

On a turbulent system with unbounded eddy viscosities

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Abstract

We study an elliptic coupled system arising in turbulence modelling in fluid dynamics with unbounded eddy viscosities. We introduce a notion of “energy solution” adapted to this system and we prove the existence of such an energy solution.

1 Introduction

We consider in this paper the following coupled system satisfied by the unknown scalar functions u and k :

$$(S) \quad \begin{cases} -\operatorname{div}(\nu(k)\nabla u) = f & \text{in } \Omega, & (1.1) \\ -\operatorname{div}(a(k)\nabla k) = \nu(k)|\nabla u|^2 & \text{in } \Omega, & (1.2) \\ u = 0 & \text{on } \partial\Omega, & (1.3) \\ k = 0 & \text{on } \partial\Omega. & (1.4) \end{cases}$$

In this system, the functions ν and a are real valued functions of k which represent eddy viscosities. This system is a mathematical subproduct of the large scale one degree closure Reynolds system used by engineers, oceanographers, meteorologists and others for simulating turbulent flows. (The reader can find details concerning modelization in [10], [13] and [3].) The variable k is the turbulent kinetic energy and the variable u an “idealization” of the velocity of the flow. The interest in studying the mathematical system (S) lies in a better understanding of the interaction between an eddy diffusion

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term like $\operatorname{div}(\nu(k)\nabla u)$ and the transfer of kinetic energy from large scales to small scales defined by the term $\nu(k)|\nabla u|^2$.

Note that in equation (1.2) the term modelizing the dissipation of small scales in the right hand side of the equation was ignored. This term is equal to $-\frac{k^{3/2}}{\ell}$ where $\ell \neq 0$ is the mixing length of the turbulence. This simplification gives rise to a simplified mathematical presentation which does not change the nature of the difficulties and the mathematical structure of the equations. (Of course, this is not the case when ℓ goes to zero. This question is not the aim of the present paper where $\ell \neq 0$ is fixed and the problems concerned with variations of ℓ are investigated in [3].)

When a and ν are continuous bounded functions of k , existence of a solution (u, k) to system (S) understood in distributional sense has been shown in [5],[9],[10],[11]. Moreover, the full incompressible system with a pressure term where u is the mean velocity field of a flow is considered in [11]. In this work, existence of a distributional solution is proved in the 2D evolution case and in the 3D steady state case. The 3D evolution case remains an open problem, even in the case of small initial data and small source terms. In all situations, uniqueness is an open problem, except in [3] where uniqueness of the solution for the scalar system (S) is proved under the condition that the eddy viscosities remain bounded and close to a constant.

When the eddy viscosities are not bounded functions of k , the question is more delicate from the mathematical point of view, but more realistic from the physical point of view. Indeed, both eddy viscosities ν and a are of the form $C + \ell\sqrt{k}$ for C a constant, and ℓ the turbulence mixing length. Existence of a solution (u, k) to the scalar system (S) has been shown by S. Clain and R. Touzani in the 2D case [6]. The main point there consists in proving that under a growth condition for ν and a (which is satisfied in the physical case), the kinetic turbulent energy k remains bounded, which reconduces one to the case of bounded eddy viscosities. Unfortunately, the result is not proved in the vector case with pressure, and it seems that there is no hope to prove the same estimate in the 3D case, even for the scalar system (S) .

In [10] (chapter 5), R. Lewandowski and F. Murat proved the existence of a renormalized solution to the scalar system (S) without neither growth conditions on the eddy viscosities nor restrictions on the dimension. The notion of renormalized solution adapted to (S) is derived from that used by P.-L. Lions and F. Murat for elliptic equations ([12]), following the work of R. DiPerna and P.-L. Lions (see for instance [7] and [8], and more references

therein). Unfortunately, this notion cannot be extended to a realistic fluid dynamic problem because of the pressure term. Thus, it remains to find a natural notion of solution to system (S) which can be extended to real situations as those described by Navier-Stokes equations. This is the aim of the present paper.

The natural notion is what we call “energy solutions” (see Definition 2.2 below). The idea is derived first from the natural estimate $\nu(k) |\nabla u|^2 \in L^1(\Omega)$, which holds for any “a priori solution” (u, k) of (S) , and from the fact that for every function v such that $\nu(k) |\nabla v|^2 \in L^1(\Omega)$, one has $\nu(k) \nabla u \nabla v \in L^1(\Omega)$. Thus the natural function space for equation (1.1) is the space of those functions $v \in H_0^1(\Omega)$ such that $\nu(k) |\nabla v|^2 \in L^1(\Omega)$, equipped with the obvious norm, from which the natural variational formulation for equation (1.1) follows, whereas equation (1.2) holds in the sense of the distributions (see Definition 2.2 below). The surprising feature in Definition 2.2 is that the space of test functions depends on the variable k . This space should be large enough, and in particular we want it to be the adherence of smooth functions with compact support. For that, we remark that under condition (2.9) (see below) on the eddy viscosities a and ν , added to the hypothesis that $\nu'(k)$ is bounded for k near 0 (see (2.10) below), one has $\sqrt{\nu(k)} \in H^1(\Omega)$. We note that condition (2.9) is satisfied by a and ν of the form $C + \ell\sqrt{k}$, but unfortunately the condition (2.10) is not verified. Thus one has to replace $C + \ell\sqrt{k}$ by $C + \ell\sqrt{\varepsilon + k}$, $\varepsilon > 0$, for small values of k . On the other hand, we prove that if $b \geq 1$ and $\sqrt{b} \in H^1(\Omega)$, then smooth functions with compact support are dense in the weighted Sobolev space

$$V = \left\{ v \in H_0^1(\Omega), b|\nabla v|^2 \in L^1(\Omega) \right\}$$

equipped with the norm

$$\left(\int_{\Omega} b|\nabla v|^2 dx \right)^{\frac{1}{2}}$$

(see Theorem 3.1 below). This result was suggested by a work of P. Cattiaux and M. Fradon [4], where an analytic proof of the analogous result in the full space is given. The proof we give here holds for every bounded domain with Lipschitz continuous boundary, and is rather different and shorter than the proof given in [4].

The main result of the present paper is Theorem 2.1 below, where existence of an energy solution (u, k) to system (S) is proved. The study

of energy solutions in the case of an incompressible fluid equation with a pressure term is in progress. Uniqueness is an open problem and seems to be a very difficult question, even if we conjecture that the solution is unique.

The paper is organized as follows. In Section 2, we define the notion of energy solution and we sketch the proof of the existence result. In Section 3, we prove the density of smooth functions with compact support in weighted Sobolev spaces under the assumptions that the weight is greater than 1 and has its square root in $H^1(\Omega)$. In Section 4, we prove the a priori estimates we need for solving the problem. In Section 5 we pass to the limit in the equations and complete the proof of the existence result.

2 Energy solutions to system (S): definition and main results

Troughout this paper, we assume that Ω denotes a bounded domain in \mathbb{R}^d ($d \geq 1$) with Lipschitz continuous boundary ($d = 3$ is the physical relevant dimension), and we assume that

$$f \in L^2(\Omega), \quad a \in C^0(\mathbb{R}), \quad \nu \in C^1(\mathbb{R}),$$

and

$$a(k), \nu(k) \geq \delta > 0, \quad \forall k \in \mathbb{R}. \quad (2.1)$$

We denote by T_n the truncation operator defined by

$$T_n(u)(x) \stackrel{\text{def}}{=} \min(n, \max(-n, u(x))).$$

Definition 2.1 *Let $b : \Omega \rightarrow \mathbb{R}$ be a measurable function. For $1 < p < +\infty$, let*

$$X^p(b, \Omega) \stackrel{\text{def}}{=} \left\{ u \in W^{1,p}(\Omega) : b \nabla u \in (L^p(\Omega))^3 \right\},$$

equipped with the norm

$$\|u\|_{X^p}^p \stackrel{\text{def}}{=} \|u\|_{L^p}^p + \|b \nabla u\|_{(L^p)^3}^p.$$

When $b \in L^p(\Omega)$ we observe that $C_c^\infty(\Omega) \subset X^p(b, \Omega)$, and in this case we define the space $X_0^p(b, \Omega)$ as the closure of $C_c^\infty(\Omega)$ in $X^p(b, \Omega)$.

□

Definition 2.2 Let $d \leq 4$. The couple (u, k) is an “energy solution” of (S) if and only if

$$(D) \quad \left\{ \begin{array}{l} k \in W_0^{1,q}(\Omega), \quad \forall q < \frac{d}{d-1}, \\ \sqrt{\nu(k)} \in H^1(\Omega), \\ k \geq 0, \\ T_n(k) \in H_0^1(\Omega), \quad \forall n \in \mathbb{R}^+ \\ u \in X_0^2\left(\sqrt{\nu(k)}, \Omega\right), \\ \int_{\Omega} \nu(k) \nabla u \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in X_0^2\left(\sqrt{\nu(k)}, \Omega\right), \\ \int_{\Omega} a(k) \nabla k \nabla \phi \, dx = \int_{\Omega} \nu(k) |\nabla u|^2 \phi \, dx, \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega). \end{array} \right. \quad \begin{array}{l} (2.2) \\ (2.3) \\ (2.4) \\ (2.5) \\ (2.6) \\ (2.7) \\ (2.8) \end{array}$$

□

Observe that when $d \leq 4$, one has $H^1(\Omega) \subset L^4(\Omega)$, so that $\sqrt{\nu(k)} \in L^2(\Omega)$, which allows us to define $X_0^2\left(\sqrt{\nu(k)}, \Omega\right)$.

The main result of the paper is the following:

Theorem 2.1 Assume that $d \leq 4$ and that there exists $C_1 > 0$ and $\gamma > \frac{1}{2}$ such that

$$\left| \frac{\nu'(k)}{2\sqrt{a(k)\nu(k)}} \right| \leq \frac{C_1}{k^\gamma}, \quad \forall k \geq 1. \quad (2.9)$$

Moreover, assume that

$$\exists C_2 \in \mathbb{R}^+; \quad \forall k \in [0, 1], \quad |\nu'(k)| \leq C_2. \quad (2.10)$$

Then the system (S) admits at least one energy solution (u, k) . □

Remark 2.1 Note that condition (2.9) is consistent with the physical model for which (see for instance [10], [13]) one has

$$\nu(k) \approx a(k) \approx C + \ell k^{\frac{1}{2}}.$$

Indeed, in this case, one has, for large values of k

$$\frac{\nu'(k)}{2\sqrt{a(k)\nu(k)}} \approx \frac{1}{4k},$$

so that (2.9) holds with $\gamma = 1$. On the other hand, for $k \in [0, 1]$, $\nu'(k)$ is not bounded. Then, one have to replace $\nu(k) \approx C + \ell k^{\frac{1}{2}}$ by $\nu(k) \approx C + \ell(k + \varepsilon)^{\frac{1}{2}}$ when $k \in [0, 1]$ and for some $\varepsilon > 0$.

The proof of Theorem 2.1 is based on the compactness method and the existence results for bounded eddy viscosities. Thanks to the already mentioned works, one knows that there exists (u_n, k_n) such that

$$(S_n) \quad \begin{cases} - \operatorname{div}(\nu_n(k_n)\nabla u_n) = f & \text{in } \Omega, & (2.11) \\ - \operatorname{div}(a_n(k_n)\nabla k_n) = T_n[\nu_n(k_n)|\nabla u_n|^2] & \text{in } \Omega, & (2.12) \\ (u_n)|_{\partial\Omega} = 0 & & (2.13) \\ (k_n)|_{\partial\Omega} = 0 & & (2.14) \end{cases}$$

where

$$a_n(k) \stackrel{\text{def}}{=} T_n(a(k)) \quad \text{and} \quad \nu_n(k) \stackrel{\text{def}}{=} T_n(\nu(k)).$$

The first and second equations in (S_n) holds in the classical weak sense. Notice that $k_n \geq 0$ a.e in Ω (see [10]). The main work is taking the limit in the equations. To do this, it is crucial to prove the strong convergence in $L^1(\Omega)$ of the sequence

$$T_n[\nu_n(k_n)|\nabla u_n|^2],$$

a point which constitutes the major difficulty in those kind of problems. As already mentioned in the introduction, the main tools to prove this strong convergence are

- the boundedness of $\|\sqrt{\nu_n(k_n)}\|_{H^1}$ under assumptions (2.1), (2.10) and (2.9) (see Lemma 4.1),

- and the weighted Sobolev space characterization (see section 3) which states that

$$X_0^2(b, \Omega) = H_0^1(\Omega) \cap X^2(b, \Omega)$$

for any weight $b \in H^1(\Omega)$, satisfying

$$b(x) \geq \delta > 0.$$

3 A density result for a weighted Sobolev space

The goal of this section is to prove that if

$$b \in H^1(\Omega) \quad \text{and} \quad b(x) \geq \delta > 0 \quad \forall x \in \Omega, \quad (3.1)$$

then

$$X_0^2(b, \Omega) = H_0^1(\Omega) \cap X^2(b, \Omega). \quad (3.2)$$

Troughout this section, (3.1) is assumed to be satisfied. For the sake of simplicity, we shall put

$$V_b(\Omega) = X^2(b, \Omega) \cap H_0^1(\Omega).$$

Notice that thanks to Poincaré inequality, the space V_b can be equipped with the equivalent norm

$$\|u\|_b = \|b\nabla u\|_{(L^2(\Omega))^d}.$$

When we are only using the space V_b , we shall use this norm without mentioning it.

We remark at first that if b is larger than δ , then

$$X_0^2(b, \Omega) \subset V_b(\Omega). \quad (3.3)$$

Indeed, it is clear that

$$\delta \|\nabla \phi\|_{(L^2)^d} \leq \|b\nabla \phi\|_{(L^2)^d} \quad \forall \phi \in \mathcal{C}_C^\infty(\Omega).$$

The inverse embedding is more delicate to prove and is given by the following result.

Theorem 3.1 *Let Ω be a Lipschitz open subset in \mathbb{R}^d and let b be a $H^1(\Omega)$ function such that*

$$b(x) \geq \delta > 0 \quad \forall x \in \Omega.$$

Then $\mathcal{C}_c^\infty(\Omega)$ is dense in $V_b(\Omega)$ with respect to the $\|\cdot\|_b$ norm. □

The proof of Theorem 3.1 is divided in three steps (and subsections).

1. Density of $L^\infty(\Omega) \cap V_b(\Omega)$ in $V_b(\Omega)$.
2. Density in $L^\infty(\Omega) \cap V_b(\Omega)$ of $L^\infty(\Omega) \cap V_b(\Omega)$ functions with compact support included in Ω .
3. Density of $\mathcal{C}_c^\infty(\Omega)$ in the set of $L^\infty(\Omega) \cap V_b(\Omega)$ functions with compact support included in Ω .

3.1 First step: density of $L^\infty(\Omega) \cap V_b(\Omega)$ in $V_b(\Omega)$

Lemma 3.1 *Let Ω be a Lipschitz open subset in \mathbb{R}^d and let b be a measurable function such that*

$$b(x) \geq \delta > 0 \quad \forall x \in \Omega.$$

Then $L^\infty(\Omega) \cap V_b(\Omega)$ is dense in $V_b(\Omega)$. □

PROOF. Let $u \in V_b(\Omega)$ and set

$$u_n \stackrel{\text{def}}{=} T_n(u).$$

For proving Lemma 3.1, we will show the two following points:

- $u_n \in L^\infty(\Omega) \cap V_b(\Omega)$,
- $\{u_n\}_{n \in \mathbb{N}}$ converges to u with respect to the $\|\cdot\|_{X^2}$ -norm.

Firstly, by definition of the truncature one has

$$|u_n| = \begin{cases} |u| & \text{if } |u| \leq n \\ n & \text{otherwise.} \end{cases} \quad \text{and} \quad \nabla u_n = \begin{cases} \nabla u & \text{if } |u| \leq n \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$|u_n| \leq n \quad \text{and} \quad b|\nabla u_n| \leq b|\nabla u| \in L^2(\Omega) \quad (3.4)$$

almost every where in Ω . It is easily deduced that $u_n \in L^\infty(\Omega) \cap V_b(\Omega)$.

The sequence $\{u_n\}_{n \in \mathbb{N}}$ is now shown to converge to u . Because $0 < \delta \leq b(x)$, $V_b(\Omega) \subset H_0^1(\Omega)$. In particular, $|\nabla u|$ is finite almost everywhere in Ω . Consequently,

$$\nabla u_n \xrightarrow{a.e.} \nabla u. \quad (3.5)$$

By (3.4), (3.5) and Lebesgue dominated convergence theorem, we deduce that $\{u_n\}_{n \in \mathbb{N}}$ converges to u with respect to the norm $\|\cdot\|_b$. \square

3.2 Second step: density of functions with compact support

The density of functions in $L^\infty \cap V_b(\Omega)$ having a compact support is proved in this subsection. For the sake of the simplicity, we denote by $L_c^\infty(\Omega)$ the set of all $L^\infty(\Omega)$ functions with compact support in Ω .

Lemma 3.2 *Assume that Ω is an open bounded Lipschitz subset of \mathbb{R}^d , and that b satisfies (3.1). Then $L_c^\infty(\Omega) \cap V_b(\Omega)$ is dense in $L^\infty(\Omega) \cap V_b(\Omega)$.* \square

Before proving Lemma 3.2, let us verify the following technical Lemma.

Lemma 3.3 *If $b \in H^1(\Omega)$ and $u \in L^\infty(\Omega) \cap V_b(\Omega)$ then $bu \in H_0^1(\Omega)$.* \square

PROOF. The proof is based on the fact that $\phi \in H_0^1(\Omega)$ if and only if its extension by zero outside Ω belongs to $H^1(\mathbb{R}^d)$ (see for instance Theorem IX.17, p.171 in [2]). Set $\Omega^c = \mathbb{R}^d \setminus \Omega$ and

$$\bar{u}(x) \stackrel{def}{=} \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c. \end{cases}$$

We know also that all $v \in H^1(\Omega)$ can be extended into a function in $H^1(\mathbb{R}^d)$ denoted by \tilde{v} . Lemma 3.3 is proved by extending respectively u and b on

\mathbb{R}^d and by proving first that the product of the extensions is equal to \overline{bu} , and secondly that it belongs to $H^1(\mathbb{R}^d)$.

• *Product of the extensions coincides with \overline{bu} .* Indeed \bar{u} is equal to zero in Ω^c then $\tilde{b}\bar{u}$ is also equal to zero in Ω^c . Moreover

$$\bar{u}|_{\Omega} = u \quad \text{and} \quad \tilde{b}|_{\Omega} = b.$$

Therefore

$$\bar{u}\tilde{b}(x) = \begin{cases} bu & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c \end{cases}$$

which means exactly that $\tilde{b}\bar{u} = \overline{bu}$.

• *Product of the extensions belongs to $H^1(\mathbb{R}^d)$.* The product of two functions of $H^1(\mathbb{R}^d)$ belongs to $W^{1,1}(\mathbb{R}^d)$, so that the gradient of $\tilde{b}\bar{u}$ is well defined as a function. Moreover

$$\nabla(\bar{u}\tilde{b})(x) = \bar{u}\nabla\tilde{b} + \tilde{b}\nabla\bar{u} = \begin{cases} u\nabla b + b\nabla u & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^c. \end{cases}$$

Obviously

$$\|\bar{u}\tilde{b}\|_{L^2(\mathbb{R}^d)} = \|bu\|_{L^2(\Omega)} \leq \|u\|_{L^\infty} \|b\|_{L^2(\Omega)}$$

and

$$\|\nabla(\bar{u}\tilde{b})\|_{L^2(\mathbb{R}^d)} = \|\nabla(bu)\|_{L^2(\Omega)} \leq \|u\|_{L^\infty} \|\nabla b\|_{L^2(\Omega)} + \|u\|_{V_b(\Omega)}.$$

Because $b \in H^1(\Omega)$ and $u \in L^\infty(\Omega) \cap V_b(\Omega)$, one deduces that $\tilde{b}\bar{u}$ belongs to $H^1(\mathbb{R}^d)$.

Until now, we have proved that when $b \in H^1(\Omega)$ and $u \in L^\infty(\Omega) \cap V_b(\Omega)$, then $\overline{bu} \in H^1(\mathbb{R}^d)$. Using once more the characterization of $H_0^1(\Omega)$, we conclude that $bu \in H_0^1(\Omega)$.

Now we prove Lemma 3.2. Let $u \in L^\infty(\Omega) \cap V_b(\Omega)$. Construct a $L_c^\infty(\Omega)$ sequence converging to u as follows. Let $f \in C^\infty([0, \infty[; [0, 1])$ be such that

$$f(x) = 0 \quad \forall x \in [0, 1], \quad \text{and} \quad f(x) = 1 \quad \forall x \geq 2,$$

and set

$$u_n \stackrel{\text{def}}{=} u(x) f(nd(x)),$$

where $d(x)$ denotes the distance to the boundary of Ω . We Prove lemma 3.2 in two steps:

- $u_n \in L_c^\infty(\Omega) \cap V_b(\Omega)$,
- $\{u_n\}_{n \in \mathbb{N}}$ converges to u in V_b .

1. Because $\text{supp } u_n \subset \left\{x \in \Omega, d(x) \geq \frac{1}{n}\right\}$, one has

$$\overline{\text{supp } u_n} \subset \Omega \quad (3.6)$$

which implies that u_n has a compact support included in Ω . Moreover, because $u \in L^\infty(\Omega)$ and $f \in L^\infty(\mathbb{R})$, $u_n \in L^\infty(\Omega)$.

In other words,

$$u_n \in L_c^\infty(\Omega). \quad (3.7)$$

It remains to prove that $u_n \in V_b(\Omega)$. By a direct calculation, one has

$$b \nabla u_n = b \nabla u \cdot f(n d) + n b u f'(n d) \nabla d. \quad (3.8)$$

The two terms of the right hand side of (3.8) are bounded in $L^2(\Omega)$. Indeed, $\|f\|_{L^\infty} = 1$ and $u \in V_b(\Omega)$ leads to

$$b |\nabla u| f(n d) \leq \|f\|_{L^\infty} b |\nabla u| = b |\nabla u| \in L^2(\Omega). \quad (3.9)$$

On the other hand,

$$b u n f'(n d) \nabla d = \frac{b u}{d} (n d f'(n d)) \nabla d.$$

It is well known that

- Ω being Lipschitz, $d \in W^{1,\infty}(\Omega)$,
- $u b \in H_0^1(\Omega)$ (lemma 3.3) implies

$$\frac{b u}{d} \in L^2(\Omega).$$

Finally, because $f \in \mathcal{C}^\infty([0, +\infty[; [0, 1])$ and $f'(x) = 0 \quad \forall x \in [0, 1] \cup [2, +\infty[$, there exists a constant $C > 0$ which does not depend on n such that $|n d f'(n d)| \leq C$. We deduce that

$$|b u n f'(n d) \nabla d| \leq C \|\nabla d\|_{L^\infty} \left| \frac{u b}{d} \right| \in L^2(\Omega). \quad (3.10)$$

Combining (3.8), (3.9) and (3.10) one obtain

$$b|\nabla u_n| \leq b|\nabla u| + C \|\nabla d\|_{L^\infty} \left| \frac{ub}{d} \right| \in L^2(\Omega). \quad (3.11)$$

By (3.6), (3.7) and (3.11), one concludes that $u_n \in L_c^\infty \cap V_b(\Omega)$.

2. It is easy checked that

$$\sqrt{b}\nabla u_n \rightarrow \sqrt{b}\nabla u \text{ a.e.} \quad (3.12)$$

because

$$\forall x \in \Omega, \forall n \geq \frac{2}{d(x)}, \quad f(nd(x)) = f'(nd(x)) = 0.$$

The proof is completed by (3.11), (3.12) and Lebesgue dominated convergence theorem. □

3.3 Third step: density of $\mathcal{C}_0^\infty(\Omega)$

Lemma 3.4 *Assume $b \in H^1(\Omega)$. Then the closure of $\mathcal{C}_c^\infty(\Omega)$ in $V_b(\Omega)$ contains $L_c^\infty(\Omega) \cap V_b(\Omega)$.* □

PROOF. One uses a mollifier and equivalence between strong and weak density. Let $u \in L_c^\infty(\Omega) \cap V_b(\Omega)$ and $\rho \in \mathcal{C}_c^\infty(\Omega)$ be such that

$$\text{supp } \rho \subset \overline{B(0,1)}, \quad \rho(x) \geq 1 \text{ on } B\left(0, \frac{1}{2}\right),$$

$$\rho(x) \geq 0 \text{ for } x \in \mathbb{R}^d, \quad \text{and} \quad \int_{B(0,1)} \rho(x) dx = 1.$$

One defines

$$\rho_\varepsilon \stackrel{\text{def}}{=} \frac{1}{\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right)$$

and

$$u_\varepsilon \stackrel{\text{def}}{=} u * \rho_\varepsilon.$$

Because $u \in L_c^\infty(\Omega)$, one has $u_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ for small values of ε .

One shall prove that because (3.1) is satisfied and $u \in L_c^\infty(\Omega) \cap V_b(\Omega)$, the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ converges weakly to u . One starts by proving first the boundedness of $\{u_\varepsilon\}_{\varepsilon>0}$ in V_b , which is the main difficult point. For that, one writes

$$b\nabla u_\varepsilon = A_\varepsilon + B_\varepsilon$$

where

$$A_\varepsilon = \int_{\Omega} \nabla \rho_\varepsilon(x-y) \cdot b(y) u(y) dy \quad \text{and} \quad B_\varepsilon = \int_{\Omega} \nabla \rho_\varepsilon(x-y) [b(x) - b(y)] u(y) dy.$$

Each term A_ε and B_ε is then studied independently, beginning with A_ε .

Because $u \in L^\infty(\Omega) \cap V_b(\Omega)$, by Lemma 3.3, $ub \in H_0^1(\Omega)$. Then, as $A_\varepsilon = \nabla(bu) * \rho_\varepsilon$, one has $A_\varepsilon \rightarrow \nabla(bu)$ in $(L^2(\Omega))^d$. In particular, A_ε is bounded in $(L^2(\Omega))^d$.

On the other hand, because $u \in L^\infty(\Omega)$, B_ε is bounded in $(L^2(\Omega))^d$ if

$$C_\varepsilon \stackrel{\text{def}}{=} \int_{\Omega} |\nabla \rho_\varepsilon(x-y)| |b(x) - b(y)| dy$$

is bounded in $(L^2(\Omega))^d$. By Cauchy-Schwarz inequality,

$$\begin{aligned} C_\varepsilon &= \int_{B(0,\varepsilon)} |\nabla \rho_\varepsilon(h)| |b(x) - b(x+h)| dh \leq \\ &\left(B_d \varepsilon^d \right)^{\frac{1}{2}} \left(\int_{B(0,\varepsilon)} |\nabla \rho_\varepsilon(h)|^2 |b(x) - b(x+h)|^2 dh \right)^{\frac{1}{2}}, \end{aligned}$$

where B_d is the volume of the unit sphere of \mathbb{R}^d . Therefore, C_ε is bounded in $(L^2(\Omega))^d$ if

$$D_\varepsilon \stackrel{\text{def}}{=} B_d \varepsilon^d \int_{\Omega} \left(\int_{B(0,\varepsilon)} |\nabla \rho_\varepsilon(h)|^2 |b(x) - b(x+h)|^2 dh \right) dx$$

is bounded in \mathbb{R} . By Fubini's theorem

$$D_\varepsilon = B_d \varepsilon^d \int_{B(0,\varepsilon)} |\nabla \rho_\varepsilon(h)|^2 \|b(x) - b(x+h)\|_{L^2(\Omega)}^2 dh,$$

and because $b \in H^1(\Omega)$,

$$\|b(x) - b(x+h)\|_{L^2(\Omega)} \leq C_2 |h|$$

for a given constant C_2 . It follows that

$$D_\varepsilon \leq C_2 B_d \varepsilon^d \int_{B(0,\varepsilon)} |h|^2 |\nabla \rho_\varepsilon(h)|^2 dh.$$

Finally, because

$$\nabla \rho_\varepsilon(h) = \frac{1}{\varepsilon^{d+1}} \nabla \rho\left(\frac{h}{\varepsilon}\right),$$

one also has

$$D_\varepsilon \leq C_2 B_d \varepsilon^{d+2} \int_{B(0,\varepsilon)} \frac{1}{\varepsilon^{d+2}} |\nabla \rho(h)|^2 dh = C_2 B_d \int_{B(0,1)} |\nabla \rho(y)|^2 dy < \infty,$$

which proves the boundedness of $\{u_\varepsilon\}_{\varepsilon>0}$ in V_b .

Using this bound for the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ and the fact that $V_b(\Omega)$ is an Hilbert space, one can extract a subsequence (still denoted $\{u_\varepsilon\}_{\varepsilon>0}$) converging weakly in V_b to some l . It remains to prove that $l = u$. For this, let $\phi \in C_c^\infty(\Omega)$ and remark first that the weak convergence of u_ε implies that

$$\int_{\Omega} u_\varepsilon \phi dx \rightarrow \int_{\Omega} l \phi dx. \quad (3.13)$$

Moreover, $u \in L^2(\Omega)$ and it is already known that

$$u_\varepsilon \rightarrow u \text{ strongly in } L^2(\Omega)$$

which implies

$$\int_{\Omega} u_\varepsilon \phi dx \rightarrow \int_{\Omega} u \phi dx. \quad (3.14)$$

From (3.13) and (3.14) is deduced

$$\int_{\Omega} u \phi dx = \int_{\Omega} l \phi dx \quad \forall \phi \in C_c^\infty(\Omega).$$

Because $u, l \in L^1_{loc}(\Omega)$, it implies that

$$u \stackrel{a.e.}{=} l$$

almost everywhere.

The proof of lemma 3.4 is now complete because weakly closed subspaces and strongly closed subspaces coincide. \square

Combining lemmas 3.1, 3.2 and 3.4, one deduces that the strong closure of $C_c^\infty(\Omega)$ in $V_b(\Omega)$ coincides with its weak closure, which is $V_b(\Omega)$; therefore every element of $V_b(\Omega)$ can be approached in norm by a sequence of functions in $C_c^\infty(\Omega)$ and theorem 3.1 is proved. \square

4 Approximate solutions and estimates

Recall that in Section 2 we introduced the approximate system

$$(S_n) \quad \begin{cases} - \operatorname{div}(\nu_n(k_n)\nabla u_n) = f & \text{in } \Omega, & (4.1) \\ - \operatorname{div}(a_n(k_n)\nabla k_n) = T_n[\nu_n(k_n)|\nabla u_n|^2] & \text{in } \Omega, & (4.2) \\ (u_n)|_{\partial\Omega} = 0 & & (4.3) \\ (k_n)|_{\partial\Omega} = 0 & & (4.4) \end{cases}$$

where

$$a_n(k) \stackrel{\text{def}}{=} T_n(a(k)) \quad \text{and} \quad \nu_n(k) \stackrel{\text{def}}{=} T_n(\nu(k)).$$

Notice that $k_n \geq 0$ a.e in Ω (see [10]).

The variational formulation of (S_n) is:

$$(u_n, k_n) \in (H_0^1(\Omega))^2, \quad (4.5)$$

$$\forall v \in H_0^1(\Omega), \quad \int_{\Omega} \nu_n(k_n)\nabla u_n \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad (4.6)$$

$$\forall \phi \in H_0^1(\Omega), \quad \int_{\Omega} a_n(k_n)\nabla k_n \cdot \nabla \phi = \int_{\Omega} T_n[\nu_n(k_n)|\nabla u_n|^2] \phi. \quad (4.7)$$

Usual estimates on $\{u_n\}_{n \in N}$ and $\{k_n\}_{n \in N}$ are obtained as follows.

- Taking $v = u_n$ as test function in (4.5), one obtains

$$\|\nu_n(k_n)|\nabla u_n|^2\|_{L^1} \leq C_4(\delta, f) \quad (4.8)$$

where the constant $C_4(\delta, f)$ only depends on f and δ .

- Using Boccardo-Gallouët estimates (see [1]) one concludes that

$$\forall p < \frac{d}{d-1}, \quad \int_{\Omega} |a(k_n)\nabla k_n|^p \, dx \leq C_5(p, f), \quad (4.9)$$

where $\lim_{p \rightarrow d'} C_5(p, f) = \infty$.

The main and essential result of this section is the estimate for the eddy viscosity.

Lemma 4.1 *Let $d = 3$. Assume that $\nu(k)$ is absolutely continuous on \mathbb{R}^+ and*

- *there exist $K \in \mathbb{R}^+$ and $\gamma > \frac{1}{2}$ such that*

$$\left| \frac{\nu'(k)}{2\sqrt{a(k)\nu(k)}} \right| \leq \frac{C_1}{|k|^\gamma} \quad \text{for almost every } k \geq 1, \quad (4.10)$$

- *for almost every $k \in \mathbb{R}^+$,*

$$0 < \delta \leq \min(a(k), \nu(k)), \quad (4.11)$$

- *there is a bound for ν' for small values of k ,*

$$\exists C_2 \in \mathbb{R}^+; \quad \forall k \in [0, 1], \quad |\nu'(k)| \leq C_2. \quad (4.12)$$

Then there exist a constant $C(\gamma, \nu, \Omega, f) > 0$ such that

$$\|\sqrt{\nu_n(k_n)}\|_{H^1} \leq C(\gamma, \nu, \Omega, f).$$

□

PROOF. The estimate of $\{\nabla\sqrt{\nu_n(k_n)}\}_{n \in \mathbb{N}}$ in $(L^2(\Omega))^d$ follows from the similar estimate where ν_n is replaced by ν . By definition, one has

$$\sqrt{T_n(\nu(k_n))} = \begin{cases} \sqrt{\nu(k_n)} & \text{if } \nu(k_n) < n \\ \sqrt{n} & \text{otherwise.} \end{cases}$$

Then, because $k_n \in W_0^{1,1}(\Omega)$, the truncature T_n is a contraction and $\nu_n(0) = \nu(0)$ for n large enough, there exists a constant $c(\Omega)$ depending only on Ω , such that

$$\|\sqrt{\nu_n(k_n)} - \sqrt{\nu(0)}\|_{L^2} \leq c(\Omega) \|\nabla\sqrt{\nu_n(k_n)}\|_{(L^2)^d}.$$

It is also clear that

$$|\nabla\sqrt{\nu_n(k_n)}| \leq |\nabla\sqrt{\nu(k_n)}| \quad \text{on } \Omega.$$

Therefore there exists a constant $C(\nu, \Omega)$, depending only upon ν and Ω such that

$$\|\sqrt{\nu_n(k_n)}\|_{H^1}^2 \leq C(\nu, \Omega) + \|\nabla\sqrt{\nu(k_n)}\|_{(L^2)^d}^2. \quad (4.13)$$

Thanks to (4.13), it is possible to deal with $\sqrt{\nu(k_n)}$ instead of $\sqrt{\nu_n(k_n)}$.

Write now

$$\int_{\Omega} |\nabla\sqrt{\nu(k_n)}|^2 dx = \sum_{j \geq 0} B_j \quad \text{where} \quad B_j \stackrel{\text{def}}{=} \int_{\{j \leq k_n \leq j+1\}} |\nabla\sqrt{\nu(k_n)}|^2 dx$$

and notice that

$$\nabla\sqrt{\nu(k_n)} = \frac{\nu'(k_n)}{2\sqrt{\nu(k_n)a(k_n)}} \sqrt{a(k_n)} \nabla k_n.$$

According to (4.10),

$$\left| \frac{\nu'(k_n)}{2\sqrt{a(k_n)\nu(k_n)}} \right| \leq \frac{C_1}{k_n^\gamma}$$

if $k_n \geq 1$. In particular, if $j \geq 1$

$$B_j \leq C_1^2 \int_{\{j \leq k_n \leq j+1\}} \frac{1}{k_n^{2\gamma}} a(k_n) |\nabla k_n|^2 dx.$$

One deduces that, if $j \geq 1$, one has

$$B_j \leq \frac{C_1^2}{j^{2\gamma}} \int_{\{j \leq k_n \leq j+1\}} a(k_n) |\nabla k_n|^2 dx.$$

Using the classical estimate satisfied for all $j \in \mathbb{N}$,

$$\int_{\{j \leq k_n \leq j+1\}} a(k_n) |\nabla k_n|^2 dx \leq C(f) \quad (4.14)$$

(see for instance [10]), one deduce that

$$\sum_{j=1}^{\infty} B_j \leq C_1^2 C(f) \left(\sum_{j \geq 1} \frac{1}{j^{2\gamma}} \right) < +\infty, \quad (4.15)$$

this because $\gamma > \frac{1}{2}$. It remains to study the term B_0 . For it, one combines (4.12), (4.14) and (4.11) to obtain

$$|B_0| \leq \frac{C_2 C(f)}{\sqrt{\delta}} \quad (4.16)$$

By (4.16) and (4.15) one has

$$\int_{\Omega} |\nabla \sqrt{\nu(k_n)}|^2 dx \leq C_1^2 C(f) \left(\sum_{j \geq 1} \frac{1}{j^{2\gamma}} \right) + \frac{C_2 C(f)}{\sqrt{\delta}} < +\infty. \quad (4.17)$$

Combining (4.13) and (4.17) leads to

$$\|\sqrt{\nu_n(k_n)}\|_{H^1} \leq C(\gamma, \nu, \Omega, f),$$

and the proof is achieved. □

5 Passing to the limit: proof of Theorem 2.1

In order to prove theorem 2.1, one has to ensure that the sequence $\{u_n, k_n\}$ converges to an energy solution of system (S). This is the subject of this section. Firstly, thanks to the a priori estimates, one extracts converging subsequences. Then one take the limit in the equations.

Estimate (4.8) combined to $\delta \|u_n\|_{H^1} \leq \|\sqrt{a_n(k_n)} \nabla u_n\|_{L^2}$ allows to extract a subsequence (still denoted $\{u_n\}_{n \in N}$) such that

$$\begin{cases} u_n \rightarrow u & \text{weakly in } H_0^1, \\ & \text{strongly in } L^p(\Omega), \quad p < \frac{2d}{d-2}, \\ & \text{a.e. in } \Omega. \end{cases} \quad (5.1)$$

Results in [1] and the condition $a(k) \geq \delta$ leads to the extraction of a subsequence (still denoted $\{k_n\}_{n \in N}$) such that

$$\begin{cases} A(k_n) \rightarrow A(k) & \text{weakly in } W^{1,p}(\Omega), \quad p < \frac{d}{d-1}, \\ & \text{strongly in } L^q(\Omega), \quad q < \frac{d}{d-2}, \\ & \text{a.e. in } \Omega, \end{cases} \quad (5.2)$$

where $A(k) = \int_0^k a(k') dk'$ and

$$\begin{cases} k_n \rightarrow k & \text{weakly in } W_0^{1,p}(\Omega), \quad p < \frac{d}{d-1}, \\ & \text{strongly in } L^q(\Omega), \quad q < \frac{d}{d-2}, \\ & \text{a.e. in } \Omega. \end{cases} \quad (5.3)$$

Finally, by Lemma 4.1, one has

$$\begin{cases} \sqrt{\nu_n(k_n)} \rightarrow g & \text{weakly in } H^1(\Omega) \\ & \text{strongly in } L^q(\Omega), \quad q < \frac{2d}{d-2} \\ & \text{a.e. in } \Omega \end{cases} \quad (5.4)$$

From the a.e. convergence in (5.3) and (5.4), one deduces that $g = \sqrt{\nu(k)} \in H^1(\Omega)$.

Taking the limit in the equations is performed in two steps. The first one concerns the first equation (lemma 5.1 below and its application), and as anyone can guess, the second one concerns the second equation (see lemma 5.2 below and its application).

Lemma 5.1 *Let $\{b_n\}_{n \in \mathbb{N}} \subset L^2(\Omega)$ be a sequence converging strongly to b in $L^2(\Omega)$ and such that $b_n(x) \geq \delta$ on Ω . Let $\{u_n\}_{n \in \mathbb{N}} \subset H_0^1(\Omega)$ be a sequence such that $b_n \nabla u_n$ is bounded in $(L^2(\Omega))^d$. Then there exists $u \in V_b$ and a subsequence (written with the same index) such that*

$$\int_{\Omega} b_n^2 \nabla u_n \nabla \psi \, dx \rightarrow \int_{\Omega} b^2 \nabla u \nabla \psi \, dx, \quad \forall \psi \in C_c^\infty(\Omega). \quad (5.5)$$

□

PROOF. From $b_n \nabla u_n$ one can extract a subsequence which converges weakly in $(L^2(\Omega))^d$ to some $\vec{\phi}$. The hypothesis $b_n \geq \delta$ tells that the sequence u_n is bounded in $H^1(\Omega)$: one can extract a subsequence which converges to some $u \in H^1(\Omega)$. As $b_n \rightarrow b$ strongly in $L^2(\Omega)$, one deduces that

$$\int_{\Omega} b_n \nabla u_n \vec{\psi} \, dx \rightarrow \int_{\Omega} b \nabla u \vec{\psi} \, dx, \quad \forall \vec{\psi} \in (L^\infty(\Omega))^d.$$

This leads to

$$\int_{\Omega} (\vec{\phi} - b \nabla u) \vec{\psi} \, dx = 0, \quad \forall \vec{\psi} \in (L^\infty(\Omega))^d.$$

Because $\vec{\phi} \in (L^2(\Omega))^d \subset (L^1(\Omega))^d$ and

$$\|b\nabla u\|_{(L^1(\Omega))^d} \leq \|b\|_{L^2(\Omega)} \|\nabla u\|_{(L^2(\Omega))^d},$$

it follows that $\vec{\phi} = b\nabla u$ almost everywhere on Ω and $u \in V_b$. The strong convergence of b_n to b in $L^2(\Omega)$ leads to

$$\int_{\Omega} b_n^2 \nabla u_n \nabla \phi \, dx = \int_{\Omega} (b_n \nabla u_n)(b_n \nabla \phi) \, dx \rightarrow \int_{\Omega} b^2 \nabla u \nabla \phi \, dx$$

because $b_n \nabla u_n \rightarrow b\nabla u$ weakly in $(L^2(\Omega))^d$ and $b_n \nabla \phi \rightarrow b\nabla \phi$ strongly in $(L^2(\Omega))^d$, and the proof of (5.5) is complete. \square

One applies Lemma 5.1 with $b_n = \sqrt{\nu_n(k_n)}$ and one concludes that $u \in V_{\sqrt{\nu(k)}}(\Omega)$ and that

$$\int_{\Omega} \nu(k) \nabla u \nabla \phi \, dx = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in X_0^2(\sqrt{\nu(k)}, \Omega) = V_{\sqrt{\nu(k)}}(\Omega),$$

the last equality being a consequence of (3.2).

Lemma 5.2 *Let $b \in H^1(\Omega)$ and $f \in L^2(\Omega)$. Let be a sequence $\{b_n\}_{n \in \mathbb{N}} \subset L^\infty(\Omega)$, larger than δ , and converging to b*

- *strongly in $L^4(\Omega)$ and*
- *weakly in $H^1(\Omega)$.*

Let $u_n \in H_0^1(\Omega)$, solution of

$$\int_{\Omega} b_n^2 \nabla u_n \nabla \phi, \, dx = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in H_0^1(\Omega). \quad (5.6)$$

There exists $u \in V_b$ such that

$$b_n \nabla u_n \rightarrow b\nabla u \quad \text{strongly in } (L^2(\Omega))^d.$$

\square

PROOF. Taking u_n as test function in 5.6 leads to a bound in $L^2(\Omega)$ for $b_n \nabla u_n$. We apply lemma 5.1 and consider u as the weak limit in V_b of the sequence $(u_n)_{n \in \mathbb{N}}$. The difficult point consists on proving the strong convergence. Let $\phi \in \mathcal{C}_c^\infty(\Omega)$. By Lemma 5.1,

$$\int_{\Omega} b_n^2 \nabla u_n \nabla \phi \, dx \rightarrow \int_{\Omega} b^2 \nabla u \nabla \phi \, dx.$$

Then

$$\int_{\Omega} b^2 \nabla u \nabla \phi \, dx = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega).$$

By Theorem 3.1, we have

$$\int_{\Omega} b^2 \nabla u \nabla \phi \, dx = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in X_0^2(b, \Omega) = V_b(\Omega). \quad (5.7)$$

Taking $\phi = u$ in (5.7), we see that

$$\int_{\Omega} b^2 |\nabla u|^2 \, dx = \int_{\Omega} f u \, dx. \quad (5.8)$$

On the other hand, because $u_n \rightarrow u$ strongly in $L^2(\Omega)$,

$$\int_{\Omega} b_n^2 |\nabla u_n|^2 \, dx \rightarrow \int_{\Omega} f u \, dx. \quad (5.9)$$

Combining (5.8) and (5.9) leads to

$$\int_{\Omega} b_n^2 |\nabla u_n|^2 \, dx \rightarrow \int_{\Omega} b^2 |\nabla u|^2 \, dx. \quad (5.10)$$

By (5.10) and Lemma 5.1, we conclude that $b_n \nabla u_n \rightarrow b \nabla u$ strongly in $(L^2(\Omega))^d$ \square

One Applies lemma 5.2 for taking the limit in the second equation. For $\phi \in \mathcal{C}_c^\infty(\Omega)$, (5.2) implies

$$\int_{\Omega} a_n(k_n) \nabla k_n \nabla \phi \, dx \rightarrow \int_{\Omega} a(k) \nabla k \nabla \phi \, dx, \quad \forall \phi \in W_0^{1,q}(\Omega), \quad \forall q > d. \quad (5.11)$$

One applies Lemma 5.2 once more with $b_n = \sqrt{\nu_n(k_n)}$ and one obtains

$$\nu_n(k_n) |\nabla u_n|^2 \rightarrow \nu(k) |\nabla u|^2 \quad \text{strongly in } L^1(\Omega). \quad (5.12)$$

It follows

$$\int_{\Omega} \nu_n(k_n) |\nabla u_n|^2 \phi \, dx \rightarrow \int_{\Omega} \nu(k) |\nabla u|^2 \phi \, dx, \quad \forall \phi \in L^\infty(\Omega).$$

Using (4.3) one concludes that

$$\int_{\Omega} a(k) \nabla k \nabla \phi \, dx = \int_{\Omega} \nu(k) |\nabla u|^2 \phi \, dx, \quad \forall q > d \quad \text{and} \quad \forall \phi \in W_0^{1,q}(\Omega),$$

and the proof of Theorem 2.1 is now complete.

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