω_1 and Ω_2 are rectangles and that the functions α_i and γ_i, $i = 1$ and 2, are continuously differentiable with bounded derivatives. Let $\boldsymbol{f}_i$, $i = 1$ and 2, belong to $L^2(\Omega_i)^2$, and let (W_1, W_2) be a solution of system (1.1), with $W_i = (\boldsymbol{u}_i, p_i, k_i)$. If the $\boldsymbol{u}_i$, $i = 1$ and 2, belong to $H^{s^*}(\Omega_i)^2$ for some $s^* > 1$, the W_i, $i = 1$ and 2, belong to $H^{s_0}(\Omega_i)^d \times H^{s_0-1}(\Omega_i) \times H^{s_0}(\Omega_i)$.*

Proof: Let (W_1, W_2) be any solution of problem (1.1). The proof relies on a boot-strap argument and is performed in two steps.

1) From the assumption on the $\boldsymbol{u}_i$, each pair $(\boldsymbol{u}_i, p_i)$ belongs to $H^{s^*}(\Omega_i)^2 \times H^{s^*-1}(\Omega_i)$. So, applying Lemma 7.4 yields that each ℓ_i belongs to $H^{s^*}(\Omega_i)$. Finally, from [24], the function $\frac{p_i}{\tilde{\alpha}_i(\ell_i)}$ also belongs to $H^{s^*-1}(\Omega_i)$.
2) Assume now that both triples $(\boldsymbol{u}_i, \frac{p_i}{\tilde{\alpha}_i(\ell_i)}, \ell_i)$ belong to $H^t(\Omega_i)^2 \times H^{t-1}(\Omega_i) \times H^t(\Omega_i)$, for a real number t, $1 < t \le s_0$. Then the pair $(\boldsymbol{u}_i, q_i)$, with $q_i = \frac{p_i}{\tilde{\alpha}_i(\ell_i)}$, is a solution of the Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u}_i + \mathbf{grad}\, q_i = \frac{\boldsymbol{f}_i}{\tilde{\alpha}_i(\ell_i)} + \frac{\tilde{\alpha}_i'(\ell_i)}{\tilde{\alpha}(\ell_i)} \nabla \ell_i \cdot \nabla \boldsymbol{u}_i - \frac{\tilde{\alpha}_i'(\ell_i)}{\tilde{\alpha}(\ell_i)^2} \nabla \ell_i\, q_i & \text{in } \Omega_i, \\ \operatorname{div} \boldsymbol{u}_i = 0 & \text{in } \Omega_i, \\ \boldsymbol{u}_i = \mathbf{0} & \text{on } \Gamma_i, \\ \partial_{n_i} \boldsymbol{u}_i - q_i\, \boldsymbol{n}_i = -\frac{1}{\tilde{\alpha}_i(\ell_i)} (\boldsymbol{u}_i - \boldsymbol{u}_j) |\boldsymbol{u}_i - \boldsymbol{u}_j| & \text{on } \Gamma, \end{cases} \tag{7.8}$$

In this equation, $\nabla \ell_i$ and q_i belongs to $H^{t-1}(\Omega_i)$, together with each component of $\nabla \boldsymbol{u}_i$. Since $H^{t-1}(\Omega_i)$ is included in $L^{\frac{2}{2-t}}(\Omega_i)$, the right-hand side in the first line of (7.10) belongs to $L^{\frac{1}{2-t}}(\Omega_i)^2$, hence to $L^2(\Omega_i)^2$ when t is $\ge \frac{3}{2}$ and to $(X_i^{3-2t})'$ when t is $< \frac{3}{2}$. It is also clear that the right-hand side in the fourth line of (7.8) belongs to $H^{t-\frac{1}{2}}(\Gamma)^2$, hence to $H^{2t-\frac{3}{2}}(\Gamma)^2$. So using Corollary 7.3 (or Lemma 7.2) yields that $(\boldsymbol{u}_i, q_i)$ belongs to $H^{t^*}(\Omega_i)^2 \times H^{t^*-1}(\Omega_i)$, with $t^* = \min\{2t-1, s_0\}$, and applying Lemma 7.4 yields that ℓ_i belongs to $H^{t^*}(\Omega_i)$.
Iterating n times this argument, where n is the smallest integer such that $s^* + n(s^* - 1)$ is $\ge s_0$, we obtain that the triple $(\boldsymbol{u}_i, q_i, \ell_i)$, hence $(\boldsymbol{u}_i, p_i, \ell_i)$, belongs to $H^{s_0}(\Omega_i)^2 \times H^{s_0-1}(\Omega_i) \times H^{s_0}(\Omega_i)$. Then, k_i also belongs to $H^{s_0}(\Omega_i)$.

Remark: A more technical proof, relying on Meyers' argument [19] (see also [8]), allows for replacing , in the statement of Theorem 7.5, the assumption "the $\boldsymbol{u}_i$ belong to $H^{s^*}(\Omega_i)^2$" by the modified one "the k_i (or ℓ_i) belong to $H^{s^*}(\Omega_i)$", also for any $s^* > 1$, however this new assumption seems stronger.

Of course, these regularity properties can be extended to any convex polygons Ω_i, since the corresponding values of s_0 can easily be computed from [22]. By a boot-strap argument, they also hold in the case where convection terms are added in the system. But, when the interface conditions are replaced by Manning's law, the regularity of the solution of the basic Stokes problem, *a fortiori* of the present system, seems unknown.

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