On a Continuous Deconvolution Equation for Turbulence Models

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Abstract

We introduce in this paper the notion of "Continuous Deconvolution Equation" in a 3D periodic case. We first show how to derive this new equation from the Van Cittert algorithm. Next we show many mathematical properties of the solution to this equation. Finally, we show how to use it to introduce a new turbulence model for high Reynolds numbers flows.

1 Introduction and main facts

1.1 General orientation

It is well known since Kolmogorov's work [25], that to simulate an incompressible 3D turbulent flow using the Navier-Stokes equations,

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0, \end{cases} \tag{1.1}$$

requires about $Nb = Re^{9/4}$ points in a numerical grid (details are available in [19] or [35]). Here, $Re = UL/\nu$ denotes the Reynolds number (see a rigorous definition in [IV.i], below in the text).

For realitic flows, such as those involved in mechanical engineering or in geophysics, Re is of order $10^8 - 10^{10}$, sometimes much more. Therefore, the number of points Nb necessary for the simulation is hudge and yields computational algorithms, the memory they need exceeding too much the memory size of the most powerfull modern computer. This is why one needs "turbulent models" to reduce the appropriate grid points number, to simulate at least averages a turbulent flows.

There exists two main famillies of turbulent models : statistical models, such as the well known $k - \varepsilon$ model (see in [31] and [35]), and Large Eddy Simulations models (see in [9] and [36]), known as "LES models". This paper deals with LES models familly. The idea behind LES, is to simulate the "large scale" of the flow, trying to keep energy informations on "small" scales. Eddy viscosities are mostly involved in those models.

Many models also emerged without eddy viscosity, such as Bardina's models [3] or related (see in [32], [28], [27] [26]), as well as the familly of α-models and related (see for instance in [17], [21], [24], [13], [12]). All of them are still consider as LES models. They mainly aim to regularize the nonlinear term $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$ in the Navier-Stokes Equations.

This idea takes inspiration in the work of Jean Leray in 1934 [30]. At this time, computers did not exist and people were not thinking at numerical simulations for flows around an aircraft wing or for wether forecasts. They mostly were trying to find analytical solutions to the 3D Navier-Stokes equations in cases of laminar flows or when geometrical symmetries occur as well as when special 2D approximations were legitimate, the general case remaining out of reach. Such calculus are well explained in the famous book of G. Batchelor [4]. Therefore, the question arises to know if the Navier-Stokes Equations have a solution or not in the general case, even if it is not possible to give analytical formula for these solutions.

Jean Leray shown the existence of what we call today "a dissipative weak solution" to the Navier-Stokes equation in the whole space $\mathbb{R}^3$ (see the definition 4.1 below in the text). To do this, he first constructed approximated smooth solutions to the Navier-Stokes Equations. Using secondly some compactness arguments, he considered the limit of a subsequence, showing that this limit is a dissipative weak solution, called formerly "Turbulent solution". We mean by "dissipative" solution, a distributional solution satisfying the energy inequality (see (4.6) below in the text).

We still do not know if there is a unique dissipative solution in the general case, and also if it does or not develop singularities in finite time. The question of singularities for particular dissipative solutions called "suitable weak solutions", is studied in the very famous paper by Caffarelli-Kohn-Nirenberg [11].

1.2 Towards the models

To build approximated smooth solutions, J. Leray got the idea to replace the nonlinear term $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$ by $((\mathbf{u}\star\rho_\varepsilon)\cdot\nabla)\,\mathbf{u}$, where $(\rho_\varepsilon)_{\varepsilon>0}$ is a sequence of mollifiers: doing like this, he introduced the first LES models without knowing it, a long time before Smagorinsky published his first paper in 1953 [37], Smagorinsky being often considered as a main pioneer of LES. This idea of smoothing the nonlinear term can be generalized in many other cases, such as the periodic case that we consider in this paper. In this case, one can regularize the Navier-Stokes equations by using the so called "Helmholtz equation".

Let $\mathbf{u}$ be an incompressible periodic field $\mathbf{u}$ ($\nabla\cdot\mathbf{u}=0$), the mean value of which, $m(\mathbf{u})$ (see (2.2) below in the text), being equal to zero. Notice that in the remainder, all fields we consider will have a zero mean value for compatibility reasons. We do not shall mention it every time so far no risk of confusion occurs. Such a field $\mathbf{u}$ being given, let us consider the Stokes Problem

$$\left\{\begin{array}{l} A\overline{\mathbf{u}} = -\alpha^2\Delta\overline{\mathbf{u}} + \overline{\mathbf{u}} + \nabla\pi = \mathbf{u}, \\ \nabla\cdot\overline{\mathbf{u}} = 0. \end{array}\right. \tag{1.2}$$

The parameter α is the "small parameter". It is generally agreed that α must be taken about the numerical grid size in numerical simulations, even if this claim is sometimes subject to caution.

The Leray-α model is the one where the nonlinear term in the Navier-Stokes Equations is regularized by taking $(\overline{\mathbf{u}}\cdot\nabla)\,\mathbf{u}$ in place of $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$. The Bardina's model of order zero is the one where one replaces the nonlinear term by $\overline{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}$. The solutions of these approximated Navier-Stokes Equations are supposed to compute approximations of mean values of pressure and velocity fields. To see this, let us take the average of (1.1). We get the following "true" equation for $\overline{\mathbf{u}}$,

$$\begin{cases} \partial_t \overline{\mathbf{u}} + \overline{(\mathbf{u}\cdot\nabla)\,\mathbf{u}} - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} = \overline{\mathbf{f}}, \\ \nabla\cdot\overline{\mathbf{u}} = 0, \\ \overline{\mathbf{u}}(0,\mathbf{x}) = \overline{\mathbf{u}}_0, \end{cases} \tag{1.3}$$

an equation that we can rephrase as

$$\begin{cases} \partial_t \overline{\mathbf{u}} + B_\alpha(\overline{\mathbf{u}},\overline{\mathbf{u}}) - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} = \overline{\mathbf{f}} + B_\alpha(\overline{\mathbf{u}},\overline{\mathbf{u}}) - \overline{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}, \\ \nabla\cdot\overline{\mathbf{u}} = 0, \\ \overline{\mathbf{u}}(0,\mathbf{x}) = \overline{\mathbf{u}}_0, \end{cases} \tag{1.4}$$

where $B_\alpha(\overline{\mathbf{u}},\overline{\mathbf{u}})$ is a nonlinear term depending on α and "regular enough". In the model, $B_\alpha(\overline{\mathbf{u}},\overline{\mathbf{u}})$ must replace $\overline{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}$, and $R_\alpha = B_\alpha(\overline{\mathbf{u}},\overline{\mathbf{u}}) - \overline{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}$ is a residual stress that we neglect for more or less good physical or numerical reasons. Then the principle of the model consists in simulating flows by computing an approximation of $\overline{\mathbf{u}}$ and $\overline{p}$, denoted by $\mathbf{u}_\alpha$ and p_α, solution of

$$\begin{cases} \partial_t \mathbf{u}_\alpha + B_\alpha(\mathbf{u}_\alpha,\mathbf{u}_\alpha) - \nu\Delta\mathbf{u}_\alpha + \nabla p_\alpha = \overline{\mathbf{f}}, \\ \nabla\cdot\mathbf{u}_\alpha = 0, \\ \mathbf{u}_\alpha(0,\mathbf{x}) = \overline{\mathbf{u}}_0. \end{cases} \tag{1.5}$$

Such a model is relevant when

- B_α correctly filters high frequencies and describes with accuracy low frequencies,
- System (1.5) has a unique "smooth enough" solution when $\mathbf{u}_0 \in L^2_{loc}$ (therefore $\overline{\mathbf{u}}_0 \in H^2_{loc}$). We mean by "smooth enough", $\mathbf{u} \in L^\infty([0,T],(H^1_{loc})^3) \cap L^2([0,T],(H^2_{loc})^3)$, $p \in L^2([0,T],H^1_{loc})$, on any time interval $[0,T]$,
- the unique solution $(\mathbf{u}_\alpha, p_\alpha)$ to (1.5) satisfies an energy balance like (4.15) (and not only an energy inequality like (4.6), see below in the text), for α fixed,
- there is a subsequence of the sequence $(\mathbf{u}_\alpha, p_\alpha)_{\alpha>0}$ which converges (in a certain meaning) to a dissipative weak solution to (1.1) when α goes to zero.

We must say that there are many B_α such that the last three points of the previous program are satisfied. But so far we want to use these equations to simulate realistic flows, we must check the first point. Unfortunatly, it does not exist a rigorous definition to make this notion precise, also linked to the notion of "cut frequency".

1.3 Approximate Deconvolution Models

In 1999 and later, Adams and Stolz ([1], [39], [38], [2]) were considering "the Bardina's model of order zero" where $B_\alpha(\mathbf{u},\mathbf{u}) = \nabla\cdot\overline{(\mathbf{u}\otimes\mathbf{u})} = \overline{(\mathbf{u}\cdot\nabla)\mathbf{u}}$. In order to improve the rebuilding of the true field in numerical simulations, they got the idea to apply a "deconvolution operator D_N". To do it, they introduced a parameter N of deconvolution, using the discrete "Van Cittert algorithm" (see in [10]),

$$\begin{cases} \mathbf{w}_0 = \overline{\mathbf{u}}, \\ \mathbf{w}_{N+1} = \mathbf{w}_N + (\overline{\mathbf{u}} - A^{-1}\mathbf{w}_N), \end{cases} \tag{1.6}$$

where the operator A is defined in (1.2). The deconvolution operator is defined by $H_N(\mathbf{u}) = \mathbf{w}_N = D_N(\overline{\mathbf{u}})$. It is fixed such that for a given incompressible field $\mathbf{u}$, $H_N(\mathbf{u}) = D_N(\overline{\mathbf{u}})$ goes to $\mathbf{u}$ in a certain meaning (see section 3.1 below). Therefore, the model consists in replacing the nonlinear term by

$$B_{\alpha,N}(\mathbf{u},\mathbf{u}) = \overline{\nabla \cdot (D_N(\mathbf{u}) \otimes D_N(\mathbf{u}))},$$

yielding the model

$$\text{(1.7)} \qquad \begin{cases} \partial_t \mathbf{u}_{\alpha,N} + \overline{\nabla \cdot (D_N(\mathbf{u}_{\alpha,N}) \otimes D_N(\mathbf{u}_{\alpha,N}))} - \nu\Delta\mathbf{u}_{\alpha,N} + \nabla p_{\alpha,N} = D_N(\overline{\mathbf{f}}), \\ \nabla \cdot \mathbf{u}_{\alpha,N} = 0, \\ \mathbf{u}_{\alpha,N}(0,\mathbf{x}) = D_N(\overline{\mathbf{u}}_0) = H_N(\mathbf{u}_0). \end{cases}$$

This model is called an "Approximate Deconvolution Model". Existence, regularity and uniqueness of a solution to this model for general deconvolution order N, were proved by Dunca-Epshteyn in 2006 [16]. The case $N = 0$ was already studied before in details in [26], [27], [32]. Questions of accuracy and error estimates were also studied in [28] for general order of deconvolution N.

The exciting point in model (1.7), is that it formally "converges" to the true averaged Navier-Stokes Equations (1.3) when N goes to infinity and α is fixed. A suitable choice of the deconvolution order N combined with a suitable choice of α, make hope that we can approach with a good accuracy the true average of the real field, defined by the Navier-Stokes Equations (expecting uniqueness of the dissipative solution).

Therefore, we had to investigate the problem of the convergence of $(\mathbf{u}_{\alpha,N}, p_{\alpha,N})_{N\in\mathbb{N}}$ to a solution of the mean Navier-Stokes Equations (1.3) when N goes to infinity. This problem is very tough, and we got very recently ideas to solve it [8]. Earlier, we got the idea in [29] to introduce a simplified deconvolution model, where the nonlinear term is $(H_N(\mathbf{u}) \cdot \nabla)\, \mathbf{u}$, yielding the model

$$\text{(1.8)} \qquad \begin{cases} \partial_t \mathbf{u}_{\alpha,N} + (H_N(\mathbf{u}_{\alpha,N}) \cdot \nabla)\, \mathbf{u}_{\alpha,N} - \nu\Delta\mathbf{u}_{\alpha,N} + \nabla p_{\alpha,N} = H_N(\mathbf{f}), \\ \nabla \cdot \mathbf{u}_{\alpha,N} = 0, \\ \mathbf{u}_{\alpha,N}(0,\mathbf{x}) = H_N(\mathbf{u}_0). \end{cases}$$

We proved in [29] existence, uniqueness and regularity of a solution $(\mathbf{u}_{\alpha,N}, p_{\alpha,N})$ to (1.8), and also that a subsequence of the sequence $(\mathbf{u}_{\alpha,N}, p_{\alpha,N})_{N\in\mathbb{N}}$ converges in a certain meaning, to a dissipative weak solution of the Navier-Stokes Equations for a fixed α, when N goes to infinity.

1.4 The deconvolution equation and ouline of the remainder

All the models we displaid above have been well studied in the periodic case. This calls for the question of adapting them in cases of realistic boundary conditions.

We have considered an ocean forced by the atmosphere, under the rigid lid hypothesis with a mean flux condition at the surface (see in [31]). As we started working on this question, it appears soon that we were not able to do the job for the Adams-Stolz deconvolution model (1.7), often known as ADM model. Indeed, if we keep the natural boundary condition at the surface, we cannot write an identity like

$$\int_\Omega \overline{\nabla \cdot (D_N(\mathbf{u}) \otimes D_N(\mathbf{u}))} \cdot \mathbf{u} = 0,$$

though it is the key to get the $L^2([0,T],(H^2)^3)\cap L^\infty([0,T],(H^1)^3)$ estimate in the periodic case. Therefore, even the modelization of the boundary condition remains an open problem, to derive an ADM model which fits with the physics and having good mathematical properties.

Facing the difficulty that infers the question of boundary conditions in model (1.7), we turned to the other deconvolution model we have in hand, the model (1.8), although we take ADM model (1.7) for the best one in this model's class. Indeed, (1.7) really approaches the averaged Navier-Stokes Equations for large deconvolution's order making it a real LES model, at least formally, when model (1.8) approaches the real Navier-Stokes Equations, fading the role of α, a fact we cannot physically interpret, although it shows a good numerical behavior (see in [5]).

We next thought that fixing the Van Cittert algorithm with realistic boundary conditions would be easy. Unfortunatly, we had troubles when writting it under the form (1.6), because precisely of the boundary conditions. This is why we decided to replace the Van Cittert Algorithm by a continuous variational problem. Our key observation is that this algorithm can be written under the form

$$-\alpha^2\left(\frac{\Delta \mathbf{w}_{N+1}-\Delta \mathbf{w}_N}{\delta\tau}\right)+\mathbf{w}_{N+1}+\nabla\pi_{N+1}=\mathbf{u}, \tag{1.9}$$

with $\delta\tau = 1$. This is precisely the finite difference equation corresponding to the continuous equation

$$\begin{cases} -\alpha^2\Delta\left(\dfrac{\partial \mathbf{w}}{\partial\tau}\right)+\mathbf{w}+\nabla\pi=\mathbf{u}, \\ \nabla\cdot\mathbf{w}=0, \\ \mathbf{w}(0,\mathbf{x})=\overline{\mathbf{u}}. \end{cases} \tag{1.10}$$

We set

$$H_\tau(\mathbf{u})=\mathbf{w}(\tau,\mathbf{x}).$$

The parameter τ is a non dimensional parameter. We call it "deconvolution parameter". Equation (1.10) is called the "deconvolution equation". The corresponding LES model becomes

$$\begin{cases} \partial_t\mathbf{u}_{\alpha,\tau}+\left(H_\tau(\mathbf{u}_{\alpha,\tau})\cdot\nabla\right)\mathbf{u}_{\alpha,\tau}-\nu\Delta\mathbf{u}_{\alpha,\tau}+\nabla p_{\alpha,\tau}=H_\tau(\mathbf{f}), \\ \nabla\cdot\mathbf{u}_{\alpha,\tau}=0, \\ \mathbf{u}_{\alpha,\tau}(0,\mathbf{x})=H_\tau(\mathbf{u}_0). \end{cases} \tag{1.11}$$

This model appears first in [7] and [6], in the case of the ocean. It also constitutes a part of the PhD thesis of A. -C. Bennis [5], who made very good numerical tests in 2D cases with the software FreeFem++ [23], showing that this model deserves constant numerical investigations in realistic 3D situations, compared with *in situ data*, a work which remains to be done.

The goal of the remainder of this paper is to study in details the deconvolution equation and the related model (1.11) in the 3D periodic case. For pedagogical reasons and for the simplicity, we study the deconvolution equation in the scalar case. Thanks we are in the periodic case, we can express the solution of this equation in term of Fourier's series. The same analysis holds for incompressible 3D fields.

We next show the existence and the uniqueness of a solution $(\mathbf{u}_{\alpha,\tau}, p_{\alpha,\tau})$ to problem (1.11) for α and τ fixed, solution "regular enough". We finish this report by showing that there exists a sequence τ_n which goes to infinity when n goes to infinity, and such that the sequence $(\mathbf{u}_{\alpha,\tau_n}, p_{\alpha,\tau_n})_{n\in\mathbb{N}}$ converges to a dissipative weak solution to the Navier-Stokes Equations when n goes to infinity, always when α is fixed.

The rest of the paper is organized as follows. We start by giving some mathematical tools such as the space functions we are working with, the Helmholtz equation. We next turn to the study of the continuous deconvolution equation. As we already said, we show facts in the scalar case for the sake of simplicity and clarity, so far the generalization to incompressible fields is straightforward. In a last section, we study the model (1.11) and prove the claimed results.

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2 Mathematical tools

2.1 General Background

Let $L \in \mathbb{R}_+^\star$, $\Omega = [0, L]^3 \subset \mathbb{R}^3$. We denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the orthonormal basis of $\mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ the standard point in $\mathbb{R}^3$. Let us first start with some basic definitions.

[II.i] A function $u : \mathbb{R}^3 \to \mathbb{C}$ is said to be Ω-periodic if and only if for all $\mathbf{x} \in \mathbb{R}^3$, for all $(p, q, r) \in \mathbb{Z}^3$ one has $u(\mathbf{x} + L(p\,\mathbf{e}_1 + q\,\mathbf{e}_2 + r\,\mathbf{e}_3)) = u(\mathbf{x})$.

[II.ii] $\mathcal{D}_{per}$ denotes all functions Ω-periodic of class C^∞.

[II.iii] We put $\mathcal{T}_3 = 2\pi\mathbb{Z}^3/L$. Let $\mathbb{T}_3$ be the torus defined by $\mathbb{T}_3 = \big(\mathbb{R}^3/\mathcal{T}_3\big)$.

[II.iv] When $p \in [1, \infty[$, we denote by $L^p(\mathbb{T}_3)$ the space function defined by

$$L^p(\mathbb{T}_3) = \{u : \mathbb{R}^3 \to \mathbb{C},\, u \in L^p_{loc}(\mathbb{R}^3),\, u \text{ is } \Omega-\text{periodic}\},$$

equipped with the norm

$$||u||_{0,p} = \left(\frac{1}{L^3}\int_{\mathbb{T}_3} |u(\mathbf{x})|^p d\mathbf{x}\right)^{\frac{1}{p}}.$$

When $p = 2$, $L^2(\mathbb{T}_3)$ is an Hermitian space with the hermitian product

$$(u, v) = \frac{1}{L^3}\int_{\mathbb{T}_3} u(x)\overline{v}(x)dx. \tag{2.1}$$

[II.v] Let $u \in L^1(\mathbb{T}_3)$. We put $m(u) = \int_\Omega u(\mathbf{x})d\mathbf{x}$.

[II.vi] Let $s \in \mathbb{R}^+$ We denote by $H^s_{per,0}(\mathbb{R}^3)$, the space

$$H^s_{per,0}(\mathbb{R}^3) = \{u : \mathbb{R}^3 \to \mathbb{C},\, u \in H^s_{loc}(\mathbb{R}^3),\, u \text{ is } \Omega-\text{periodic},\, m(u) = 0\}. \tag{2.2}$$

The space $H^s_{per,0}(\mathbb{R}^3)$ is equipped by the induced topology of the classical space $H^s(\mathbb{T}_3)$.

[II.vii] For $\mathbf{k} = (k_1, k_2, k_3) \in \mathcal{T}_3$, we put

$$|\mathbf{k}|^2 = k_1^2 + k_2^2 + k_3^2, \quad |k|_\infty = \sup_i |k_i|, \tag{2.3}$$

$I_n = \{\mathbf{k} \in \mathcal{T}_3;\ |\mathbf{k}|_\infty \leq n\}$.

[II.viii] We say that a Ω-periodic function P is a trigonometric polynomial if there exists $n \in \mathbb{N}$ and coefficients $a_\mathbf{k}$, $\mathbf{k} \in I_n$, and such that $P = \sum_{\mathbf{k}\in I_n} a_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}$. The degree of P is the greatest q such that there is a $\mathbf{k}$ with $|\mathbf{k}|_\infty = q$ and $a_\mathbf{k} \neq 0$.

[II.ix] We denote by V_n the finite dimensional space of all trigonometric polynomial of degree less than n with mean value equal to zero,

$$V_n = \{u = \sum_{\mathbf{k}\in I_n} u_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}},\, u_\mathbf{0} = 0\},$$

and $\mathbb{P}_n$ the orthogonal projection from $L^2(\mathbb{T}_3)$ onto his closed subspace V_n.

[II.x] Let us put $\mathcal{I}_3 = \mathcal{T}_3^\star = (2\pi\mathbb{Z}^3/L) \setminus \{0\}$.

A real number s being given, we consider the space function $\mathbb{H}_s$ defined by

$$\mathbb{H}_s = \left\{ u : \mathbb{R}^3 \to \mathbb{C}, \quad u = \sum_{\mathbf{k}\in\mathcal{T}_3} u_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}, \quad u_\mathbf{0} = 0, \quad \sum_{\mathbf{k}\in\mathcal{T}_3} |\mathbf{k}|^{2s}|u_\mathbf{k}|^2 < \infty \right\}, \tag{2.4}$$

We put

$$||u||_{s,2} = \left(\sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{2s}|u_\mathbf{k}|^2 \right)^{\frac{1}{2}}, \quad (u, v)_s = \sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{2s} u_\mathbf{k} \overline{v}_\mathbf{k}. \tag{2.5}$$

In the formula above, $\overline{v}_\mathbf{k}$ stands for the complex conjugate of $v_\mathbf{k}$. The following can be proved (see in [33])

- For all $s \geq 0$, the space $\mathbb{H}_s$ is an hermitian space, isomorphic to the space $H^s_{per,0}(\mathbb{R}^3)$.
- One always has $(\mathbb{H}_s)' = \mathbb{H}_{-s}$

[II.xi] Let $s \geq 0$ and $\mathbb{H}_s^\mathbb{R}$ be the closed subset of $\mathbb{H}_s$ made of all real valued functions $u \in \mathbb{H}_s$,

$$\mathbb{H}_s^\mathbb{R} = \{u \in \mathbb{H}_s, \quad \forall\mathbf{x} \in \mathbb{T}_3,\ u(\mathbf{x}) = \overline{u(\mathbf{x})}\}. \tag{2.6}$$

2.2 Basic Helmholtz Filtration

Let $\alpha > 0$, $s \geq 0$, $u \in \mathbb{H}_s$ and let $\overline{u} \in \mathbb{H}_{s+2}$ be the unique solution to the equation

$$-\alpha^2 \Delta \overline{u} + \overline{u} = u. \tag{2.7}$$

We are aware that $\overline{u}$ could be mixed up with the complex conjugate of u instead of the solution of the Helmholtz equation (2.7). Unfortunatly, this is also the usual notation used by many authors working on the topic. This is why we decided to keep the notations like that, expecting that no confusion will occur. We also shall denote by A the operator

$$A : \left\{ \begin{array}{l} \mathbb{H}_{s+2} \longrightarrow \mathbb{H}_s, \\ w \longrightarrow -\alpha^2 \Delta w + w. \end{array} \right. \tag{2.8}$$

Therefore, one has

$$\overline{u} = A^{-1} u. \tag{2.9}$$

It is easy checked that when $u = \sum_{\mathbf{k} \in \mathcal{T}_3} u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}$, then

$$\overline{u} = \sum_{\mathbf{k} \in \mathcal{T}_3} \frac{u_{\mathbf{k}}}{1 + \alpha^2 |\mathbf{k}|^2} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \tag{2.10}$$

Formula (2.10) yields easely the estimates

$$||\overline{u}||_{s+2,2} \leq \frac{1}{\alpha^2} ||u||_{s,2}, \qquad ||\overline{u} - u||_{s,2} \leq \alpha ||u||_{s+1,2}. \tag{2.11}$$

We shall sometimes denote $\overline{u}_\alpha$ instead of $\overline{u}$, when we need to recall the dependance on the α parameter.

Theorem 2.1 *Assume $u \in \mathbb{H}_s$. Then the sequence $(\overline{u}_\alpha)_{\alpha > 0}$ converges strongly to u in the space $\mathbb{H}_s$.*

Proof. By definition, one has

$$||\overline{u} - u||_{s,2}^2 = \sum_{\mathbf{k} \in \mathcal{I}_3} \left(\frac{\alpha^2 |\mathbf{k}|^2}{1 + \alpha^2 |\mathbf{k}|^2} \right)^2 |\mathbf{k}|^{2s} |u_{\mathbf{k}}|^2.$$

Let $\varepsilon > 0$. As $u \in \mathbb{H}_s$, there exists N be such that

$$\sum_{\mathbf{k} \in \mathcal{I}_3 \backslash I_N} |\mathbf{k}|^{2s} |u_{\mathbf{k}}|^2 \leq \frac{\varepsilon}{2},$$

and since $\alpha^2 |\mathbf{k}|^2 / (1 + \alpha^2 |\mathbf{k}|^2) \leq 1$,

$$\mathbf{I}_N = \sum_{\mathbf{k} \in \mathcal{I}_3 \backslash I_N} \left(\frac{\alpha^2 |\mathbf{k}|^2}{1 + \alpha^2 |\mathbf{k}|^2} \right)^2 |\mathbf{k}|^{2s} |u_{\mathbf{k}}|^2 \leq \frac{\varepsilon}{2}.$$

On the other hand, because the set I_N is finite,

$$\lim_{\alpha \to 0} \sum_{\mathbf{k} \in I_N^\star} \left(\frac{\alpha^2 |\mathbf{k}|^2}{1 + \alpha^2 |\mathbf{k}|^2} \right)^2 |\mathbf{k}|^{2s} |u_{\mathbf{k}}|^2 = 0.$$

Therefore, there exists $\alpha_0 > 0$ be such that for each $\alpha \in]0, \alpha_0[$ one has

$$\mathbf{J}_N = \sum_{\mathbf{k} \in I_N^\star} \left(\frac{\alpha^2 |\mathbf{k}|^2}{1 + \alpha^2 |\mathbf{k}|^2} \right)^2 |\mathbf{k}|^{2s} |u_\mathbf{k}|^2 \leq \frac{\varepsilon}{2}.$$

As $||\overline{u} - u||_{s,2}^2 = \mathbf{I}_N + \mathbf{J}_N$, then for all $\alpha \in]0, \alpha_0[$, one has $||\overline{u} - u||_{s,2}^2 \leq \varepsilon$ ending the proof like that. ■

3 From discrete to continuous deconvolution operator

3.1 The Van-Cittert Algorithm

Let us consider the operator

$$D_N = \sum_{n=0}^{N} (I - A^{-1})^n.$$

We introduce the operator

$$H_N(u) = D_N(\overline{u}). \tag{3.1}$$

A straightforward calculation yields

$$H_N \left(\sum_{\mathbf{k} \in \mathcal{I}_3} u_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} \right) = \sum_{\mathbf{k} \in \mathcal{I}_3} \left(1 - \left(\frac{\alpha^2 |\mathbf{k}|^2}{1 + \alpha^2 |\mathbf{k}|^2} \right)^{N+1} \right) u_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \tag{3.2}$$

One can prove the following (see in [29]):

- Let $s \in \mathbb{R}$, $u \in \mathbb{H}_s$. Then $H_N(u) \in \mathbb{H}_{s+2}$ and $||H_N(u)||_{s+2,2} \leq C(N, \alpha) ||u||_{s,2}$, where $C(N, \alpha)$ blows up when α goes to zero and/or N goes to infinity. This is due to the fact
$$\left(1 - \left(\frac{\alpha^2 |\mathbf{k}|^2}{1 + \alpha^2 |\mathbf{k}|^2} \right)^{N+1} \right) \approx \frac{N+1}{\alpha^2 |\mathbf{k}|^2} \quad \text{as} \quad |\mathbf{k}|_\infty \to \infty.$$
- The operator H_N maps continuously $\mathbb{H}_s$ into $\mathbb{H}_s$ and $||H_N||_{\mathcal{L}(\mathbb{H}_s)} = 1$.
- Let $u \in \mathbb{H}_s$. Then the sequence $(H_N(u))_{N \in \mathbb{N}}$ converges strongly to u in $\mathbb{H}_s$ when N goes to infinity.

Let us put $w_0 = \overline{u}$, $w_N = H_N(u)$. We now show how one can compute each w_N thanks to the Van-Cittert algorithm (see also in [10]), starting from the definition

$$w_N = \sum_{n=0}^{N} (I - A^{-1})^n \overline{u}. \tag{3.3}$$

When A^{-1} acts on both sides in (3.3), one gets

$$\begin{aligned} A^{-1} w_N &= \sum_{n=0}^{N} A^{-1} (I - A^{-1})^n \overline{u} \\ &= - \sum_{n=0}^{N} (I - A^{-1})^{n+1} \overline{u} + \sum_{n=0}^{N} (I - A^{-1})^n \overline{u} \\ &= - w_{N+1} + \overline{u} + w_N. \end{aligned}$$

In summary, the Van-Cittert algorithm is the following:

$$\begin{cases} w_0 = \overline{u}, \\ w_{N+1} = w_N + (\overline{u} - A^{-1}w_N). \end{cases} \tag{3.4}$$

3.2 The continuous deconvolution equation

Performing A in both sides of (3.4) yields

$$Aw_{N+1} - Aw_N + w_N = A\overline{u} = u.$$

Using the definition of A, $Aw = -\alpha^2\Delta w + w$, one deduces the equality

$$-\alpha^2(\Delta w_{N+1} - \Delta w_N) + w_{N+1} = u. \tag{3.5}$$

Here is the analogy. Let $\delta\tau > 0$ be a real number and consider the equation

$$-\alpha^2\left(\frac{\Delta w_{N+1} - \Delta w_N}{\delta\tau}\right) + w_{N+1} = u. \tag{3.6}$$

We notice the following facts

- Equation (3.5) is the special cas of equation (3.6) when $\delta\tau = 1$,
- equation (3.6) is a finite difference scheme that corresponds to the equation satisfied by the variable $w = w(\tau, \mathbf{x})$, $\tau > 0$,

$$\begin{cases} -\alpha^2\Delta\left(\dfrac{\partial w}{\partial\tau}\right) + w = u, \\ w(0, \mathbf{x}) = \overline{u}(\mathbf{x}), \end{cases} \tag{3.7}$$

with the zero mean condition $m(w) = 0$ so far u also satisfies $m(u) = 0$ as well as $m(\overline{u}) = 0$. We call equation (3.7) *the continuous deconvolution equation*. The parameter τ is dimensionless. We call it *the deconvolution parameter*.

Before doing anything, we first perform the change of variable $v(\tau, \mathbf{x}) = w(\tau, \mathbf{x}) - u(\mathbf{x})$. The variable v is solution of the equation

$$\begin{cases} -\alpha^2\Delta\left(\dfrac{\partial v}{\partial\tau}\right) + v = 0, \\ v(0, \mathbf{x}) = \overline{u}(\mathbf{x}) - u(\mathbf{x}), \end{cases} \tag{3.8}$$

with periodic boundary conditions. We also keep in mind that we impose all variables to have a zero mean value on a cell, a fact we shall not recall every time.

In the rest of this section, we shall study with accuracy the solution of problem (3.8) and thus problem (3.7) that we shall solve completly. To do this, we shall express the solution in terms of Fourier Series.

We search for a solution $v(\tau, \mathbf{x})$ as

$$v(\tau, \mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{I}_3} v_{\mathbf{k}}(\tau)e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}, \tag{3.9}$$

with initial condition, with obvious notations,

$$v_{\mathbf{k}}(0) = -\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}u_{\mathbf{k}} = (\overline{u}-u)_{\mathbf{k}}. \tag{3.10}$$

We deduce that each mode at frequency $\mathbf{k}$ satisfies the differential equation

$$\begin{cases} \alpha^2|\mathbf{k}|^2\dfrac{dv_{\mathbf{k}}}{d\tau} + v_{\mathbf{k}} = 0, \\ v_{\mathbf{k}}(0) = (\overline{u}-u)_{\mathbf{k}}. \end{cases} \tag{3.11}$$

We deduce that

$$v_{\mathbf{k}}(\tau) = (\overline{u}-u)_{\mathbf{k}} e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2}}. \tag{3.12}$$

Therefore, the general solution to problem (3.7) is

$$w(\tau,\mathbf{x}) = u(\mathbf{x}) - \sum_{\mathbf{k}\in\mathcal{I}_3}\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)u_{\mathbf{k}}e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2}+\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}, \tag{3.13}$$

where

$$u = \sum_{\mathbf{k}\in\mathcal{I}_3} u_{\mathbf{k}}e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}.$$

3.3 Various properties of the deconvolution equation

We now prove general properties satisfied by the solution of the deconvolution equation, using either Equation (3.7) itself, either formula (3.13).

In the following, we put

$$H_\tau(u) = H_\tau(u)(\tau,\mathbf{x}) = w(\tau,\mathbf{x}), \tag{3.14}$$

where $v(\tau,\mathbf{x})$ is the solution of Equation (3.7).

Lemma 3.1 *Let $s\in\mathbb{R}$, $u\in\mathbb{H}_s$. Then for all $\tau\geq 0$, $H_\tau(u)\in\mathbb{H}_s$ and*

$$||H_\tau(u)||_{s,2} \leq 2||u||_{s,2}. \tag{3.15}$$

Proof. Since one has for every $\tau\geq 0$ and every $\mathbf{k}\in\mathcal{I}_3$

$$0 \leq \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2}} \leq 1,$$

the claimed result is a direct consequence of (3.13). ■

Lemma 3.2 *Let $\alpha>0$ be fixed, $s\in\mathbb{R}$ and $u\in\mathbb{H}_s$. Then $(H_\tau(u))_{\tau>0}$ converges strongly to u in $\mathbb{H}_s$, when $\tau\to\infty$.*

Proof. One has

$$u - H_\tau(u) = \sum_{\mathbf{k}\in\mathcal{I}_3}\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)u_{\mathbf{k}}e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2}+\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}},$$

which yields

$$||u-H_\tau(u)||_{s,2}^2 = \sum_{\mathbf{k}\in\mathcal{I}_3}|\mathbf{k}|^{2s}\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^2|u_{\mathbf{k}}|^2e^{-\frac{2\tau}{\alpha^2|\mathbf{k}|^2}} \leq e^{-\frac{2\tau}{\alpha^2}}||u||_{s,2}.$$

Therefore, $\lim_{\tau\to 0}||u-H_\tau(u)||_{s,2}=0$, and the proof is finished. ■

Lemma 3.3 *Let $\alpha > 0$ and $\tau \geq 0$ be fixed, $s \in \mathbb{R}$ and $u \in \mathbb{H}_s$. Then $H_\tau(u) \in \mathbb{H}_{s+2}$ and one has*

$$||H_\tau(u)||_{s+2,2} \leq \frac{C(L)(1+\tau)}{\alpha^2}||u||_{s,2}, \tag{3.16}$$

where $C(L)$ is a constant which only depends on the box size L.

Proof. Let us write Equation (3.7) under the form

$$-\alpha^2 \Delta \frac{\partial H_\tau(u)}{\partial \tau} = u - H_\tau(u).$$

Since we already know that $u - H_\tau(u) \in \mathbb{H}_s$, we deduce from standard elliptic theory that

$$\frac{\partial H_\tau(u)}{\partial \tau} \in \mathbb{H}_{s+2}, \qquad \left|\left|\frac{\partial H_\tau(u)}{\partial \tau}\right|\right|_{s+2,2} \leq \frac{C(L)}{\alpha^2}||u - H_\tau(u)||_{s,2} \leq \frac{3C(L)}{\alpha^2}||u||_{s,2} \tag{3.17}$$

We now write

$$H_\tau(u) = \overline{u} + \int_0^\tau \frac{\partial H_{\tau'}(u)}{\partial \tau'} d\tau',$$

The result is a consequence of (3.17) combined with (2.11).. ■

3.4 An additional convergence result

We finish this section devoted to the Continuous deconvolution equation by a convergence result. Indeed, when one studies existence result for some variational problem such as the Navier-Stokes Equations and related, we usually must prove some compactness or continuity result. In all cases, there is one moment when one faces the question of studying a sequence $(u_n)_{n\in\mathbb{N}}$ of approximated solutions which converges to some u in a certain meaning, and one must identify the equation satisfied by u.

The problem we are working with uses the operator $u \to H_\tau(u)$. Among many compactness results that we potentially can prove, we shall restric ourself to the one we shall use in the next section.

As we are looking at evolutions problems. Therefore, the fonctions (and later the fields) we consider are time dependent, that means $u = u(t, \mathbf{x})$ for $\mathbf{x} \in \mathbb{T}_3$ and t belonging to a time interval $[0, T]$. Let $s \geq 0$; the space $L^2([0, T], \mathbb{H}_s)$ can easely be described to be the set of all fonction $u : \mathbb{T}_3 \to \mathbb{C}$ that can be decompose as Fourier series (see in [33])

$$u = \sum_{\mathbf{k}\in\mathcal{I}_3} u_\mathbf{k}(t) e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} \quad \text{be such that} \quad ||u||^2_{L^2([0,T],\mathbb{H}_s)} = \sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{2s} \int_0^T |u_\mathbf{k}(t)|^2 dt < \infty.$$

Lemma 3.4 *Let $\alpha > 0$ and $\tau > 0$ be fixed. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2([0, T], \mathbb{H}_s)$ which converges strongly to u in the space $L^2([0, T], \mathbb{H}_s)$. Therefore, $(H_\tau(u_n))_{n\in\mathbb{N}}$ converges to $H_\tau(u)$ strongly in $L^2([0, T], \mathbb{H}_s)$ when $n \to \infty$.*

Proof. We use formula (3.13) to estimate $H_\tau(u_n) - H_\tau(u)$. Therefore one with obvious notations

$$H_\tau(u_n) - H_\tau(u) = u_n - u + \sum_{\mathbf{k}\in\mathcal{I}_3} \left(\frac{\alpha^2|\mathbf{k}|^2}{1 + \alpha^2|\mathbf{k}|^2}\right)(u_{\mathbf{k},n} - u_\mathbf{k}) e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2} + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \tag{3.18}$$

This yields the estimate

$$
(3.19) \qquad ||H_\tau(u_n) - H_\tau(u)||_{L^2([0,T],\mathbb{H}_s)} \leq 2||u_n - u||_{L^2([0,T],\mathbb{H}_s)},
$$

because

$$
\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right) e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2}} \leq 1.
$$

The result is then a direct consequence of (3.19). ■

4 Application to the Navier-Stokes Equations

4.1 Dissipative solutions to the Navier-Stokes Equations

Let us start by writting again the Navier-Stokes Equations:

$$
(4.1) \qquad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \\ m(\mathbf{u}) = \mathbf{0}, \quad m(p) = 0, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0 \end{cases}
$$

Here, $\mathbf{u}$ stands for the velocity and p for the pressure, and they are both the unknowns. Since the field are real valued and periodic, one can consider them as fields from $\mathbb{T}_3$ towards $\mathbb{R}^3$ for the velocity, from $\mathbb{T}_3$ to $\mathbb{R}$ for the pressure. The second hand side $\mathbf{f}$ is a data of the problem as well as the kinematic viscosity $\nu > 0$. Recall that

$$
m(\mathbf{u}) = \int_\Omega \mathbf{u}(t,\mathbf{x})d\mathbf{x}, \quad m(p) = \int_\Omega p(t,\mathbf{x})d\mathbf{x}.
$$

Recall that for fields satisfying $\nabla\cdot\mathbf{u} = 0$, one always has $(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot(\mathbf{u}\otimes\mathbf{u})$. We shall use sometimes this identity when we need it, without special warnings. Let us recall some facts and notations.

[IV.i] The Reynolds number Re is defined as $Re = UL/\nu$, where L is the box size, U is a typical velocity scale, for instance

$$
U = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\frac{1}{L^3}\int_\Omega |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x}\right)^{1/2} dt,
$$

where "lim" stands for the generalized Banach Limit (see in [14], [15] and [18]).

[IV.ii] Let $s \geq 0$. We set

$$
\mathbf{IH}_s = \{\mathbf{u} \in (\mathbb{H}_s^{\mathbb{R}})^3, \quad \nabla\cdot\mathbf{u} = 0\}.
$$

The space $\mathbf{IH}_s$ is a close subset of $(\mathbb{H}_s)^3$ and is made of real valued vector fields, see [II.xi]), equipped with the hermitian product, for $\mathbf{u} = (u_1,u_2,u_3)$, $\mathbf{v} = (v_1,v_2,v_3)$, $(\mathbf{u},\mathbf{v})_s = (u_1,v_1)_s + (u_1,v_1)_s + (u_1,v_1)_s$ (see (2.5)). We still denote $||\mathbf{u}||_{s,2} = (||u_1||_{s,2}^2 + ||u_2||_{s,2}^2 + ||u_3||_{s,2}^2)^{1/2}$.

[IV.iii] We put $W^{-1,p'}(\mathbb{T}_3) = (W^{1,p}(\mathbb{T}_3))'$ for $1/p + 1/p' = 1$, $p \geq 1$. We also put $\mathbf{IH}_{-s} = (\mathbf{IH}_s)'$ for $s \geq 0$.

[IV.iv] The usual case we keep in mind for the data in the Navier-Stokes Equations, is the case $\mathbf{u}_0 \in \mathbf{IH}_0$ and $\mathbf{f} \in L^2([0,T],(H^1(\mathbb{T}_3)^3)')$, noting that $(H^1(\mathbb{T}_3)^3)' \subset \mathbf{IH}_{-1}$.

Definition 4.1 *We say that* $(\mathbf{u}, p)$ *is a dissipative solution to the Navier-Stokes Equations* (4.1) *on the time interval* $[0,T]$ *if:*

1) The following holds:

$$\mathbf{u} \in L^2([0,T], \mathbb{H}_1) \cap L^\infty([0,T], \mathbb{H}_0), \tag{4.2}$$

$$p \in L^{5/3}([0,T] \times \mathbb{T}_3), \tag{4.3}$$

$$\partial_t \mathbf{u} \in L^{5/3}([0,T], (W^{-1,5/3}(\mathbb{T}_3))^3) \tag{4.4}$$

2) $\lim_{t\to 0} ||\mathbf{u}(t,\cdot) - \mathbf{u}_0(\cdot)||_{0,2} = 0,$

3) $\forall\, \mathbf{v} \in L^{5/2}([0,T], W^{1,5/2}(\mathbb{T}_3)^3)$ *one has for all* $t \in [0,T]$,

$$\left\{ \begin{array}{l} (\partial_t \mathbf{u}, \mathbf{v}) - \displaystyle\int_0^t \int_{\mathbb{T}_3} \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} + \nu \int_0^t \int_{\mathbb{T}_3} \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} dt' - \\ \qquad\qquad\qquad \displaystyle\int_0^t \int_{\mathbb{T}_3} p \, (\nabla \cdot \mathbf{v}) = \int_0^t (\mathbf{f}, \mathbf{v}), \end{array} \right. \tag{4.5}$$

where $(\cdot,\cdot)$ *stands here for the duality product between* $W^{1,5/2}(\mathbb{T}_3)^3$ *and* $W^{-1,5/3}(\mathbb{T}_3)^3$, *noting that* $(H^1(\mathbb{T}_3)^3)' \subset W^{-1,5/3}(\mathbb{T}_3)^3$.

4) The energy inequality holds, for all $t \in [0,T]$,

$$\frac{1}{2} \int_{\mathbb{T}_3} |\mathbf{u}(t,\mathbf{x})|^2 + \nu \int_0^t \int_{\mathbb{T}_3} |\nabla \mathbf{u}(t', \mathbf{x})|^2 d\mathbf{x} dt \leq \frac{1}{2} \int_{\mathbb{T}_3} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + \int_0^t (\mathbf{f}, \mathbf{u}) dt', \tag{4.6}$$

where $(\cdot,\cdot)$ *stands here for the duality product between* $\mathbb{H}_1$ *and* $\mathbb{H}_{-1}$, *noting that* $(H^1(\mathbb{T}_3)^3)' \subset \mathbb{H}_{-1}$.

Remark 4.1 *This definition makes sense as soon as* $\mathbf{u}_0 \in \mathbb{H}_0$ *and* $\mathbf{f} \in L^2([0,T], (H^1(\mathbb{T}_3)^3)')$, *giving a sense to the integrals in the right hand sides of* (4.5)*and* (4.6)*. Moreover, by interpolation we see that the regularity conditions in point 1) in the definition above, make sure that* $\mathbf{u} \in L^{10/3}([0,T] \times \mathbb{T}_3)^3$*. When combining this fact with the regularity for* $\partial_t \mathbf{u}$*, we see that all integrals in the right hand side of* (4.5) *are well defined.*

Remark 4.2 *The condition imposed on the pressure,* $p \in L^{5/3}([0,T] \times \mathbb{T}_3)$*, is directly satisfied when we already have the estimate* $\mathbf{u} \in L^\infty([0,T], \mathbb{H}_0) \cap L^2([0,T], \mathbb{H}_1)$*. Indeed, when one takes the divergence of the momentum equation formally using* $\nabla \cdot \mathbf{u} = 0$ *(included in the definition of the space functions* $\mathbb{H}_s$ *in [IV.ii]) said, we get the following equation for the pressure*

$$-\Delta p = \nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u})). \tag{4.7}$$

Now, by using Hölder's inequality, it easy checked that $\mathbf{u} \in L^\infty([0,T], \mathbb{H}_0) \cap L^2([0,T], \mathbb{H}_1)$ *implies* $\mathbf{u} \in L^{10/3}([0,T] \times \mathbb{T}_3)^3$*. Therefore,*

$$\nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u})) \in L^{5/3}([0,T], W^{-2,5/3})$$

and by standard elliptic theory, $p \in L^{5/3}([0,T] \times \mathbb{T}_3)$*.*

Let us recall the result due to J. Leray [30].

Theorem 4.1 *Assume that* $\mathbf{u}_0 \in \mathbb{H}_0$ *and* $\mathbf{f} \in L^2([0,T], (H^1(\mathbb{T}_3)^3)')$*. Then the Navier-Stokes Equations* (4.1) *have a dissipative solution.*

We still do not know if

- this solution is unique,
- if it develops singularities in finite time, even if $\mathbf{u}_0$ and $\mathbf{f}$ are smooth.

4.2 The deconvolution model

The deconvolution equation for incompressible fields takes the form

$$
\text{(4.8)} \qquad \begin{cases} -\alpha^2 \Delta \left(\dfrac{\partial \mathbf{w}}{\partial \tau} \right) + \mathbf{w} + \nabla \pi = \mathbf{u}, \\ \nabla \cdot \mathbf{w} = 0, \\ m(\mathbf{w}) = 0, \quad m(\pi) = 0, \\ \mathbf{w}(0, \mathbf{x}) = \overline{\mathbf{u}}, \end{cases}
$$

where $\mathbf{u}$ is such that $m(\mathbf{u}) = \nabla \cdot \mathbf{u} = 0$, and $\overline{\mathbf{u}}$ is the solution of the Stokes problem

$$
\text{(4.9)} \qquad \begin{cases} A\overline{\mathbf{u}} = -\alpha^2 \Delta \overline{\mathbf{u}} + \overline{\mathbf{u}} + \nabla \xi = \mathbf{u}, \\ \nabla \cdot \overline{\mathbf{u}} = 0, \\ m(\overline{\mathbf{u}}) = 0, \quad m(\xi) = 0. \end{cases}
$$

In the equations above, π and ξ are necessary Lagrange multipliers, involved because of the zero divergence constraint. We set in the following

$$
H_\tau(\mathbf{u})(t, \mathbf{x}) = w(\tau, t, \mathbf{x}),
$$

where $w(\tau, t, \mathbf{x})$ is the solution at the deconvolution parameter τ for a fixed time t. Of course $H_0(\mathbf{u}) = \overline{\mathbf{u}}$. A straightforward adaptation of the results of section 3.3 combined with classical results related to the Stokes problem (see [22]) yield Lagrange multpliers π and ξ are both equal to zero, and that the following facts are satisfied.

(iv.i) Let $\mathbf{u} \in L^\infty([0, T], \mathbb{H}_0)$. Then for all $\tau \geq 0$, $H_\tau(\mathbf{u}) \in L^\infty([0, T], \mathbb{H}_2)$ and one has

$$
\text{(4.10)} \qquad \sup_{t \geq 0} ||H_\tau(\mathbf{u})||_{2,2} \leq C \sup_{t \geq 0} ||\mathbf{u}||_{0,2},
$$

where the constant C depends on τ and blows up when τ goes to infinity. Thanks to Sobolev injection Theorem, we deduce from (4.10) that in addition

$$
\text{(4.11)} \qquad H_\tau(\mathbf{u}) \in L^\infty([0, T] \times \mathbb{T}_3)^3, \quad ||H_\tau(\mathbf{u})||_{L^\infty([0,T] \times \mathbb{T}_3)^3} \leq C(\tau, \alpha, \sup_{t \geq 0} ||\mathbf{u}||_{0,2}).
$$

(iv.ii) Let $\mathbf{u} \in L^2([0, T], \mathbb{H}_1)$. Then the following estimate holds:

$$
\text{(4.12)} \qquad \int_0^T ||\mathbf{u}(t, \cdot) - H_\tau(\mathbf{u})(t, \cdot)||_{1,2}^2 dt \leq e^{-\frac{2\tau}{\alpha^2}} \int_0^T ||\mathbf{u}(t, \cdot)||_{1,2}^2 dt.
$$

In particular, the sequence $(H_\tau(\mathbf{u}))_{\tau > 0}$ goes strongly to $\mathbf{u}$ in the space $L^2([0, T], \mathbb{H}_1)$ when τ goes to infinity and $\alpha > 0$ is fixed.

Let us consider the problem

$$
\text{(4.13)} \qquad \begin{cases} \partial_t \mathbf{u}_{\alpha,\tau} + (H_\tau(\mathbf{u}_{\alpha,\tau}) \cdot \nabla)\, \mathbf{u}_{\alpha,\tau} - \nu \Delta \mathbf{u}_{\alpha,\tau} + \nabla p_{\alpha,\tau} = H_\tau(\mathbf{f}), \\ \nabla \cdot \mathbf{u}_{\alpha,\tau} = 0, \\ m(\mathbf{u}_{\alpha,\tau}) = 0, \quad m(p_{\alpha,\tau}) = 0, \\ \mathbf{u}_{\alpha,\tau}(0, \mathbf{x}) = H_\tau(\mathbf{u}_0), \end{cases}
$$

with periodic boundary conditions.

Definition 4.2 *We say that* $(\mathbf{u}_{\alpha,\tau}, p_{\alpha,\tau})$ *is a weak solution to Problem* (4.13) *if the following properties are satisfied:*

1) $\mathbf{u}_{\alpha,\tau} \in L^\infty([0,T], \mathbb{H}_1) \cap L^2([0,T], \mathbb{H}_1)$, $\partial_t \mathbf{u}_{\alpha,\tau} \in (L^2([0,T] \times \mathbb{T}_3))^3$, $p \in L^2([0,T], \mathbb{H}_1)$,

2) $\lim_{t \to 0} ||\mathbf{u}_{\alpha,\tau}(t,\cdot) - H_\tau(\mathbf{u}_0)||_{0,2} = 0$,

3) $\forall\, \mathbf{v} \in L^2([0,T], (H^1(\mathbb{T}_3)^3))$,

$$
(4.14) \quad \left\{ \begin{array}{l} \displaystyle\int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{u}_{\alpha,\tau} \cdot \mathbf{v} + \int_0^T \int_{\mathbb{T}_3} (H_\tau(\mathbf{u}_{\alpha,\tau}) \cdot \nabla)\, \mathbf{u}_{\alpha,\tau} \cdot \mathbf{v} + \nu \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{u}_{\alpha,\tau} : \nabla \mathbf{v} + \\ \displaystyle\qquad\qquad \int_0^T \int_{\mathbb{T}_3} \nabla p \cdot \mathbf{v} = \int_0^T \int_{\mathbb{T}_3} H_\tau(\mathbf{f}) \cdot \mathbf{v}. \end{array} \right.
$$

4) The following energy balance holds for all $t \in [0,T]$,

$$
(4.15) \quad \left\{ \begin{array}{l} \displaystyle\frac{1}{2} \int_{\mathbb{T}_3} |\mathbf{u}_{\alpha,\tau}(t,\mathbf{x})|^2 d\mathbf{x} + \nu \int_0^t \int_{\mathbb{T}_3} |\nabla \mathbf{u}_{\alpha,\tau}(t',\mathbf{x})|^2 d\mathbf{x} dt' = \\ \displaystyle\qquad \frac{1}{2} \int_{\mathbb{T}_3} |H_\tau(\mathbf{u}_0)(\mathbf{x})|^2 d\mathbf{x} + \int_0^t \int_{\mathbb{T}_3} H_\tau(\mathbf{f}) \cdot \mathbf{u}_{\alpha,\tau} d\mathbf{x} dt'. \end{array} \right.
$$

We now prove the two following results.

Theorem 4.2 *Assume that* $\mathbf{u}_0 \in \mathbb{H}_0$ *and* $\mathbf{f} \in L^2([0,T], (H^1(\mathbb{T}_3)^3)')$. *Then Problem* (4.13) *admits a unique weak solution* $(\mathbf{u}_{\alpha,\tau}, p_{\alpha,\tau})$.

Theorem 4.3 *Assume that* $\mathbf{u}_0 \in \mathbb{H}_0$ *and* $\mathbf{f} \in L^2([0,T], (H^1(\mathbb{T}_3)^3)')$. *Then there exists a sequence* $(\tau_n)_{n\in\mathbb{N}}$ *which goes to infinity when* n *goes to infinity and such that the sequence* $(\mathbf{u}_{\alpha,\tau_n}, p_{\alpha,\tau_n})_{n\in\mathbb{N}}$ *goes to a dissipative weak solution of the Navier-Stokes Equations.*

Proof of Theorem 4.2. A complete proof of Theorem 4.2 would use the Galerkin method. We construct approximations as solutions of variational problems set on the finite dimensional space V_n, thanks to the Cauchy-Lipchitz Theorem. We next derive estimates to finally pass to the limit. To make this report easy and not too heavy, we bypass the construction of approximations on finite dimensional space, a procedure we often already have done for close models (see for instance in [29]). The general Galerkin method is well explained in the famous book of J. -L. Lions published in 1969 [34]. Therefore, we concentrate our efforts on the two main points making the result true:

- A priori estimates,
- The compactness property and how to pass to the limit in the equations

• *A priori estimates*. For the simplicity, we write $(\mathbf{u}, p)$ instead of $(\mathbf{u}_{\alpha,\tau}, p_{\alpha,\tau})$. We perform computations assuming that $(\mathbf{u}, p)$ has enough regularity to validate the integrations by parts we do. We also keep in mind that the boundary terms compensate each other in the integrations by parts, thanks to the periodicity. Therefore no boundary terms occur in these computations.

As usual, we take $\mathbf{u}$ as test function in (4.13) and we integrate by parts on $\mathbb{T}_3$ and on the time interval $[0,t]$ for some $t \in [0,T]$, using $\nabla \cdot \mathbf{u} = 0$ as well as $\nabla \cdot (H_\tau(\mathbf{u})) = 0$. We get in particular

$$
\int_{\mathbb{T}_3} (H_\tau(\mathbf{u}) \cdot \nabla)\, \mathbf{u} \cdot \mathbf{u} = 0,
$$

and therefore

$$\frac{1}{2}\int_{\{t\}\times\mathbb{T}_3}|\mathbf{u}|^2+\nu\int_{[0,t]\times\mathbb{T}_3}|\nabla\mathbf{u}|^2=\frac{1}{2}\int_{\mathbb{T}_3}|H_\tau(\mathbf{u}_0)|^2+\int_{[0,t]\times\mathbb{T}_3}H_\tau(\mathbf{f})\cdot\mathbf{u}. \tag{4.16}$$

As $\mathbf{u}_0\in\mathbb{H}_0$, $H_\tau(\mathbf{u}_0)\in\mathbb{H}_2$, and recall that $||H_\tau(\mathbf{u}_0)||_{0,2}\le 2||\mathbf{u}_0||_{0,2}$. Similarly,

$$\left|\int_{[0,t]\times\mathbb{T}_3}H_\tau(\mathbf{f})\cdot\mathbf{u}\right|\le C||\mathbf{f}||_{-1,2}||\mathbf{u}||_{1,2},$$

where again C do not depend on τ and α. Here and in the remainder, we still denote by $||\cdot||_{-1,2}$ the norm on $(H^1(\mathbb{T}_3)^3)'$. Therefore, (4.16) yields

$$\sup_{t\in[0,T]}\int_{\{t\}\times\mathbb{T}_3}|\mathbf{u}|^2\le C(||\mathbf{u}_0||_{0,2},||\mathbf{f}||_{-1,2}), \tag{4.17}$$

$$\int_{[0,t]\times\mathbb{T}_3}|\nabla\mathbf{u}|^2\le C(||\mathbf{u}_0||_{0,2},||\mathbf{f}||_{-1,2},\nu). \tag{4.18}$$

Next, we use fact (iv.i) ($H_\tau(\mathbf{u})\in L^\infty([0,T]\times\mathbb{T}_3)^3$) and estimate (4.11) together with (4.17). This yields in particular

$$\mathbb{A}=(H_\tau(\mathbf{u})\cdot\nabla)\mathbf{u}\in L^2([0,T]\times\mathbb{T}_3)^3,\quad ||\mathbb{A}||_{L^2([0,T]\times\mathbb{T}_3)^3}\le C(\tau,\alpha,||\mathbf{u}_0||_{0,2},||\mathbf{f}||_{-1,2}). \tag{4.19}$$

Let us now take $\partial_t\mathbf{u}$ as test function in equation (4.13) and we integrate on $[0,t]\times\mathbb{T}_3$, using $\nabla\cdot(\partial_t\mathbf{u})=0$. Therefore we get

$$\left\{\begin{array}{l}\displaystyle\int_{[0,t]\times\mathbb{T}_3}|\partial_t\mathbf{u}|^2+\frac{1}{2}\int_{\{t\}\times\mathbb{T}_3}|\nabla\mathbf{u}|^2=\\ \displaystyle\frac{1}{2}\int_{\mathbb{T}_3}|\nabla H_\tau(\mathbf{u}_0)|^2+\int_{[0,t]\times\mathbb{T}_3}\mathbb{A}\cdot\partial_t\mathbf{u}+\int_{[0,t]\times\mathbb{T}_3}H_\tau(\mathbf{f})\cdot\partial_t\mathbf{u}\end{array}\right. \tag{4.20}$$

Since $H_\tau(\mathbf{u}_0)\in\mathbb{H}_2$ and $H_\tau(\mathbf{f})\in L^2([0,T],H^1(\mathbb{T}_3)^3)$, using (4.19) combined with Cauchy-Schwarz and Young inequalities, we deduce from (4.20)

$$\int_{[0,t]\times\mathbb{T}_3}|\partial_t\mathbf{u}|^2\le C(\tau,\alpha,||\mathbf{u}_0||_{0,2},||\mathbf{f}||_{-1,2}), \tag{4.21}$$

$$\sup_{t\in[0,T]}\int_{\{t\}\times\mathbb{T}_3}|\nabla\mathbf{u}|^2\le C(\tau,\alpha,||\mathbf{u}_0||_{0,2},||\mathbf{f}||_{-1,2}). \tag{4.22}$$

In other words $\partial_t\mathbf{u}\in L^2([0,T]\times\mathbb{T}_3)^3$ and $\mathbf{u}\in L^\infty([0,T],\mathbb{H}_1)$. In fact, one easely verifies that $\partial_t\mathbf{u}\in L^2([0,T],\mathbb{H}_0)$.

We now get a bound for $\mathbf{u}$ in the space $L^2([0,T],\mathbb{H}_2)$. For it, let us consider a fixed $t\in[0,T]$ and let us write the Navier-Stokes equations (4.13) under the form of a Stokes Problem

$$\left\{\begin{array}{l}-\nu\Delta\mathbf{u}+\nabla p=H_\tau(\mathbf{f})-\mathbb{A}-\partial_t\mathbf{u},\\ \nabla\cdot\mathbf{u}=0,\\ m(\mathbf{u})=0,\quad m(p)=0.\end{array}\right. \tag{4.23}$$

Classical facts on the Stokes Problem yield the estimate

$$\begin{array}{r}||\mathbf{u}||^2_{\mathbb{H}_2}+||p||^2_{H^1(\mathbb{T}_3)}\le C_1(\nu)||H_\tau(\mathbf{f})-\mathbb{A}-\partial_t\mathbf{u}||^2_{L^2(\mathbb{T}_3)}\le\\ C_2(\nu)(||H_\tau(\mathbf{f})||^2_{L^2(\mathbb{T}_3)}+||\mathbb{A}||^2_{L^2(\mathbb{T}_3)}+||\partial_t\mathbf{u}||^2_{L^2(\mathbb{T}_3)})\end{array} \tag{4.24}$$

We now quietly integrate (4.24) with respect to time. We get

$$(4.25)\qquad ||\mathbf{u}||_{L^2([0,T],\mathbb{H}_2)} + ||p||_{L^2([0,T],H^1(\mathbb{T}_3))} \le C(\nu,\tau,\alpha,||\mathbf{u}_0||_{0,2},||\mathbf{f}||_{-1,2}),$$

where we have used the regularizing effect of H_τ and estimates (4.19) and (4.21).

In summary, we get

[IV.v] $\mathbf{u}$ is in $L^2([0,T],\mathbb{H}_1)\cap L^\infty([0,T],\mathbb{H}_0)$ and therefore $p\in L^{5/3}([0,T],\mathbb{L}_{5/3})$ and $\partial_t\mathbf{u}\in L^{5/3}([0,T],W^{-1,5/3}(\mathbb{T}_3)^3)$. *The bounds only depend on the data ν, $||\mathbf{u}_0||_{0,2}$ and $||\mathbf{f}||_{-1,2}$*

[IV.vi] $\mathbf{u}\in L^2([0,T],\mathbb{H}_2)\cap L^\infty([0,T],\mathbb{H}_1)$ and $p\in L^2([0,T],H^1(\mathbb{T}_3))$. The bounds depend on the data ν, $||\mathbf{u}_0||_{0,2}$ and $||\mathbf{f}||_{-1,2}$ as well as the deconvolution parameter τ and the filtration parameter α. In particular these bounds blow up when τ goes to infinity and/or α goes to zero.

[IV.vii] $\partial_t\mathbf{u}\in L^2([0,T],\mathbb{H}_0)$. The bounds depend on the data ν, $||\mathbf{u}_0||_{0,2}$ and $||\mathbf{f}||_{-1,2}$ as well as the deconvolution parameter τ and the filtration parameter α.

• *Compactness property.* Let us now consider a sequence $(u_n,p_n)_{n\in\mathbb{N}}$ of "smooth" solutions to problem (4.13). We aim to show that we can extract from this sequence a subsequence which converges in a certain meaning to a solution of problem (4.13), when n goes to infinity.

Fact [IV.vi] makes sure that we can extract a subsequence, still denoted $(u_n,p_n)_{n\in\mathbb{N}}$, be such that

$$(4.26)\qquad \begin{aligned} \mathbf{u}_n &\longrightarrow \mathbf{u} \quad \text{weakly in } L^2([0,T],\mathbb{H}_2),\\ \mathbf{u}_n &\longrightarrow \mathbf{u} \quad \text{weakly-star in } L^\infty([0,T],\mathbb{H}_1),\\ p_n &\longrightarrow p \quad \text{weakly in } L^2([0,T],H^1(\mathbb{T}_3)).\end{aligned}$$

Let us now find a strong compactness property. We have the following

$$\mathbb{H}_2\subset\mathbb{H}_1\subset\mathbb{H}_0,$$

the injections being continous, compact and dense. We know that $(\partial_t\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T],\mathbb{H}_0)$ while $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T],\mathbb{H}_2)$ We deduce from Aubin-Lions Lemma (see in [34]) that

$$(4.27)\qquad \mathbf{u}_n\longrightarrow\mathbf{u}\quad\text{strongly in } L^2([0,T],\mathbb{H}_1).$$

Finally, it is easy checked that we can extract an other subsequence such that

$$(4.28)\qquad \partial_t\mathbf{u}_n\longrightarrow\partial_t\mathbf{u}\quad\text{weakly in } L^2([0,T]\times\mathbb{T}_3)^3.$$

Notice that the limit (u,p) satisfies facts [IV.v], [IV.vi] and [IV.vii].

It remains to show that (u,p) is a solution to problem (4.13). Let us start with the initial data, writting

$$\mathbf{u}_n(t) = H_\tau(\mathbf{u}_0) + \int_0^t \partial_t\mathbf{u}_n dt'.$$

It is easy to pass to limit here in $L^2([0,T]\times\mathbb{T}_3)^3$, to get for free

$$\mathbf{u}(t) = H_\tau(\mathbf{u}_0) + \int_0^t \partial_t\mathbf{u}\, dt',$$

also telling us that $\mathbf{u} \in C^0([0,T], \mathbb{H}_0)$ and that $\mathbf{u}(0,\mathbf{x}) = H_\tau(\mathbf{u}_0(\mathbf{x}))$. We have in fact much better, since $\mathbf{u} \in C^0([0,T], \mathbb{H}_1)$. The proof is left to the reader.

Let us now pass to the limit in the momentum equation. Let $\mathbf{v} \in L^2([0,T], \mathbb{H}_1)$ be given as test vector field. One obvioulsy has when n goes to infinity,

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{u}_n \cdot \mathbf{v} \longrightarrow \int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{u} \cdot \mathbf{v}, \\
&\int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{u}_n : \nabla \mathbf{v} \longrightarrow \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{u} : \nabla \mathbf{v}, \\
&\int_0^T \int_{\mathbb{T}_3} p_n (\nabla \cdot \mathbf{u}) \longrightarrow \int_0^T \int_{\mathbb{T}_3} p\, (\nabla \cdot \mathbf{u}),
\end{aligned}
\tag{4.29}
$$

where we have use the identity

$$\int_{\mathbb{T}_3} p\,(\nabla \cdot \mathbf{u}) = -\int_{\mathbb{T}_3} \nabla p \cdot \mathbf{v}.$$

It remains to treat the term $(H_\tau(\mathbf{u}_n) \cdot \nabla)\mathbf{u}_n$ which constitues the novelty. This is why we focus our attention on it. Let us remark that $(\nabla \mathbf{u}_n)_{n\in\mathbb{N}}$ goes strongly to $\nabla \mathbf{u}$ in the space $L^2([0,T] \times \mathbb{T}_3)^9$. On the other hand, applying Lemma 3.4, we get that $(H_\tau(\mathbf{u}_n))_{n\in\mathbb{N}}$ converges to $H_\tau(\mathbf{u})$ in $L^2([0,T] \times \mathbb{T}_3)^9$ when n goes to infinity. Therefore the sequence $((H_\tau(\mathbf{u}_n) \cdot \nabla)\mathbf{u}_n)_{n\in\mathbb{N}}$ goes strongly to $(H_\tau(\mathbf{u}) \cdot \nabla)\mathbf{u}$ in $L^1([0,T] \times \mathbb{T}_3)^3$. Finally, since the sequence $((H_\tau(\mathbf{u}_n) \cdot \nabla)\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T] \times \mathbb{T}_3)^3$, it converges weakly, up to a subsequence, to some $\mathbf{g}$ in $L^2([0,T] \times \mathbb{T}_3)^3$. The result above and uniqueness of the limit, allows us to claim that $\mathbf{g} = (H_\tau(\mathbf{u}) \cdot \nabla)\mathbf{u}$. Consequently

$$\int_0^T \int_{\mathbb{T}_3} (H_\tau(\mathbf{u}_n) \cdot \nabla)\mathbf{u}_n \cdot \mathbf{v} \longrightarrow \int_0^T \int_{\mathbb{T}_3} (H_\tau(\mathbf{u}) \cdot \nabla)\mathbf{u} \cdot \mathbf{v}.$$

In summary, (u,p) satisfies

(iv.iii) $u \in L^2([0,T], \mathbb{H}_2) \cap L^\infty([0,T], \mathbb{H}_1)$, $p \in L^2([0,T], H^1(\mathbb{T}_3))$,

(iv.iv) $\lim_{t\to 0} ||u(t,\cdot) - H_\tau(\mathbf{u}_0)||_{0,2} = 0$,

(iv.v) $\forall \mathbf{v} \in L^2([0,T], \mathbb{H}_1)$,

$$\int_{[0,T]\times\mathbb{T}_3} [\partial_t \mathbf{u} \cdot \mathbf{v} + (H_\tau(\mathbf{u}) \cdot \nabla)\mathbf{u} \cdot \mathbf{v} + \nu \nabla \mathbf{u} \cdot \nabla \mathbf{v}] + \int_{[0,T]\times\mathbb{T}_3} \nabla p \cdot \mathbf{v} = \int_{[0,T]\times\mathbb{T}_3} H_\tau(\mathbf{f}) \cdot \mathbf{v}.$$

Uniqueness is proven exactly like in [29], and we skip the details. Moreover, taking $\mathbf{u}$ as test vector field, which is a legal operation, and integrating in space and time using $\nabla \cdot \mathbf{u} = 0$ yields the energy equality

$$\frac{1}{2}\int_{\{t\}\times\mathbb{T}_3} |\mathbf{u}|^2 + \nu \int_{[0,T]\times\mathbb{T}_3} |\nabla \mathbf{u}|^2 = \frac{1}{2}\int_{\mathbb{T}_3} |\mathbf{u}_0|^2 + \int_{[0,T]\times\mathbb{T}_3} \mathbf{f} \cdot \mathbf{u}.$$

Therefore, (u,p) is a smooth weak solution to problem (4.13), which concludes the proof of Theorem 4.2. ■

Proof of Theorem 4.3. We finish this report by proving the convergence result when τ goes to infinity. We note (u_τ, p_τ) the corresponding solution so far the grid parameter α is fixed. In this case, we only can use estimates (4.17) and (4.18). We also will use estimate (4.12). Let us first write the equation for the pressure:

$$-\Delta p_\tau = \nabla \cdot (\nabla \cdot (H_\tau(\mathbf{u}_\tau) \otimes \mathbf{u}_\tau)). \tag{4.30}$$

This yields, by interpolation combining (4.17), (4.18) and (4.12), the existence of a constant $C = C(\nu, ||\mathbf{u}_0||_{0,2}, ||f||_{-1,2})$ be such that

$$||p_\tau||_{L^{5/3}([0,T]\times\mathbb{T}_3)} \le C. \tag{4.31}$$

When writting

$$\partial_t \mathbf{u}_\tau = -\nabla \cdot (H_\tau(\mathbf{u}_\tau) \otimes \mathbf{u}_\tau) + \nu\Delta\mathbf{u}_\tau - \nabla p_\tau + H_\tau(\mathbf{f}), \tag{4.32}$$

we obtain the existence of a constant $C = C(\nu, ||\mathbf{u}_0||_{0,2}, ||f||_{-1,2})$ such that

$$||\partial_t \mathbf{u}_\tau||_{L^{5/3}([0,T],W^{-1,5/3}(\mathbb{T}_3)^3)} \le C. \tag{4.33}$$

We are now well equiped to pass to the limit. Thanks to all these bounds, there exists $(\tau_n)_{n\in\mathbb{N}}$ which goes to infinity when n goes to infinity and such that there exists $\mathbf{u} \in L^2([0,T], \mathbf{IH}_1) \cap L^\infty([0,T], \mathbf{IH}_0)$ and $p \in L^{5/3}([0,T] \times \mathbb{T}_3)$ be such that

$$\begin{cases} \mathbf{u}_{\tau_n} \longrightarrow \mathbf{u} \quad \text{weakly in} \quad L^2([0,T], \mathbf{IH}_1), \\ \mathbf{u}_{\tau_n} \longrightarrow \mathbf{u} \quad \text{weakly star in} \quad L^\infty([0,T], \mathbf{IH}_0), \\ p_{\tau_n} \longrightarrow p \quad \text{weakly in} \quad L^{5/3}([0,T] \times \mathbb{T}_3), \end{cases} \tag{4.34}$$

when n goes to infinity. We must prove that (u, p) is a dissipative weak solution to the Navier-Stokes equations.

Let us start with the compactness result issued from Aubin-Lions Lemma. We have

$$H^1(\mathbb{T}_3) \subset L^{10/3}(\mathbb{T}_3) \subset W^{-1,5/3}(\mathbb{T}_3),$$

the injections being dense and continous, the first one being compact (since $10/3 < 6$, 6 being the critical exponent in the 3D case). Therefore, applying again Aubin-Lions Lemma using the bound on $(\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ in $L^2([0,T], \mathbf{IH}_1) \subset L^2([0,T], H^1(\mathbb{T}_3)^3)$ and the bound of $(\partial_t \mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ in $L^{5/3}([0,T], W^{-1,5/3}(\mathbb{T}_3)^3)$, we see that $(\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ is compact in $L^{5/3}([0,T], L^{10/3}(\mathbb{T}_3)^3)$. Then we have in particular

$$\mathbf{u}_{\tau_n} \longrightarrow \mathbf{u} \quad \text{strongly in} \quad L^{5/3}([0,T] \times \mathbb{T}_3)^3. \tag{4.35}$$

Using Egorov's Theorem combined with Lebesgue inverse Theorem, we deduce from (4.35) combined to the bound in $L^{10/3}$ that

$$\forall\, q < 10/3, \quad \mathbf{u}_{\tau_n} \longrightarrow \mathbf{u} \quad \text{strongly in} \quad L^q([0,T] \times \mathbb{T}_3)^3. \tag{4.36}$$

Let us again consider $(\partial_t \mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$. The bound (4.33) authorizes to extract a subsequence (still using the same notation) and such that

$$\partial_t \mathbf{u}_{\tau_n} \longrightarrow \mathbf{g} \quad \text{weakly in} \quad L^{5/3}([0,T], W^{-1,5/3})^3. \tag{4.37}$$

We must prove that $\mathbf{g} = \partial_t \mathbf{u}$. Let φ be a C^∞ field defined on $[0,T] \times \mathbb{T}_3$ and such that $\varphi(0, \mathbf{x}) = \varphi(T, \mathbf{x}) = 0$. Then one has

$$\int_{[0,T]\times\mathbb{T}_3} \partial_t \mathbf{u}_{\tau_n} \cdot \varphi = -\int_{[0,T]\times\mathbb{T}_3} \mathbf{u}_{\tau_n} \cdot \partial_t \varphi.$$

Passing to the limit in this equality using (4.37) yields

$$\int_{[0,T]\times\mathbb{T}_3} \mathbf{g}\cdot\varphi = -\int_{[0,T]\times\mathbb{T}_3} \mathbf{u}\cdot\partial_t\varphi,$$

which tells us that $\mathbf{g} = \mathbf{u}$ in the distributional sense, and also in L^p sense by uniqueness of the limit.

From now, $\mathbf{v} \in L^{5/2}([0,T], W^{1,5/2}(\mathbb{T}_3)^3)$ is a fixed test vector field. We have the obvious following convergences when n goes to infinity,

$$\left\{ \begin{array}{l} \displaystyle\int_Q \partial_t \mathbf{u}_{\tau_n}\cdot\mathbf{v} \longrightarrow (\partial_t\mathbf{u}\cdot\mathbf{v}), \quad \int_Q \nabla\mathbf{u}_{\tau_n} : \nabla\mathbf{v} \longrightarrow \int_Q \nabla\mathbf{u} : \nabla\mathbf{v}, \\ \displaystyle\int_Q p_{\tau_n}(\nabla\cdot\mathbf{v}) \longrightarrow \int_Q p\,(\nabla\cdot\mathbf{v}), \quad \int_Q H_{\tau_n}(\mathbf{f})\cdot\mathbf{v} \longrightarrow \int_Q \mathbf{f}\cdot\mathbf{v}, \end{array}\right. \tag{4.38}$$

where $Q = [0,T]\times\mathbb{T}_3$ for the simplicity, $(\cdot,\cdot)$ stands for the duality product between $L^{5/2}([0,T], W^{1,5/2}(\mathbb{T}_3)^3)$ and $L^{5/3}([0,T], W^{-1,5/3}(\mathbb{T}_3)^3)$, and where we also have used Lemma 3.4.

We now have to deal with the nonlinear term. We first notice that $(H_{\tau_n}(\mathbf{u}_n)\otimes\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ is bounded in $L^{5/3}(Q)^9$. Thus -up to a subsequence- it converges weakly in $L^{5/3}(Q)^9$ to a guy named $\mathbf{h}$ for the time being. That means

$$\int_Q H_{\tau_n}(\mathbf{u}_n)\otimes\mathbf{u}_{\tau_n} : \nabla\mathbf{v} \longrightarrow \int_Q \mathbf{h} : \nabla\mathbf{v}. \tag{4.39}$$

The challenge is to prove that $\mathbf{h} = \mathbf{u}\otimes\mathbf{u}$. We already know that $\mathbf{u}_{\tau_n}$ converges to $\mathbf{u}$ strongly in $L^{10/3-\varepsilon}(Q)$ ($\varepsilon > 0$ and as usual "small"). Let us study the sequence $H_{\tau_n}(\mathbf{u}_n)$. It obviously converges to $\mathbf{u}$ but we must precise in which space and for which topology. We shall work in a L^2 space type ($2 < 10/3$...). We can write

$$H_{\tau_n}(\mathbf{u}_n) - \mathbf{u} = H_{\tau_n}(\mathbf{u}_n - \mathbf{u}) + H_{\tau_n}(\mathbf{u}) - \mathbf{u}.$$

Thanks to (3.15), we have for any fixed time t,

$$||H_{\tau_n}(\mathbf{u}_n - \mathbf{u})(t,\cdot)||_{0,2}^2 \le 2||(\mathbf{u}_n - \mathbf{u})(t,\cdot)||_{0,2}^2,$$

an inequality that we integrate on the time interval $[0,T]$. This ensures that the sequence $(H_{\tau_n}(\mathbf{u}_n - \mathbf{u}))_{n\in\mathbb{N}}$ converges to zero in $L^2(Q)^3$ when n goes to infinity. Applying Lemma 3.4, we deduce that the sequence $(H_{\tau_n}(\mathbf{u}) - \mathbf{u})_{n\in\mathbb{N}}$ converges to zero in $L^2(Q)^3$ when n goes to infinity.

In summary, we obtain the convergence of $(H_{\tau_n}(\mathbf{u}_n)\otimes\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ to $\mathbf{u}\otimes\mathbf{u}$ in $L^1(Q)^3$, making sure that $\mathbf{h} = \mathbf{u}\otimes\mathbf{u}$ and also thanks to (4.39),

$$\int_Q H_{\tau_n}(\mathbf{u}_n)\otimes\mathbf{u}_{\tau_n} : \nabla\mathbf{v} \longrightarrow \int_Q \mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{v}. \tag{4.40}$$

In conclusion, $(\mathbf{u}, p)$ satisfies (4.5). Point 1) in definition (4.1) is already checked. To conclude our proof, it remains to prove points 2) (initial data) and 4) (energy inequality). We start with the energy inequality.

We already know that $(\mathbf{u}_{\tau_n}, p_{\tau_n})$ satisfy the energy equality (4.15). Let $0 \le t_1 < t_2 \le T$, and integrate (4.15) on the time interval $[t_1, t_2]$. We get

$$
(4.41) \qquad \left\{ \begin{array}{l} \displaystyle \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}_3} |\mathbf{u}_{\tau_n}(t,\mathbf{x})|^2 d\mathbf{x}dt + \nu \int_{t_1}^{t_2}\int_0^t \int_{\mathbb{T}_3} |\nabla \mathbf{u}_{\tau_n}(t',\mathbf{x})|^2 d\mathbf{x}dt'dt = \\ \displaystyle \quad \frac{t_2-t_1}{2}\int_{\mathbb{T}_3} |H_{\tau_n}(\mathbf{u}_0)(\mathbf{x})|^2 d\mathbf{x} + \int_{t_1}^{t_2}\int_0^t \int_{\mathbb{T}_3} H_{\tau_n}(\mathbf{f}) \cdot \mathbf{u}_{\tau_n} d\mathbf{x}dt'dt. \end{array} \right.
$$

Because $(H_{\tau_n}(\mathbf{f}))_{n\in\mathbb{N}}$ converges strongly to $\mathbf{f}$ in $L^2([0,T],(H^1(\mathbb{T}_3)^3)')$ while $(\mathbf{u}_{\tau_n})_{n\in N}$ converges weakly to $\mathbf{u}$ in $L^2([0,T],\mathbb{H}_1)$, standard arguments yields

$$
(4.42) \qquad \int_{t_1}^{t_2}\int_0^t \int_{\mathbb{T}_3} H_{\tau_n}(\mathbf{f}) \cdot \mathbf{u}_{\tau_n} d\mathbf{x}dt'dt \longrightarrow \int_{t_1}^{t_2} (\mathbf{f},\mathbf{u})dt.
$$

Analogous arguments also tell

$$
(4.43) \qquad \frac{t_2-t_1}{2}\int_{\mathbb{T}_3} |H_{\tau_n}(\mathbf{u}_0)(\mathbf{x})|^2 d\mathbf{x} \longrightarrow \frac{t_2-t_1}{2}\int_{\mathbb{T}_3} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x}.
$$

As we know that $(\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ goes to $\mathbf{u}$ strongly in $L^2(Q)^3$, we have

$$
(4.44) \qquad \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}_3} |\mathbf{u}_{\tau_n}(t,\mathbf{x})|^2 d\mathbf{x}dt \longrightarrow \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}_3} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x}dt.
$$

Finally, standard arguments in analysis (see for instance in [31]), the weak convergence of $(\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ to $\mathbf{u}$ in $L^2([0,T],\mathbb{H}_1)$ yields

$$
(4.45) \qquad \int_{t_1}^{t_2}\int_0^t \int_{\mathbb{T}_3} |\nabla \mathbf{u}(t',\mathbf{x})|^2 d\mathbf{x}dt'dt \le \liminf_{n\in\mathbb{N}} \int_{t_1}^{t_2}\int_0^t \int_{\mathbb{T}_3} |\nabla \mathbf{u}_{\tau_n}(t',\mathbf{x})|^2 d\mathbf{x}dt'dt.
$$

When one combines (4.41) together with (4.42),(4.43), (4.44) and (4.44), we obtain

$$
(4.46) \qquad \left\{ \begin{array}{l} \displaystyle \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}_3} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x}dt + \nu \int_{t_1}^{t_2}\int_0^t \int_{\mathbb{T}_3} |\nabla \mathbf{u}(t',\mathbf{x})|^2 d\mathbf{x}dt'dt \le \\ \displaystyle \qquad\qquad \frac{t_2-t_1}{2}\int_{\mathbb{T}_3} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + \int_{t_1}^{t_2} (\mathbf{f},\mathbf{u})dt, \end{array} \right.
$$

an inequality which holds for every t_1, t_2 such that $0 \le t_1 < t_2 \le T$. We deduce that $\mathbf{u}$ satisfies the energy inequality (4.15).

To finish the proof, we have to study the initial data. Let us first notice that $\mathbf{u}(t,\cdot)_{t>0}$ is bounded in $L^2(\mathbb{T}_3)^3$. Therefore, we can find a sequence $(t_n)_{n\in\mathbb{N}}$ which converges to 0 and a field $\mathbf{k} \in L^2(\mathbb{T}_3)^3$ be such that $\mathbf{u}(t_n,\cdot)_{n\in N}$ converges weakly in $L^2(\mathbb{T}_3)^3$ to $\mathbf{k}$. The first task is to prove that $\mathbf{k} = \mathbf{u}_0$. We start from the equality

$$
(4.47) \qquad \mathbf{u}_{\tau_n}(t,\cdot) = H_{\tau_n}(\mathbf{u}_0) + \int_0^t \partial_t \mathbf{u}_{\tau_n} dt',
$$

an equality that we consider in the space $W^{-1,5/3}(\mathbb{T}_3)^3$. Using a straightforward variant of Lemma 3.4 and the convergence results proved above, we can pass to the limit in (4.47), to get in $W^{-1,5/3}(\mathbb{T}_3)^3$,

$$
(4.48) \qquad \mathbf{u}(t,\cdot) = \mathbf{u}_0 + \int_0^t \partial_t \mathbf{u}\, dt'.
$$

Because $\partial_t \mathbf{u} \in L^{5/3}([0,T], W^{-1,5/3}(\mathbb{T}_3)^3) \subset L^1([0,T], W^{-1,5/3}(\mathbb{T}_3)^3)$, this last equality says that $\mathbf{u}(0,\cdot) = \mathbf{u}_0$ at least in $W^{-1,5/3}(\mathbb{T}_3)^3$, and consequently in $L^2(\mathbb{T}_3)^3$. Therefore we have $\mathbf{k} = \mathbf{u}_0$. The limit being unique, we deduce that the whole sequence $\mathbf{u}(t,\cdot)_{t>0}$ converges weakly in $\mathbb{H}_0$ to $\mathbf{u}_0$ when t goes to zero. Moreover, one has

$$||\mathbf{u}||_{0,2} \leq \liminf_{t\to 0} ||\mathbf{u}(t,\cdot)||_{0,2}. \tag{4.49}$$

On the other hand, when one let t go to zero in the energy inequality, we get

$$\limsup_{t\to 0} ||\mathbf{u}(t,\cdot)||_{0,2} \leq ||\mathbf{u}_0||_{0,2}. \tag{4.50}$$

We deduce that

$$\lim_{t\to 0} ||\mathbf{u}(t,\cdot)||_{0,2} = ||\mathbf{u}_0||_{0,2},$$

which combined with the weak convergence yields

$$\lim_{t\to 0} ||\mathbf{u}_0 - \mathbf{u}(t,\cdot)||_{0,2} = 0. \tag{4.51}$$

This concludes the question of the initia data and also the proof of Theorem 4.3. ■

Remark 4.3 *With not too much sweat, one can prove that the approximated velocity in model (4.13) is in the space $C([0,T], \mathbb{H}_1)$. Concerning the Navier-Stokes equation, it is well known that the trajectories are continuous in $L^2(\mathbb{T}_3)^3$ with respect to its weak topology. Nevertheless, one may wonder about the strong continuity of the trajectory at $t = 0$ that we have proved here. This approach seems indeed to be not usual in the folklore of the Navier-Stokes Equations. However, it fits with the famous local regularity result due to Fujita-Kato [20].*

References

[1] N. A. Adams and S. Stolz, *Deconvolution methods for subgrid-scale approximation in large eddy simulation*, Modern Simulation Strategies for Turbulent Flow, R.T. Edwards, 2001.

[2] ——, *A subgrid-scale deconvolution approach for shock capturing*, Journal of Computational Physics, 178 (2002), pp. 391–426.

[3] J. Bardina, J. Ferziger, and W. Reynolds, *Improved subgrid scale models for large eddy simulation*, AIAA paper, 80 (1980), p. 1357.

[4] G. Batchelor, *The theory of homogeneous turbulence*, Cambridge University Press, 1953.

[5] A. C. Bennis, *Étude de quelques modèles de turbulence pour l'océanographie*, Thèse de l'Université Rennes 1, n° 3762 (2008).

[6] A. C. Bennis, R. Lewandowski, and E. Titi, *A generalized Leray-deconvolution model of turbulence*, In Progress, (2009).

[7] ——, *Simulations de l'écoulement turbulent marin avec un modèle de déconvolution*, Comptes-Rendus de l'Académie des Sciences de Paris, under press, arXiv:0809.2655, (2009).

[8] L. Berselli and R. Lewandowski, *Convergence of ADM models to Navier-Stokes Equations*, In progress, (2009).

[9] L. C. Berselli, T. Iliescu, and W. Layton, *Mathematics of Large Eddy Simulation of Turbulent Flows*, Springer, Berlin, 2005.

[10] M. Bertero and B. Boccacci, *Introduction to Inverse Problems in Imaging*, IOP Publishing Ltd, 1998.

[11] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. on Pure and applied maths, 35 (1982), pp. 771–831.

[12] S. Chen, C. Foias, D. Holm, E. Olson, E. S. Titi, and S. Wynne, *The Camassa-Holm equations and turbulence*, Physica D, D133 (1999), pp. 49–65.

[13] V. V. Chepyzhov, E. S. Titi, and M. I. Vishik, *On the convergence of the leray-alpha model to the trajectory attractor of the 3D Navier-Stokes system*, Matematicheskii Sbornik, 12 (2007), pp. 3–36.

[14] C. Doering and C. Foias, *Energy dissipation in body-forced turbulence*, J. Fluid Mech., 467 (2002), pp. 289–306.

[15] C. Doering and J. Gibbon, *Applied analysis of the Navier-Stokes equations*, Cambridge University Press, 1995.

[16] A. Dunca and Y. Epshteyn, *On the stolz-adams de-convolution model for the large eddy simulation of turbulent flows*, SIAM J. Math. Anal., 37 (2006), pp. 1890–1902.

[17] C. Foias, D. D. Holm, and E. S. Titi, *The Navier-Stokes-alpha model of fluid turbulence*, Physica D, 152 (2001), pp. 505–519.

[18] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, 2001.

[19] U. Frisch, *Turbulences*, Cambridge, 1995.

[20] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem*, Arch. Rat. Mach. Anal., 16 (1964), pp. 269–315.

[21] B. J. Geurts and D. D. Holm, *Leray and LANS-alpha modeling of turbulent mixing*, Journal of Turbulence, 00 (2005), pp. 1–42.

[22] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, 1986.

[23] F. Hecht, O. Pironneau, A. L. Hyaric, and K. Ohtsua, *Freefem++ manual v2.21*, http://www.freefem.org/ff++/, (2006).

[24] A. A. Ilyin, E. M. Lunasin, and E. S. Titi, *A modified leray-alpha subgrid-scale model of turbulence*, Nonlinearity, 19 (2006), pp. 879–897.

[25] A. N. Kolmogorov, *The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers*, Dokl. Akad. Nauk SSSR, 30 (1941), pp. 301–305.

[26] W. Layton and R. Lewandowski, *A simple and stable scale similarity model for large eddy simulation: energy balance and existence of weak solutions*, Applied Math. letters, 16 (2003), pp. 1205–1209.

[27] ——, *On a well posed turbulence model*, Continuous Dynamical Systems series B, 6 (2006), pp. 111–128.

[28] ——, *Residual stress of approximate deconvolution large eddy simulation models of turbulence*, Journal of Turbulence, 7 (2006), pp. 1–21.

[29] ——, *A high accuracy Leray-deconvolution model of turbulence and its limiting behavior*, Analysis and Applications, 6 (2008), pp. 23–49.

[30] J. Leray, *Sur les mouvements d'une liquide visqueux emplissant l'espace*, Acta Math., 63 (1934), pp. 193–248.

[31] R. Lewandowski, *Analyse Mathématique et Océanographie*, collection RMA, Masson, Paris, 1997.

[32] ——, *Vorticities in a LES model for 3D periodic turbulent flows*, Journ. Math. Fluid. Mech., 8 (2006), pp. 398–42.

[33] ——, *Approximations to the Navier-Stokes Equations*, In progress, 2009.

[34] J. L. Lions, *Quelques méthodes de résolution de problèmes aux limites non linéaires*, Gauthiers-Villard, 1969.

[35] B. Mohammadi and O. Pironneau, *Analysis of the k-Epsilon model*, collection RMA, Masson, 1994.

[36] P. Sagaut, *Large eddy simulation for Incompressible flows*, Springer, Berlin, 2001.

[37] J. Smagorinsky, *The dynamical influence of large-scale heat sources and sinks on the quasi-stationary mean motions of the atmosphere*, Quarterly Journal of the Royal Meteorological Society, 79 (1953), pp. 342–366.

[38] S. Stolz and N. A. Adams, *An approximate deconvolution procedure for large eddy simulation*, Phys. Fluids, 10 (1999), pp. 1699–1701.

[39] S. Stolz, N. A. Adams, and L. Kleiser, *An approximate deconvolution model for large-eddy simulation with application to incompressible wall-bounded flows*, Physics of fluids, 13 (2001), pp. 997–1015.