# RESIDUAL STRESS OF APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE

William Layton [1]

Department of Mathematics, University of Pittsburgh, Pittsburgh,

PA 15260, USA, wjl@pitt.edu, http://www.math.pitt.edu/~wjl, partially supported by NSF Grant DMS0207627 and 0508260,

Roger Lewandowski (corresponding author)

IRMAR, UMR CNRS 6625, Universite Rennes 1, F-35000, Rennes, France,

Roger.Lewandowski@univ-rennes1.fr, http://name.math.univ-rennes1.fr/roger.lewandowski

**Abstract**

We consider the case of a homogeneous, isotropic, fully developed, turbulent flow. We show analytically by using the $-5/3$ Kolmogorov's law that the time averaged consistency error of the $N^{\text{th}}$approximate deconvolution LES model converges to zero following a law as the cube root of the averaging radius, independently of the Reynolds number. The consistency error is measured by the residual stress. The filter under consideration is a second order differential filter, but the 1/3 law is still valid in the case of the Gaussian filter and large class of filters used in LES. We also show how the 1/3 error law can be derived by a dimensional analysis.

**Key words :** large eddy simulation, approximate deconvolution model, turbulence

[1] Author's names are in alphabetical order

# 1 Introduction

Direct numerical simulation of turbulent flows of incompressible, viscous fluids is often not computationally economical or even feasible. Thus, various turbulence models are used for simulations seeking to predict flow statistics or averages. In LES (large eddy simulation) the evolution of local, spatial averages is sought. Broadly, there are two types of LES models of turbulence: *descriptive* or phenomenological models (e.g., eddy viscosity models) and *predictive* models (considered herein). The accuracy of a model, meaning the relative error

$$\frac{||\text{filteredNSEsolution-LESsolution}||}{||\text{filteredNSEsolution}||} \tag{1}$$

can be assessed in several experimental and analytical ways[2]. In a posteriori testing, a DNS is performed and the relative error calculated by the quotient (1) above. The other common approach in LES is a' priori testing, Sagaut[35]. We study here an analytic form of a' priori testing. To present this, let $\boldsymbol{\tau}$ denote the subfilter scale stress tensor

$$\boldsymbol{\tau}(\mathbf{u},\mathbf{u}) = \overline{\mathbf{u}\otimes\mathbf{u}} - \overline{\mathbf{u}}\otimes\overline{\mathbf{u}}, \tag{2}$$

where $\mathbf{u}$ denotes the velocity of the flow. The filters can be the classical Gaussian filter of parameter $\delta$ or others. In this paper (as in early work, see Layton-Lewandowski[20],[22]), we

[2]Here the "filteredNSEsolution" stands for the filtered solution to the Navier-Stokes equation deduced from the LES filter under consideration and $||\cdot||$ denotes any norm, defining the sense given to the notion of "accuracy".

shall work with the approximated Gaussian filter, filtering wave numbers higher than $\delta^{-1}$, called a differential filter and having for transfer function the function

$$\widehat{G}(k) = \frac{1}{\delta^2 k^2 + 1}. \tag{3}$$

The parameter $\delta$ is the averaging radius (for instance the size of the numerical grid used to simulate a isotropic turbulent flow). We denote by $A\mathbf{u} = \overline{\mathbf{u}}$ the filtered velocity field, and for a tensor $\mathbf{T}$, $A\mathbf{T} = \overline{\mathbf{T}}$ denotes the coresponding filtered tensor.
LES models are replace this tensor by one that depends only on $\overline{\mathbf{u}}$. For example, the simplest model in the family of Approximate Deconvolution Models (ADM) we study is given by

$$\boldsymbol{\tau}_{\text{model}}(\overline{\mathbf{u}}, \overline{\mathbf{u}}) = \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} - \overline{\mathbf{u}} \otimes \overline{\mathbf{u}}.$$

The difference, evaluated at the true solution to the Navier-Stokes Equations, is the consistency error. À priori testing of accuracy proceeds by computing $\mathbf{u}$ by DNS, filtering $\mathbf{u}$ then computing

$$||\overline{\boldsymbol{\tau}_0}|| = ||\boldsymbol{\tau}_{\text{model}}(\overline{\mathbf{u}}, \overline{\mathbf{u}}) - \boldsymbol{\tau}(\mathbf{u}, \mathbf{u})|| = ||\overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} - \overline{\mathbf{u} \otimes \mathbf{u}}||, \tag{4}$$

where $\boldsymbol{\tau}_0 = \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} - \mathbf{u} \otimes \mathbf{u}$ is the residual stress. It is like testing consistency error of a finite difference method. If $\mathbf{w}$ denotes the "LESsolution", an analysis of the error in a Large Eddy Simulation (LES) model [22] has shown that the error $\mathbf{u} - \mathbf{w}$ satisfies an equation driven only by $\overline{\boldsymbol{\tau}_0}$. Thus, $||\overline{\mathbf{u}} - \mathbf{w}||$ being small requires small consistency error, $||\boldsymbol{\tau}_0||$ small, and stability of the LES model.

One important approach (for which there are currently few results) is to study analytically the model's *consistency error* (defined precisely below) as a function of the averaging radius $\delta$ and the Reynolds number Re. The inherent difficulties are that

(i) consistency error bounds for regular functions hardly address essential features of turbulent flows such as irregularity and richness of scales,

(ii) worst case bounds for solutions of the Navier-Stokes equations are so pessimistic as to yield little insight.

However, it is known that after time or ensemble averaging, turbulent velocity fields are often observed to have intermediate regularity as predicted by the Kolmogorov theory (often called the K41 theory), see in Frisch[16], Berselli *et al.*[4], Pope[33], Sagaut[35] and Lesieur[24]. This case is often referred to as homogeneous isotropic turbulence and various norms of flow quantities can be estimated in this case using the K41 theory. We mention Lilly's famous paper[29] as an early and important example.

In this paper we consider this third way begun in Layton-Lewandowski[21]: error bounds are developed for *time averaged, fully developed, homogeneous, isotropic turbulence*. We are seeking for estimates of, on one hand

- of the time average of the $L_2$ norm of the filtered residual second order stress $\boldsymbol{\tau}_0$ defined by equation (4) and which appears in the approximate deconvolution model of order 0. The precise quantity we have in mind is the limit when the time $T$ goes to infinity of the integral

$$\frac{1}{L^{3/2}U^2}\left(\frac{1}{T}\int_0^T \left\{\int_{\mathbb{R}^3} |\overline{\boldsymbol{\tau}_0}(\mathbf{x},t)|^2 d\mathbf{x}\right\}^{1/2} dt\right), \tag{5}$$

a dimensionless quantity which will be also denoted by

$$\frac{1}{L^{3/2}U^2} < ||\overline{\boldsymbol{\tau}_0}||_{L_2} >$$

in the remainder[3] and which can be considered as an averaged relative error. We denote by $U$ a typical size of the velocity's modulus and $<\cdot>$ is the time average. Finally, $L$ denotes a typical length scale of the flow (the size of an obstacle if there is one, the fundamental wavelength of an $L$-periodic box...). Notice that this model is not the one introduced by Bardina *et al.*[2] as we shall see in the remainder.

And on the other hand

- of the time average of the $L_2$ norm of the residual stress $\boldsymbol{\tau}_N$ appearing in the general approximate deconvolution introduced by Stolz and Adams[36], that means the limit when the time $T$ goes to infinity

$$\frac{1}{L^{3/2}U^2}\lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \left\{\int_{\mathbb{R}^3} |\overline{\boldsymbol{\tau}_N}(\mathbf{x},t)|^2 d\mathbf{x}\right\}^{1/2} dt\right) = \frac{1}{L^{3/2}U^2} < ||\overline{\boldsymbol{\tau}_N}||_{L_2} >, \tag{6}$$

[3]Recall that the residual stress is defined by $\boldsymbol{\tau}_0 = (\tau_0^{ij})_{1\le ij\le 3} = \overline{u^i}\,\overline{u^j} - u^i u^j$, $|\overline{\boldsymbol{\tau}_0}(\mathbf{x},t)| = (\overline{\tau_0^{ij}}(\mathbf{x},t).\overline{\tau_0^{ij}}(\mathbf{x},t))^{1/2}$, the velocity $\mathbf{u}$ is the vector field $\mathbf{u} = (u^1,u^2,u^3)$.

where $(\tau_N)_{ij} = G_N\overline{u^i}G_N\overline{u^j} - u^i u^j$, $G_N = \sum_{n=0}^{n=N}(I-A)^n$ being the deconvolution operator.

Such bounds are inherently interesting and they also help answer two important related questions of *accuracy* and *feasibility* of LES. Indeed, numerical simulations lead to these two following theoretical questions:

- How small must $\delta$ be with respect to the Reynolds number Re to have the average consistency relative error $<< O(1)$? (accuracy)
- Can consistency relative error $<< O(1)$ be attained for the cutoff length-scale $\delta$ within the inertial range? (feasibility)

These questions can be rephrased as: is the models solution close to the true flow averages? And, does solution of the model require fewer degrees of freedom than a DNS? The rest of this paper is an attempt to give partial answers to these two crucial questions. In particular, we obtain by an analytical way the following bounds, under suitable assumptions (see below assumptions 2.1 and 2.2),

$$\frac{1}{L^{3/2}U^2} < ||\overline{\boldsymbol{\tau}_N}||_{L_2} > \leq C_N \left(\frac{\delta}{L}\right)^{1/3}, \tag{7}$$

for $N = 0, 1.....$ (the l.h.s is defined by the equality (6) above). These bounds do not depend on the Reynolds number Re and $C_N$ is a dimensionless constant which depends on the total dissipation rate $\varepsilon$, and which is bounded with respect to $N$. Using this bound, we deduce that the error $\overline{\mathbf{u}} - \mathbf{w}$ satisfies the following, in the case $N = 0$,

$$< ||\nu\nabla(\overline{\mathbf{u}} - \mathbf{w})||_{L_2}^2 >= O(\delta^{1/3}). \tag{8}$$

The main observation to derive (7) is that $< ||\overline{\boldsymbol{\tau}_N}||_{L_2} >$ is driven by the averaged value of the $L_2$ norm of the filtered error $G_N\overline{\mathbf{u}} - \mathbf{u}$, $K_N =< ||G_N\overline{\mathbf{u}} - \mathbf{u}||_{L_2}^2/L^3 >$. When $N = 0$, $K_N$ is the turbulent kinetic energy (TKE) and for general integer $N$, we shall call it the generalized TKE[4]. The next obervation is that, after averaging, the Kolmogorov $-5/3$'s law can be used to evaluate the TKE by the way of the transfer function of the filter $A$. As we shall see, these bounds still hold when one replaces the differential, second order filter[5] by the Gaussian filter. We show next that the 1/3 exponent in (7) can also be predicted by dimensional analysis. We finish by an interpretation of the bound in terms of accuracy and feasibility of the models and bring also an attempt of a physical interpretation of it.

**Remark 1.1** *We notice that it is easily shown, arguing as below, that the $L_1$ norm of $\overline{\boldsymbol{\tau}_N}$ is also bounded by a constant times $\delta^{1/3}$.*

**Remark 1.2** *Notice that the general philosophy of the present paper is related to the one of papers of Chen* et all[9],[7],[8].

The paper is written to be as self contained as possible and is organized as follows. The technical mathematical details are given in the appendix, sections 7, 8 and 9. In the section

[4]recall that $G_0\overline{\mathbf{u}} = \overline{\mathbf{u}}$ and one has formally $G_\infty\overline{\mathbf{u}} = \mathbf{u}$

[5]defined by its transfer function (3)

2 we review the Navier-Stokes equations and the Kolmogorov $-5/3$'s and "K41 phenomenology". This allows us to set our notation and to write precisely the assumptions we make on the flow (see assumptions 2.1 and 2.2 below). Section 3 is devoted to the presentation of the ADM models and points out the analytical link between the modeling error and the norms of the residual stress. In the section 4, we show in details how we obtain the $\delta^{1/3}$'s bound analytically using the $-5/3$ Kolmogorov's law. Next, the 1/3 law is also qualtitatively derived by a dimensional analysis. We finish by conclusions and discussions in section 5, by exploring a connection between the 1/3 law we found with other classical laws used in the turbulence modeling.

In the first appendix, one shows how to derive a $\mathrm{Re}^{1/2}\delta$ bound without K41 and compare it to the $\delta^{1/3}$'s bound.

# 2 Navier-Stokes equations and K41 phenomenology

## 2.1 The Navier-Stokes equations

Let the velocity $\mathbf{u}(\mathbf{x}, t) = (u^1(\mathbf{x}, t), u^2(\mathbf{x}, t), u^3(\mathbf{x}, t))$ , $\mathbf{x} = (x_1, x_2, x_3)$ and pressure $p(\mathbf{x}, t)$ be a solution to the underlying Navier Stokes equations (NSE for short)

$$\frac{\partial u^i}{\partial t} + \frac{\partial u^j u^i}{\partial x_i} + \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u^i}{\partial x_k \partial x_k} = f^i, \tag{9}$$

with the continuity equation

$$\frac{\partial u^i}{\partial x_i} = 0, \tag{10}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $\mathbf{f} = (f^1, f^2, f^3)$ is the body force and $\mathbb{R}^3$ is the flow domain. The Navier-Stokes equations are rewritten under the simplest vectorial form

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p - \nu \triangle \mathbf{u} = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0, \text{ in } \mathbb{R}^3 \times (0, T). \tag{11}$$

Generally speaking, any vector fields $\mathbf{v} = (v^1, v^2, v^2)$ being given, $\mathbf{v} \otimes \mathbf{v}$ is the second order tensor $v^i v^j$.

The above Navier-Stokes Equations are supplemented by the initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$. One must describe the spatial boundary conditions. The boundary conditions are generally best described by the space functions where the fields are subject to be. Usually no boundary conditions are prescribed for the pressure which is not a prognostic variable. As considered in Lions [30], among many different cases studied in the litterature, there are three main situations.

**1.** The flow is periodic with $\Omega = [0, L]^3$ for period and we impose the compatibility conditions

$$\int_\Omega \mathbf{u}(\mathbf{x}, t) d\mathbf{x} = 0, \quad \int_\Omega p(\mathbf{x}, t) d\mathbf{x} = 0. \tag{12}$$

The velocity is subject to be in the space

$$\mathbf{V} = \{\mathbf{v} \in (H^1_{\text{loc}}(\mathrm{I\!R}^3))^3, \quad \mathbf{v} \quad \Omega-\text{periodic}, \quad \nabla\cdot\mathbf{v}=0\}. \tag{13}$$

In this case $\mathbf{u}_0 \in (L^2_{\text{loc}}(\mathrm{I\!R}^3))^2$, is $\Omega$-periodic and satisfies $\nabla\cdot\mathbf{u}_0 = 0$.

**2.** The fluid flows in the whole space $\Omega = \mathrm{I\!R}^3$. As shown originaly by Leray [23], no special decay conditions at infinity are needed. The solution constructed by Leray lies in the distributional space

$$\mathbf{V} = \{\mathbf{v} \in (L^6(\mathrm{I\!R}^3))^3; \quad \nabla\mathbf{v} \in (L^2(\mathrm{I\!R}^3))^9, \quad \nabla\cdot\mathbf{v}=0 \quad \text{in } \mathcal{D}'(\mathrm{I\!R}^3)\}, \tag{14}$$

as shown in Lions[30] for instance. In this case $\mathbf{u}_0 \in (L^2(\mathrm{I\!R}^3))^2$ and satisfies $\nabla\cdot\mathbf{u}_0 = 0$. In fact, decay conditions at infinity are prescribed in some sense by the fact that $\mathbf{u}\in\mathbf{V}$.

**3.** The fluid flows in a bounded smooth domain $\Omega$ in $\mathrm{I\!R}^3$ and the no slip condition holds, that is $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, $\partial\Omega$ being the boundary of $\Omega$. In this case, the velocity lies in the space

$$\mathbf{V} = \{\mathbf{v} \in (H^1_0(\Omega))^3, \quad \nabla\cdot\mathbf{v}=0\}, \tag{15}$$

and $\mathbf{u}_0 \in \mathbf{H}$ where

$$\mathbf{H} = \{\mathbf{v} \in (L^2(\Omega))^3, \quad \mathbf{u}.\mathbf{n}|_{\partial\Omega} = 0 \text{ in } (H^{-1/2}(\partial\Omega))^3, \quad \nabla\cdot\mathbf{v}=0 \text{ in } (H^{-1}(\Omega))^3\} \tag{16}$$

In all cases, the source term $\mathbf{f}$ is such that $\mathbf{f} \in L^2(\mathbf{I}; \mathbf{V}')$, where $\mathbf{I} = [0, T]$ or $\mathbf{I} = [0, \infty[$. In these cases, as shown in Theorem 3.2 and 3.4 pages 81 and 82 in Lions[30], one can construct a weak solution to the Navier-Stokes equations, called a Leray-Hopf solution, which satisfies the following energy inequalities, for all $t > 0$,

$$\begin{gathered} \frac{1}{2}\int_\Omega |\mathbf{u}(x,t)|^2 d\mathbf{x} + \nu \int_0^t \int_\Omega |\nabla \mathbf{u}(\mathbf{x},\tau)|^2 d\mathbf{x} d\tau \leq \\ \int_0^t {}_{\mathbf{V}'}< \mathbf{f}(\mathbf{x},\tau), \mathbf{u}(\mathbf{x},\tau) >_{\mathbf{V}} d\tau + \frac{1}{2}\int_\Omega |\mathbf{u}_0(x)|^2 d\mathbf{x}, \end{gathered} \tag{17}$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\mathbf{u}(x,t)|^2 d\mathbf{x}\right) + \nu \int_\Omega |\nabla \mathbf{u}(\mathbf{x},\tau)|^2 d\mathbf{x} d\tau \leq {}_{\mathbf{V}'}< \mathbf{f}(\mathbf{x},t), \mathbf{u}(\mathbf{x},t) >_{\mathbf{V}}, \tag{18}$$

this last inequality being satisfied in $\mathcal{D}'(\mathbf{I})$.

Note that it is not known if the Leray-Hopf solution is unique and if there is one Leray-Hopf solution satisfying the energy equality as suggested by the physics of fluids. There is currently no mathematical resolution of this question.

In the remainder, we consider the cases 1 or 2, periodic case or case of the whole space.

## 2.2 The K41 phenomenology

To fix the ideas, we focus on the case of the whole space $\mathbb{R}^3$. The most important components of the K41 theory are the time (or ensemble) averaged energy dissipation rate, $\varepsilon$, and the distribution of the flows kinetic energy across wave numbers, $E(k)$. Recall that $< \cdot >$ denote long time averaging, that means for any tensor $\boldsymbol{\phi}$ related to the turbulence, the limit, when it exists, for large time T of the time average $(1/T) \displaystyle\int_0^T \boldsymbol{\phi}(\mathbf{x}, t) dt$, denoted by

$$< \boldsymbol{\phi} > (\mathbf{x}) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \boldsymbol{\phi}(\mathbf{x}, t) dt. \tag{19}$$

As we shall see in the remainder, assumptions 2.1 and 2.2 below ensure the existence of averages under considerations.

Time averaging is the original approach to turbulence of Reynolds[34]. It satisfies the following Cauchy-Schwartz inequality where $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are any fields,

$$< \int_{\mathbb{R}^3} |\boldsymbol{\phi}(\mathbf{x})| \, |\boldsymbol{\psi}(\mathbf{x})| d\mathbf{x} > \;\; \leq \;\; < \int_{\mathbb{R}^3} |\boldsymbol{\phi}(\mathbf{x})|^2 d\mathbf{x} >^{\frac{1}{2}} < \int_{\mathbb{R}^3} |\boldsymbol{\psi}(\mathbf{x})|^2 d\mathbf{x} >^{\frac{1}{2}}, \tag{20}$$

an inequality that we will use in the remainder, and which can be founded in Zeidman[40] or in Layton[19].

Moreover, it can be easely proven that we also have

$$< \int_{\mathbb{R}^3} |\boldsymbol{\phi}(\mathbf{x})| \, |\boldsymbol{\psi}(\mathbf{x})| d\mathbf{x} > \leq < ||\boldsymbol{\phi}||_{L_\infty}^2 >^{\frac{1}{2}} < ||\boldsymbol{\psi}(\mathbf{x})||_{L_1}^2 >^{\frac{1}{2}}, \tag{21}$$

where for a fixed time $t$, $||\boldsymbol{\phi}||_{L_\infty} = \sup_{\mathbf{x} \in \mathbb{R}^3} |\boldsymbol{\phi}(\mathbf{x}, t)|$.
Given the velocity field of a particular flow $\mathbf{u}(\mathbf{x}, t)$, the (time averaged) energy dissipation rate of that flow is defined to be, when the limit exists,

$$\varepsilon := \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \frac{\nu}{L^3} \int_{\mathbb{R}^3} |\nabla \mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x} \right\} dt = < \varepsilon(t) >, \tag{22}$$

where $|\nabla \mathbf{u}(\mathbf{x}, t)|^2 = \dfrac{\partial u^i}{\partial x_j}(\mathbf{x}, t) \cdot \dfrac{\partial u^i}{\partial x_j}(\mathbf{x}, t)$ and

$$\varepsilon(t) = \frac{\nu}{L^3} \int_{\mathbb{R}^3} |\nabla \mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x}.$$

In the remainder, we shall assume that the source term $\mathbf{f}$ is such that $< ||\mathbf{f}||_{L_2} >$ as well as $< t||\mathbf{f}||_{L_2} >$ exist. We are not able to deduce from the Navier-Stokes Equations the existence of $\varepsilon$, but we prove in appendix 10 that there exists a function $\theta(t)$ which converges to 0 when $t$ goes to infinity and such that

$$\begin{aligned} &\limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x} \right\} dt \leq \\ &< ||\mathbf{f}||_{L_2} > (||\mathbf{u}_0||_{L_2} + < t||\mathbf{f}||_{L_2} >) + < t\theta(t)||\mathbf{f}||_{L_2} > . \end{aligned} \tag{23}$$

In the remainder we shall assume the existence of $\varepsilon$. It is known for many turbulent flows that the energy dissipation rate $\varepsilon$ scales like $\dfrac{U^3}{L}$. This estimate follows for homogeneous, isotropic turbulence from the K41 phenomenology (see in Frisch[16], Lesieur[24] and Pope[33]) and has been proven by Constantin and Doering[10] and also Wang[39] as an upper bound directly from the Navier-Stokes equations for turbulent flows in bounded domains driven by persistent shearing of a moving boundary (rather than a body force). The same estimate has been proven by Foias[13] when the flow is driven by a persistent body force, the boundary conditions are periodic and the forcing acts on the largest modes.

If $\widehat{\mathbf{u}}(\mathbf{k},t)$ denotes the Fourier transform of $\mathbf{u}(\mathbf{x},t)$ where $\mathbf{k}$ is the wave-number vector and $k = |\mathbf{k}|$ is its magnitude, then the kinetic energy of the flow can be evaluated in physical space or in wave number space using the Fourier transform $\widehat{\mathbf{u}}$ of $\mathbf{u}$ at time $t$

$$\frac{1}{2}||\mathbf{u}||_{L_2}^2 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x},t)|^2 d\mathbf{x} = \frac{1}{2}\int_{\mathbf{R}^3} |\widehat{\mathbf{u}}(\mathbf{k},t)|^2 d\mathbf{k}. \tag{24}$$

Time averaging and rewriting the last integral in spherical coordinates gives

$$< \frac{1}{2L^3}||\mathbf{u}||_{L_2}^2 > = \int_0^\infty E(k)dk, \tag{25}$$

where $E(k)$ is the energy density which has for dimension the square of a velocity times a length. This is the amount *in time average* of kinetic energy for wave vectors $\mathbf{k}$ such that

$k \leq |\mathbf{k}| \leq k + dk$. It can also be defined by the formula

$$E(k) := \frac{1}{2L^3} \int_{|\mathbf{k}|=k} < |\widehat{\mathbf{u}}(\mathbf{k}, t)|^2 > d\boldsymbol{\sigma}. \tag{26}$$

The case of homogeneous, isotropic turbulence includes the assumption that (after time or ensemble averaging) $\mathbf{R}(\widehat{u}(\mathbf{k}, t))$ for any correlation tensor $\mathbf{R}$ depend only on $k$ and thus not the angles $\theta$ or $\varphi$. Thus for the simplicity, one may write $E$ under the form

$$E(k) = 2\pi k^2 < |\widehat{\mathbf{u}}(k)|^2 > . \tag{27}$$

Further, the K41 theory states that at high enough Reynolds numbers there is a range of wave numbers

$$0 < k_{\min} := U\nu^{-1} \leq k \leq \varepsilon^{\frac{1}{4}} \nu^{-\frac{3}{4}} =: k_{\max} < \infty, \tag{28}$$

known as the inertial range, beyond which the kinetic energy in u is negligible, and in this range

$$E(k) \doteq \alpha\, \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{29}$$

where $\alpha$ (in the range 1.4 to 1.7) is the universal Kolmogorov constant, $k$ is the wave number and $\varepsilon$ is the particular flow's energy dissipation rate. The energy dissipation rate $\varepsilon$ is the only parameter which differs from one flow to another. Outside the inertial range the kinetic energy in the small scales decays exponentially when $\varepsilon(t)$ is bounded, as shown in Foais and Temam[15] and also in Doering and Titi[11]. We shall make this assumption troughout the paper. Thus, we still have $E(k) \leq \alpha\,\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ since, after time averaging the energy in those scales is negligable, $E(k) \simeq 0$ for $k \geq k_{\max}$ and $E(k) \leq E(k_{\min})$ for $k \leq k_{\min}$. The fundamental assumption underlying our consistency error estimates is Assumption 2.2 below that over all wave numbers $E(k) \leq \alpha\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$. Indeed, in figure 6.14 page 235 of Pope's book[33] the power spectrums of 17 different turbulent flows are plotted and the above bound is obvious in the plot. To summerize, we will use the following elements of the K41 theory.

**Assumption 2.1** *The energy dissipation rate $\varepsilon(t)$ is bounded and the time averaged energy dissipation rate $\varepsilon$ is well defined and satisfies*

$$\varepsilon \leq C_1 \frac{U^3}{L}.$$

**Assumption 2.2** *The energy spectrum of the flow satisfies*

$$E(k) \leq \alpha\,\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}. \tag{30}$$

We show in the appendix in section 7 that a direct consequence of the assumptions 2.1 and 2.2 combined with the results of Doering and Titi[11], is the fact that the velocity field $\mathbf{u}$ has a $L_\infty$ norm in average, and that we have

$$< ||\mathbf{u}||^2_{L^\infty} >^{\frac{1}{2}} \leq C_2 U, \tag{31}$$

where $C_2 = C_2(\varepsilon_{\rm sup}, \varepsilon)$, $\varepsilon_{\rm sup} = \sup_{t\in\mathbb{R}^+} \varepsilon(t)$.

**Remark 2.1** *It is possible to obtain in a simplest way a bound for* $< ||\boldsymbol{\tau}_0||_{L_1} >$. *For this, we do not need to assume* $\varepsilon(t)$ *bounded.*

**Remark 2.2** *It is possible to define* $E$ *in the case of periodic boundary solutions as done in Olson and Titi*[32]*, without changing our results.*

# 3 Description of the ADM models

## 3.1 About the filter

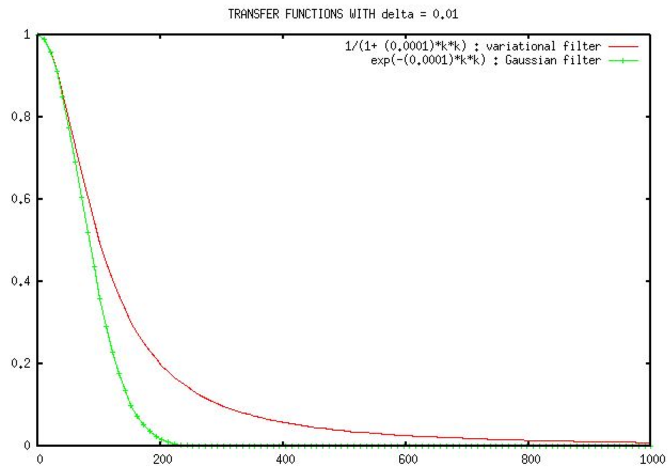
We study a model for spacial averages of the fluid velocity with the following differential filter. Let $\delta$ denote the averaging radius; given any field related to the turbulence $\boldsymbol{\phi}$ its average, denoted $\overline{\boldsymbol{\phi}}$, is the solution of the following problem

$$A^{-1}\overline{\boldsymbol{\phi}} := -\delta^2 \triangle \overline{\boldsymbol{\phi}} + \overline{\boldsymbol{\phi}} = \boldsymbol{\phi}, \quad \overline{\boldsymbol{\phi}} = A(\boldsymbol{\phi}), \tag{32}$$

which reads component by component if $\boldsymbol{\phi} = (\phi^{j_1 \dots j_q}_{i_1,\dots,i_p})$

$$-\delta^2 \frac{\partial^2 (\overline{\phi^{j_1 \dots j_q}_{i_1,\dots,i_p}})}{\partial x_i \partial x_i} + \overline{\phi^{j_1 \dots j_q}_{i_1,\dots,i_p}} = \phi^{j_1 \dots j_q}_{i_1,\dots,i_p}.$$

Figure 3.1: Transfer function of the variational filter together with Gaussian filter




Differential filters are well-established in LES, starting with work of Germano[18] and continuing in Galdi and Layton[17], Sagaut[35], and have many connections to regularization processes such as the Yoshida regularization of semigroups and the very interesting work of Foias *et al.*[14] (and others) on Lagrange averaging of the Navier-Stokes equations. As mentioned in the introduction, the transfer function of the filter $A$ is the function $\widehat{G}(k) = 1/(\delta^2 k^2 + 1)$.

## 3.2 Filtered motion equations

Averaging the NSE shows that the true flow averages satisfy the (non-closed) equations

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u} \otimes \mathbf{u}}) - \nu \triangle \overline{\mathbf{u}} + \nabla \overline{p} = \overline{\mathbf{f}} \quad \text{and} \quad \nabla \cdot \overline{\mathbf{u}} = 0. \tag{33}$$

The zeroth order model arises from the first order Taylor expansion $\mathbf{u} \simeq \overline{\mathbf{u}} + O(\delta^2)$, giving $\overline{\mathbf{u} \otimes \mathbf{u}} \simeq \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} + O(\delta^2)$. Calling $\mathbf{w}, q$ the resulting approximations to $\overline{\mathbf{u}}, \overline{p}$, we obtain the model studied in Layton-Lewandowski[20],[22]:

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot (\overline{\mathbf{w} \otimes \mathbf{w}}) - \nu \triangle \mathbf{w} + \nabla q = \overline{\mathbf{f}} \quad \text{and} \quad \nabla \cdot \mathbf{w} = 0. \tag{34}$$

## 3.3 Consistency error in the zeroth-order case

This zeroth order model's consistency error order two tensor $\boldsymbol{\tau}_0$ is given by, as mentionned in the introduction already:

$$\boldsymbol{\tau}_0 := \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} - \mathbf{u} \otimes \mathbf{u}, \tag{35}$$

where one recallls that $\boldsymbol{\tau}_0 = (\tau_0^{ij})_{1 \leq ij \leq 3}$, $(\tau_0^{ij}) = \overline{u^i}\,\overline{u^j} - u^i u^j$, $|\boldsymbol{\tau}_0|^2 = \tau_0^{ij}.\tau_0^{ij}$. It is worth pointing out that $\boldsymbol{\tau}_0$ is a function of $\delta/L$. The subsequent analysis will reveal its explicit dependence. Notice that our model differs from the one introduced by Bardina *et al*[2] where the following approximation is used: $\overline{\mathbf{u} \otimes \mathbf{u}} \simeq \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} - \overline{\overline{\mathbf{u}}} \otimes \overline{\overline{\mathbf{u}}} + \overline{\mathbf{u}} \otimes \overline{\mathbf{u}}$.

Subtracting the model (34) from the averaged NSE (33), on obtains the model's error equation satisfied by $\mathbf{e} = \overline{\mathbf{u}} - \mathbf{w}$, satisfies $\mathbf{e}(\mathbf{x}, 0) = 0, \nabla \cdot \mathbf{e} = 0$ and

$$\frac{\partial \mathbf{e}}{\partial t} + \nabla \cdot (\overline{\overline{\mathbf{u}} \otimes \mathbf{e} + \mathbf{e} \otimes \mathbf{w}}) - \nu \triangle \mathbf{e} + \nabla(\overline{p} - q) = \nabla \cdot \overline{\boldsymbol{\tau}_0}. \tag{36}$$

This equation is driven by the model's consistency error $\boldsymbol{\tau}_0$ through the term $\nabla \cdot \overline{\boldsymbol{\tau}_0}$. If the term $\nabla \cdot \overline{\boldsymbol{\tau}_0}$ is considered as a force and $\mathbf{e}$ as a displacement, the virtual work in the motion is the integral

$$\int_{\mathbb{R}^3} \mathbf{e} \cdot (\nabla \cdot \overline{\boldsymbol{\tau}_0}) = -\int_{\mathbb{R}^3} \overline{\boldsymbol{\tau}_0} : \nabla \mathbf{e},$$

where the equality follows from the Stokes formula. Such a remark combined with assumptions 2.1 and 2.2 and results in Doering-Titi[11] yields the following inequality

$$< \nu ||\nabla \mathbf{e}||_{L^2}^2 >^{1/2} \leq C < ||\boldsymbol{\tau}_0||_{L_2} > \tag{37}$$

where $C = C(\varepsilon_{\text{sup}}, \nu)$. The proof of the inequality (37) is given in the appendix of section 8 in the periodic case. To prove it, we have to assume that the velocity field is ergodic to replace the time average by the statistic average. We think that this technical point should be removable. However, we have to assume that $\varepsilon_{\text{sup}}$ is not too large. This means for instance that the source term $\mathbf{f}$ is not too large in $L_2$ averaged norm and also that the fluctuation of the field around the average is also not too large. This is the usual price to pay for obtaining rigorous regularity estimates in the context of the Navier-Stokes equations, as shown for instance in Temam[38]. Such an estimate can be viewed as an "à posteriori" estimate in the sense given in Bernardi *et all*[3]

Finally, notice that a small sharpening of a result in Layton-Lewandowski[22] yields on a time interval $[0, T]$ and without any particular assumption that $\nu ||\nabla \mathbf{e}||_{L^2}^2 = O(e^T ||\boldsymbol{\tau}_0||_{L_2})$. Unfortunatly, the time increasing rate $e^T$ does not allow to take time average in this estimate.

For such reasons, we consider that the modeling error is actually driven by $\boldsymbol{\tau}_0$ rather than $\nabla \cdot \overline{\boldsymbol{\tau}_0}$. Since the model is stable to perturbations[22], the *accuracy* of the model is governed by the size of various norms of its consistency error tensor $\boldsymbol{\tau}_0$. We choose in the remainder to seek for estimates of the non-dimensionalized $L_2$ norm of the residual stress because of estimate (37), considered as a *relative error*.

### 3.4 Generalized ADM models

The example above is the simplest (hence zeroth order) model in many families of LES models. We consider herein a family of Approximate Deconvolution Models (or ADM's) whose use in LES was pioneered by Stolz and Adams[1],[36]. The size of the $N^{\text{th}}$ models consistency error tensor directly determines the model's accuracy for these higher order model's as well as shown in Dunca and Epshteyn[12]. Let $G_N$ $(N = 0, 1, 2, ...)$ denote the van Cittert approximate deconvolution operator (see in Bertero and Boccacci[5]) given by

$$G_N \boldsymbol{\phi} = \sum_{n=0}^{N} (I - A)^n \boldsymbol{\phi} \qquad (38)$$

where $\boldsymbol{\phi}$ denotes any tensor related to the turbulence and the operator $A$ is defined in (32). It satisfies at first order $\mathbf{u} = G_N \overline{\mathbf{u}} + O(\delta^{2N+2})$, non uniformly in the wave numbers.

The models studied by Adams and Stolz[1],[36] (see also Stolz *et al.*[37]) are given by

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot (\overline{G_N \mathbf{w} \otimes \ G_N \mathbf{w}}) - \nu \triangle \mathbf{w} + \nabla q + \mathbf{w}' = \overline{\mathbf{f}} \quad \text{and} \quad \nabla \cdot \mathbf{w} = 0. \tag{39}$$

The $\mathbf{w}'$ term is included to damp strongly the temporal growth of the fluctuating component of w driven by noise, numerical errors, inexact boundary conditions and so on. The consistency error induced by adding the $\mathbf{w}'$ term is larger than that of the nonlinear term in smooth flow regions but is smaller than it in region of fully developed turbulence. While it does affect the model's dynamics, it does not affect the overall consistency error estimate. Thus, herein we drop the $\mathbf{w}'$ term.

For example, the induced closure model's corresponding to $N = 0$ and 1 are

$$G_0 \overline{\mathbf{u}} \ = \ \overline{\mathbf{u}}, \text{ so } \quad \overline{\mathbf{u} \otimes \mathbf{u}} \simeq \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} + O(\delta^2), \tag{40}$$

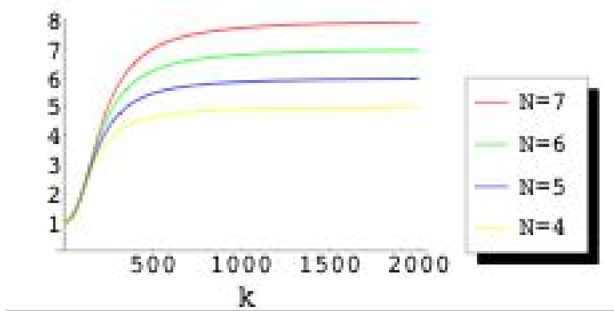
$$G_1 \overline{\mathbf{u}} \ = \ 2\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}}, \quad \text{so} \quad \overline{\mathbf{u} \otimes \mathbf{u}} \simeq \overline{(2\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}}) \otimes (2\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}})} + O(\delta^4). \tag{41}$$

The transfer function of the operator $G_N$ is the function

$$\widehat{G}_N = (1 + \delta^2 k^2) \left[ 1 - \left( \frac{\delta^2 k^2}{1 + \delta^2 k^2} \right)^{N+1} \right]. \tag{42}$$

The corresponding residual stress is defined by $\boldsymbol{\tau}_N = \overline{G_N \overline{\mathbf{u}} \otimes G_N \overline{\mathbf{u}}} - \overline{\mathbf{u} \otimes \mathbf{u}}$. Notice that Stolz and Adams recommend $N = 5$.

Figure 3.2: Plots of the transfer functions $\widehat{G}_4$, $\widehat{G}_5$, $\widehat{G}_6$, $\widehat{G}_7$ with $\delta = 0.01$




# 4 Proof of the main result

## 4.1 Error estimate in terms of the TKE

In this section, we prove the consistency error is $O(\delta^{1/3})$ uniformly in Re as claimed in the introduction. For clarity, we first consider the case of the zeroth order model which yields the system of equations (34).

We begin by showing that the error estimate we are looking for when $N = 0$, that is $< ||\boldsymbol{\tau}_0||_{L_2}/U^2L^{3/2} >$, is driven by the turbulent kinetic energy (TKE) $< ||\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2}/L^3 >$. The same method yields that the $N^{\text{th}}$-error $< ||\boldsymbol{\tau}_N||_{L_2}/U^2L^{3/2} >$ is driven by the generalized TKE, $< ||G_N\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2}/L^3 >$.

We recall here the following inequality proven in Lewandowski[27] for the periodic case.

$$||\overline{\mathbf{u}}||_\infty \leq ||\mathbf{u}||_\infty. \tag{43}$$

The proof in Lewandowski[27] can be easily adapted to the case where the data $\mathbf{f}$ and $\mathbf{u}_0$ have a compact support or satisfy right decreasing hypothesis at infinity. We assume now that we are in this situation. Moreover, one can also prove that for every $p \in [1, \infty[$ and every field $\boldsymbol{\varphi}$ one has

$$||\overline{\boldsymbol{\varphi}}||_{L_p} \leq ||\boldsymbol{\varphi}||_{L_p} \tag{44}$$

The fundamental consequence is the following. Note first that the following identity holds:

$$\boldsymbol{\tau}_0 = \overline{\mathbf{u}} \otimes (\overline{\mathbf{u}} - \mathbf{u}) + (\overline{\mathbf{u}} - \mathbf{u}) \otimes \mathbf{u}. \tag{45}$$

one deduces from the identity (45) and the inequality (43) above combined with the inequality (21), the inequality

$$\boxed{< ||\boldsymbol{\tau}_0||_{L_2} > \leq < ||\mathbf{u}||^2_{L_\infty} >^{1/2} < ||\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} >^{1/2}} \tag{46}$$

Therefore, Assumption (2.1) and (2.2) thru (31) yield the inequality

$$\boxed{< ||\boldsymbol{\tau}_0||_{L_2} > \leq 2C_2 U < ||\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} >^{1/2}} \tag{47}$$

Now the game consists in the evaluation the TKE, $K =< ||\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2}/L^3 >$.

## 4.2 The $\delta^{1/3}$ bound for the zeroth order model

Thanks to the definition of the energy's density $E$ and using the transfer function of the filter, one may write

$$< ||\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} > \leq \int_{k_{\min}}^{k_{\max}} \left(1 - \frac{1}{1+\delta^2 k^2}\right)^2 E(k) dk. \tag{48}$$

By using the $-5/3$ K41 law, one obtains

$$< ||\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} > \leq \alpha \varepsilon^{2/3} L^3 \int_{k_{\min}}^{k_{\max}} \left(\frac{\delta^2 k^2}{1+\delta^2 k^2}\right)^2 k^{-5/3} dk := I. \tag{49}$$

We have to evaluate the integral $I$ which appears in r.h.s of the previous inequality. It requires different treatments for small and large wave numbers. The transition point is the cutoff wave number $\delta$. Thus we break the integral $I$ into to integrals

$$I = I_{\text{low}} + I_{\text{high}}, \quad I_{\text{low}} = \int_{k_{\min}}^{1/\delta} ...dk, \quad I_{\text{high}} = \int_{1/\delta}^{k_{\max}} ...dk.$$

For the low frequency components we have $\delta^2 k^2/(1+\delta^2 k^2) \leq \delta^2 k^2$. Therefore

$$I_{\text{low}} \leq \alpha L^3 \varepsilon^{\frac{2}{3}} \delta^4 \int_{k_{\min}}^{1/\delta} k^{\frac{7}{3}} dk \leq \alpha L^3 \varepsilon^{\frac{2}{3}} \delta^4 \int_0^{1/\delta} k^{\frac{7}{3}} dk = \frac{3}{10} \alpha L^3 \varepsilon^{\frac{2}{3}} \delta^{\frac{2}{3}}. \tag{50}$$

For the high frequency components we have $\delta^2 k^2/(1+\delta^2 k^2) \le 1$. We deduce the inequalities

$$I_{\text{high}} \le \alpha L^3 \varepsilon^{\frac{2}{3}} \int_{1/\delta}^{k_{\max}} k^{-5/3} dk \le \alpha L^3 \varepsilon^{\frac{2}{3}} \int_{1/\delta}^{\infty} k^{-5/3} dk = \frac{3}{2} \alpha L^3 \varepsilon^{\frac{2}{3}} \delta^{\frac{2}{3}}, \tag{51}$$

where $\alpha$ is the Komogorov constant whose value is in the range $[1.4, 1.7]$. Using Assumption 2.1 combined with the above inequalities gives

$$\boxed{< \frac{1}{L^{3/2} U^2} ||\boldsymbol{\tau}_0||_{L_2} > \le 3.6\, C_2 C_1^{\frac{1}{3}} \left( \frac{\delta}{L} \right)^{\frac{1}{3}}} \tag{52}$$

In the inequality above, the upper estimate $\alpha \le 1.7$ was used and $C_1$ is the $O(1)$ constant in Assumption 2.1.

## 4.3 General ADM model

Recall that $\boldsymbol{\tau}_N = G_N \overline{\mathbf{u}} \otimes G_N \overline{\mathbf{u}} - \mathbf{u} \otimes \mathbf{u}$ is the residual stress corresponding to the general ADM model. One derives here a bound for the quantity $< ||\boldsymbol{\tau}_N||_{L_1}/U^2 L^3 >$. One may write $\boldsymbol{\tau}_N$ under the form

$$\boldsymbol{\tau}_N = (G_N \overline{\mathbf{u}} - \mathbf{u}) \otimes G_N \overline{\mathbf{u}} + \mathbf{u} \otimes (G_N \overline{\mathbf{u}} - \mathbf{u})$$

We show in appendix 9 the following generalization of (43):

$$||G_N\overline{\mathbf{u}}||_{L_\infty} \leq ||\mathbf{u}||_{L_\infty}. \tag{53}$$

Therefore

$$< ||\tau_N||_{L_2} >\leq 2 < ||\mathbf{u}||^2_{L_\infty} >^{\frac{1}{2}} < ||G_N\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} >^{\frac{1}{2}} . \tag{54}$$

The game now consists in evaluating $L^3 K_N =< ||G_N\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} >$. As before and due to the knowledge of the $\widehat{G}_N$ transfer function (see the formula (42)) multiplied by the inverse of transfer function of the operator $A$, $< ||G_N\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} >$ is computed thanks to the integral

$$< ||G_N\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} >= I = L^3 \int_{k_{\min}}^{k_{\max}} \left( \frac{\delta^2 k^2}{1 + \delta^2 k^2} \right)^{2N+2} E(k)dk.$$

the integral $I$ is as before broken into to parts, $I_{\text{low}}$ for the frequency component less than $\delta^{-1}$ and $I_{\text{high}}$ for the frequency component greater than $\delta^{-1}$. We use the same inequalities as we did before combined with the K41. We skip the technical details, but just mention that we find

$$I_{\text{low}} \leq \left( \frac{1}{4N + \frac{10}{3}} \right) C_1^{\frac{2}{3}} \alpha U^2 L^3 \left( \frac{\delta}{L} \right)^{\frac{2}{3}}, \quad I_{\text{high}} \leq \frac{3}{2} C_1^{\frac{2}{3}} \alpha U^2 L^3 \left( \frac{\delta}{L} \right)^{\frac{2}{3}}, \tag{55}$$

which yields the inequality

$$< ||G_N\overline{\mathbf{u}} - \mathbf{u}||^2_{L_2} >\leq \left( \frac{3}{2} + \frac{1}{4N + \frac{10}{3}} \right) C_1^{\frac{2}{3}} \alpha U^2 L^3 \left( \frac{\delta}{L} \right)^{\frac{2}{3}} \tag{56}$$

One deduces from this the bound

$$\boxed{< \frac{1}{U^2 L^{3/2}} ||\boldsymbol{\tau}_N||_{L_2} > \leq 2\Psi_N \left(\frac{\delta}{L}\right)^{\frac{1}{3}}} \tag{57}$$

$\Psi_N^2 = \left(\frac{3}{2} + \frac{1}{4N + \frac{10}{3}}\right) C_2 C_1^{\frac{2}{3}} \alpha$. Notice that $\Psi_\infty \approx 1.5\, C_2 C_1^{1/3} \leq \Psi_N \leq \Psi_0 \approx 1.8\, C_2 C_1^{1/3}$ for $\alpha \in [1.4, 1.7]$.

**Remark 4.1** *The main analytical fact in the bound above is the fact that the transfer function $\widehat{G}$ satisfies $|1 - \widehat{G}(k)| \leq \delta^2 k^2$ for the low frequencies and $|1 - \widehat{G}(k)| \leq 1$ for the high frequencies. We remark that the Gaussian filter ($e^{-\delta^2 k^2}$) satisfies the same formal properties. Therefore the same bound holds and this is the case for any second order filter having the same characteristics.*

## 4.4 Dimensional analysis

The bound above are obtained thanks to the $-5/3$ Komogov's law. We show now that one can give a physical sense to the $\delta^{1/3}$ law to make it consistent as a feature of turbulence. The computation above shows that under this law, the bound for $< ||\boldsymbol{\tau}_N||_{L_2} >$ is of order $\varepsilon^{1/3}\delta^{1/3}$ and is driven by $\sqrt{L^3 K_N}$. This lead us to postulate a law of the form $\mathcal{F}(\sqrt{K}_N, \varepsilon, \delta) = 0$. By the $\Pi$-Theorem, we see that there exists a nondimensional number $\chi_N$ be such that

$$\sqrt{K_N} = \chi_N\, \varepsilon^{1/3} \delta^{1/3}. \tag{58}$$

The basic inequality (54) can be rewritten under the form

$$< ||\boldsymbol{\tau}_N||_{L_2} > \leq C_2 U L^{3/2} \sqrt{K}_N \tag{59}$$

We obtain then the inequality

$$< \frac{||\boldsymbol{\tau}_N||_{L_2}}{L^{3/2}U^2} > \leq \frac{\chi_N C_2}{U} \varepsilon^{1/3} \delta^{1/3}.$$

This is exactly the form of the bound (57). The analytical considerations above show that $\chi_N$ is bounded with respect to $N$. It must be stressed that this law, when it is derived by dimensional considerations, is obtained without the $-5/3$ law and is valid for every kind of filter, as for instance a statistical filter. The link to the K41 and the particular form of the LES filter allows for direct computations. However, this dimensional analysis argument suggests existence of a deeper physical principle.

# 5 Conclusions and discussion

## 5.1 From the initial questions : first observations

As suggested in the introduction, this work has been generated following the mathematical analysis of the ADM zeroth order model in Layton-Lewandowski[20],[22]. In those work, we were able to prove that the zeroth order ADM model converges in some abstract mathematical sense to the Navier-Stokes equations when $\delta$ goes to zero (this has been generalized for every fixed $N$ by Dunca and Epshteyn[12]). Therefore, the problem was to evaluate the rate of convergence to know if the model is "consistent" and simulation with the model "feasible". Consistent asks "how does $\delta$ be small such that the relative error is small with respect to 1". Feasibility asks "are fewer degrees of freedom required to simulate the model than required by a DNS" and to know if this number is compatible with the actual computer's power.

We have choosen to study this question analytically with the variational filter we have studied in [20] and [22] together with the $-5/3$ Kolmogorov's law. This yields a $\delta^{1/3}$ law satisfied by the consistency error bound. This law seems to be also satisfied in the case of the Gaussian filter and probably for a large class of second order filters.

We note first that the constant involved in this law does not depend on the Reynolds number. This constant also remains bounded with respect to $N$, a bound which depends on the features of the flow. Unfortunatly, this bound does not goes to zero when $N$ goes to infinity[6]. Analytical study of consistency error leads to splitting the contribution of the TKE into two parts. The first one concerns the low frequencies component, $I_{\text{low}}$. The second one concerns the high frequency component, $I_{\text{high}}$. The inequalities (55) show that the component $I_{\text{low}}$ goes to zero when $N$ goes to infinity. The component $I_{\text{high}}$ remains bounded but there is no reason to tell that it goes to zero when $N$ goes to infinty[7].

[6] One must say here that we have tried to prove the mathematical convergence of the ADM models to the Navier-Stokes equations when $N$ goes to infinity for a fixed $\delta$. We failed with the classical tools mathematical fluid dynamics. This is mainly due to the lack of informations on the fields in the high frequency components.

[7] The obstruction is of the same type when trying to use the functional analysis to solve this question of

## 5.2 About consistency and feasibility

Recall first that the Reynolds number is not involved in the bound above. The model is consistent when $\delta^{1/3} = o(1) \approx 10^{-1}$ (when $L = 1$), that is $\delta = O(10^{-3})$. Let us see what is the practical consequence in terms of pratical simulations.

Computers todays are 8-bytes words. Assume that the code used is based on a finite elements method. The main computational concern is the central memory loading of the corresponding stiffness matrix. Assume that $\delta \approx 2\Delta x \approx 10^p$, where $\Delta x$ is the mesh size. Then the matrix is about $10^{3p}$ lines and each line has about 200 non zero coefficients. Therefore, the required bytes numbers is of order $8 \times 200 \times 10^{3p}$, that is about $10^{3p+1}$. Consistency requires $p = 3$. The consideration above yield that such a calculation will require a computer having a central memory of about 10 Go. If such a computer is not a classical PC, there exists today clusters with such a power, yielding practical feasibility.

## 5.3 A physical interpretation : the link to usual laws

When trying to seek for a law under the form $\mathcal{F}(\sqrt{K_N}, \varepsilon, \delta) = 0$, we are lead to the law (58), that is $\sqrt{K_N} = \chi_N \, \varepsilon^{1/3}\delta^{1/3}$. Now the question arises what can $\delta$ physically be, independly of any computational considerations. This suggests a connection with the natural Prandtl mixing lenght usually denoted by $\ell$. Therefore, taking $\delta = \ell$ this law (58) can be rewritten

convergence: we are not able to keep a control on the high frequency components, a difficulty well illustrated by the shapes of the transfer functions $\widehat{G}_N$ given in figure 3.2.

under the form

$$\varepsilon = \chi_N^{-3} \frac{K_N \sqrt{K_N}}{\ell}. \tag{60}$$

Here we recognize the classical law used when one closes the $k - \varepsilon$ system to avoid the doubtful $\varepsilon$-equation (see in Lewandowski[25],[26], in Brossier-Lewandowski[6] or in Mohammadi-Pironneau[31]). Indeed, this law supposes that $\varepsilon$, $\ell$ and $K$ are linked and this law follows from the classical Π-Theorem. One may object that the considered $\varepsilon$ in the $k - \varepsilon$ model is the average of the dissipation due to the fluctuations and here $\varepsilon$ is the total dissipation. But the difference between these two objects is also controled by the residual stress, yielding the same laws.

The dimensional analysis is a useful way to predict the 1/3 law that we have derived analytically. This leads to more questions about the nature of the $-5/3$ Kolmogorov's law, eddy viscosity and how the small scales (smaller than $O(\delta)$) act on the large scales (the "large eddies").

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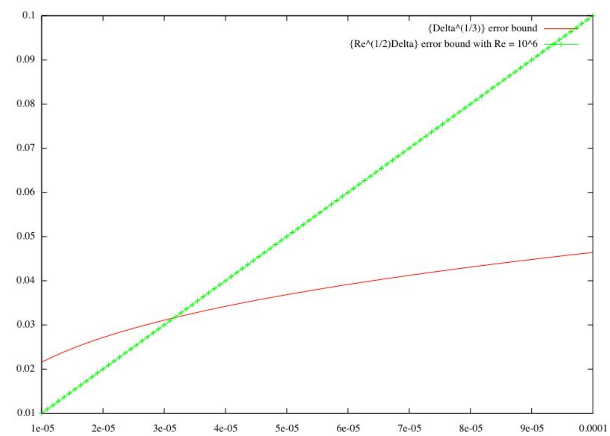
# 6 Appendix : bound without the K41 Phenomenology

## 6.1 Analytical $\delta\,\mathrm{Re}^{1/2}$ bound

In this appendix, we give an analytical bound without using the $-5/3$ K41 law. The result is a bound in $\delta\mathrm{Re}^{1/2}$. In the next subsection, we compare this bound to the $\delta^{1/3}$ bound.

Figure 6.1: Curves comparing the bound in $\delta^{\frac{1}{3}}$ together with the bound in $\mathrm{Re}^{\frac{1}{2}}\delta$ when $\mathrm{Re} = 10^6$.



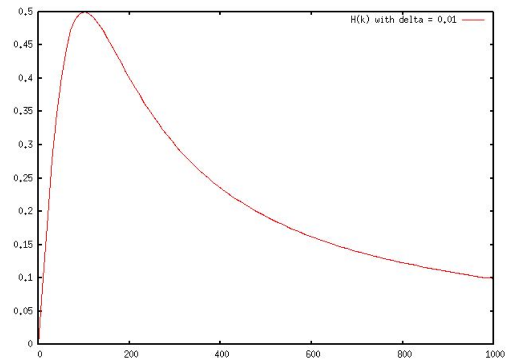

As before, we need to estimate the TKE $< ||\overline{\mathbf{u}} - \mathbf{u}||_{L_2}^2 / L^3 >$. By a direct computation one has

$$\widehat{\overline{\mathbf{u}}}(\mathbf{k}) - \widehat{\mathbf{u}}(\mathbf{k}) = \delta H(k) k \widehat{\mathbf{u}}(\mathbf{k}), \tag{61}$$

where $k = |\mathbf{k}|$, $H(k) = \delta k/(1+\delta^2 k^2)$, and satisfies $|H(k)| \leq 0.5$. By noticing that $||\nabla \mathbf{u}||_{L_2} = k\widehat{\mathbf{u}}(\mathbf{k})\,.k\widehat{\mathbf{u}}(\mathbf{k})$, one deduces that

$$||\overline{\mathbf{u}} - \mathbf{u}||_{L_2} \leq \frac{\delta}{2} ||\nabla \mathbf{u}||_{L_2}. \tag{62}$$

Figure 6.2: Plotting of the function $H(k)$ with $\delta = 0.01$




Therefore, thanks to the basic inequality (47) the following holds,

$$< ||\boldsymbol{\tau}_0||_{L_2} > \leq C_2 \delta L^{3/2} U \nu^{-1/2} < \frac{1}{L^3} \nu ||\nabla \mathbf{u}||_{L_2}^2 >^{1/2} \leq C_2 \delta \nu^{-\frac{1}{2}} L^{3/2} U \varepsilon^{\frac{1}{2}}. \tag{63}$$

which yields by using assumption 2.1,

$$\boxed{< \frac{1}{L^{3/2} U^2} ||\boldsymbol{\tau}_0||_{L_2} > \leq C_2 C_1^{\frac{1}{2}} \frac{\delta}{L} Re^{\frac{1}{2}}} \tag{64}$$

## 6.2 Comparison to the $\delta^{1/3}$ bound

We have plotted in figure 6.1 both curves comparing the $\delta^{1/3}$ law with the $\delta \mathrm{Re}^{1/2}$ bound. The $\delta^{1/3}$ law gives a better bound until a critical $\delta_c$ where the curves intersect each other. When $\delta < \delta_c$, one observes that the $\delta \mathrm{Re}^{1/2}$ gives a better result. A simple computation yields

$\delta_c = O(\mathrm{Re}^{-3/4})$, which fits perfectly with the $k_{\max} \approx \varepsilon^{1/4}\nu^{-3/4}$ predicted by the Kolmogorov's law. That means that the bound obtained without the $-5/3$ law begins to be better when the flow is fully resolved. We do not have any real explanation for this, but it has been so striking to us that it must be mentioned.

# 7 Appendix : A regularity result

One shows here how hypothesis 2.1 together with (30) yields inequality (31).

Because $\varepsilon(t)$ is bounded, one knows from Doering-Titi[11] that $E(k)$ decreases exponentially for $k \geq \varepsilon_{\sup}^{1/4}\nu^{-3/4}$. This informtaion together with (30) makes sure that for every $s > 0$,

$$\int_0^\infty (1+k^2)^{\frac{s}{2}} E(k)dk \leq C(s, \varepsilon, \varepsilon_{\sup}) < \infty.$$

Therefore, for each $s > 0$, one has

$$< ||\mathbf{u}||_{H^s}^2 > \leq C(s, \varepsilon, \varepsilon_{\sup}).$$

By Sobolev embedding Theorem in the case $s = 2$ and in a non dimensional form we get

$$< ||\mathbf{u}||_{L_\infty}^2 / U^2 > \leq \tilde{C}(s, \varepsilon, \varepsilon_{\sup}),$$

and (31) follows.

# 8 Appendix : Estimation of the error thru the residual stress

We prove inequality (37).

To fix the ideas, we focus here on the case of periodic boundary conditions, $\Omega = [0, L]^3$ being the cell period.

We assume that the field is ergodic. Therefore, one can replace the long time average by the statistical mean. The point of this is that now the filter commutes with time and space derivative. In the context of a fully developed isotropic and homgeneous turbulence this mean is time independant.

All over this work and following Doering-Titi[11] and Foias-Temam[15] we have assumed that $\varepsilon(t)$ is bounded, a reasonable assumption as soon as we are in the case of a steady-state developed turbulence. Notice that this question is far to be solved from the mathematical view point and is one of the subject of the famous "millenium price". Indeed, it is known that to be in $L^\infty(H^1)$ for the velocity yields uniqueness and the energy balance (17) is an equality. Morever, following arguments in Lewandowski[26] [25], it can be proven that this assumption leads to strong convergence of $\mathbf{e}$ to zero in $H^1$ space type (and not only weak convergence) for $\delta$ going to zero. For this reason, we conjecture here that this assumption hold also for the approximation $\mathbf{w}$, that is

$$\nu \int_\Omega |\nabla \mathbf{w}(\mathbf{x},t)|^2 d\mathbf{x} \le \varepsilon_{\text{sup}}, \tag{65}$$

Proof of (65) can be given on a fixed time interval $[0,T]$ [28], and we take it for grant here on $[0,\infty[$. Notice that it leads directly to

$$\nu \int_\Omega |\nabla \mathbf{e}(\mathbf{x},t)|^2 d\mathbf{x} \le 4\,\varepsilon_{\text{sup}}. \tag{66}$$

Let us first recall the equation satified by the error $\mathbf{e}$:

$$\frac{\partial \mathbf{e}}{\partial t} + \nabla \cdot (\overline{\overline{\mathbf{u}} \otimes \mathbf{e} + \mathbf{e} \otimes \mathbf{w}}) - \nu \triangle \mathbf{e} + \nabla(\overline{p} - q) = \nabla \cdot \overline{\boldsymbol{\tau}_0}, \quad \nabla \cdot \mathbf{e} = 0, \tag{67}$$

Take $A^{-1}\mathbf{e}$ as test vector field in (67) and integrate by parts, where $A$ is defined by (32) and $\overline{\phi} = A\phi$. We obtain, arguing as in Layton-Lewandowski[22],

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{||\mathbf{e}||^2 + \delta^2 ||\nabla \mathbf{e}||^2\} + \nu\{||\nabla \mathbf{e}||^2 + \delta^2 ||\Delta \mathbf{e}||^2\} \\ &+ (\nabla \cdot (\overline{\mathbf{u}} \otimes \overline{\mathbf{u}} - \mathbf{w} \otimes \mathbf{w}), \mathbf{e}) = -(\boldsymbol{\tau}, \nabla \mathbf{e}), \end{aligned} \tag{68}$$

where $(\cdot,\cdot)$ is the $L_2$ scalar product and $||\cdot||$ is the $L_2$ norm.

Writing $\overline{\mathbf{u}} \otimes \overline{\mathbf{u}} - \mathbf{w} \otimes \mathbf{w} = \mathbf{w} \otimes \mathbf{e} + \mathbf{e} \otimes \overline{\mathbf{u}}$ and using $(\nabla \cdot (\mathbf{w} \otimes \mathbf{e}), \mathbf{e}) = 0$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{||\mathbf{e}||^2 + \delta^2||\nabla \mathbf{e}||^2\} + \nu\{||\nabla \mathbf{e}||^2 + \delta^2||\Delta \mathbf{e}||^2\} \\ &= -(\boldsymbol{\tau}, \nabla \mathbf{e}) - (\mathbf{e}\,.\nabla \overline{\mathbf{u}}, \mathbf{e}), \end{aligned} \tag{69}$$

The inequality (69) combined to (66) and the Sobolev inequality (combined with Poincaré's inequality on $\Omega$), leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{||\mathbf{e}||^2 + \delta^2||\nabla \mathbf{e}||^2\} + \nu\{||\nabla \mathbf{e}||^2 + \delta^2||\Delta \mathbf{e}||^2\} \\ &\leq \sqrt{\frac{\varepsilon_{\text{sup}}}{\nu}} \left(2||\boldsymbol{\tau}_0|| + C||\nabla \mathbf{e}||^2\right), \end{aligned} \tag{70}$$

One takes the average of (69) taking into account that the average commutes with the time derivative (ergodicity) and that the average is stationnary. One get

$$\nu < ||\nabla \mathbf{e}||^2 > \leq \sqrt{\frac{\varepsilon_{\text{sup}}}{\nu}} \left(2 < ||\boldsymbol{\tau}_0|| > + C < ||\nabla \mathbf{e}||^2 >\right), \tag{71}$$

We assume now that $\varepsilon_{\text{sup}}$ is such that

$$0 < \nu - C\sqrt{\frac{\varepsilon_{\text{sup}}}{\nu}} = \kappa$$

Therefore

$$\nu < ||\nabla \mathbf{e}||^2 > \leq 2\frac{\sqrt{\nu \varepsilon_{\text{sup}}}}{\kappa} < ||\boldsymbol{\tau}_0|| >$$

which is inequality (37).

# 9 Appendix : $L_\infty$ estimate of $G_N\overline{\mathbf{u}}$

We prove here the inequality (53). The proof here is different than the proof of estimates (43) and (44) in Lewandowski [27], and works only for $L_\infty$ norm and not for every $L_p$ norms. Thanks to a result of Dunca and Epshteyn[12], one knows that $G_N\overline{\mathbf{u}}$ satisfies

$$\mathbf{u} - G_N\overline{\mathbf{u}} = (-1)^{N+1}\left(\frac{\delta}{L}\right)^{2N+2}\Delta^{N+1}\overline{\mathbf{u}} \tag{72}$$

which can be rewritten under the form

$$AG_N\overline{\mathbf{u}} + G_N\overline{\mathbf{u}} = \mathbf{u}, \tag{73}$$

where

$$A = B \circ G_N^{-1}, \quad B = (-1)^{N+1}\left(\frac{\delta}{L}\right)^{2N+2}\Delta^{N+1}.$$

The operators $B$ and $G_N^{-1}$ are self adjoint pseudo differential operators which commute. Moreover, as shown in Dunca-Epshteyn[12], $G_N^{-1}$ is a non negative operator, as well as the operator $B$. Finally, if $w$ is a scalar field, $w^+$ its positive part and $w^-$ its negative part, it ie easy checked that $< Aw^+, w^- >=< Aw^-, w^+ >= 0$. Therefore, arguing as in Lewandowski[25] chapter 4, there is a maximum principle to equation of type (73), that means that if $w$ is a solution to

$$Aw + w = f,$$

where $f$ is a bounded function, one has

$$\inf f \leq w \leq \sup f, \quad \text{almost everywhere.}$$

In particular $||G_N \overline{\mathbf{u}}||_{L_\infty} \leq ||\overline{\mathbf{u}}||_{L_\infty}$, which is inequality (53).

# 10 Appendix : estimate of the total dissipation

The aim of this appendix is to prove the inequality 23. We are in the case of the whole space. We assume that $\mathbf{f}$ has the required regularity to ensure the validity of the computations below. From (18) one get

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}(\cdot,t)||_{L_2}^2 \leq ||\mathbf{f}(\cdot,t)||_{L_2}||\mathbf{u}(\cdot,t)||_{L_2}. \tag{74}$$

Then one has

$$||\mathbf{u}(\cdot,t)||_{L_2} \leq ||\mathbf{u}_0||_{L_2} + \int_0^t ||\mathbf{f}(\cdot,\tau)||_{L_2} d\tau \tag{75}$$

Now from (17) again we deduce

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}(\cdot,t)||_{L_2}^2 + \nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \leq ||\mathbf{f}(\cdot,t)||_{L_2}||\mathbf{u}(\cdot,t)||_{L_2} \tag{76}$$

which combined to (75) yields

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}(\cdot,t)||_{L_2}^2+\nu\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2 d\mathbf{x}\leq||\mathbf{f}(\cdot,t)||_{L_2}\left(||\mathbf{u}_0||_{L_2}+\int_0^t||\mathbf{f}(\cdot,\tau)||_{L_2}d\tau\right) \tag{77}$$

We integrate with respect to the time and observe that for large $t$,

$$\int_0^t||\mathbf{f}(\cdot,\tau||_{L_2}d\tau=t<||\mathbf{f}||_{L_2}>+t\theta(t),$$

where $\theta(t)$ goes to zero when $t$ goes to infinity. Therefore one has

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}(\cdot,t)||_{L_2}^2+\nu\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2\leq||\mathbf{f}(\cdot,t)||_{L_2}\left(||\mathbf{u}_0||_{L_2}+t<||\mathbf{f}||_{L_2}>+t\theta(t)\right). \tag{78}$$

One integrates on the time interval $[0,t]$, divide by $t$ and obtain

$$\begin{aligned}&\frac{1}{t}\int_0^t\left(\nu\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2(\mathbf{x},\tau)d\mathbf{x}\right)d\tau\leq||\mathbf{u}_0||_{L_2}\frac{1}{t}\int_0^t||\mathbf{f}(\cdot,\tau)||_{L_2}d\tau+\\&<||\mathbf{f}||_{L_2}>\frac{1}{t}\int_0^t\tau||\mathbf{f}(\cdot,\tau)||_{L_2}d\tau+\frac{1}{t}\int_0^t\tau\theta(\tau)||\mathbf{f}(\cdot,\tau)||_{L_2}d\tau+\frac{1}{t}||\mathbf{u}_0||_{L_2}^2.\end{aligned} \tag{79}$$

When $t$ goes to infinity one get

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\{\nu\int_{\mathbb{R}^3}|\nabla\mathbf{u}(\mathbf{x},t)|^2d\mathbf{x}\right\}dt\leq\\&<||\mathbf{f}||_{L_2}>(||\mathbf{u}_0||_{L_2}+<t||\mathbf{f}||_{L_2}>)+<t\theta(t)||\mathbf{f}||_{L_2}>.\end{aligned} \tag{80}$$