Vorticities in a LES model for 3D periodic turbulent flows

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Abstract

This paper is devoted to the study of a LES model for simulate turbulent 3D periodic flows. We focus our attention to the equation satisfied by the vorticity issued from this LES model for small values of the numerical grid size δ. We obtain entropy inequalities for the sequence of corresponding vorticities and corresponding pressures independant of δ when the initial data $\mathbf{u}_0$ is in L^2_x while the initial vorticity $\omega_0 = \nabla \times \mathbf{u}_0$ is in L^1_x. We show convergence in a distributional sense of the corresponding equations for the vorticities when δ tends to zero to the classical 3D equation for the vorticity.

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Dedicated to Haïm Brézis for his 60^{th} birthday

1 Introduction and main results

Let consider the Navier-Stokes equations, posed in a Q-periodic case in $\mathbb{R}^3$ ($Q = [0, 2\pi]^3$),

$$(1.1)\qquad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}_{t=0} = \mathbf{u}_0, \quad \nabla\cdot\mathbf{u}_0 = 0, \end{cases}$$

subject to the constrain that the unknown field $(\mathbf{u}, p)$ has zero mean on Q. Equations (1.1) are known to be the idealized physical model to compute Newtonian fluid flows. They are also known to be unstable in numerical simulations when the Reynolds number is high, thus when the flow is turbulent. Therefore, numerical turbulent models are needed for real simulations of turbulent flows with accuracy governed by a grid size $\delta > 0$. This paper deals with such a model.

In Layton-Lewandowski [10] one has introduced the following Large Scale Model considered as a Large Eddy Simulation model:

$$(1.2)\qquad \begin{cases} \partial_t \mathbf{w} + \nabla\cdot(A_\delta^{-1}(\mathbf{w}\mathbf{w})) - \nu\Delta\mathbf{w} + \nabla q = 0, \\ \nabla\cdot\mathbf{w} = 0, \\ \mathbf{w}_{t=0} = A_\delta^{-1}\mathbf{u}_0, \end{cases}$$

where $A_\delta\phi := -\delta^2\Delta\phi + \phi$, $(\mathbf{w}\mathbf{w})_{ij} = w^i w^j$ for $\mathbf{w} = (w^1, w^2, w^3)$; A_δ^{-1} is called the LES filter. The boundary conditions are periodic conditions with mean value equal to zero.

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The main goal in using such a model is to filter eddies of scale less than the numerical grid size δ in numerical simulations. Moreover, we are looking for a simple and tractable LES model which yields accurate simulations using simple numerical algorithms. For a general overview of LES models, the reader is refered to Layton [9] or Sagaut [19].

In Layton-Lewandowski [11] and [10] we have proved that (1.2) has a unique regular solution $(\mathbf{w}_\delta, q_\delta)$. We also have shown that there exists a sequence $(\delta_n)_{n\in I\!N}$ which converges to zero and such that the sequence $(\mathbf{w}_{\delta_n}, q_{\delta_n})_{n\in I\!N} = (\mathbf{w}_n, q_n)$ converges to a distributional solution $(\mathbf{u}, p)$ of the Navier-Stokes equations (1.1) (see the recall of Theorem 2.1 and Theorem 2.2 below). We shall say that (u, p) is a LES solution to the Navier-Stokes equations and $(\mathbf{w}_n, q_n)$ is a sequence of LES approximations.

This paper has two main correlated objects. The first one is the study of the corresponding sequence of vorticities $(\omega_n = \nabla \times \mathbf{w}_n)_{n\in I\!N}$, solution of the familly of equations:

$$\left\{ \begin{array}{l} \partial_t \omega_n + A_{\delta_n}^{-1}((\mathbf{w}_n \nabla)\omega_n) - \nu\Delta\omega_n = A_{\delta_n}^{-1}(((\nabla \times \mathbf{w}_n)\nabla)\mathbf{w}_n). \\ (\omega_n)_{t=0} = A_{\delta_n}^{-1}(\omega_0) = \nabla \times (A_{\delta_n}^{-1}\mathbf{u}_0), \end{array} \right. \tag{1.3}$$

We are looking for sharp estimates satisfied by ω_n independant of δ_n. The second second object of this paper is the question of how to pass to the limit in equations (1.3).

These questions are pertinent for three reasons. First because several numerical studies of turbulent flows are based on the study of the vorticity (see for instance in Wickerhauser et al [22] or in Lesieur [12]). Indeed, the vorticity is one of the best indicator for the understanding of coherent structures in the turbulence. Next because several codes developed in oceanography are using directly the equation for the vorticity coupled with potential equations to simulate general oceanic circulations (see for instance in Bermejo [4]). Then, LES models for this equation are needed. Finally as we shall see in the remainder, because it leads to a very interesting and surprinsing open question on the vorticity's equation linked to uniqueness.

One notes that the only avaible bound for the r.h.s in (1.3) is in $L^1_{t,x}$ on Q. Thus, the natural environment for the study of this sequence of equations is when the initial data $\omega_0 = \nabla \times \mathbf{u}_0$ is Q-periodic, in L^1_x on Q, while $\mathbf{u}_0$ is always in L^2_x on Q, the usual space for initial datas in (1.1). Thus the natural space for the initial datas in our study is

$$\mathrm{H} = \left\{ \mathbf{v} \;\; Q-\text{periodic},\; \mathbf{v}|_Q \in L^2(Q),\; \nabla\cdot\mathbf{v} = 0\,,\, \int_Q \mathbf{v} = 0,\; \nabla\times\mathbf{v}|_Q \in L^1(Q) \right\}. \tag{1.4}$$

For a similar reason, we are lead to seek for entropy estimates. Generally speaking, the entropies in this context are function of the considered field. These entropies will be constructed by using the following functions's set:

$$\mathcal{G} = \left\{ \begin{array}{l} g : I\!R \to I\!R,\;\text{ bounded Lipchitz with a finite number} \\ \text{ of discontinuities ;}\quad (g')^2 = g, \quad g'' = 0 \end{array} \right\}. \tag{1.5}$$

In the following, $\mathbf{u}_0 \in \mathrm{H}$ and $\mathbf{u}$ is the corresponding LES velocity constructed as a limit of the $\mathbf{w}_n$ and $\omega_n = \nabla \times \mathbf{w}_n$. Our main first result is the following.

Theorem 1.1 *The sequence* $(\omega_n)_{n\in I\!N}$ *satisfies the following estimates:*

$$||\omega_n||_{L^\infty_t(L^1_x)} \le C\left(||\mathbf{u}_0||_{L^2_x}, ||\omega_0||_{L^1_x}, \nu\right), \tag{1.6}$$

$$\forall\, g \in \mathcal{G}, \quad ||g(\omega_n)||_{L^2_t(H^1_x)} \le C(||\mathbf{u}_0||_{L^2_x}, ||\omega_0||_{L^1_x}, \nu, ||g||_\infty), \tag{1.7}$$

$$\forall\, p < 5/4, \quad ||\omega_n||_{L^p_t(W^{1,p}_x)} \le C(||\mathbf{u}_0||_{L^2_x}, ||\omega_0||_{L^1_x}, \nu, p), \tag{1.8}$$

where $g(\omega_n) = (g(\omega_n^1), g(\omega_n^2), g(\omega_n^3))$ *for* $\omega_n = (\omega_n^1, \omega_n^2, \omega_n^3)$.

Here, $C(a, b, ...)$ is a constant which depends on $a, b....$ To simplify the notations, L^2_x will now denote the space of all the Q-periodic functions (vector fields or tensors) with restriction to Q in $L^2(Q)$ (with their components restricted to Q in $L^2(Q)$) and such that their mean value on Q are equal to zero, that is $\int_Q \mathbf{v} = 0$. In the context of regularity, we treat functions, vector fields and any tensor with the same formalism. Generally speaking, if $E(X)$ is a generic space function on X, E_x denotes all the Q-periodic functions (vector fields or tensors) with their restriction on Q in $E(Q)$ and with zero mean value on Q, $L^p_t(E_x) = L^p(I\!R_+, E(Q))$.

Our second main result is the following.

Theorem 1.2 *The sequence* $(\omega_n)_{n\in I\!N}$ *satisfies the following properties:*
(i) $(\omega_n)_{n\in I\!N}$ *converges weakly to* $\omega = \nabla \times \mathbf{u}$ *in* $L^p_t(W^{1,p}_x)$ *for all* $p < 5/4$*, strongly in* $L^1_{t,x}$ *and almost everywhere in space and time,*
(ii) the sequence of equations (1.3) *converges in the sense of the distributions to the equation*

$$\left\{\begin{array}{l} \partial_t\omega + (\mathbf{u}\nabla)\omega - \nu\Delta\omega = ((\nabla \times \mathbf{u}\nabla))\mathbf{u}, \\ \omega_{t=0} = \omega_0 = \nabla \times \mathbf{u}_0 \in L^1_x, \end{array}\right. \tag{1.9}$$

(iii) $\omega = \nabla \times \mathbf{u}$ *has the following regularity:*

$$\omega \in L^\infty_t(L^1_x), \tag{1.10}$$

$$\forall\, g \in \mathcal{G}, \quad g(\omega) \in L^2_t(H^1_x), \tag{1.11}$$

$$\omega \in \bigcap_{p<5/4} L^p_t(W^{1,p}). \tag{1.12}$$

Notice that we do not know if the LES solution to (1.1) is unique. The results above hold for any choice of a LES solution.

The estimates (1.6), (1.7) and (1.8) are new and not trivial to obtain in reason of the nature of the filter A^{-1}_δ. Estimate (1.8) is a consequence of (1.7) for a suitable choice of functions in $\mathcal{G}$ combined with the Boccardo-Gallouët's inequality [6].

Points (i) and (ii) in Theorem 1.2 are also new as far as we know as well as the entropy property (1.11).

The property (1.10) has been obtained first in Constantin [7] for any Leray's solutions to the Navier-Stokes equations (see also in Bardos-Nikolaenko [2]).

Property (1.12) is a consequence of (1.8) which is one of the key to pass to the limit in (1.3). This process follows here the scheme used before in Lewandowski [14] for the study of the $k - \varepsilon$ model and in Lewandowski [13] for the study of closure models for simulate the general oceanic circulation. It has also for consequence the fact that the velocity field $\mathbf{u}$ in any LES solution satisfies $\mathbf{u} \in L^p_t(W^{2,p}_x)$ for all $p < 5/4$. In Lions [17], it is proved that when $\nabla\mathbf{u}_0$ is a bounded measure, then every Leray's solution to the Navier-Stokes equations is such that its second derivative $D^2\mathbf{u}$ is in $L^{4/3,\infty}_{x,t}$, where $L^{p,\infty}$ is the classical Lorentz space. Thus, (1.12) as a corollary yields a same type regularity property for the second derivative of $\mathbf{u}$ than in [17]. It has been obtained in this work with a different approach, a slightly different hypothesis on the initial data (we assume here $\omega_0 \in L^1_x$) and for an other type of solutions to the Navier-Stokes equations, as far as uniqueness of solution for (1.1) is not known.

Now an interesting and surprising open question. Results in Blanchard-Murat [5] suggest that equation (1.9) admits a unique renormalized solution $\tilde{\omega}$. We do not know if $\tilde{\omega} = \omega$.

This question is open. The difficulty is mainly due to the lack of strong compactness in $L^1_{t,x}$ of the sequence $(((\nabla \times \mathbf{w}_n)\nabla \mathbf{w}_n))_{n\in I\!N}$ which can present oscillations. We suppose that the notion of H-measures introduced in Tartar [21] can be adapted to study this question but we did not have explored yet this direction. Nevertheless, the Kolmogorov's laws say that frequencies larger than $O(R^{-3/4})$ in the flow are damped, where R is the Reynolds number (see for in instance in Frisch [8]). Therefore, such oscillations do not arise physically.

Let conclude this introduction by mention the result that we have obtained concerning the sequence of LES pressures $(q_n)_{n\in I\!N}$. The equation satisfied by q_n can be written under the form (see formula (4.5) in the remainder):

$$-\Delta q_n = A^{-1}_{\delta_n}(\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n). \tag{1.13}$$

We note that the r.h.s in (1.13) is also bounded in $L^1_{t,x}$ and not more. We prove in the paper an entropy's inequality result for the pressure

Lemma 1.1 *For all $g \in \mathcal{G}$, one has*

$$||g(q_n)||_{L^2_t(H^1_x)} \leq C||g||_\infty \frac{||\mathbf{u}_0||^2_{L^2_x}}{2\nu}, \tag{1.14}$$

where C is a constant.

Unfortunatly, we are not able to pass to the limit in the sequence of equations (1.13) in a reasonnable sense. It is the second problem opened by this work (see Remark 4.2 in the remainder).

The paper is organized as follows. We start by the recall of the main results of [11] and [10] and make clear the notion of LES solution. Then we prove that any LES solution satisfies the classical energy inequality, a result not yet proved by the past for this model. We deduce regularity properties for the LES solution deduced from interpolation, usefull for the remainder. Next, we study the equation for the pressure. We begin by proving additional regularity properties that we need for q_n when n is fixed. Afterwards we prove Lemma 1.1. Then, we consider the equations for the vorticities and give complete proofs of Theorem 1.1 and Theorem 1.2.

2 LES Approximations

2.1 Definition of an LES filter

The mean operator A_δ is defined by

$$A_\delta \overline{\phi} := -\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi, \tag{2.1}$$

with periodic conditions, and fields with mean value equal to zero. This defines an operator

$$A_\delta : W^{1,p}_x \to W^{-1,p}_x.$$

One easily sees that A_δ is self-adjoint and has the regularity property

$$\forall\, r, \quad \forall\, \phi \in H^r_x, \quad \overline{\phi} = A^{-1}_\delta \phi \in H^{r+2}_x. \tag{2.2}$$

We shall call he operator A^{-1}_δ the LES filter. Note that A^{-1}_δ commutes with every differential operator thanks to the periodic conditions. That means for instance

$$\partial_i A^{-1}_\delta \phi = A^{-1}_\delta \partial_i \phi \tag{2.3}$$

2.2 Approximations : existence and uniqueness

One consider the following problem, where $(\mathbf{w}, q)$ is the unknown Q-periodic field:

$$
\left\{
\begin{array}{l}
\partial_t \mathbf{w} + \nabla \cdot (A_\delta^{-1}(\mathbf{w}\mathbf{w})) - \nu \Delta \mathbf{w} + \nabla q = 0, \\
\nabla \cdot \mathbf{w} = 0, \\
\mathbf{w}_{t=0} = A_\delta^{-1} \mathbf{u}_0, \\
\displaystyle\int_Q \mathbf{w} = 0, \quad \int_Q q = 0.
\end{array}
\right.
\tag{2.4}
$$

Here we note $\mathbf{w}\mathbf{w} = \mathbf{w} \otimes \mathbf{w} = (\mathbf{w}_i \mathbf{w}_j)_{1 \le i,j \le 3}$ and we recall that when $\nabla \cdot \mathbf{w} = 0$, one has $\nabla \cdot (\mathbf{w}\mathbf{w}) = (\mathbf{w}\nabla)\mathbf{w}$.

Results of [11] and [10] can be summerized as follows, where Let

$$
V = \left\{ \mathbf{v} \in L_x^2,\ \nabla \cdot \mathbf{v} = 0 \right\},
\tag{2.5}
$$

Theorem 2.1 *Assume that* $\mathbf{u}_0 \in V$. *Then the model* (2.4) *has a unique weak solution*

$$
(\mathbf{w}_\delta, q_\delta) \in [L_t^2(H_x^2) \cap L_t^\infty(H_x^1)] \times L_t^2(L_x^2)
\tag{2.6}
$$

and the energy equality holds for all $t > 0$*:*

$$
\left\{
\begin{array}{l}
\displaystyle\frac{1}{2} \int_Q (|\mathbf{w}_\delta(t,x)|^2 dx + \delta^2 \int_Q |\nabla \mathbf{w}_\delta(t,x)|^2)\, dx + \\
\qquad\qquad \displaystyle\nu \int_0^t \int_Q (|\nabla \mathbf{w}_\delta(t',x)|^2 + \delta^2 |\Delta \mathbf{w}_\delta(t',x)|^2)\, dx dt' = \\
\qquad\qquad\qquad\qquad \displaystyle\frac{1}{2} \int_Q (|\overline{\mathbf{u}_0}(x)|^2 + \delta^2 |\nabla \overline{\mathbf{u}_0}(x)|^2)\, dx,
\end{array}
\right.
\tag{2.7}
$$

where $\overline{\mathbf{u}_0} = A_\delta^{-1} \mathbf{u}_0$. *In addition if* $\mathbf{u}_0 \in V \cap H^{k-1}$ $(k \ge 1)$*, then*

$$
(\mathbf{w}_\delta, q_\delta) \in [L_t^2(H_x^{k+2}) \cap L_t^\infty(H_x^{k+1})] \times L_t^2(H_x^k).
\tag{2.8}
$$

2.3 Convergence towards Navier-Stokes equations

Recall one result proved in [11].

Theorem 2.2 *There is a sequence* $\delta_n \to 0$ *as* $n \to \infty$ *such that*

$$
(\mathbf{w}_{\delta_n}, q_{\delta_n}) \to (\mathbf{u}, p) \quad \textit{as} \quad n \to \infty
$$

where

$$
(\mathbf{u}, p) \in [L_t^\infty(L_x^2) \cap L_t^2(H_x^1)] \times L_t^{\frac{4}{3}}(L_x^2)
\tag{2.9}
$$

is a solution of the Navier-Stokes equations (1.1) *in the sense of the distributions. The sequence* $(w_{\delta_n})_{n \in I\!N}$ *converges strongly to* $\mathbf{u}$ *in* $L_t^2(L_x^2)$, a.e *and weakly in* $L_t^2(H_x^1)$ *while the sequence* $(q_{\delta_n})_{n \in I\!N}$ *converges weakly to* p *in the space* $L_t^{\frac{4}{3}}(L_x^2)$.

Definition 2.1 *We shall say that* $\mathbf{u} = \mathbf{u}(t,x)$ *is in the set* $\mathcal{V}(\mathbf{u}_0)$ *and* $p = p(t,x)$ *in the set* $\mathcal{P}(\mathbf{u}_0)$ *if and only if*

$$
(\mathbf{u}, p) \in [L_t^\infty(L_x^2) \cap L_t^2(H_x^1)] \times L_t^{\frac{4}{3}}(L_x^2)
$$

is a distributional solution to the Navier-Stokes equations (1.1) *and is a limit of a subsequence of the sequence* $(\mathbf{w}_\delta, q_\delta)$ *solution of* (2.4). *We shall say that* $(\mathbf{u}, p)$ *is a LES solution to the Navier-Stokes equations.*

Notice that Theorem 2.2 makes sure that $\mathcal{V}(\mathbf{u}_0) \neq \emptyset$, $\mathcal{P}(\mathbf{u}_0) \neq \emptyset$.

3 Classical estimates

3.1 Energy estimates

In the following one writes $\mathbf{u} = (u_1, u_2, u_3)$ and uses the convention of repeated index sommation. For a fixed $\mathbf{u}_0 \in V$, one puts

$$||\mathbf{u}_0||^2_{L^2_x} = E_0. \tag{3.1}$$

Given $\mathbf{u} = \mathbf{u}(t,x) \in \mathcal{V}(\mathbf{u}_0)$, one notes

$$E(t,\mathbf{u}) = \frac{1}{2}\int_Q |\mathbf{u}(t,x)|^2 dx + \nu \int_0^t \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt'. \tag{3.2}$$

Theorem 3.1 *Let $\mathbf{u}_0 \in V$, $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$. Then $t \rightarrow E(t,\mathbf{u})$ is a non increasing function of t and one has in particular*

$$\forall t \in I\!R, \quad E(t,\mathbf{u}) \leq E_0. \tag{3.3}$$

Corollary 3.1 *The following inequalities hold:*

$$||\mathbf{u}||_{L^\infty_t(L^2_x)} \leq \sqrt{E_0}, \tag{3.4}$$

$$||\mathbf{u}||_{L^2_t(H^1_x)} \leq \sqrt{\frac{E_0}{2\nu}}. \tag{3.5}$$

Proof of Theorem 3.1. The field $\mathbf{u}$ is limit of a subsequence of the sequence $(\mathbf{w}_\delta)_{\delta>0}$, subsequence still denoted by $(\mathbf{w}_\delta)_{\delta>0}$. One starts from the energy balance (2.7). Let

$$D_\delta(t) = \delta^2 \left(\int_Q |\nabla \mathbf{w}_\delta(t,x)|^2 dx + \nu \int_0^t \int_Q |\Delta \mathbf{w}_\delta(t',x)|^2 dx dt' \right).$$

Integrating (2.7) with respect to the time on the interval $[0,\tau]$ yields

$$\left\{ \begin{array}{l} \dfrac{1}{2}\displaystyle\int_0^\tau \int_Q |\mathbf{w}_\delta(t,x)|^2 dx dt + \nu \int_0^\tau \int_0^t \int_Q |\nabla \mathbf{w}_\delta(t',x)|^2 dx dt' dt + \int_0^\tau D_\delta(t)\, dt = \\ \dfrac{1}{2}\tau \displaystyle\int_Q (|\overline{\mathbf{u}_0}(x)|^2 + \delta^2 |\nabla \overline{\mathbf{u}_0}(x)|^2)\, dx, \end{array} \right. \tag{3.6}$$

where one has noted $\overline{\mathbf{u}_0} = A_\delta^{-1}\mathbf{u}_0$. One studies first the term $I_\delta = \delta^2 \int_Q |\nabla \overline{\mathbf{u}_0}(x)|^2 dx \geq 0$. Recall that one has

$$-\delta^2 \Delta \overline{\mathbf{u}_0} + \overline{\mathbf{u}_0} = \mathbf{u}_0. \tag{3.7}$$

Thus, taking $\overline{\mathbf{u}_0}$ as test function yields

$$\delta^2 \int_Q |\nabla \overline{\mathbf{u}_0}(x)|^2 dx + \int_Q |\overline{\mathbf{u}_0}(x)|^2 dx = \int_Q \overline{\mathbf{u}_0}(x).\mathbf{u}_0(x)\, dx. \tag{3.8}$$

By putting $\overline{\mathbf{u}_0} = \mathbf{u}_{0,\delta}$, (3.8) makes sure that the sequence $(\mathbf{u}_{0,\delta})_{\delta>0}$ is bounded in L^2_x. Thus its converges weakly (up to a subsequence) to some $\mathbf{g} \in L^2_x$. One takes a smooth test vector field $\mathbf{v}$ in (3.7) and one computes doing two part integrations on a cell. One has

$$-\delta^2 \int_Q \mathbf{u}_{0,\delta}.\Delta \mathbf{v} + \int_Q \mathbf{u}_{0,\delta}.\mathbf{v} = \int_Q \mathbf{u}_0.\mathbf{v}.$$

When δ goes to zero, one obtains $\displaystyle\int_Q \mathbf{u}_0.\mathbf{v} = \int_Q \mathbf{g}.\mathbf{v}$. Therefore, $\mathbf{g} = \mathbf{u}_0$. The limit being unique, all the sequence converges. One let δ go to zero in (3.8) and therefore

$$\liminf I_\delta + \liminf \int_Q |\mathbf{u}_{0,\delta}|^2 = \int_Q |\mathbf{u}_0|^2 = \limsup I_\delta + \limsup \int_Q |\mathbf{u}_{0,\delta}|^2.$$

On one hand, $\liminf I_\delta \geq 0$. On the other hand, $\displaystyle\liminf \int_Q |\mathbf{u}_{0,\delta}|^2 \geq \int_Q |\mathbf{u}_0|^2$. Consequently, each term being non negative,

$$\liminf I_\delta = \limsup I_\delta = 0 = \lim_{\delta\to 0} \delta^2 \int_Q |\nabla \overline{\mathbf{u}_0}(x)|^2 dx \tag{3.9}$$

Notice that in addition, one has proved that the sequence $(\mathbf{u}_{0,\delta})_{\delta>0}$ converges strongly towards $\mathbf{u}_0$ in L^2_x because of the weak convergence, combined to the convergence of the norms, consequence of the previous argument, that means

$$\lim_{\delta\to 0} \int_Q |\mathbf{u}_{0,\delta}(x)|^2 dx = \int_Q |\mathbf{u}_0(x)|^2 dx. \tag{3.10}$$

Let now consider the term $\displaystyle\int_0^\tau D_\delta(t)\,dt$ in equality (3.6). Notice first that $D_\delta(t) \geq 0$. Each term in the r.h.s of (3.6) being non negative, one can conclude that the sequence $(D_\delta(t))_{\delta>0}$ is bounded in $L^1([0,T])$, T being any non negative real number that one fixes until the end of the proof. Thus, up to a subsequence, it converges weakly in the sense of measures to a non negative Radon measure $\mu_1(t)$ and one has

$$\lim_{\delta\to 0} \int_0^\tau D_\delta(t)\,dt = \int_0^\tau d\mu_1(t). \tag{3.11}$$

Moreover, equality (2.7) combined with (3.9) and (3.10) makes sure that the sequence $(|\nabla \mathbf{w}_\delta|^2)_{\delta>0}$ is bounded in $L^1_{t,x}$. The weak convergence of $(\mathbf{w}_\delta)_{\delta>0}$ towards $\mathbf{u}$ in $L^2_t(H^1_x)$ guaranties the existence of a non negative Radon space periodic defect measure $\mu_2(t,x)$ such that

$$\left\{\begin{array}{l} \displaystyle\lim_{\delta\to 0} \int_0^\tau \int_0^t \int_Q |\nabla \mathbf{w}_\delta(t',x)|^2 dx dt' dt = \\ \displaystyle\int_0^\tau \int_0^t \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt' dt + \int_0^\tau \int_0^t \int_Q d\mu_2(t',x). \end{array}\right. \tag{3.12}$$

Finally, by the strong convergence of $(\mathbf{w}_\delta)_{\delta>0}$ towards $\mathbf{u}$ in $L^2_{t,x}$, one has

$$\lim_{\delta\to 0} \int_0^\tau \int_Q |\mathbf{w}_\delta(t,x)|^2 dx dt = \int_0^\tau \int_Q |\mathbf{u}(t,x)|^2 dx dt. \tag{3.13}$$

By putting together (3.9), (3.10), (3.11), (3.12), (3.13), one obtains (for $\tau \in [0,T]$),

$$\int_0^\tau E(t,\mathbf{u})\,dt + \int_0^\tau \int_0^t \int_Q d\mu_2(t',x) + \int_0^\tau d\mu_1(t) = \tau E_0, \tag{3.14}$$

yielding

$$E(t,\mathbf{u}) + \int_0^t \int_Q d\mu_2(t',x) + \mu_1(t) = E_0. \tag{3.15}$$

One deduces (3.3), $E(t,\mathbf{u}) \leq E_0$, thanks to the positiveness of μ_1 and μ_2. Notice that (3.15) does not depend on T and is true for each $t \in I\!\!R$. Inequalities (3.4) and (3.5) are direct consequences of (3.3). Classical results make sure that $\mathbf{u} \in C^0_t(L^2_{x,w})$. This tells that $\mathbf{u}$ is continuous with respect to the time into L^2_x equipped with its weak topology.

It remains to prove that $t \to E(t, \mathbf{u})$ is a non increasing function of t. Let $t_0 \in I\!R$. As $\mathbf{u} \in C^0_t(L^2_{x,w})$, one have $\mathbf{u}(t_0, x) \in L^2_x$, and $\mathbf{u}(t_0, x) \in V$. Then one can solve (1.1) by replacing $\mathbf{u}_0$ by $\mathbf{u}(t_0, x)$ in (2.4) for $t \geq t_0$. Clearly, when substituting 0 by t_0 one has for $t \geq t_0$, $\mathbf{u} \in \mathcal{V}(\mathbf{u}(t_0, x))$. The previous reasoning applies and then

$$\forall\, t \geq t_0, \quad \frac{1}{2}\int_Q |\mathbf{u}(t,x)|^2 dx + \nu \int_{t_0}^{t} \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt' \leq \int_Q |\mathbf{u}(t_0,x)|^2 dx. \tag{3.16}$$

Adding $\nu \displaystyle\int_0^{t_0} \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt'$ in both side of (3.16) yields $E(t, \mathbf{u}) \leq E(t_0, \mathbf{u})$ for each $t \geq t_0$, which means that $E(t, \mathbf{u})$ is a non increasing function of t, finishing the proof of Theorem 3.1.

Remark 3.1 *Beside the scope of the previous proof is a property of singular perturbation equations due to the operator* A_δ *for small values of* δ*. The reader may attempt that boundary layers can appear. But we are here in a periodic case and looking for* L^p *properties. This class of problems has been studied in Lions [15] in the Hilbert case. In our case, one can prove a more general result.*

Lemma 3.1 *Let* $\varphi \in L^p_x$, $1 \leq p < \infty$. *Then one have*

$$||A_\delta^{-1}\varphi||_{L^p_x} \leq ||\varphi||_{L^p_x}. \tag{3.17}$$

Moreover, when $p > 1$, $(A_\delta^{-1}\varphi)_{\delta>0}$ *converges towards* φ *strongly in* L^p_x.

Proof. Put $\overline{\varphi} = A_\delta^{-1}\varphi$. Recall that

$$-\delta^2 \Delta \overline{\varphi} + \overline{\varphi} = \varphi. \tag{3.18}$$

Take $\psi(\overline{\varphi}) = \overline{\varphi}|\overline{\varphi}|^{p-2}$ as test function in (3.18) when $p > 1$, $\psi(\overline{\varphi}) = \text{sgn}(\overline{\varphi})$ when $p = 1$ and integrate by part (eventually use truncations and pass to the limit; we skip here this kind of details, developed in a similar context in part 5). This yields to

$$\delta^2 \int_Q \psi'(\overline{\varphi})|\nabla \overline{\varphi}|^2 + \int_Q |\overline{\varphi}|^p = \int_Q \varphi \psi(\overline{\varphi}). \tag{3.19}$$

Because ψ is a non decreasing function, we can deduce from (3.19) that

$$\int_Q |\overline{\varphi}|^p \leq \int_Q \varphi \psi(\overline{\varphi}). \tag{3.20}$$

Inequality (3.17) is directly deduced from (3.20) when $p = 1$. Assume now that $p > 1$. Then (3.20) yields

$$\int_Q |\overline{\varphi}|^p \leq \int_Q |\varphi|\, |\overline{\varphi}|^{p-1}. \tag{3.21}$$

We use Hölder inequality in the r.h.s of (3.21). Then (3.21) becomes

$$||\overline{\varphi}||^p_{L^p_x} \leq ||\varphi||_{L^p_x} ||\overline{\varphi}||^{p-1}_{L^p_x}, \tag{3.22}$$

yielding (3.17). When $p > 1$, (3.17) tells that from the sequence $(A_\delta^{-1}\varphi)_{\delta>0}$, we can extract a subsequence (still denote by the same) which converges weakly in L^p_x towards some $g \in L^p_x$. In (3.18), take v a smooth test function and integrate by part:

$$-\delta^2 \int_Q \overline{\varphi}\Delta v + \int_Q v\overline{\varphi} = \int_Q v\varphi. \tag{3.23}$$

In (3.23), the term $\int \overline{\varphi}\Delta v$ converges towards $\int g\Delta v$ when δ goes to zero. Thus, when δ goes to zero, one have

$$\int_Q vg = \int_Q v\varphi. \tag{3.24}$$

Then, $g = \varphi$ *a.e.* The space L^p_x being uniformely convex for $p > 1$, we deduce from (3.18) the strong convergence in L^p_x. Finally, by uniqueness of the limit, all the sequence converges and the proof is complete.

3.2 Interpolation

Notice first that by Sobolev imbedding theorem, (3.5) yields

$$||\mathbf{u}||_{L^2_t(L^6_x)} \le C_{s6}\sqrt{\frac{E_0}{2\nu}}, \tag{3.25}$$

where C_{s6} is the Sobolev constant.

For the sake of the simplicity, one notes for $1 \le p \le \infty$ and $1 \le q \le \infty$,

$$n_{p,q} = ||\mathbf{u}||_{L^p_t(L^q_x)}.$$

Lemma 3.2 *One has*

$$\forall\, r \in [2,6], \quad n_{\frac{4r}{3(r-2)},r} \le n_{\infty,2}^{\frac{6-r}{2r}}\, n_{2,6}^{\frac{3(r-2)}{2r}}. \tag{3.26}$$

Corollary 3.2 *The following holds*

$$\forall\, r \in [2,6], \quad n_{\frac{4r}{3(r-2)},r} \le \frac{C_{s6}^{\frac{3(r-2)}{2r}}}{(2\nu)^{\frac{3(r-2)}{4r}}}\sqrt{E_0}. \tag{3.27}$$

Corollary 3.2 and in particular (3.27) follows from (3.26) combined with (3.25), (3.4) and (3.5)

Proof of lemma 3.2. Let $r \in [2,6]$ and write $r = 2\theta + 6(1-\theta)$. By Hölder inequality one has

$$\int_Q |\mathbf{u}|^r \le \int_Q \left(|\mathbf{u}|^2\right)^\theta \left(\int_Q |\mathbf{u}|^6\right)^{1-\theta} \le n_{\infty,2}^{2\theta}||\mathbf{u}||_{L^6_x}^{6(1-\theta)}. \tag{3.28}$$

Writing $\theta = \frac{6-r}{4}$ yields

$$\left(\int_Q |\mathbf{u}|^r\right)^{\frac{4}{3(r-2)}} \le n_{\infty,2}^{\frac{2(6-r)}{3(r-2)}}||\mathbf{u}||^2_{L^6_x}, \tag{3.29}$$

that is

$$||\mathbf{u}||_{L^r_x}^{\frac{4r}{3(r-2)}} \le n_{\infty,2}^{\frac{2(6-r)}{3(r-2)}}||\mathbf{u}||^2_{L^6_x}. \tag{3.30}$$

Inequality (3.26) is deduced from (3.30) after integrating with respect to the time and an easy algebraic calculation. In what follows, one puts

$$t(r) = \frac{4r}{3(r-2)}, \quad r \in [2,6]. \tag{3.31}$$

3.3 Regularity of the convective term

Lemma 3.3 *The convective term satisfies*

$$\forall\, r \in [2,6], \quad ||(\mathbf{u}\nabla)\mathbf{u}||_{L_t^{\frac{4r}{5r-6}}(L_x^{\frac{2r}{r+2}})} \le C_{s6}^{\frac{3(r-2)}{2r}} (2\nu)^{\frac{6-5r}{4r}} E_0. \tag{3.32}$$

Proof of corollary 3.3. An easy computation combined with Hölder inequality yields

$$||(\mathbf{u}\nabla)\mathbf{u}||_{L_t^{\frac{4r}{5r-6}}(L_x^{\frac{2r}{r+2}})} \le ||\mathbf{u}||_{L_t^{t(r)}(L_x^r)} ||\nabla\mathbf{u}||_{L_t^2(L_x^2)}.$$

Then (3.32) is a consequence of (3.5) combined to (3.27). Notice that in particular

$$(\mathbf{u}\nabla)\mathbf{u} \in L_t^1(L_x^{3/2}) \cap L_t^2(L_x^1).$$

4 On the equation for the pressure

4.1 Orientations

Let $\mathbf{u}_0 \in V$, $(\mathbf{u}, p) \in \mathcal{V}(\mathbf{u}_0) \times \mathcal{P}(\mathbf{u}_0)$. We start by classical estimates on p.

Next, because $(\mathbf{u}, p)$ is a LES solution, it is limit of a sequence $(\mathbf{w}_n, q_n)_{n\in I\!N}$ such that $(\mathbf{w}_n, q_n)$ is the unique solution to

$$\begin{cases} \partial_t \mathbf{w}_n + \nabla\cdot(A_{\delta_n}^{-1}(\mathbf{w}_n\mathbf{w}_n)) - \nu\Delta\mathbf{w}_n + \nabla q_n = 0, \\ \nabla\cdot\mathbf{w}_n = 0, \\ (\mathbf{w}_n)_{t=0} = A_{\delta_n}^{-1}\mathbf{u}_0, \end{cases} \tag{4.1}$$

where $(\delta_n)_{n\in I\!N}$ is a sequence of non negative numbers which converges to 0.

We seek in this section for more estimates for q_n when n is fixed. Next, we derive entropy estimates satisfied by the sequence $(q_n)_{n\in I\!N}$.

4.2 Direct estimates

We start with direct estimates on the pressure. Taking the divergence of the motion's equation in (1.1) yields the following equation for the pressure

$$-\Delta p = -\nabla\cdot((\mathbf{u}\nabla)\mathbf{u}). \tag{4.2}$$

Lemma 4.1 *One has*

$$\forall\, r \in [2,6], \quad ||p||_{L_t^{\frac{4r}{5r-6}}(W_x^{1,\frac{2r}{r+2}})} \le C_{s6}^{\frac{3(r-2)}{2r}} (2\nu)^{\frac{6-5r}{4r}} E_0. \tag{4.3}$$

Proof This is a direct consequence of the classical elliptic theory combined with (3.32) and the fact that we are working with periodic conditions. Notice that one has in particular

$$p \in L_t^1(W_x^{1,3/2}). \tag{4.4}$$

Remark 4.1 *Beside de scope of estimate* (4.3) *we have used the fact that* $-\Delta$ *is an isomorphism between* $W_x^{1,p}$ *and* $W_x^{-1,p}$*, a fact that we shall use again in the remainder. This is mainly due to regularity results in Agmon-Douglis-Nirenberg [1]. We mention also a proof of a similar result in Lions-Magenes. [16].*

4.3 Improved estimates for the approximations

One give here an improvement to the results of [11] needed for future applications in the paper. Fix n and denote by q_n the corresponding pressure in (2.4).

One first remark that for any non compressible field $\mathbf{v}$ (with $\nabla \cdot \mathbf{v} = 0$), one has

$$\nabla \cdot ((\mathbf{v}\nabla)\mathbf{v}) = \nabla \mathbf{v} : \nabla \mathbf{v}^t. \tag{4.5}$$

Therefore, equation (4.2) can be rewrited as

$$-\Delta p = -\nabla \mathbf{u} : \nabla \mathbf{u}^t \in L^1_{t,x}. \tag{4.6}$$

Lemma 4.2 *For a fixed n one have*

$$q_n \in L^\infty_t (W^{4,3}_x). \tag{4.7}$$

Proof. Thanks to (4.5) and (2.3), the equation for q_n can be written under the form

$$-\Delta q_n = A^{-1}_{\delta_n}(\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n). \tag{4.8}$$

Due to the regularity result (2.8), $\nabla \mathbf{w}_n \in L^\infty_t(H^1_x)$ and thus $\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n \in L^\infty_t(L^3_x)$. Consequently, $A^{-1}_{\delta_n}(\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n) \in L^\infty_t(W^{2,3}_x)$ and (4.7) is obvious.

When one combines (2.8) with (4.7), one obtains for a fixed n,

$$\partial_t \mathbf{w}_n \in L^2_t(H^1_x) \cap L^\infty_t(L^2_x). \tag{4.9}$$

4.4 Entropy estimates

We prove here lemme 1.1 announced in the introduction. Recall that

$$\mathcal{G} = \left\{ \begin{array}{l} g : I\!R \to I\!R, \text{ bounded Lipchitz with a finite number} \\ \text{of discontinuities ;} \quad (g')^2 = g, \quad g'' = 0 \end{array} \right\}. \tag{4.10}$$

Lemma 4.3 *For all $g \in \mathcal{G}$ one has*

$$||g(q_n)||_{L^2_t(H^1_x)} \le C||g||_\infty \frac{E_0}{2\nu}, \tag{4.11}$$

where C is a generic constant.

Proof. Thanks to the regularity property (4.7), all the manipulations below are justified, thanks to results in Stampacchia [20]. Take $A_{\delta_n} q_n = -\delta^2_n \Delta g(q_n) + g(q_n)$ as test function in (4.8). Notice that $\Delta g(q_n) = g'(q_n)\Delta q_n$ because of (**??**). Using the fact that A_δ is self-adjoint, this leads to

$$\int_Q g'(q_n)|\nabla q_n|^2 + \delta^2_n \int_Q g'(q_n)|\Delta q_n|^2 = \int_Q (\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n)\, g(q_n). \tag{4.12}$$

Because of (**??**),

$$\int_Q g'(q_n)|\Delta q_n|^2 = \int_Q |g'(q_n)\Delta q_n|^2 = \int_Q |\Delta g(q_n)|^2 \ge 0.$$

in the same way

$$\int_Q g'(q_n)|\nabla q_n|^2 = \int_Q |g'(q_n)\nabla q_n|^2 = \int_Q |\nabla g(q_n)|^2.$$

Finally,

$$\left|\int_Q (\nabla \mathbf{w}_n : \nabla \mathbf{w}_n^t)\, g(q_n)\right| \leq ||g||_\infty ||\nabla \mathbf{w}_n||^2_{L^2_x}.$$

Therefore, (4.12) yields

$$\int_Q |\nabla g(q_n)|^2 \leq ||g||_\infty ||\nabla \mathbf{w}_n||^2_{L^2_x}. \tag{4.13}$$

Estimate (4.11) is obtained when one integrates (4.13) with respect to the time and one uses (2.7), (3.9) and (3.10).

Remark 4.2 *When one takes $g = T_k$ (which satisfies (**??**)), one sees that the sequence $(T_k q_n)_{n\in I\!N}$ is bounded in $L^2_t(H^1_x)$. Therefore, one can extract from this sequence a subsequence which converges weakly in $L^2_t(H^1_x)$ to a function $\tilde{p}_k$. Arguing as in Murat [18], we can prove that there is a measurable function $\tilde{p}$ such that $\tilde{p}_k = T_k(\tilde{p})$. Despite the fact that $(q_n)_{n\in I\!N}$ converges weakly to p in $L^{4/3}_{t,x}$, we have no reason to claim that $p = \tilde{p}$. Indeed, if the sequence $(q_n)_{n\in I\!N}$ devellops high oscillations at infinity (what it probably does), then one can have $p \neq \tilde{p}$.*

5 The vorticity equation

5.1 Prelude on the vorticity and program

Let $\mathbf{u}_0 \in V$, $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$. One defines the vorticity by

$$\omega = \nabla \times \mathbf{u}. \tag{5.1}$$

Notice that one deduces from estimate (3.5) the natural estimate for the vorticity

$$||\omega||_{L^2_t(L^2_x)} \leq C\sqrt{\frac{E_0}{2\nu}}, \tag{5.2}$$

C being a generic constant.

Recall the classical formula for incompressible vector fields $\mathbf{v}$,

$$\nabla \times \nabla \cdot (\mathbf{v}\mathbf{v}) = (\mathbf{v}\nabla)(\nabla \times \mathbf{v}) - ((\nabla \times \mathbf{v})\nabla)\mathbf{v}. \tag{5.3}$$

Taking formally the *curl* of the motion equation in (1.1) yields the equation for ω (see for instance in Batchelor [3]):

$$\begin{cases} \partial_t \omega + (\mathbf{u}\nabla)\omega - \nu\Delta\omega = (\omega\nabla)\mathbf{u}, \\ \omega_{t=0} = \omega_0 = \nabla \times \mathbf{u}_0. \end{cases} \tag{5.4}$$

We shall focus our attention on the following equation:

$$\begin{cases} \partial_t \omega + (\mathbf{u}\nabla)\omega - \nu\Delta\omega = ((\nabla \times \mathbf{u})\nabla)\mathbf{u}, \\ \omega_{t=0} = \omega_0 = \nabla \times \mathbf{u}_0, \end{cases} \tag{5.5}$$

where ω is space periodic and have a zero mean value. We do not know if for any $\mathbf{u}_0 \in V$ and $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$ and a suitable assumption on ω_0, equation (5.5) admits a unique distributional solution ω which satisfies estimates (5.2) and is also solution to equation (5.4).

We assume throughout the remainder of the paper that

$$\mathbf{u}_0 \in \mathrm{H} = \{\mathbf{v} \in L^2_x, \ \ \nabla \times \mathbf{v} \in L^1_x\} \tag{5.6}$$

We first remark that (3.5) makes sure that

$$((\nabla \times \mathbf{u})\nabla)\mathbf{u} \in L^1_{t,x} \tag{5.7}$$

Then, (5.5) is an equation with "*a second hand side in* L^1".

We are now in poistion to give complete proofs to Theorem 1.1 and Theorem 1.2. We first recall the situation we are considering and then give the plan of the remainder.

Let $(\mathbf{u}, p) \in \mathcal{V}(\mathbf{u}_0) \times \mathcal{P}(\mathbf{u}_0)$. Thus it is a limit of a sequence $(\mathbf{w}_n, q_n)_{n\in I\!N}$ solution of

$$\begin{cases} \partial_t \mathbf{w}_n + \nabla \cdot (A^{-1}_{\delta_n}(\mathbf{w}_n\mathbf{w}_n)) - \nu\Delta\mathbf{w}_n + \nabla q_n = 0, \\ \nabla \cdot \mathbf{w}_n = 0, \\ (\mathbf{w}_n)_{t=0} = A^{-1}_{\delta_n}\mathbf{u}_0, \end{cases} \tag{5.8}$$

where δ_n goes to zero when n goes to infinity. Let

$$\omega_n = \nabla \times \mathbf{w}_n. \tag{5.9}$$

Using (2.3) and (5.3), we know that ω_n satisfies the equation

$$\begin{cases} \partial_t \omega_n + A^{-1}_{\delta_n}((\mathbf{w}_n\nabla)\omega_n) - \nu\Delta\omega_n = A^{-1}_{\delta_n}(((\nabla \times \mathbf{w}_n)\nabla)\mathbf{w}_n), \\ (\omega_n)_{t=0} = A^{-1}_{\delta_n}(\omega_0) = \nabla \times (A^{-1}_{\delta_n}\mathbf{u}_0). \end{cases} \tag{5.10}$$

Notice that by (2.8) and (4.9), we know that for fixed n, $\mathbf{w}_n \in L^2_t(H^3_x) \cap L^\infty_t(H^2_x)$, $\partial_t \mathbf{w} \in L^2_t(H^1_x)$. Thus one has for a fixed n,

$$\omega_n \in L^2_t(H^2_x) \cap L^\infty_t(H^1_x), \tag{5.11}$$

$$\partial_t \omega_n \in L^2_t(L^2_x). \tag{5.12}$$

In the following we

- prove the $L^\infty_t(L^1_x)$ bound (1.6),
- prove the entropy estimate (1.7) and the estimate (1.8) and finish the proof of Theorem 1.1,
- show compactness properties of the sequence $(\omega_n)_{n\in I\!N}$ and pass to the limit in sequence of equations (5.10) in order to finish the proof of Theorem 1.1,
- conclude by further remarks of the regularity of any LES solution.

5.2 $L^\infty_t(L^1_x)$ bound

Lemma 5.1 *The sequence* $(\omega_n)_{n\in I\!N}$ *is bounded in* $L^\infty_t(L^1_x)$ *and there exists a constant* C *such that*

$$||\omega_n||_{L^\infty_t(L^1_x)} \le C\zeta_n, \tag{5.13}$$

where $\lim_{n\to\infty} \zeta_n = \dfrac{E_0}{2\nu} + ||\omega_0||_{L^1_x}$.

Proof. Put $\omega_n = (\omega_n^1, \omega_n^2, \omega_n^3)$. Writing (5.10) component by component yields

$$\partial_t \omega_n^j + A^{-1}_{\delta_n}((\mathbf{w}_n\nabla)\omega_n^j) - \nu\Delta\omega_n^j = A^{-1}_{\delta_n}((\nabla\times\mathbf{w}_n)^i\partial_i \mathbf{w}_n^j). \tag{5.14}$$

Let $\varepsilon > 0$ and φ_ε the Lipchitz function defined on $I\!R$ by

$$\forall\ 0\le x\le\varepsilon,\ \varphi_\varepsilon(x) = \frac{x}{\varepsilon},\quad \forall\, x\ge\varepsilon,\ \varphi_\varepsilon(x)=1,\quad \varphi_\varepsilon(-x) = -\varphi_\varepsilon(x).$$

Let

$$\psi_\varepsilon(x) = \int_0^x \varphi_\varepsilon(x')\,dx'.$$

Take $A_{\delta_n}\varphi_\varepsilon(\omega_n^j) = -\delta^2\Delta\varphi_\varepsilon(\omega_n^j) + \varphi_\varepsilon(\omega_n^j)$ as test function in (5.10) and integrate by part. This operation makes sense thanks to the regularity properties (5.11) and (5.12). One obtains:

$$\left\{\begin{array}{l} \dfrac{d}{dt}\displaystyle\int_Q \psi_\varepsilon(\omega_n^j) - \delta_n^2\int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j + \int_Q A^{-1}_{\delta_n}((\mathbf{w}_n\nabla)\omega_n^j)A_{\delta_n}\varphi_\varepsilon(\omega_n^j) + \\ \nu\displaystyle\int_Q \varphi'_\varepsilon(\omega_n^j)|\nabla\omega_n^j|^2 + \nu\delta_n^2\int_Q \varphi'_\varepsilon(\omega_n^j)|\Delta\omega_n^j|^2 = \\ \displaystyle\int_Q A^{-1}_{\delta_n}((\nabla\times\mathbf{w}_n)^i\partial_i\mathbf{w}_n^j)A_{\delta_n}\varphi_\varepsilon(\omega_n^j). \end{array}\right. \tag{5.15}$$

We have used the fact that $\Delta\varphi_\varepsilon(\omega_n^j) = \varphi'_\varepsilon(\omega_n^j)\omega_n^j + \varphi''_\varepsilon(\omega_n^j)|\nabla\omega_n^j|^2$ and $\varphi''_\varepsilon = 0$. These computations are justified by the results in Stampacchia [20] combined with (5.11) and (5.12).

Because A_{δ_n} is self-adjoint and $\mathbf{w}_n$ has a zero divergence,

$$\int_Q A^{-1}_{\delta_n}((\mathbf{w}_n\nabla)\omega_n^j)A_{\delta_n}\varphi_\varepsilon(\omega_n^j) = \int_Q ((\mathbf{w}_n\nabla)\omega_n^j)\varphi_\varepsilon(\omega_n^j) = 0. \tag{5.16}$$

Moreover, because $|\varphi_\varepsilon|\le 1$, one has

$$\left\{\begin{array}{l} \left|\displaystyle\int_Q A^{-1}_{\delta_n}(((\nabla\times\mathbf{w}_n)^i\partial_i\mathbf{w}_n^j)A_{\delta_n}\varphi_\varepsilon(\omega_n^j)\right| = \\ \left|\displaystyle\int_Q ((\nabla\times\mathbf{w}_n)^i\partial_i\mathbf{w}_n^j)\varphi_\varepsilon(\omega_n^j)\right| \le C||\nabla\mathbf{w}_n||^2_{L^2_x}, \end{array}\right. \tag{5.17}$$

where C denotes a generic constant. Finally, one has

$$\int_Q \varphi'_\varepsilon(\omega_n^j)|\nabla\omega_n^j|^2 \ge 0,\quad \int_0^t\int_Q \varphi'_\varepsilon(\omega_n^j)|\Delta\omega_n^j|^2 \ge 0. \tag{5.18}$$

When one combines (5.15), (5.16), (5.17) and (5.18), one obtains

$$\frac{d}{dt}\int_Q \psi_\varepsilon(\omega_n^j) - \delta_n^2\int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j \le C||\nabla\mathbf{w}_n||^2_{L^2_x}. \tag{5.19}$$

One integrates (5.19) with respect to the time on the interval $[0,t]$:

$$\int_Q \psi_\varepsilon(\omega_n^j(t,x))\,dx - \delta_n^2 \int_0^t \int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j \le C||\nabla \mathbf{w}_n||^2_{L^2_x} + \int_Q \psi_\varepsilon((\omega_n^j)_0). \tag{5.20}$$

Note that it is easy seen that $\omega_n^j \in C^0_t(L^2_x)$ thanks to (5.12). Then because

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(x) = |x|, \quad 0 \le \psi_\varepsilon(x) \le C|x|,$$

it is obvious that

$$\lim_{\varepsilon\to 0}\int_Q \psi_\varepsilon(\omega_n^j(t,x))\,dx = \int_Q |\omega_n^j(t,x)|\,dx, \quad \lim_{\varepsilon\to 0}\int_Q \psi_\varepsilon((\omega_n^j)_0)\,dx = ||(\omega_n^j)_0||_{L^1_x}. \tag{5.21}$$

We now consider the term $-\delta_n^2 \int_0^t \int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j$ in (5.23). Because $\varepsilon(\varphi'_\varepsilon)^2 = \varphi'_\varepsilon$,

$$\int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j = \varepsilon\int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\varphi'_\varepsilon(\omega_n^j)\Delta\omega_n^j = -\varepsilon\frac{d}{2dt}\int_Q |\nabla\varphi_\varepsilon(\omega_n^j)|^2. \tag{5.22}$$

Hence (5.23) yields

$$\left\{\begin{array}{l} \int_Q \psi_\varepsilon(\omega_n^j(t,x))\,dx + \delta_n^2\varepsilon\int_Q |\nabla\varphi_\varepsilon(\omega_n^j)|^2 \le \\ C||\nabla \mathbf{w}_n||^2_{L^2_x} + \int_Q \psi_\varepsilon((\omega_n^j)_0) + \delta_n^2\varepsilon\int_Q |\nabla\varphi_\varepsilon((\omega_n^j))_0|^2. \end{array}\right. \tag{5.23}$$

which yields

$$\int_Q \psi_\varepsilon(\omega_n^j(t,x))\,dx \le C||\nabla \mathbf{w}_n||^2_{L^2_x} + \int_Q \psi_\varepsilon((\omega_n^j)_0) + \delta_n^2\varepsilon\int_Q |\nabla\varphi_\varepsilon((\omega_n^j))_0|^2. \tag{5.24}$$

Assume first that $|(\omega^j))_0| \ge \rho > 0$ and $(\omega^j))_0$ is continuous. Hence by the maximum principle, $|(\omega_n^j))_0| \ge \rho > 0$. When one let ε go to zero in (5.24) by using (5.21), one has for such initial datas

$$\int_Q |\omega_n^j(t,x)|\,dx \le C||\nabla \mathbf{w}_n||^2_{L^2_t(L^2_x)} + ||(\omega_n^j)_0||_{L^1_x}. \tag{5.25}$$

This inequality remains true for every initial datas by a density argument. Then (5.13) is a consequence of (5.25) combined to (2.7), (3.9) and (3.10). The proof of Lemma 5.1 is complete as well as estimate (1.6) is proved.

5.3 Entropy inequalities and consequences

The goal of this part is the proof of entropy inequalities in order to prove (1.7) and its consequence (1.8).

Let

$$g \in \mathcal{G} = \left\{\begin{array}{l} g : I\!\!R \to I\!\!R, \text{ bounded Lipchitz with a finite number} \\ \text{of discontinuities ;} \quad (g')^2 = g, \quad g'' = 0 \end{array}\right\}, \tag{5.26}$$

$$G(x) = \int_0^x g(x')\,dx'. \tag{5.27}$$

If $\omega = (\omega^1, \omega^2, \omega^3)$, $g(\omega)$ is the vector field $(g(\omega^1), g(\omega^2), g(\omega^3))$. We shall note in the following

$$B^k_{n,j} = \left\{ (t,x) \in I\!R \times Q;\; k \leq |\omega^j_n(t,x)| \leq k+1 \right\}. \tag{5.28}$$

Moreover, T_k denotes the truncation function at height k, that is

$$T_k(x) = x \;\text{ if } |x| \leq k, \quad T_k(x) = k\frac{x}{|x|} \;\text{ if } |x| \geq k. \tag{5.29}$$

The main result of this part is the following.

Lemma 5.2 *The sequence* $(\omega^j_n)_{n\in I\!N}$ *satisfies for all* t

$$\left\{ \begin{array}{l} \displaystyle\int_Q G(\omega^j_n(t,x))\,dx \;+ \delta_n^2 \int_Q |\nabla g(\omega^j_n(t,x))|^2 dx + \\ \displaystyle\nu \int_0^t \int_Q g'(\omega^j_n(t',x)) |\nabla \omega^j_n(t',x)|^2 dx dt' + \nu\delta_n^2 \int_0^t \int_Q |\Delta g(\omega^j_n(t,x))|^2 dx dt' \leq \\ \displaystyle C||g||_\infty E_0 + \int_Q G(\overline{\omega_0}^j(x))\,dx + \delta_n^2 \int_Q |\nabla g(\overline{\omega_0}^j(x)|^2 dx, \end{array} \right. \tag{5.30}$$

where $\overline{\omega_0}^j = {A_{\delta_n}}^{-1}{\omega_0}^j$.

Lemma 5.3 *One has*

$$\delta_n^2 \int_Q |\nabla g(\overline{\omega_0}^j(x)|^2 dx \leq 2||g||_\infty ||\omega^j_0||_{L^1_x}. \tag{5.31}$$

When one remarks that when $g \in \mathcal{G}$,

$$\int_0^t \int_Q g'(\omega^j_n(t',x)) |\nabla \omega^j_n(t',x)|^2 dx dt' = \int_0^t \int_Q |\nabla g(\omega^j_n(t',x))|^2 dx dt',$$

one deduces from (5.30) combined with Lemme 5.3 that

$$||g(\omega_n)||_{L^2_t(H^1_x)} \leq C(||\mathbf{u}_0||_{L^2_x}, ||\omega_0||_{L^1_x}), \tag{5.32}$$

which is estimate (1.7) announced in the introduction.

Corollary 5.1 *The following inequalities hold*

$$\forall\, k > 0, \quad ||T_k(\omega^j_n)||^2_{L^2_x(H^1_x)} \leq CkE_0 + k^2 mes(Q) + (2k+1)||\omega^j_0||_{L^1_x}, \tag{5.33}$$

$$\forall\, k > 0, \quad \int\int_{B^k_{n,j}} |\nabla \omega^j_n|^2 \leq CE_0 + 2||\omega^j_0||_{L^1_x} + mes(Q), \tag{5.34}$$

$$\forall\, p < 5/4, \quad ||\omega^j_n||_{L^p_t(W^{1,p}_x)} \leq C(p, E_0, ||\omega^j_0||_{L^1_x}), \tag{5.35}$$

where $C(p, E_0, ||\omega^j_0||_{L^1_x})$ *goes to infinity when* p *goes to* $5/4$.

Proof of Lemma 5.2. Due to the regularity results (5.11) and (5.12) for a fixed n, the fact that g is Lipchitz with a derivative with a finite number of discontinuity, the results in Stampacchia [20] apply and validate all the following manipulations below.

In equation (5.14) take $A_{\delta_n} g(\omega^j_n) = -\delta_n^2 \Delta g(\omega^j_n) + g(\omega^j_n)$ as test function. Notice that because $g \in \mathcal{G}$, $\Delta g(\omega^j_n) = g'(\omega^j_n)\Delta\omega^j_n$. Moreover, as we have several used before by the fact that A_{δ_n} is self-adjoint,

$$\int_Q A^{-1}_{\delta_n}((\nabla \times \mathbf{w}_n)^i \partial_i \mathbf{w}^j_n) A_{\delta_n} g(\omega^j_n) = \int_Q (\nabla \times \mathbf{w}_n)^i \partial_i \mathbf{w}^j_n g(\omega^j_n). \tag{5.36}$$

Finally, because $\nabla\cdot\mathbf{w}_n = 0$,

$$\int_Q A_{\delta_n}^{-1}((\mathbf{w}_n\nabla)\omega_n^j)A_{\delta_n}g(\omega_n^j) = \int_Q (\mathbf{w}_n\nabla)\omega_n^j g(\omega_n^j) = 0. \tag{5.37}$$

Then one obtains

$$\left\{\begin{array}{l} \dfrac{d}{dt}\displaystyle\int_Q G(\omega_n^j) - \delta_n^2\int_Q g'(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j + \nu\int_Q g'(\omega_n^j)|\nabla\omega_n^j|^2 + \\ \delta_n^2\nu\displaystyle\int_Q g'(\omega_n^j)|\Delta(\omega_n^j)|^2 \le ||g||_\infty||\nabla\mathbf{w}_n||_{L_x^2}^2. \end{array}\right. \tag{5.38}$$

Because $g\in\mathcal{G}$, $g' = (g')^2$. Therefore,

$$\left\{\begin{array}{l} \displaystyle\int_Q g'(\omega_n^j)\,\partial_t\omega_n^j\,\Delta\omega_n^j = \int_Q g'(\omega_n^j)\,\partial_t\omega_n^j\,g'(\omega_n^j)\Delta\omega_n^j\; = \\ \displaystyle\int_Q g'(\omega_n^j)\,\partial_t\omega_n^j\,\Delta g(\omega_n^j) = \int_Q \partial_t g(\omega_n^j)\,\Delta g(\omega_n^j) = -\frac{d}{dt}\int_Q|\nabla g(\omega_n^j)|^2. \end{array}\right. \tag{5.39}$$

For the same reason,

$$\int_Q g'(\omega_n^j)|\Delta(\omega_n^j)|^2 = \int_Q g'(\omega_n^j)\Delta(\omega_n^j)g'(\omega_n^j)\Delta(\omega_n^j) = \int_Q|\Delta(g(\omega_n^j))|^2. \tag{5.40}$$

When one combines (5.38), (5.39) and (5.40) one obtains

$$\left\{\begin{array}{l} \dfrac{d}{dt}\displaystyle\int_Q G(\omega_n^j) + \delta_n^2\frac{d}{dt}\int_Q|\nabla g(\omega_n^j)|^2 + \nu\int_Q g'(\omega_n^j)|\nabla\omega_n^j|^2 + \\ \delta_n^2\nu\displaystyle\int_Q|\Delta(g(\omega_n^j))|^2 \le ||g||_\infty||\nabla\mathbf{w}_n||_{L_x^2}^2. \end{array}\right. \tag{5.41}$$

Inequality (5.30) follows by integrating (5.35) with respect to the time and by using (2.7), (3.9) and (3.10).

Proof of Lemma 5.3. Recall that

$$-\delta_n^2\Delta\overline{\omega_0}^j + \overline{\omega_0}^j = \omega_0^j. \tag{5.42}$$

Taking $g(\overline{\omega_0}^j)$ as test function in (5.42) yield

$$\delta_n^2\int_Q g'(\overline{\omega_0}^j)|\nabla\overline{\omega_0}^j|^2 + \int_Q g(\overline{\omega_0}^j)\overline{\omega_0}^j = \int_Q g(\overline{\omega_0}^j)\omega_0^j \tag{5.43}$$

Always because $g\in\mathcal{G}$, (5.43) yields

$$\delta_n^2\int_Q|\nabla g(\overline{\omega_0}^j)|^2 \le ||g||_\infty\left(\int_Q|\overline{\omega_0}^j| + \int_Q|\omega_0^j|\right). \tag{5.44}$$

Then (5.31) is a consequence of (5.44) combined to (3.17) in Lemma 3.1.

Remark 5.1 *We do not know if*

$$\lim_{n\to\infty}\delta_n^2\int_Q|\nabla g(A_{\delta_n}^{-1}\omega_0{}^j)|^2 = 0.$$

Proof of Corollary 5.1. Inequality (5.33) follows from (5.30) by taking $g(x) = T_k(x)\in\mathcal{G}$. Inequality (5.34) is obtained when one takes the odd function $g = g_k\in\mathcal{G}$ be such that $g_k(x) = 0$ for $0\le x\le k$, $g_k(x) = x-k$ for $k\le x\le k+1$ and $g_k(x) = 1$ for $x\ge k+1$. The inequalities are consequences of (5.13), (5.31), (3.17) and the reasonning made in Lewandowski [13] page 131 and 132. Finally, (5.35) is a consequence of (5.33) and (5.34) combined with the Boccardo-Gallouët's inequality [6].

5.4 Passing to the limit : proof of theorem 1.2

We are now in position to prove Theorem 1.1 that we recall below

Theorem 5.1 *The sequence* $(\omega_n)_{n\in\mathbb{N}}$ *satisfies the following properties:*
(i) $(\omega_n)_{n\in\mathbb{N}}$ *converges weakly to* $\omega = \nabla\times\mathbf{u}$ *in* $L^p_t(W^{1,p}_x)$ *for all* $p < 5/4$*, strongly in* $L^1_{t,x}$ *and almost everywhere in space and time,*
(ii) the sequence of equations (5.10) *converges in the sense of the distributions to the equation*

$$\begin{cases} \partial_t\omega + (\mathbf{u}\nabla)\omega - \nu\Delta\omega = ((\nabla\times\mathbf{u}\nabla))\mathbf{u}, \\ \omega_{t=0} = \omega_0 = \nabla\times\mathbf{u}_0 \in L^1_x, \end{cases} \tag{5.45}$$

(iii) $\omega = \nabla\times\mathbf{u}$ *has the following regularity:*

$$\omega \in L^\infty_t(L^1_x), \tag{5.46}$$

$$\forall\, g \in \mathcal{G}, \quad g(\omega) \in L^2_t(H^1_x), \tag{5.47}$$

$$\omega \in \bigcap_{p<5/4} L^p_t(W^{1,p}). \tag{5.48}$$

We have to prove compactness properties for the sequence $(\omega_n)_{n\in\mathbb{N}}$ and pass to the limit in equation (5.10) to prove that ω is a distributional solution to (5.5) and show that (5.46), (5.47) and (5.48) hold.

We do this task step by step.

Step 1: Weak compactness in $L^p_t(W^{1,p}_x)$. Thanks to (5.35) and arguing like in [13], one can extract from the sequence $(\omega_n)_n$ a subsequence (still denoted by the same) such that $(\omega_n)_n$ converges weakly to some $\tilde\omega$ in $L^p_t(W^{1,p}_x)$ for all $p < 5/4$. Because $\omega_n = \nabla\times\mathbf{w}_n$, we also know that $(\omega_n)_{n\in\mathbb{N}}$ converges weakly to ω in $L^2_{t,x}$. Thus $\tilde\omega = \omega$. In particular (5.48) is proved. Moreover, $(-\Delta\omega_n)_{n\in\mathbb{N}}$ converges weakly to $-\Delta\omega$ in $L^p(W^{-1,p})$ for each $p < 5/4$. Notice that it is obvious here that all the sequence is conecerned by uniqueness of the limit.

Step 2: Strong compactness property. We have in view to estimate $(\partial_t\omega_n)_{n\in\mathbb{N}}$. Let consider now the transport term. For φ a smooth test vector field, one has

$$< A_\delta^{-1}((\mathbf{w}_n\nabla)\omega_n), \varphi >= \int_Q \nabla\cdot A_{\delta_n}^{-1}(\mathbf{w}_n\omega_n)\varphi = -\int_Q A_{\delta_n}^{-1}(\mathbf{w}_n\omega_n)\nabla\varphi.$$

Combining (3.17), (3.27) and (5.2), one remarks that the sequence $(A_{\delta_n}^{-1}(\mathbf{w}_n\omega_n))_{n\in\mathbb{N}}$ is bounded in $L^{8/7}_t(L^{3/4}_x)$. Thus, $(A_{\delta_n}^{-1}((\mathbf{w}_n\nabla)\omega_n))_{n\in\mathbb{N}}$ is bounded in $L^{8/7}_t(W^{-1,3/4}_x)$. Let $\varepsilon > 0$. Because of the strong compactness of $(\mathbf{w}_n)_{n\in\mathbb{N}}$ in $L^{8/3-\varepsilon}_t(L^{4-\varepsilon}_x)$ and the weak compactness of $(\omega_n)_{n\in\mathbb{N}}$, it is easy seen that $(A_{\delta_n}^{-1}((\mathbf{w}_n\nabla)\omega_n))_{n\in\mathbb{N}}$ converges weakly in $L^{8/7-\varepsilon}_t(W^{-1,3/4+\varepsilon}_x)$ to $(\mathbf{u}\nabla)\omega$.

Thanks to the $L^1_{t,x}$ bound of $(A_{\delta_n}^{-1}(((\nabla\times\mathbf{w}_n)\nabla)\mathbf{w}_n))_{n\in\mathbb{N}}$, we deduce from the considerations above that $(\partial_t\omega_n)_{n\in\mathbb{N}}$ is bounded in $L^1(W^{-3,p})$ for some $p > 1$. Therefore, by using an Aubin-Lions lemma adapted to the L^1 time case (see in [13]), we know that $(\omega_n)_{n\in\mathbb{N}}$ is compact in $L^1_{t,x}$. Thererefore, from $(\omega_n)_{n\in\mathbb{N}}$ one can extract a subsequence (still denoted by the same) which converges *a.e.* in space-time to ω a and with its modulus dominated by a $L^1_{t,x}$-function. Notice that at this step, point (i) in Theorem 1.2 is proved.

Step 3: $L^\infty_t(L^1_x)$ *bound for the limit.* Now, there exists sets $A_t \subset \mathbb{R}$ and $A_x \subset Q$, with $mes(A^c_x) = mes(A^c_t) = 0$ and for all $(t,x) \in A_t\times A_x$, $(\omega_n(t,x))_{n\in\mathbb{N}}$ converges to $\omega(t,x)$.

Therefore, when one fixes $t \in A_t$, $(\omega_n(t,\cdot))_{n\in\mathbb{N}}$ converges *a.e.* to $\omega(t,\cdot)$. By Fatou's Lemma combined with (5.13) one has for almost every time t,

$$\int_Q |\omega(t,\cdot)| \leq \liminf_{n\in\mathbb{N}} \int_Q |\omega_n(t,\cdot)| \leq C\left(\frac{E_0}{2\nu} + ||\omega_0||_{L^1_x}\right).$$

Thereore, $\omega \in L^\infty_t(L^1_x)$ and one has

$$||\omega||_{L^\infty_t(L^1_x)} \leq C\left(\frac{E_0}{2\nu} + ||\omega_0||_{L^1_x}\right). \tag{5.49}$$

Then (5.46) is proved.

Step 4: $L^2_t(H^1_x)$ bound for the entropies. Let $g \in \mathcal{G}$. Thanks to (5.32), one knows that the sequence $(g(\omega_n))_{n\in\mathbb{N}}$ is bounded in $L^2_t(H^1_x)$. Thus from this sequence one can extract a subsequence wich weakly converges in $L^2_t(H^1_x)$ to a vector field $\tilde{g} \in L^2_t(H^1_x)$. But, one also know from the above arguments that $(g(\omega_n))_{n\in\mathbb{N}}$ converges *a.e.* to $g(\omega)$, and therefore thanks to Lebesgue's Theorem, strongly in $L^2_{x,t}$ $(|g(\omega_n)| \leq ||g||_\infty \in L^\infty(\mathbb{R}\times Q))$. Therefore, $\tilde{g} = g(\omega) \in L^2_t(H^1_x)$ and (5.47) is proved. This finishes the proof of point (iii) in Theorem 1.2.

Step 5: Passing to the limit in the equation in the sense of the distributions. One treats each term after each other in (5.10).

It is now easy seen that $(\partial_t\omega_n)_{n\in\mathbb{N}}$ converges to $\partial_t\omega$ in the distributional sens. It remains to treat the source term in the equations.

Remark now that for φ a smooth test vector field, one has

$$< A^{-1}_{\delta_n}(((\nabla\times\mathbf{w}_n)\nabla)\mathbf{w}_n), \varphi >= \int_Q \nabla\cdot A^{-1}_{\delta_n}((\nabla\times\mathbf{w}_n)\mathbf{w}_n).\varphi = -\int_Q A^{-1}_{\delta_n}((\nabla\times\mathbf{w}_n)\mathbf{w}_n) : \nabla\varphi.$$

Thus, the same argument as above applies, and one concludes that $(A^{-1}_{\delta_n}(((\nabla\times\mathbf{w}_n)\nabla)\mathbf{w}_n))_{n\in\mathbb{N}}$ converges to $(\nabla\times\mathbf{u})\nabla)\mathbf{u} = (\omega\nabla)\mathbf{u}$ in the sense of the distribution.

Considerations above yields the following conclusion when one passes to the limit in (5.10): $\forall\,\varphi$ periodic, with C^∞ class in space and time and with compact support in time,

$$< \partial_t\omega, \varphi > + < (\mathbf{u}\nabla)\omega, \varphi > +\nu\int_0^\infty\int_Q \nabla\omega : \nabla\varphi = \int_0^\infty\int_Q (\omega\nabla)\mathbf{u}.\varphi. \tag{5.50}$$

Step 6: Initial data. To finish the proof of Theorem 1.2 (point (ii)), one has to deal with the initial data. Notice that we cannot make sure that the sequence $(A^{-1}_{\delta_n}\omega_0)_{n\in\mathbb{N}}$ converges strongly in L^1 to ω. We do not need this information. Indeed, let φ a C^∞ in space-time vector field, but not with time compact support and such that $\varphi(T,\cdot) = 0$ for some $T > 0$. One has

$$< \partial_t\omega_n, \varphi >= \int_Q A^{-1}_{\delta_n}\omega_0\,.\varphi - \int_0^T\int_Q \omega_n.\partial_t\varphi.$$

Of course,

$$\lim_{n\to\infty}\int_0^T\int_Q \omega_n.\,\partial_t\varphi = \int_0^T\int_Q \omega.\,\partial_t\varphi.$$

Remark now that

$$\int_Q A^{-1}_{\delta_n}\omega_0\,.\,\varphi = -\int_Q A^{-1}_{\delta_n}\mathbf{u}_0\,.\nabla\times\varphi.$$

Thanks to Lemma 3.1 and $\mathbf{u}_0 \in L^2_x$, $(A^{-1}_{\delta_n}\mathbf{u}_0)_{n\in\mathbb{N}}$ converges strongly to $\mathbf{u}_0$ in L^2_x. Therefore,

$$\lim_{n\to\infty} \int_Q A^{-1}_{\delta_n}\mathbf{u}_0 \,.\nabla \times \varphi = \int_Q \mathbf{u}_0 \,.\nabla \times \varphi = -\int_Q \omega_0.\varphi,$$

yielding

$$\lim_{n\to\infty} < \partial_t \omega_n, \varphi >= \int_Q \omega_0.\varphi - \int_0^T \int_Q \omega.\, \partial_t \varphi.$$

Now we are totaly sure that ω is a distributional solution to (5.5) realized as limit of the approximations (5.10). The proof of Theorem 1.2 is now complete.

5.5 Additional regularity property of LES solutions

We note now that

$$\nabla \times \nabla \times \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \Delta \mathbf{v}. \tag{5.51}$$

Thus, combined with the incompressible constrain, (5.1) yields

$$-\Delta \mathbf{u} = \nabla \times \omega. \tag{5.52}$$

Therefore, as a consequence of (5.48), for any $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$, one has

$$\mathbf{u} \in \bigcap_{p<5/4} L^p_t(W^{2,p}_x).$$

Remark also that by (4.3), for any $p \in \mathcal{P}(\mathbf{u}_0)$, $p \in L^{5/4}_t(W^{1,5/4}_x)$. Therefore, one deduces

$$\partial_t u \in \bigcap_{p<5/4} L^p_{t,x}.$$

Remark 5.2 *Thanks to the hypothesis "$\omega_0 \in L^1_x$", we have gain 2 space-derivative for the velocity but we did not have gain anything for the pressure, we mean better than* (4.3)*. This is due to the fact that we did not have gain enough regularity in time.*

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