The Kolmogorov Law of turbulence

What can rigorously be proved ?

Roger LEWANDOWSKI

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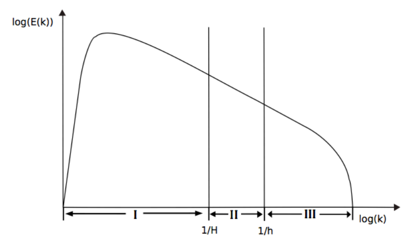
Introduction



Aim: Mathematical framework for the Kolomogorov laws.

Table of contents

1. Incompressible Navier-Stokes Equations (NSE),
2. Probabilistic framework, Reynolds Stress, Correlations,
3. Homogeneity, Turbulent Kinetic Energy, Dissipation,
4. Isotropy, Energy spectrum, Similarity and law of the -5/3.

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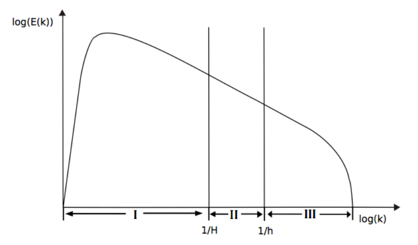

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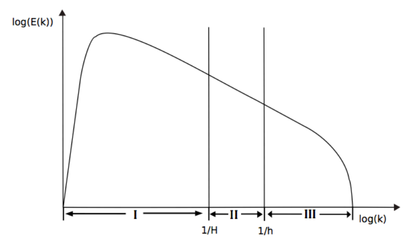

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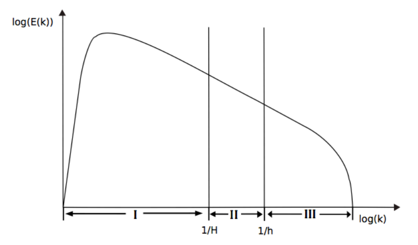

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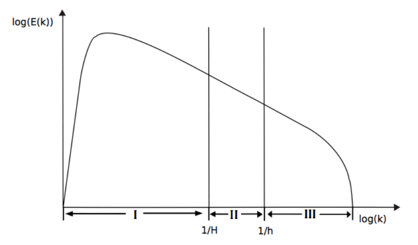

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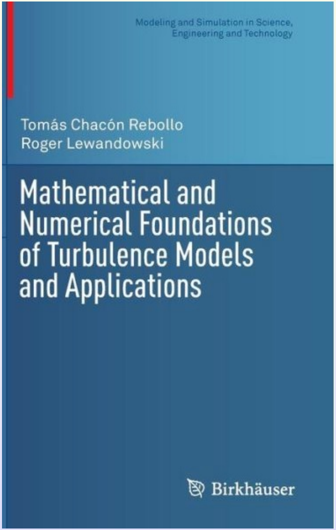

Authors in Maths publications are always by alphabetical order

Solutions to the NSE

1) Incompressible 3D Navier-Stokes Equations (NSE)

$$Q = [0, T] \times \Omega \quad \text{or} \quad Q = [0, \infty[\times \Omega, \quad \Omega \subset \mathbb{R}^3, \quad \Gamma = \partial\Omega,$$

with the no slip boundary condition and $\mathbf{v}_0$ as initial data:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p = \mathbf{f} & \text{in } Q, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } Q, \\ \mathbf{v} = 0 & \text{on } \Gamma, \\ \mathbf{v} = \mathbf{v}_0 & \text{at } t = 0, \end{cases}$$

where $\nu > 0$ is the kinematic viscosity, $\mathbf{f}$ is any external force,

$$D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t), \quad ((\mathbf{v} \cdot \nabla)\,\mathbf{v})_i = v_j \frac{\partial v_i}{\partial x_j}, \quad \nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}.$$

Remark

$\nabla \cdot \mathbf{v} = 0 \Rightarrow (\mathbf{v} \cdot \nabla)\,\mathbf{v} = \nabla \cdot (\mathbf{v} \otimes \mathbf{v}), \quad \mathbf{v} \otimes \mathbf{v} = (v_i v_j)_{1 \le i,j \le 3}.$

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Solutions to the 3D NSE

2) Two types of solutions to the 3D NSE

1. Strong solutions over a small time interval $[0, T_{max}[$ "à la" Fujita-Kato,
2. Weak solutions (also turbulent), global in time, "à la" Leray-Hopf.

Strong solutions are $C^{1,\alpha}$ over $[0, T_{\mathrm{max}}[\times\Omega$ for smooth data,

$$T_{\mathrm{max}} = T_{\mathrm{max}}(||\mathbf{v}_0||, ||\mathbf{f}||, \nu),$$

the corresponding solution is unique, yielding the writing

$$\mathbf{v} = \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0), \quad p = p(t, \mathbf{x}, \mathbf{v}_0).$$

Remark

Strong solutions are defined over $[0, \infty[$ when $\mathbf{v}_0$ and/or $||\mathbf{f}||$ are "small enough", ν is "large enough", which means that the flow is rather laminar.

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Solutions to the 3D NSE

Weak solutions are defined through an appropriate variational formulation set in the sequence of function spaces

$$\{\mathbf{v} \in H_0^1(\Omega)^3, \nabla\cdot\mathbf{v} = 0\} \hookrightarrow \mathbf{V} = \{\mathbf{v} \in L^2(\Omega)^3, \mathbf{v}\cdot\mathbf{n}|_\Gamma = 0, \nabla\cdot\mathbf{v} = 0\}$$

such that the trajectory

$$\mathbf{v} = \mathbf{v}(t) \in \mathbf{V}$$

is weakly continuous from $[0, T] \to \mathbf{V}$, $T \in]0, \infty]$ ($\forall \boldsymbol{\eta} \in \mathbf{V}$, $t \to \langle \mathbf{v}(t), \boldsymbol{\eta}\rangle$ is a continuous function of t, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbf{V}$).

Remark

For $\mathbf{v}_0 \in \mathbf{V}$ and $\mathbf{f} \in L^2(\Omega)^3$, there exists a global weak solution $\mathbf{v} = \mathbf{v}(t)$, $t \in \mathbb{R}^+$. Because of lack of uniqueness result, we can't write

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1) Long Time Average

Let $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel σ-algebra on $\mathbb{R}_+$, λ the Lebesgue measure, and let μ denotes the probability measure

$$\forall\, A \in \mathcal{B}(\mathbb{R}_+), \quad \mu(A) = \lim_{t\to\infty} \frac{1}{t}\lambda(A \cap [0,t]),$$

Let $\mathbf{v} \in L^1(\mathbb{R}^+ \to \mathbf{V}; \mu)$,

$$E(\mathbf{v}) = \overline{\mathbf{v}} = \int_{\mathbb{R}^+} \mathbf{v}(t)d\mu(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{v}(t)dt.$$

Let $\mathbf{v}$ be a Leray-Hopf solution to the NSE. It is not known wether for a given $\mathbf{x} \in \Omega$, $\mathbf{v}(t,\mathbf{x}) \in L^1(\mathbb{R}^+ \to \mathbf{V}; \mu)$. However, it can be proved that in some sense when $\mathbf{f}$ is steady [1],

$$E(\mathbf{v}) = \overline{\mathbf{v}} = \overline{\mathbf{v}}(\mathbf{x}) \in W^{2,5/4}(\Omega)^3$$

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2) Ensemble Average

The source term $\mathbf{f}$ and the viscosity ν are fixed. Let $\mathbb{K} \subset \mathbf{V}$ be a compact,

$$T_{\mathbb{K}} = \inf_{\mathbf{v}_0 \in \mathbb{K}} T_{\max}(||\mathbf{v}_0||, ||\mathbf{f}||, \nu) > 0, \quad Q = [0, T_{\mathbb{K}}] \times \Omega$$

Let $\{\mathbf{v}_0^{(1)}, \cdots, \mathbf{v}_0^{(n)}, \cdots\}$ be a countable dense subset of $\mathbb{K}$,

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Probabilistic framework

Up to a subsequence

$$\mu_n \to \mu \quad \text{weakly in the sense of the measures,} \quad ||\mu|| = 1.$$

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It is not known if the probability measure μ is unique or not

Probabilistic framework

Up to a subsequence

$$\mu_n \to \mu \quad \text{weakly in the sense of the measures,} \quad ||\mu|| = 1.$$

so that, $\forall\,(t, \mathbf{x}) \in Q$,

$$\overline{\mathbf{v}}(t, \mathbf{x}) = E_\mu(\mathbf{v}(t, \mathbf{x}, \cdot)) = \int_{\mathbb{K}} \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0) d\mu(\mathbf{v}_0) = \lim_{n\to\infty} \overline{\mathbf{v}}_n(t, \mathbf{x}).$$

$$\overline{p}(t, \mathbf{x}) = E_\mu(p(t, \mathbf{x}, \cdot)) = \int_{\mathbb{K}} p(t, \mathbf{x}, \mathbf{v}_0) d\mu(\mathbf{v}_0)$$

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Reynolds Stress

1) Reynolds decomposition We can decompose $(\mathbf{v}, p)$ as follows:

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad p = \overline{p} + p',$$

which is the Reynolds decomposition, $\mathbf{v}'$ and p' are the fluctuations. Either for long time or ensemble averages:

$$\overline{\partial_t \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)} = \partial_t \overline{\mathbf{v}}(t, \mathbf{x}), \tag{1}$$

$$\overline{\nabla \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)} = \nabla \overline{\mathbf{v}}(t, \mathbf{x}), \tag{2}$$

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called the Reynods rules. From $\overline{\overline{\mathbf{v}}} = \overline{\mathbf{v}}$ and $\overline{\overline{p}} = \overline{p}$, one gets:

Lemma

$$\forall (t, \mathbf{x}) \in Q, \quad \overline{\mathbf{v}'(t, \mathbf{x}, \mathbf{v}_0)} = 0, \quad \overline{p'(t, \mathbf{x}, \mathbf{v}_0)} = 0.$$

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2) Averaged NSE

Note that $E_\mu(\mathbf{f}) = \mathbf{f}\,E_\mu(1) = \mathbf{f}$. By the Reynolds rules and the previous lemma:

$$\left\{\begin{array}{rcll} \partial_t\overline{\mathbf{v}} + (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}} - \nu\Delta\overline{\mathbf{v}} + \nabla\overline{p} & = & -\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})} + \mathbf{f} & \text{in } Q, \\ \nabla\cdot\overline{\mathbf{v}} & = & 0 & \text{in } Q, \\ \overline{\mathbf{v}} & = & 0 & \text{on } \Gamma, \\ \overline{\mathbf{v}} & = & \overline{\mathbf{v}_0} & \text{at } t=0, \end{array}\right.$$

where

$$\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}'\otimes\mathbf{v}'}$$

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Standard correlation tensors

Standard correlation tensor Let $M_1, \dots, M_n \in Q$, $M_k = (t_k, \mathbf{x}_k)$,

$$\mathbb{B}_n = \mathbb{B}_n(M_1, \dots, M_n) = (B_{i_1 \dots i_n}(M_1, \dots, M_n))_{1 \le i_1 \dots i_n \le 3}$$

at these points is defined by

$$B_{i_1 \dots i_n}(M_1, \dots, M_n) = \overline{\prod_{k=1}^{n} v_{i_k}(t_k, \mathbf{x}_k, \mathbf{v}_0)} =$$
$$\int_{\mathbb{K}} \left(\prod_{k=1}^{n} v_{i_k}(t_k, \mathbf{x}_k, \mathbf{v}_0) \right) d\mu(\mathbf{v}_0) = E_\mu \left(\prod_{k=1}^{n} v_{i_k}(t_k, \mathbf{x}_k, \mathbf{v}_0) \right),$$

where $\mathbf{v} = (v_1, v_2, v_3)$.

Extension of the test family

1. field family:

$$\mathcal{G} = \left\{ \begin{array}{c} v_1, v_2, v_3, p, \partial_j v_i \, (1 \le i, j \le 3)\}, \partial_t v_i \, (1 \le i \le 3), \\ \partial_i p \, (1 \le i \le 3), \partial^2_{ij} v_k \, (1 \le i, j, k \le 3) \end{array} \right\},$$

2. fluctuations field family:

$$\mathcal{H} = \left\{ \begin{array}{c} v'_1, v'_2, v'_3, p', \partial_j v'_i \, (1 \le i, j \le 3)\}, \partial_t v'_i \, (1 \le i \le 3), \\ \partial_i p' \, (1 \le i \le 3), \partial^2_{ij} v'_k \, (1 \le i, j, k \le 3) \end{array} \right\},$$

each element of

$$\mathcal{F} = \mathcal{G} \cup \mathcal{H}$$

is Hölder continuous with respect to $(t, \mathbf{x}) \in Q$, and continuous with respect to $\mathbf{v}_0 \in \mathbb{K}$.

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Let $D = I \times \omega \subset Q$ open and connected subset, such that $I \subset]0, T_{\mathbb{K}}[$ and $\overline{\omega} \subset \mathring{\Omega}$.

Aim To introduce different concepts of homogeneity in D, reflected in the local invariance under spatial translations of the correlation tensors based on the families $\mathcal{G}$ and/or $\mathcal{F} = \mathcal{G} \cup \mathcal{H}$, which is essential in the derivation of models such as $k - \mathscr{E}$.

Let $M = (t, \mathbf{x}) \in D$, and denote

$$(\tau_t, r_{\mathbf{x}}) = \sup\{(t, r) \,/\,]t - \tau, t + \tau[\times B(\mathbf{x}, r) \subset D\}.$$

For simplicity, we also denote

$$(t + \tau, \mathbf{x} + \mathbf{r}) = M + (\tau, \mathbf{r}), \quad (t, \mathbf{x} + \mathbf{r}) = M + \mathbf{r}.$$

Discussion about Homogeneity

Let $D = I \times \omega \subset Q$ open and connected subset, such that $I \subset]0, T_{\mathbb{K}}[$ and $\overline{\omega} \subset \mathring{\Omega}$.

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Discussion about Homogeneity

Definition

We say that the flow is 1) homogeneous (standard definition), 2) strongly homogeneous (extended definition, suitable for $k - \mathscr{E}$) in D, if $\forall\, n \in \mathbb{N}$,

$$\begin{array}{llll} 1) & \forall\, M_1, \dots, M_n \in D, & \forall\, \psi_1, \dots, \psi_n \in \mathcal{G}, & \forall\, \mathbf{r} \in \mathbb{R}^3 \\ 2) & \forall\, M_1, \dots, M_n \in D, & \forall\, \psi_1, \dots, \psi_n \in \mathcal{F}, & \forall\, \mathbf{r} \in \mathbb{R}^3 \end{array}$$

such that $|\mathbf{r}| \leq \inf_{1 \leq i \leq n} r_{\mathbf{x}_i}$, we have

$$B(\psi_1, \dots, \psi_n)(M_1 + \mathbf{r}, \dots, M_n + \mathbf{r}) = B(\psi_1, \dots, \psi_n)(M_1, \dots, M_n),$$

where

$$B(\psi_1, \dots, \psi_n)(M_1, \dots, M_n) = \overline{\psi_1(M_1) \cdots \psi_n(M_n)},$$

Discussion about Homogeneity

Lemma

Assume that the flow is homogeneous (resp. strongly hom.). Let

$$\psi_1, \ldots, \psi_n \in \mathcal{G} \ (resp \in \mathcal{F}), \quad M_1, \ldots, M_n \in D, \quad M_i = (t_i, \mathbf{r}_i),$$

such that

$$\forall\, i = 1, \cdots, n, \quad t_i = t.$$

Let $\mathbf{r}_i$ denotes the vector such that $M_i = M_1 + \mathbf{r}_{i-1}$ $(i \geq 2)$. Then, $B(\psi_1, \ldots, \psi_n)(M_1, \ldots, M_n)$ only depends on t and $\mathbf{r}_1, \cdots, \mathbf{r}_{n-1}$.

Discussion about Homogeneity

Theorem

Assume that $\mathbf{f}$ *satisfies the compatibility condition* $\nabla \mathbf{f} = 0$ *in* D*, and the flow is strongly homogeneous in* D*. Then*

1. $\forall\, \psi \in \mathcal{F}$, $\nabla \overline{\psi} = 0$ *in* D,
2. $\nabla \boldsymbol{\sigma}^{(\mathrm{R})} = 0$ *in* D,
3. *and we have* $\forall\, t \in I$,

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}(t) = \overline{\mathbf{v}}(t_0) + \int_{t_0}^{t} \mathbf{f}(s)\, ds \;\; \text{in} \;\; D,$$

by noting $t_0 = \inf I$.

Definition

We say that a flow is mildly homogeneous in $D = I \times \omega$, if $\forall\, \psi, \varphi \in \mathcal{H}$ (fluctuations family), we have

$$\forall\, M = (t, \mathbf{x}) \in D, \quad \overline{\psi(t,\mathbf{x})\partial_i\varphi(t,\mathbf{x})} = -\overline{\partial_i\psi(t,\mathbf{x})\varphi(t,\mathbf{x})}.$$

This definition is motivated by:

Lemma

Any strongly homogeneous flow is mildly homogeneous.

Equations for the TKE and the turbulent dissipation

The turbulent kinetic energy k (TKE) and the turbulent dissipation $\mathscr{E}$ are defined by:

$$k = \frac{1}{2} tr \boldsymbol{\sigma}^{(\mathrm{R})} = \frac{1}{2} \overline{|\mathbf{v}'|^2}, \quad \mathscr{E} = 2\nu \overline{|D\mathbf{v}'|^2}.$$

Question: What equation can we find out for k and $\mathscr{E}$?

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Theorem

Assume that the flow is mildly homogeneous in D

1. *The TKE k satisfies in D:*

$$\partial_t k + \overline{\mathbf{v}} \cdot \nabla k + \nabla \cdot \overline{e'\mathbf{v}'} = -\boldsymbol{\sigma}^{(\mathrm{R})} : \nabla \overline{\mathbf{v}} - \mathscr{E}$$

2. *The turbulent dissipation $\mathscr{E}$ satisfies in D:*

$$\begin{aligned} \partial_t \mathscr{E} + \overline{\mathbf{v}} \cdot \nabla \mathscr{E} + \nabla \cdot \overline{\nu h' \mathbf{v}'} = \ & 2\nu\left(\overline{\omega' \otimes \omega'} : \nabla \overline{\mathbf{v}} + \overline{(\omega' \otimes \omega')' : \nabla \mathbf{v}'}\right) \\ & -2\nu^2 \overline{|\nabla \omega'|^2}, \end{aligned}$$

where

$$\begin{array}{lll} e = \frac{1}{2}|\mathbf{v}'|^2, & \textit{decomposed as} & e = \overline{e} + e' = k + e', \\ \omega = \nabla \times \mathbf{v}, & \textit{decomposed as} & \omega = \overline{\omega} + \omega', \\ h = |\omega'|^2, & \textit{decomposed as} & h = \overline{h} + h'. \end{array}$$

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1) Basics

Throughout what follows, we assume that the flow is homogeneous (standard definition), and for simplicity stationnary ("homogeneity in time"). Let $\mathcal{B}_n$ denotes all correlation tensors of the form:

$$\mathbb{B}_n = \mathbb{B}_n(M_1, \dots, M_n) = (B_{i_1 \dots i_n}(M_1, \dots, M_n))_{1 \leq i_1 \dots i_n \leq 3},$$
$$\psi_{i_1}, \dots, \psi_{i_n} \in \mathcal{G},$$
$$B_{i_1 \cdots i_n}(\psi_1, \dots, \psi_n)(\mathbf{r}_1, \dots, \mathbf{r}_{n-1}) = \overline{\psi_{i_1}(\mathbf{x})\psi_{i_2}(\mathbf{x} + \mathbf{r}_1) \cdots \psi_{i_n}(\mathbf{x} + \mathbf{r}_{n-1})}.$$

Let $\mathbf{a}_1, \cdots, \mathbf{a}_n \in \mathbb{R}^3$, $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3})$. We set

$$[\mathbb{B}_n(\mathbf{r}_1, \cdots, \mathbf{r}_{n-1}), (\mathbf{a}_1, \cdots, \mathbf{a}_n)] = a_{1i_1} \cdots a_{ni_n} B_{i_1 \cdots i_n}(\mathbf{r}_1, \cdots, \mathbf{r}_{n-1}),$$

using the Einstein summation convention.
We denote by $O_3(\mathbb{R})$ an orthogonal group, which means that $Q \in O_3(\mathbb{R})$ if and only if $QQ^t = Q^t Q = \mathrm{I}$.

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Discussion about isotropy

Definition

We say that the flow is isotropic in D if and only if,

$$\begin{array}{l} \forall\, n \geq 1, \quad \forall\, \mathbb{B}_n \in \mathcal{B}_n, \\ \forall\, Q \in O_3(\mathbb{R}), \quad \forall\, \mathbf{a}_1, \cdots, \mathbf{a}_n \in \mathbb{R}^3, \\ \forall\, \mathbf{x} \in \omega, \quad \forall\, (\mathbf{r}_1, \cdots, \mathbf{r}_{n-1}) \in B(0, r_{\mathbf{x}})^{n-1}, \end{array}$$

then we have

$$[\mathbb{B}_n(Q\mathbf{r}_1, \cdots, Q\mathbf{r}_{n-1}), (Q\mathbf{a}_1, \cdots, Q\mathbf{a}_n)] = [\mathbb{B}_n(\mathbf{r}_1, \cdots, \mathbf{r}_{n-1}), (\mathbf{a}_1, \cdots, \mathbf{a}_n)].$$

2) Two order tensor

We fix δ_0 once and for all and $\mathbf{x}$ satisfies $d(\mathbf{x}, \partial\omega) \geq \delta_0$ so that $\mathbb{B}_2(\mathbf{r})$ is well defined for $|\mathbf{r}| \leq \delta_0$ and at least of class C^1 with respect to $\mathbf{r}$ (and does not depend on $\mathbf{x}$).

Theorem

Assume the flow isotropic in D. Then there exist two C^1 scalar functions $B_d = B_d(r)$ and $B_n = B_n(r)$ on $[0, \delta_0[$ and such that

$$\forall\, \mathbf{r} \in B(0, \delta_0), \quad \mathbb{B}_2(\mathbf{r}) = (B_d(r) - B_n(r))\frac{\mathbf{r} \otimes \mathbf{r}}{r^2} + B_n(r)\mathrm{I}_3,$$

where $r = |\mathbf{r}|$, $\mathbf{r} \otimes \mathbf{r} = (r_i r_j)_{1 \leq i,j \leq 3}$. Moreover, B_d and B_n are linked through the following differential relation:

$$\forall\, r \in [0, \delta_0[, \quad rB_d'(r) + 2(B_d(r) - B_n(r)) = 0,$$

where $B_d'(r)$ is the derivative of B_d.

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Energy spectrum

Energy spectrum for isotropic flows Let

$$\mathbb{E} = \frac{1}{2} tr \mathbb{B}_2|_{\mathbf{r}=0} = \frac{1}{2}\overline{|\mathbf{v}|^2},$$

be the total mean kinetic energy

Theorem

There exists a measurable function $E = E(k)$, defined over $\mathbb{R}_+$, the integral of which over $\mathbb{R}_+$ is finite, and such that

$$\mathbb{E} = \int_0^\infty E(k)dk.$$

Remark

$E(k)$ is the amount of kinetic energy in the sphere $S_k = \{|\mathbf{k}| = k\}$, which physically means $E \geq 0$ in $\mathbb{R}_+$, and therefore $E \in L^1(\mathbb{R}_+)$. We cannot rigorously prove $E \geq 0$ which remains an open problem.

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be the total mean kinetic energy

Theorem

There exists a measurable function $E = E(k)$, defined over $\mathbb{R}_+$, the integral of which over $\mathbb{R}_+$ is finite, and such that

$$\mathbb{E} = \int_0^\infty E(k) dk.$$

Remark

$E(k)$ is the amount of kinetic energy in the sphere $S_k = \{|\mathbf{k}| = k\}$, which physically means $E \geq 0$ in $\mathbb{R}_+$, and therefore $E \in L^1(\mathbb{R}_+)$. We cannot rigorously prove $E \geq 0$ which remains an open problem.

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The turbulent dissipation $\mathscr{E}$ is deduced from the energy spectrum from the formula:

$$\mathscr{E} = \nu \int_0^\infty k^2 E(k) dk,$$

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1) Dimensional bases

Only length and time are involved in this frame, heat being not considered and the fluid being incompressible.

Definition

A length-time basis is a couple $b = (\lambda, \tau)$, where λ a given constant length and τ a constant time.

Definition

Let $\psi = \psi(t, \mathbf{x})$ (constant, scalar, vector, tensor...) be defined on $Q = [0, T_{\mathbb{K}}] \times \Omega$. The couple $(d_\ell(\psi), d_\tau(\psi)) \in \mathbb{Q}^2$ is such that

$$\psi_b(t', \mathbf{x}') = \lambda^{-d_\ell(\psi)} \tau^{-d_\tau(\psi)} \psi(\tau t', \lambda \mathbf{x}'),$$

where $(t', \mathbf{x}') \in Q_b = \left[0, \frac{T_{\mathbb{K}}}{\tau}\right] \times \frac{1}{\lambda}\Omega$, is dimensionless. We say that $\psi_b = \psi_b(t', \mathbf{x}')$ is the b-dimensionless field deduced from ψ.

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Similarity

2) Kolmogorov scales

Let us consider the length-time basis $b_0 = (\lambda_0, \tau_0)$, determined by

$$\lambda_0 = \nu^{\frac{3}{4}} \mathscr{E}^{-\frac{1}{4}}, \quad \tau_0 = \nu^{\frac{1}{2}} \mathscr{E}^{-\frac{1}{2}}.$$

We recall that λ_0 is called the Kolmogorov scale. The important point here is that

$$\mathscr{E}_{b_0} = \nu_{b_0} = 1.$$

Moreover, for all wave number k,

$$E(k) = \nu^{\frac{5}{4}} \mathscr{E}^{\frac{1}{4}} E_{b_0}(\lambda_0 k),$$

We must find out the universal profil E_{b_0}

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3) Assumptions

Scale separation. *Let* ℓ *be the Prandtl mixing length. Then*

$$\lambda_0 << \ell.$$

Similarity. *There exists an interval*

$$[k_1, k_2] \subset \left[\frac{2\pi}{\ell}, \frac{2\pi}{\lambda_0}\right] \;\; s.t. \;\; k_1 << k_2 \;\; and \; on \;\; [\lambda_0 k_1, \lambda_0 k_2],$$

$$\forall\, b_1 = (\lambda_1, \tau_1),\; b_2 = (\lambda_2, \tau_2) \;\; s.t. \;\; \mathscr{E}_{b_1} = \mathscr{E}_{b_2}, \;\; then \;\; E_{b_1} = E_{b_2}$$

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4) Law of the -5/3

Theorem

Scale separation and Similarity Assumptions yield the existence of a constant C such that

$$\forall\, k' \in [\lambda_0 k_1, \lambda_0 k_2] = J_r, \quad E_{b_0}(k') = C(k')^{-\frac{5}{3}}.$$

Corollary

The energy spectrum satisfies the $-5/3$ law

$$\forall\, k \in [k_1, k_2], \quad E(k) = C \mathscr{E}^{\frac{2}{3}} k^{-\frac{5}{3}},$$

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Idea of the proof Let

$$b^{(\alpha)} = (\alpha^3 \lambda_0, \alpha^2 \tau_0).$$

As

$$\mathscr{E}_{b^{(\alpha)}} = 1 = \mathscr{E}_{b_0},$$

The similarity assumption yields

$$\forall\, k' \in J_r, \quad \forall \alpha > 0, \quad E_{b^{(\alpha)}}(k') = E_{b_0}(k').$$

which leads to the functional equation,

$$\forall\, k' \in J_r, \quad \forall\, \alpha > 0, \quad \frac{1}{\alpha^5} E_{b_0}(k') = E_{b_0}(\alpha^3 k'),$$

whose unique solution is given by

$$\forall\, k' \in J_r, \quad E_{b_0}(k') = C(k')^{-\frac{5}{3}}, \quad C = \left(\frac{k_1}{\lambda_0}\right)^{\frac{5}{3}} E_0\left(\frac{k_1}{\lambda_0}\right),$$

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Theorem

Assume:

1. *The Law of the -5/3 holds true,*
2. *The turbulent dissipation holds in the inertial range,*
3. *The Boussinesq assumption holds true, i.e*

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_t D\overline{\mathbf{v}} + \frac{2}{3}k\mathrm{I}.$$

Then Smagorinsky's postulate holds true, i.e

$$\nu_t = C\delta^2|D\overline{\mathbf{v}}|.$$

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