The Kolmogorov Law of turbulence

What can rigorously be proved ?

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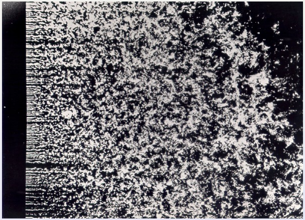
Introduction

Framework : homogeneous and isotropic turbulent flows modeling.



Typical example : turbulence behind a grid

Initial problem : Find out an estimate of the decay of the turbulence when moving away from the grid

Kolmogorov 1941 (K41), footnote:

"We may indicate here only certain general considerations speaking for advanced hypothesis. For every large Reynolds number *Re*, the turbulent flow may be thought in the following way: on the averaged flow (characterised by $\overline{\mathbf{v}}$) are superposed the pulsation of the first order consisting in disorderly displacements of separate fluid volumes, one respect to another, of diameter of the order of magnitude $\ell^{(1)}$ (where $\ell^{(1)}$ is the Prandtl mixing path); the order of magnitude of velocities of these relative velocities are denoted by $\mathbf{v}^{(1)}$. The pulsation of the first order are for very large *Re* in their turn unsteady, on which are superposed the pulsation ofthe second order with mixing path $\ell^{(2)} < \ell^{(1)}$ and relative velocities $\mathbf{v}^{(2)}$ such that $|\mathbf{v}^{(2)}| < |\mathbf{v}^{(1)}|$; such a process of successive refinement of turbulent pulsation may be carried out until the pulsation of some sufficiently large order n, the Reynolds number of which,

$$R^{(n)} = \frac{\ell^{(n)}\mathbf{v}^{(n)}}{\nu},$$

becomes small enough such that the effect of viscosity on the pulsation of the order n prevents the pulsation of the order $n+1$."

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i) Statistical description of the tubulence $\rightarrow$ mean velocity and pressure $\overline{\mathbf{v}}, \overline{p}$, defined as expectations of $\mathbf{v}, p$ (Taylor),

ii) From global (Taylor) to local (K41) notions of homogeneity and isotropy through correlation tensors,

iii) Stating the structure of the general 2 order correlation tensor $B(\mathbf{r})$, the trace of which is the mean energy (Taylor, K41),

iv) Finding out the behavior of $B(\mathbf{r})$ for $r = |\mathbf{r}|$ small, and deriving from a similarity principle the law of the 2/3 in the inertial range $[r_1, r_2]$, by introducing appropriate lenght and time scales (K41).

Remark

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Aim of the talk:

1) To discuss on which solutions to the incompressible NSE are most appropriate to set a rigorous probabilistic framework, in order to define the expectations $\overline{\mathbf{v}}$ and $\overline{p}$,

2) To perform abstract algebraic definitions of homogeneous and isotropic tensors and to fully characterize the 1 and 2 order homogeneous tensors,

3) To recall the Reynolds decomposition, to introduce the Reynolds stress $\sigma^{(\mathrm{R})}$, to define the notion of homogeneous and isotropic turbulence from the correlation tensors, to deduce from the NSE the equation for $\sigma^{(\mathrm{R})}$ in the homogeneous and isotropic case,

4) To perform an expansion of $B(r)$ near zero by using the NSE, and to derive the law of the 2/3 in the inertial range, after having rigously set the principle of similarity.

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Solutions to the NSE

1) Incompressible Navier-Stokes Equations (NSE):

$$Q = [0, T] \times \Omega \quad \text{or} \quad Q = [0, \infty[\times \Omega, \quad \Omega \subset \mathbb{R}^3, \quad \Gamma = \partial\Omega,$$

with the no slip boundary condition and $\mathbf{v}_0$ as initial data:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p &= \mathbf{f} & \text{in } Q, \\ \nabla \cdot \mathbf{v} &= 0 & \text{in } Q, \\ \mathbf{v} &= 0 & \text{on } \Gamma, \\ \mathbf{v} &= \mathbf{v}_0 & \text{at } t = 0, \end{cases}$$

where $\nu > 0$ is the kinematic viscosity, $\mathbf{f}$ is any external force,

$$D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t), \quad ((\mathbf{v} \cdot \nabla)\mathbf{v})_i = v_j \frac{\partial v_i}{\partial x_j}, \quad \nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}.$$

NB: $\nabla \cdot \mathbf{v} = 0 \Rightarrow (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \cdot (\mathbf{v} \otimes \mathbf{v}), \quad \mathbf{v} \otimes \mathbf{v} = (v_i v_j)_{1 \leq i,j \leq 3}.$

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Solutions to the NSE

2) Two types of solutions to the NSE:

i) Strong solutions over a small time interval $[0, T_{max}[$ "à la" Fujita-Kato,

ii) Weak solutions (also turbulent) , global in time, "à la" Leray-Hopf

i) Strong solutions are $C^{1,\alpha}$ over $[0, T_{\max}[\times\Omega$ for smooth data,

$$T_{\max} = T_{\max}(||\mathbf{v}_0||, ||\mathbf{f}||, \nu),$$

the corresponding solution is unique, yielding the writing

$$\mathbf{v} = \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0), \quad p = p(t, \mathbf{x}, \mathbf{v}_0).$$

NB. Strong solutions are defined over $[0, \infty[$ when $\mathbf{v}_0$ is "small enough", ν and/or $||\mathbf{f}||$ are "large enough", which means that the flow is rather laminar.

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ii) Weak solutions: defined through an appropriate variational formulation set in the sequence of function spaces

$$\{\mathbf{v} \in H_0^1(\Omega)^3, \nabla\cdot\mathbf{v} = 0\} \hookrightarrow \mathbf{V} = \{\mathbf{v} \in L^2(\Omega)^3,\ \mathbf{v}\cdot\mathbf{n}|_\Gamma = 0, \nabla\cdot\mathbf{v} = 0\}$$

such that the trajectory

$$\mathbf{v} = \mathbf{v}(t) \in \mathbf{V}$$

is weakly continuous from $[0, T] \to \mathbf{V}$, $T \in]0, \infty]$ ($\forall \boldsymbol{\eta} \in \mathbf{V}$, $t \to \langle \mathbf{v}(t), \boldsymbol{\eta}\rangle$ is a continous function of t, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbf{V}$).

Remark

For $\mathbf{v}_0 \in \mathbf{V}$ and appropriate source terms $\mathbf{f}$, the exists a global weak solution $\mathbf{v} = \mathbf{v}(t)$, $t \in \mathbb{R}^+$, however uniqueness is not known, so that at this point we can't write

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Probabilistic framework

1) Long Time Average. Let $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel σ-algebra on $\mathbb{R}_+$, λ the Lebesgue measure, and let μ denotes the probability measure

$$\forall\, A \in \mathcal{B}(\mathbb{R}_+), \quad \mu(A) = \lim_{t \to \infty} \frac{1}{t} \lambda(A \cap [0, t]),$$

Let $\mathbf{v} \in L^1(\mathbb{R}^+ \to \mathbf{V}; \mu)$,

$$E(\mathbf{v}) = \overline{\mathbf{v}} = \int_{\mathbb{R}^+} \mathbf{v}(t) d\mu(t) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{v}(t) dt.$$

Let $\mathbf{v}$ be a Leray-Hopf solution to the NSE. It is not known wether for a given $\mathbf{x} \in \Omega$, $\mathbf{v}(t, \mathbf{x}) \in L^1(\mathbb{R}^+ \to \mathbf{V}; \mu)$. However, it can be proved that in some sense when $\mathbf{f}$ is steady [1],

$$E(\mathbf{v}) = \overline{\mathbf{v}} = \overline{\mathbf{v}}(\mathbf{x}) \in W^{2,5/4}(\Omega)$$

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Probabilistic framework

2) Ensemble Average. The source term $\mathbf{f}$ and the viscosity ν are fixed. Let $\mathbb{K} \subset \mathbf{V}$ be a compact,

$$T_{\mathbb{K}} = \inf_{\mathbf{v}_0 \in \mathbb{K}} T_{\max}(||\mathbf{v}_0||, ||\mathbf{f}||, \nu) > 0, \quad Q = [0, T_{\mathbb{K}}] \times \Omega$$

Let $\{\mathbf{v}_0^{(1)}, \cdots, \mathbf{v}_0^{(n)}, \cdots\}$ be a countable dense subset of $\mathbb{K}$,

$$\overline{\mathbf{v}}_n = \overline{\mathbf{v}}_n(t, \mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0^{(i)}) = E_{\mu_n}(\mathbf{v}(t, \mathbf{x}, \cdot)),$$

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Up to a subsequence

$$\mu_n \to \mu \quad \text{weakly in the sense of the measures,} \quad ||\mu|| = 1.$$

so that, $\forall (t, \mathbf{x}) \in Q$,

$$\overline{\mathbf{v}}(t, \mathbf{x}) = E_\mu(\mathbf{v}(t, \mathbf{x}, \cdot)) = \int_{\mathbb{K}} \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0) d\mu(\mathbf{v}_0) = \lim_{n \to \infty} \overline{\mathbf{v}}_n(t, \mathbf{x}).$$

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It is not known if the probability measure μ is unique or not

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Up to a subsequence

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Reynolds Stress

1) Reynolds decomposition. We can decompose $(\mathbf{v}, p)$ as follows:

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad p = \overline{p} + p',$$

which is the Reynolds decomposition, $\mathbf{v}'$ and p' are the fluctuations. Either for long time averaged or ensemble average:

$$\overline{\partial_t \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)} = \partial_t \overline{\mathbf{v}}(t, \mathbf{x}), \tag{1}$$

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Moreover, by noting that $\overline{\overline{\mathbf{v}}} = \overline{\mathbf{v}}$ and $\overline{\overline{p}} = \overline{p}$, it easily checked that:

Lemma

The fluctuation's mean vanishes, i.e.

$$\forall (t, \mathbf{x}) \in Q, \quad \overline{\mathbf{v}'(t, \mathbf{x}, \mathbf{v}_0)} = 0, \quad \overline{p'(t, \mathbf{x}, \mathbf{v}_0)} = 0.$$

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2) Averaged NSE. Note that $E_\mu(\mathbf{f}) = \mathbf{f}E_\mu(1) = \mathbf{f}$. By the Reynolds rules and the previous lemma:

$$\left\{ \begin{array}{rcll} \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\,\overline{\mathbf{v}} - \nu\Delta\overline{\mathbf{v}} + \nabla\overline{p} & = & -\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} + \mathbf{f} & \text{in } \ Q, \\ \nabla \cdot \overline{\mathbf{v}} & = & 0 & \text{in } \ Q, \\ \overline{\mathbf{v}} & = & 0 & \text{on } \ \Gamma, \\ \overline{\mathbf{v}} & = & \overline{\mathbf{v}_0} & \text{at } \ t = 0, \end{array} \right.$$

where

$$\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}$$

is the Reynolds stress.

Correlation tensors

Let $\mathbf{x}_0, \mathbf{x} \in \Omega$, $\mathbf{r}_1, \cdots, \mathbf{r}_{n-1}$ s.t $\mathbf{x}_0 + \mathbf{r}_i \in \Omega$,

$$\mathbf{w}(t, \mathbf{x}) = \mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{x}_0), \quad \mathbf{w} = (w_1, w_2, w_3).$$

Let $B^{(n)} = (B^{(n)}_{i_1 \cdots i_n})_{1 \le i_1 \cdots i_n}$ the n-order correlation tensor:

$$B^{(n)}_{i_1 \cdots i_n}(t, \mathbf{x}_0, \mathbf{r}_1, \cdots, \mathbf{r}_{n-1}) = \overline{w_{i_1}(t, \mathbf{x}_0) w_{i_2}(t, \mathbf{x}_0 + \mathbf{r}_1) \cdots w_{i_n}(t, \mathbf{x}_0 + \mathbf{r}_{n-1})},$$

We assume that the turbulence is i) stationnary, ii) homogeneous:

i) The correlation tensors are invariant under time translation, which yields they do not depend on t,

ii) The mean field $\overline{\mathbf{w}}$ only depends on $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$, and is steady, so that $\overline{\mathbf{w}} = \overline{\mathbf{w}}(\mathbf{r})$ and the correlation tensors are invariant under spatial translations,

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Remark

For homogeneous turbulence, $B^{(n)}(\mathbf{x}_0, \mathbf{r}_1, \cdots, \mathbf{r}_{n-1})$ *do not depend on* $\mathbf{x}_0$*, so that it will be denoted by* $B^{(n)}(\mathbf{r}_1, \cdots, \mathbf{r}_{n-1})$ *.*

In the following, we will focus on

1. a proper definition of isotropic turbulence, which expresses some invariance of the turbulence under isometries,
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$$B^{(2)}_{ij}(\mathbf{r}) = B_{ij}(\mathbf{r}) = \overline{w_i(\mathbf{x}) w_j(\mathbf{x} + \mathbf{r})},$$

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1) Dual action. Let $R_n = (\mathbf{r}_1, \cdots, \mathbf{r}_{n-1})$,
$H_n = (\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_n) \in \mathbb{R}^{3n} = \mathbb{R}^3 \times \cdots \times \mathbb{R}^3$, $\mathbf{h}_i = (h_{i1}, h_{i2}, h_{i3})$.

The dual action of $B^{(n)}$ at R_n is defined by

$$[B^{(n)}(R_n), H_n] = B^{(n)}_{i_1 \cdots i_n}(R_n) h_{i_1 1} \cdots h_{i_p p},$$

or equivalently

$$[B^{(n)}(R_n), H_n] = B^{(n)}(R_n) : \mathbf{h}_1 \otimes \mathbf{h}_2 \otimes ..... \otimes \mathbf{h}_n,$$

where " : " stands for the contracted tensor product, "$\otimes$" the tensor product.

Example. In the case of $B^2(\mathbf{r}) = (B^{(2)}_{ij}(\mathbf{r}))_{1 \le ij \le 3}$, then

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2) Isotropic fields. The mean field $\overline{\mathbf{w}} = \overline{\mathbf{w}}(\mathbf{r})$ is said isotropic if

$$\forall\, Q \in O_3(\mathbb{R}), \forall\, \mathbf{r}, \mathbf{u} \in \mathbb{R}^3, \quad (\overline{\mathbf{w}}(Q\mathbf{r}), Q\mathbf{u}) = (\overline{\mathbf{w}}(\mathbf{r}), \mathbf{u}).$$

We set $r = |\mathbf{r}|$.

Theorem

Let $\overline{\mathbf{w}} = \overline{\mathbf{w}}(\mathbf{r})$ be isotropic. Then there exists a function $a = a(r)$ such that

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Definition

Homogeneous turbulence is already defined. It is said said to be isotropic, if $\overline{\mathbf{w}}$ is isotropic and for all n, $B^{(n)}$ is isotropic

Consequence: The mean field vanishes in Ω, and the NSE reduces to

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Lemma

The following relations hold:

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We deduce

$$B_d(0) = B_n(0) = 0, \tag{7}$$

$$B'_d(0) = B'_n(0) = 0, \tag{8}$$

Theorem

Assume that the mean pressure gradient is constant inside Ω. Then there exists a C^1 scalar function $E = E(r)$ such that $E(0) = E'(0) = 0$ and such that

$$B(\mathbf{r}) = E(r)\frac{\mathbf{r} \otimes \mathbf{r}}{r^2} - \frac{3}{2}E(r)\mathrm{I}_3 + o(r^3).$$

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$$B_d'(0) = B_n'(0) = 0, \tag{8}$$

Theorem

Assume that the mean pressure gradient is constant inside Ω*. Then there exists a* C^1 *scalar function* $E = E(r)$ *such that* $E(0) = E'(0) = 0$ *and such that*

$$B(\mathbf{r}) = E(r)\frac{\mathbf{r} \otimes \mathbf{r}}{r^2} - \frac{3}{2}E(r)\mathrm{I}_3 + o(r^3).$$

1) Dimensional bases. Only length and time are involved in this frame, heat being not considered and fluids being incompressible.

Definition

A length-time basis is a couple $b = (\lambda, \tau)$, where λ a given constant length and τ a constant time.

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Let $\psi = \psi(t, \mathbf{x})$ (constant, scalar, vector, tensor...) be defined on $Q = [0, T_{\mathbb{K}}] \times \Omega$. The couple $(d_\ell(\psi), d_\tau(\psi))$ is such that

$$\psi_b(t', \mathbf{x}') = \lambda^{-d_\ell(\psi)} \tau^{-d_\tau(\psi)} \psi(\tau t', \lambda \mathbf{x}'),$$

where $(t', \mathbf{x}') \in Q_b = \left[0, \frac{T_{\mathbb{K}}}{\tau}\right] \times \frac{1}{\lambda}\Omega$, is dimensionless. We say that $\psi_b = \psi_b(t', \mathbf{x}')$ is the b-dimensionless field deduced from ψ.

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Similarity

2) Kolmogorov scales. The question is the behavior of $E(r)$ when r differs from 0.

Following Kolmogorov, we assume that E is entirely driven in $[0, \ell]$, ℓ being the Mixing Prandtl Length, by the kinematic viscosity ν and the mean dissipation at $\mathbf{x}_0$, specified by

$$\mathscr{E} = \overline{2\nu|D\mathbf{v}(\mathbf{x}_0)|^2}.$$

Let $b_0 = (\lambda_0, \tau_0)$, where

$$\lambda_0 = \nu^{\frac{3}{4}}\mathscr{E}^{-\frac{1}{4}}, \quad \tau_0 = \nu^{\frac{1}{2}}\mathscr{E}^{-\frac{1}{2}}.$$

Therefore,

$$\forall\, r' \in [0, \frac{\ell}{\lambda_0}[, \quad E(\lambda_0 r') = (\nu\mathscr{E})^{\frac{1}{2}} E_{b_0}(r'),$$

where E_{b_0} is a universal profil.

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3) Similarity assumption and the law of the $2/3$. We assume the following

1. $\lambda_0 << \ell$
2. there exists r_1, r_2, s.t. $\lambda_0 << r_1 << r_2 << \ell$, and for all lenght-times bases $b_1 = (\lambda_1, \tau_1)$ and $b_2 = (\lambda_2, \tau_2)$,

$$\mathscr{E}_{b_1} = \mathscr{E}_{b_2} \Rightarrow \forall\, r' \in \left[\frac{r_1}{\lambda_1}, \frac{r_2}{\lambda_1}\right] \cap \left[\frac{r_1}{\lambda_2}, \frac{r_2}{\lambda_2}\right], \; E_{b_1}(r') = E_{b_2}(r').$$

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$$r'_{1,0} = \frac{r_1}{\lambda_0}, \quad r'_{2,0} = \frac{r_2}{\lambda_0}.$$

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Similarity

Theorem

If the similarity assumption holds, then there exists a constant C such that

$$\forall\, r' \in [r'_{1,0}, r'_{2,0}], \quad E_{b_0}(r') = C(r')^{\frac{2}{3}},$$

which yields

$$\forall r \in [r_1, r_2], \quad E(r) = C(\mathscr{E} r)^{\frac{2}{3}}.$$